

DIRECTED POLYMERS AND THE QUANTUM TODA LATTICE

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We characterize the law of the partition function of a Brownian directed polymer model in terms of a diffusion process associated with the quantum Toda lattice. The proof is via a multidimensional generalization of a theorem of Matsumoto and Yor concerning exponential functionals of Brownian motion. It is based on a mapping which can be regarded as a geometric variant of the RSK correspondence.

1. Introduction. Let $B_1(t), B_2(t), \dots, B_N(t), t \geq 0$, be a collection of independent standard one-dimensional Brownian motions and write $B_i(s, t) = B_i(t) - B_i(s)$ for $s \leq t$. Let $\beta \in \mathbb{R}, t \geq 0$, and consider the random variable

$$Z_t^N(\beta) = \int_{0 < s_1 < \dots < s_{N-1} < t} e^{\beta(B_1(s_1) + B_2(s_1, s_2) + \dots + B_N(s_{N-1}, t))} ds_1 \dots ds_{N-1}.$$

This is the partition function for a model of a 1 + 1 dimensional directed polymer in a random environment which has been introduced and studied in the papers [38, 42, 54]. The free energy density is given explicitly by

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log Z_N^N(\beta) = \inf_{t > 0} [\beta^2 t - \Psi(t)] - \log \beta^2$$

almost surely, where $\Psi(z) = \Gamma'(z) / \Gamma(z)$. The law of $Z_t^N(\beta)$ is well understood in the zero temperature limit $\beta \rightarrow \infty$, where it has close connections with random matrices. Define

$$(1) \quad \begin{aligned} M_t^N &= \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log Z_t^N(\beta) \\ &= \max_{0 \leq s_1 \leq \dots \leq s_{N-1} \leq t} (B_1(s_1) + B_2(s_1, s_2) + \dots + B_N(s_{N-1}, t)). \end{aligned}$$

Note that, by Brownian scaling, the law of M_t^N / \sqrt{t} is independent of t .

THEOREM 1.1. *The random variable M_1^N has the same distribution as the largest eigenvalue of a $N \times N$ GUE random matrix [2, 23]. In fact [7, 43], the stochastic process $(M_t^N, t \geq 0)$ has the same law as the largest eigenvalue of a*

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standard Hermitian Brownian motion, that is, it has the same law as the first coordinate of a Brownian motion conditioned (in the sense of Doob) never to exit the Weyl chamber $C_N = \{x \in \mathbb{R}^N : x_1 > \dots > x_N\}$, started from the origin. This is a diffusion process in \overline{C}_N with infinitesimal generator $\Delta/2 + \nabla \log h \cdot \nabla$ where

$$(2) \quad h(x) = \prod_{1 \leq i < j \leq N} (x_i - x_j).$$

This connection with random matrices yields very precise information concerning the distribution and asymptotic behavior of M^N when N is large. For example, it follows that

$$\lim_{N \rightarrow \infty} P(M_N^N \leq 2N + xN^{1/3}) = F_2(x),$$

where F_2 is the Tracy–Widom distribution [57].

In this paper we obtain an analogue of Theorem 1.1 for the stochastic process $(\log Z_t^N(\beta), t > 0)$. We will show that, for each $\beta > 0$, this process has the same law as the first coordinate of a diffusion process in \mathbb{R}^N which is closely related to the quantum Toda lattice. This yields an analytic description of the law of $Z_t^N(\beta)$ which should provide a good starting point for further asymptotic analysis.

2. The quantum Toda lattice. The quantum Toda lattice is a quantum integrable system with Hamiltonian given by the Schrödinger operator

$$(3) \quad H = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} - 2 \sum_{i=1}^{N-1} e^{x_{i+1} - x_i}.$$

It is closely related to the Lie algebra \mathfrak{gl}_N : the exponents in the potential correspond to the simple roots $e_i - e_{i+1}$, where e_1, \dots, e_N denote the standard basis elements in \mathbb{R}^N . More generally, the quantum Toda lattice associated with a real split semisimple (or reductive) Lie algebra \mathfrak{g} with Cartan subalgebra \mathfrak{a} has Hamiltonian given by

$$\Delta_{\mathfrak{a}} - 2 \sum_{\alpha \in \Pi} d_{\alpha} e^{-\alpha(x)},$$

where $\Delta_{\mathfrak{a}}$ is the Laplacian on \mathfrak{a} , Π is a set of simple roots in \mathfrak{a}^* and d_{α} are rational numbers with a particular property [21]. For example, if $\mathfrak{g} = \mathfrak{so}_{2N+1}$, then we can identify \mathfrak{a} with \mathbb{R}^N , take

$$\Pi = \{e_1 - e_2, e_2 - e_3, \dots, e_{N-1} - e_N, e_N\},$$

and the corresponding Hamiltonian is given by

$$\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} - 2 \sum_{i=1}^{N-1} e^{x_{i+1} - x_i} - e^{-x_N}.$$

The connection between the (generalized) quantum Toda lattice and the representation theory of the corresponding Lie algebra \mathfrak{g} was first observed by Kostant [35], who showed that its eigenfunctions can be represented as particular matrix elements of infinite-dimensional representations of \mathfrak{g} . In the simplest case when $\mathfrak{g} = \mathfrak{sl}_2$ or \mathfrak{gl}_2 , the eigenfunctions are given in terms of classical Whittaker functions (actually Macdonald functions). For this reason, they are often called \mathfrak{g} -Whittaker functions, or G -Whittaker functions, if $\mathfrak{g} = \text{Lie}(G)$. They also arise in the harmonic analysis of automorphic forms on Lie groups (see, e.g., [8]). There is a spectral decomposition theorem in the general setting due to Semenov–Tian–Shansky [53]. In this paper we will only consider the case $\mathfrak{g} = \mathfrak{gl}_N$. However, many of the constructions given throughout the paper have Lie-theoretic interpretations and extend to the more general setting. This will be indicated where appropriate.

The eigenfunctions of H have the following integral representation [19, 22, 26, 28]:

$$(4) \quad \psi_\lambda(x) = \int_{\Gamma(x)} e^{\mathcal{F}_\lambda(T)} \prod_{k=1}^{N-1} \prod_{i=1}^k dT_{k,i},$$

where $\Gamma(x)$ denotes the set of real triangular arrays $(T_{k,i}, 1 \leq i \leq k \leq N)$ with $T_{N,i} = x_i, 1 \leq i \leq N$, and

$$\mathcal{F}_\lambda(T) = \sum_{k=1}^N \lambda_k \left(\sum_{i=1}^k T_{k,i} - \sum_{i=1}^{k-1} T_{k-1,i} \right) - \sum_{k=1}^{N-1} \sum_{i=1}^k (e^{T_{k,i} - T_{k+1,i}} + e^{T_{k+1,i+1} - T_{k,i}}).$$

This integral is defined for $\lambda \in \mathbb{C}^N$ and has a recursive structure which we will now describe. Write $H = H^{(N)}, \psi_\lambda = \psi_\lambda^{(N)}$. We will drop these superscripts again later, whenever they are unnecessary. For convenience we define $H^{(1)} = d^2/dx^2$ and $\psi_\lambda^{(1)}(x) = e^{\lambda x}$. Following [19], for $N \geq 2$ and $\theta \in \mathbb{C}$, define a kernel on $\mathbb{R}^N \times \mathbb{R}^{N-1}$ by

$$Q_\theta^{(N)}(x, y) = \exp\left(\theta \left(\sum_{i=1}^N x_i - \sum_{i=1}^{N-1} y_i \right) - \sum_{i=1}^{N-1} (e^{y_i - x_i} + e^{x_{i+1} - y_i})\right).$$

Denote the corresponding integral operator by $Q_\theta^{(N)}$, defined on a suitable class of functions by

$$Q_\theta^{(N)} f(x) = \int_{\mathbb{R}^{N-1}} Q_\theta^{(N)}(x, y) f(y) dy.$$

Then

$$(5) \quad \psi_{\lambda_1, \dots, \lambda_N}^{(N)} = Q_{\lambda_N}^{(N)} \psi_{\lambda_1, \dots, \lambda_{N-1}}^{(N-1)},$$

and the integral formula (4) can be re-expressed as

$$\psi_\lambda^{(N)} = Q_{\lambda_N}^{(N)} Q_{\lambda_{N-1}}^{(N-1)} \cdots Q_{\lambda_2}^{(2)} \psi_{\lambda_1}^{(1)}.$$

Moreover, as remarked in [19], the following intertwining relation holds:

$$(6) \quad (H^{(N)} - \theta^2) \circ \mathcal{Q}_\theta^{(N)} = \mathcal{Q}_\theta^{(N)} \circ H^{(N-1)}.$$

This follows from the identity

$$(H_x^{(N)} - \theta^2) \mathcal{Q}_\theta^{(N)}(x, y) = H_y^{(N-1)} \mathcal{Q}_\theta^{(N)}(x, y),$$

which is readily verified. Combining (5) with the intertwining relation (6) yields the eigenvalue equation:

$$H^{(N)} \psi_\lambda^{(N)} = \left(\sum_{i=1}^N \lambda_i^2 \right) \psi_\lambda^{(N)}.$$

We note the following immediate consequences of the above integral representation. If $\lambda \in i\mathbb{R}^N$, then $\overline{\psi_\lambda(x)} = \psi_{-\lambda}(x)$; if $\lambda \in i\mathbb{R}^N$ and $\nu \in \mathbb{R}^N$, then $|\psi_{\lambda+\nu}(x)| \leq \psi_\nu(x)$. It is also known (combining results from [20, 25, 31]) that for each $x \in \mathbb{R}^N$, $\psi_\lambda(x)$ is an entire function of $\lambda \in \mathbb{C}^N$.

The above construction has a representation-theoretic interpretation which is described in [19]. It is closely related to the Gauss decomposition and has been extended to the other classical Lie algebras in [21]. Encoded in the integrand are the defining hyperplanes of the Gelfand–Tsetlin polytope associated with the vector x .

In the present setting (see, e.g., [31]), the spectral decomposition theorem states that the integral transform

$$(7) \quad \hat{f}(\lambda) = \int_{\mathbb{R}^N} f(x) \psi_\lambda(x) dx$$

defines an isometry from $L_2(\mathbb{R}^N, dx)$ into $L_2(i\mathbb{R}^N, s_N(\lambda) d\lambda)$, where $s_N(\lambda) d\lambda$ is the *Sklyanin measure* defined by

$$(8) \quad s_N(\lambda) = \frac{1}{(2\pi i)^N N!} \prod_{j \neq k} \Gamma(\lambda_j - \lambda_k)^{-1}.$$

There is also a Mellin–Barnes type integral formula for ψ_λ due to Kharchev and Lebedev [31–33] (see also [26]). This is a kind of dual of the above integral representation and has a similar recursive structure. For $N \geq 2$ and $z \in \mathbb{R}$, define a kernel on $\mathbb{C}^N \times \mathbb{C}^{N-1}$ by

$$\hat{Q}_z^{(N)}(\lambda, \gamma) = e^{z(\sum \lambda_i - \sum \gamma_i)} \prod_{i,j} \Gamma(\lambda_i - \gamma_j).$$

Then

$$(9) \quad \psi_\lambda^{(N)}(x) = \int \hat{Q}_{x_1}^{(N)}(\lambda, \gamma) \psi_\gamma^{(N-1)}(x_2, \dots, x_N) s_{N-1}(\gamma) d\gamma,$$

where the integral is along vertical lines with $\Re\gamma_i < \Re\lambda_j$ for all i, j . This construction also has a representation-theoretic interpretation which is described in [18]. Gerasimov et al. [20] give a clear account of the nature of the duality between the two constructions and, in particular, show how this duality yields the following identity: for $\lambda, \nu \in \mathbb{C}^N$,

$$(10) \quad \int_{\mathbb{R}^N} e^{-e^{x_1-z}} \psi_\lambda(x) \psi_\nu(x) dx = e^{z \sum (\lambda_i + \nu_i)} \prod_{i,j} \Gamma(\lambda_i + \nu_j).$$

This is closely related to a Whittaker integral identity which was conjectured by Bump [9] and later proved by Stade [56], Theorem 1.1; there is an extensive literature on similar and related identities; see, for example, [10, 27, 55].

When $N = 2$, the eigenfunctions ψ_λ are given by

$$\psi_\lambda(x) = 2 \exp\left(\frac{1}{2}(\lambda_1 + \lambda_2)(x_1 + x_2)\right) K_{\lambda_1 - \lambda_2}(2e^{(x_2 - x_1)/2}),$$

where $K_\nu(z)$ is the Macdonald function. In this case, the Givental’s formula is equivalent to the integral formula

$$K_\nu(z) = \frac{1}{2} \int_0^\infty t^{\nu-1} \exp\left(-\frac{z}{2}(t + 1/t)\right) dt,$$

the contour integral representation (9) is equivalent to

$$K_\nu(z) = \frac{1}{4\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(s)\Gamma(s - \nu) \left(\frac{z}{2}\right)^{\nu-2s} ds, \quad a > \max\{\Re\nu, 0\},$$

and the integral transform defined by (7) is essentially (up to a change of variables) the Kontorovich–Lebedev transform.

3. The main result. For $x, \nu \in \mathbb{R}^N$, denote by σ_ν^x the probability measure on the set Γ of real triangular arrays $(T_{k,i})_{1 \leq i \leq k \leq N}$ defined by

$$\int f d\sigma_\nu^x = \psi_\nu(x)^{-1} \int_{\Gamma(x)} f(T) e^{\mathcal{F}_\nu(T)} \prod_{k=1}^{N-1} \prod_{i=1}^k dT_{k,i}.$$

For $i = 1, \dots, N - 1$, and continuous $\eta : (0, \infty) \rightarrow \mathbb{R}^N$, define

$$(\mathcal{T}_i \eta)(t) = \eta(t) + \left(\log \int_0^t e^{\eta_{i+1}(s) - \eta_i(s)} ds \right) (e_i - e_{i+1}),$$

where e_1, \dots, e_N denote the standard basis vectors in \mathbb{R}^N . Let Π_1 be the identity mapping ($\Pi_1 \eta = \eta$) and, for $2 \leq k \leq N - 1$, $\Pi_k = \mathcal{T}_1 \circ \dots \circ \mathcal{T}_{k-1} \circ \Pi_{k-1}$. Finally, we define

$$T = \Pi_N = (\mathcal{T}_1 \circ \dots \circ \mathcal{T}_{N-1}) \circ \dots \circ (\mathcal{T}_1 \circ \mathcal{T}_2) \circ \mathcal{T}_1.$$

The main result of this paper is the following.

THEOREM 3.1.

(1) If $(W(t), t > 0)$ is a standard Brownian motion in \mathbb{R}^N with drift v , then $(TW(t), t > 0)$ is a diffusion process in \mathbb{R}^N with infinitesimal generator given by

$$\mathcal{L}_v = \frac{1}{2} \psi_v^{-1} \left(H - \sum_{i=1}^N v_i^2 \right) \psi_v = \frac{1}{2} \Delta + \nabla \log \psi_v \cdot \nabla.$$

(2) For each $t > 0$, the conditional law of $\{(\Pi_k W)_i(t), 1 \leq i \leq k \leq N\}$, given $\{TW(s), s \leq t; TW(t) = x\}$, is given by σ_v^x .

(3) For each $t > 0$, the conditional law of $W(t)$, given $\{TW(s), s \leq t; TW(t) = x\}$, is given by γ_v^x , where

$$\int_{\mathbb{R}^N} e^{(\lambda, y)} \gamma_v^x(dy) = \frac{\psi_{v+\lambda}(x)}{\psi_v(x)}, \quad \lambda \in \mathbb{C}^N.$$

(4) If μ_t^v denotes the law of $TW(t)$, then

$$\mu_t^v(dx) = \exp\left(-\frac{1}{2} \sum_{i=1}^N v_i^2 t\right) \psi_v(x) \vartheta_t(x) dx,$$

where

$$(11) \quad \vartheta_t(x) = \int_{i \in \mathbb{R}^N} \psi_{-\lambda}(x) e^{\sum_i \lambda_i^2 t / 2} s_N(\lambda) d\lambda.$$

It is easy to see that the process $(\{\Pi_k W\}_i(t), 1 \leq i \leq k \leq N), t > 0)$ is Markov. Indeed, setting $Z_{k,i} = (\Pi_k W)_i$, it follows from the construction that Z is a Markov process taking values in Γ which satisfies the system of stochastic differential equations: $dZ_{1,1} = dW_1$ and, for $k = 2, \dots, N$,

$$\begin{aligned} dZ_{k,1} &= dZ_{k-1,1} + e^{Z_{k,2} - Z_{k-1,1}} dt, \\ dZ_{k,2} &= dZ_{k-1,2} + (e^{Z_{k,3} - Z_{k-1,2}} - e^{Z_{k,2} - Z_{k-1,1}}) dt, \\ &\vdots \\ dZ_{k,k-1} &= dZ_{k-1,k-1} + (e^{Z_{k,k} - Z_{k-1,k-1}} - e^{Z_{k,k-1} - Z_{k-1,k-2}}) dt, \\ dZ_{k,k} &= dW_k - e^{Z_{k,k} - Z_{k-1,k-1}} dt. \end{aligned}$$

The infinitesimal generator of this process is given by

$$\mathcal{A}_v = \frac{1}{2} \sum_{1 \leq i \leq k \leq N} \frac{\partial^2}{\partial z_{k,i}^2} + \sum_{1 \leq i \leq k < l \leq N} \frac{\partial^2}{\partial z_{k,i} \partial z_{l,i}} + \sum_{1 \leq i \leq k \leq N} b_{k,i}(z) \frac{\partial}{\partial z_{k,i}},$$

where

$$\begin{aligned} b_{1,1}(z) &= v_1; \\ b_{k,k}(z) &= v_k - e^{z_{k,k} - z_{k-1,k-1}}, \quad k = 2, \dots, N; \\ b_{k,1}(z) &= e^{z_{k,1} - z_{k-1,1}}, \quad k = 2, \dots, N; \\ b_{k,i}(z) &= e^{z_{k,i+1} - z_{k-1,i}} - e^{z_{k,i} - z_{k-1,i-1}}, \quad 1 < i < k \leq N. \end{aligned}$$

The main content of Theorem 3.1 is the fact that $Z_{N,\cdot}$ is a Markov process, with respect to its own filtration. The reason it holds is because

$$(12) \quad \mathcal{L}_\nu \circ \Sigma_\nu = \Sigma_\nu \circ \mathcal{A}_\nu,$$

where Σ_ν is the Markov operator defined by

$$(13) \quad \Sigma_\nu f(x) = \int f(z) \sigma_\nu^x(dz).$$

There is an additional (and nontrivial) technical issue related to the fact that these processes start at a particular entrance law coming from “ $-\infty$,” but the intertwining relation (12) lies at the heart of the proof. Actually, the proof of Theorem 3.1 given below is based on some intermediate intertwining relations which exploit the recursive structure of the quantum Toda lattice and the intertwining relation (12) is obtained as a consequence, but it should, nevertheless, be regarded as the analytic counterpart of Theorem 3.1. As far as we are aware, the intertwining relation (12) and its intermediaries (given in Section 7 below) have not been previously considered in the literature.

The operator \mathcal{T} was introduced (using a different notation) in the paper [41], where it was surmised, based on heuristic arguments, that $\mathcal{T}W$ should be a diffusion process which has the same law as a Brownian motion conditioned, in an appropriate sense, on the asymptotic behavior of its exponential functionals. In [3] it was observed that such a conditioned Brownian motion can be defined and, moreover, is closely related to the quantum Toda lattice, thus providing the impetus for the present work. The above notation used to define \mathcal{T} follows a more general framework which has been developed in the papers [5, 6]. It is shown in [5] that the operators \mathcal{T}_i satisfy the braid relations, that is,

$$\mathcal{T}_i \circ \mathcal{T}_{i+1} \circ \mathcal{T}_i = \mathcal{T}_{i+1} \circ \mathcal{T}_i \circ \mathcal{T}_{i+1}, \quad 1 \leq i < N.$$

It follows that for each element $\sigma \in \mathfrak{S}_N$ we can uniquely define

$$\mathcal{T}_\sigma = \mathcal{T}_{i_1} \circ \dots \circ \mathcal{T}_{i_p},$$

where $\sigma = (i_1, i_1 + 1) \dots (i_p, i_p + 1)$ is any reduced decomposition of σ as a product of adjacent transpositions. The operator \mathcal{T} corresponds to the longest element of \mathfrak{S}_N , that is, $\mathcal{T} = \mathcal{T}_{\sigma_0}$, where

$$\sigma_0 = \begin{pmatrix} 1 & 2 & \dots & N \\ N & N-1 & \dots & 1 \end{pmatrix}.$$

The mapping

$$\eta_{[0,t]} \mapsto (\{(\Pi_k \eta)_i(t), 1 \leq i \leq k \leq N\}, \{\mathcal{T}\eta(s), s \leq t\})$$

is a geometric variant of the RSK (Robinson–Schensted–Knuth) correspondence. We will explain this connection later and give an interpretation of the measure γ_0^x appearing in Theorem 3.1 as a geometric analogue of the Duistermaat–Heckman

measure associated with the point x . The definition of the operator \mathcal{T} extends naturally to other Lie algebras, with \mathfrak{S}_N replaced by the corresponding Weyl group [5, 6]. It is natural to expect the analogue of Theorem 3.1 to hold in this more general setting.

4. The law of the partition function. By Brownian scaling, it is easy to see that the processes $(Z_t^N(\beta), t \geq 0)$ and $(\beta^{-2(N-1)} Z_{\beta^2 t}^N(1), t \geq 0)$ are identical in law, so for convenience we will define $Z_t^N = Z_t^N(1)$. The transformation $\mathcal{T}W$ is related to the random variable Z_t^N as follows. We first note that \mathcal{T} satisfies (cf. [5], Lemma 4.6)

$$(14) \quad (-\sigma_0) \circ \mathcal{T} = \mathcal{T} \circ (-\sigma_0),$$

where $-\sigma_0(\eta_1, \dots, \eta_N) = (-\eta_N, \dots, -\eta_1)$ and η_i denotes the i th coordinate of the path η . It is straightforward to see from the definition of \mathcal{T} that

$$\begin{aligned} (\mathcal{T}W)_N(t) &= -\log \int_{0 < s_1 < \dots < s_{N-1} < t} e^{-(W_1(s_1) + W_2(s_1, s_2) + \dots + W_N(s_{N-1}, t))} ds_1 \dots ds_{N-1}. \end{aligned}$$

From the relation (14) we have

$$(\mathcal{T}W)_1(t) = \log \int_{0 < s_1 < \dots < s_{N-1} < t} e^{W_N(s_1) + W_{N-1}(s_1, s_2) + \dots + W_1(s_{N-1}, t)} ds_1 \dots ds_{N-1}.$$

Thus, if we set $W = (B_N, \dots, B_1)$, then $\log Z_t^N = (\mathcal{T}W)_1(t)$ and we deduce the following:

COROLLARY 4.1. *The stochastic process $(\log Z_t^N, t > 0)$ has the same law as the first coordinate of the diffusion process in \mathbb{R}^N with infinitesimal generator*

$$\mathcal{L} = \frac{1}{2} \psi_0^{-1} H \psi_0 = \frac{1}{2} \Delta + \nabla \log \psi_0 \cdot \nabla,$$

started according to the entrance law

$$\mu_t(dx) = \psi_0(x) \vartheta_t(x) dx, \quad t > 0,$$

where ϑ_t is given by (11). In particular, for $u \in \mathbb{R}$, we have

$$P(\log Z_t^N \leq u) = \mu_t(\{x \in \mathbb{R}^N : x_1 \leq u\}).$$

Note that the relation (14) also implies that the probability measure μ_t is invariant under the transformation $-\sigma_0$. Combining Corollary 4.1 with the formula (10), we obtain [after shifting the contours in the integral (11) to the left in order to apply Fubini's theorem] the following:

COROLLARY 4.2. For $s > 0$,

$$E e^{-s Z_i^N} = \int s^{\sum \lambda_i} \prod_i \Gamma(-\lambda_i)^N e^{(1/2) \sum_i \lambda_i^2 t} s_N(\lambda) d\lambda,$$

where the integral is along vertical lines with $\Re \lambda_i < 0$ for all i .

The probability measure on $\iota \mathbb{R}^N$ with density proportional to

$$e^{\sum_i \lambda_i^2 t/2} s_N(\lambda) \equiv \frac{1}{(2\pi \iota)^N N!} e^{\sum_i \lambda_i^2 t/2} \prod_{i>j} (\lambda_i - \lambda_j) \prod_{i<j} \frac{\sin \pi (\lambda_i - \lambda_j)}{\pi}$$

can be interpreted (up to a factor of $\iota \pi$) as the law, at time $1/t$, of the radial part of a Brownian motion in the symmetric space of positive definite Hermitian matrices or, equivalently, the law of the eigenvalues of a “perturbed GUE random matrix” $A_N/\sqrt{t} + R_N/t$, where A_N is an $N \times N$ GUE random matrix and R_N is a diagonal matrix with entries given by the vector $\pi(N - 1, N - 3, \dots, 1 - N)$ (see, e.g., [30]). In particular, it is a determinantal point process [29]. The above expression for the moment generating function of Z_i^N can thus be written as a Fredholm determinant. It will be interesting to relate this, in a suitable scaling limit, to the “crossover distributions” recently introduced in the context of KPZ and the stochastic heat equation by Sasamoto and Spohn [49–52] and Amir, Corwin and Quastel [1], building on recent work of Tracy and Widom [58–61] on the asymmetric simple exclusion process. See also [12–15, 46] for related recent developments.

We conclude this section by remarking that the other coordinates of $\mathcal{T}W(t)$ can also be interpreted as logarithmic partition functions, as follows. Define an “up/right path” in $\mathbb{R} \times \mathbb{Z}$ to be an increasing path which either proceeds to the right or jumps up by one unit. For each sequence $0 < s_1 < \dots < s_{N-1} < t$ we can associate an up/right path ϕ from $(0, 1)$ to (t, N) which has jumps between the points (s_i, i) and $(s_i, i + 1)$, for $i = 1, \dots, N - 1$, and is continuous otherwise. Then we can write

$$(\mathcal{T}W)_1(t) = \log Z_i^N = \log \int e^{E(\phi)} d\phi,$$

where

$$E(\phi) = B_1(s_1) + B_2(s_2) - B_2(s_1) + \dots + B_N(t) - B_N(s_{N-1})$$

and the integral is with respect to the Lebesgue measure on the Euclidean set of all such paths. There is an analogue of Greene’s theorem in this context [5] (cf. [34]) which yields a similar formula for the other coordinates, namely, for each $k = 2, \dots, N$,

$$(\mathcal{T}W)_1(t) + \dots + (\mathcal{T}W)_k(t) = \log \int e^{E(\phi_1) + \dots + E(\phi_k)} d\phi_1 \dots d\phi_k,$$

where the integral is with respect to the Lebesgue measure on the Euclidean set of k -tuples of nonintersecting (disjoint) up/right paths with respective initial points $(0, 1), \dots, (0, k)$ and respective end points $(t, N - k + 1), \dots, (t, N)$. An interesting property of this formulation is that it extends naturally to the continuum setting of KPZ and the stochastic heat equation [44].

5. The case $N = 2$. When $N = 2$, the eigenfunctions ψ_ν are given by

$$\psi_\nu(x) = 2 \exp\left(\frac{1}{2}(\nu_1 + \nu_2)(x_1 + x_2)\right) K_{\nu_1 - \nu_2}\left(2e^{(x_2 - x_1)/2}\right).$$

In this case, Theorem 3.1 is equivalent to the following theorem of Matsumoto and Yor [36, 37].

THEOREM 5.1.

(1) Let $(B_t^{(\mu)}, t \geq 0)$ be a standard one-dimensional Brownian motion with drift μ , and define

$$Z_t^{(\mu)} = \int_0^t e^{2B_s^{(\mu)} - B_t^{(\mu)}} ds.$$

Then $\log Z^{(\mu)}$ is a diffusion process with infinitesimal generator

$$\frac{1}{2} \frac{d^2}{dx^2} + \left(\frac{d}{dx} \log K_\mu(e^{-x}) \right) \frac{d}{dx},$$

where K_μ is the Macdonald function.

(2) The conditional law of $B_t^{(\mu)}$, given $\{Z_s^{(\mu)}, s \leq t; Z_t^{(\mu)} = z\}$, is given by the generalized inverse Gaussian distribution

$$\frac{1}{2} K_\mu(1/z)^{-1} e^{\mu x} \exp(-\cosh(x)/z) dx.$$

(3) The law of $Z_t^{(\mu)}$ is given by

$$P(Z_t^{(\mu)} \in dz) = 2z^{-1} \theta_{1/z}(t) K_\mu(1/z) e^{-\mu^2 t/2} dz,$$

where $\theta_r(t)$ is characterized by the Kontorovich–Lebedev transform

$$2 \int_0^\infty K_\lambda(r) \theta_r(t) \frac{dr}{r} = e^{\lambda^2 t/2}, \quad \lambda \in i\mathbb{R}.$$

The above Kontorovich–Lebedev transform can be inverted to obtain

$$\theta_r(t) = \frac{1}{2\pi^2} \int_{-i\infty}^{i\infty} K_\lambda(r) e^{\lambda^2 t/2} \lambda \sin(\pi\lambda) d\lambda.$$

The probability measure $H_r^{(1)}(dt) = I_0(r)^{-1} \theta_r(t) dt$ is known as the first Hartman–Watson law [24, 37]. It is also characterized by

$$\int_0^\infty e^{-\nu^2 t/2} \theta_r(t) dt = I_\nu(r), \quad \nu > 0,$$

where I_λ is the modified Bessel function of the first kind.

6. The zero-temperature limit. By Brownian scaling, we can write down a version of Theorem 3.1 for general $\beta > 0$. We will state this in the case of zero drift. For continuous $\eta : (0, \infty) \rightarrow \mathbb{R}^N$, define

$$\begin{aligned}
 (\mathcal{T}_i^\beta \eta)(t) &= \eta(t) + \frac{1}{\beta} \log \left(\beta^2 \int_0^t e^{\beta(\eta_{i+1}(s) - \eta_i(s))} ds \right) (e_i - e_{i+1}), \\
 & \hspace{25em} i = 1, \dots, N - 1; \\
 \Pi_1^\beta &= Id.; \quad \Pi_k^\beta = \mathcal{T}_1^\beta \circ \dots \circ \mathcal{T}_{k-1}^\beta \circ \Pi_{k-1}^\beta, \quad 2 \leq k \leq N; \\
 \mathcal{T}^\beta &= \Pi_N^\beta = (\mathcal{T}_1^\beta \circ \dots \circ \mathcal{T}_{N-1}^\beta) \circ \dots \circ (\mathcal{T}_1^\beta \circ \mathcal{T}_2^\beta) \circ \mathcal{T}_1^\beta.
 \end{aligned}$$

Note that

$$\frac{1}{\beta} \log Z_i^N(\beta) = (\mathcal{T}^\beta W)_1(t) - \frac{N-1}{\beta} \log \beta^2.$$

COROLLARY 6.1.

- (1) If W is a standard Brownian motion in \mathbb{R}^N , then $\mathcal{T}^\beta W$ is a diffusion in \mathbb{R}^N with generator $\Delta/2 + \nabla \log \psi_0(\beta \cdot) \cdot \nabla$.
- (2) For each $t > 0$, the conditional law of $\{(\Pi_k^\beta W)_i(t), 1 \leq i \leq k \leq N\}$, given $\{\mathcal{T}^\beta W(s), s \leq t; \mathcal{T}^\beta W(t) = x\}$, is given by $\sigma_0^{\beta x}(\beta \cdot)$.
- (3) For each $t > 0$, the conditional distribution of $W(t)$, given $\{\mathcal{T}^\beta W(s), s \leq t; \mathcal{T}^\beta W(t) = x\}$, is given by $\gamma_0^{\beta x}(\beta \cdot)$.
- (4) The law of $\mathcal{T}^\beta W(t)$ is given by $\mu_{\beta^2 t}(\beta \cdot)$.

Letting $\beta \rightarrow \infty$, we recover the multidimensional version of Pitman’s “2M – X” theorem obtained in [5–7, 40, 43]. For continuous $\eta : (0, \infty) \rightarrow \mathbb{R}^N$, with $\eta(0) = 0$, define

$$\begin{aligned}
 (\mathcal{P}_i \eta)(t) &= \eta(t) - \inf_{0 < s < t} (\eta_i(s) - \eta_{i+1}(s))(e_i - e_{i+1}), \quad i = 1, \dots, N - 1; \\
 \Gamma_1 &= Id.; \quad \Gamma_k = \mathcal{P}_1 \circ \dots \circ \mathcal{P}_{k-1} \circ \Gamma_{k-1}, \quad 2 \leq k \leq N; \\
 \mathcal{P} &= \Gamma_N = (\mathcal{P}_1 \circ \dots \circ \mathcal{P}_{N-1}) \circ \dots \circ (\mathcal{P}_1 \circ \mathcal{P}_2) \circ \mathcal{P}_1.
 \end{aligned}$$

By the method of Laplace, as $\beta \rightarrow \infty$, $\mathcal{T}^\beta W \rightarrow \mathcal{P}W$ uniformly on compact intervals and, for each $t > 0$ and $1 \leq i \leq k \leq N$, $(\Pi_k^\beta W)_i(t) \rightarrow (\Gamma_k W)_i(t)$. For $1 \leq k \leq N$, $X^k = ((\Gamma_k W)_1, \dots, (\Gamma_k W)_k)$. By construction, the stochastic process $\mathbb{X}(t) = (X^1(t), \dots, X^N(t))$, $t \geq 0$, is Markov and takes values in the Gelfand–Tsetlin cone

$$GT_N = \{(x^1, \dots, x^N) \in \bar{C}_1 \times \dots \times \bar{C}_N : x_{i+1}^{k+1} \leq x_i^k \leq x_i^{k+1}, 1 \leq i \leq k \leq N - 1\},$$

where

$$C_k = \{x \in \mathbb{R}^k : x_1 > \dots > x_k\}.$$

It is a $N(N - 1)/2$ -dimensional Brownian motion with singular covariance reflected in GT_N via an explicit Skorohod reflection map. But we do not need these facts, and refer the reader to the papers [6, 40, 43] for details.

From the integral formula (4) we have

$$\begin{aligned} \psi_0(\beta x) &= \int_{\Gamma(\beta x)} \exp\left(-\sum_{k=1}^{N-1} \sum_{i=1}^k (e^{T_{k,i}-T_{k+1,i}} + e^{T_{k+1,i+1}-T_{k,i}})\right) \prod_{k=1}^{N-1} \prod_{i=1}^k dT_{k,i} \\ &= \beta^{N(N-1)/2} \\ &\quad \times \int_{\Gamma(x)} \exp\left(-\sum_{k=1}^{N-1} \sum_{i=1}^k (e^{\beta(T'_{k,i}-T'_{k+1,i})} + e^{\beta(T'_{k+1,i+1}-T'_{k,i})})\right) \prod_{k=1}^{N-1} \prod_{i=1}^k dT'_{k,i}. \end{aligned}$$

Write $x_i^k = T'_{k,i}$. As $\beta \rightarrow \infty$, if $x \in C_N$, the integrand converges to 1 if (x^1, \dots, x^N) lies in the Gelfand–Tsetlin polytope

$$GT_N(x) = \{(x^1, \dots, x^N) \in GT_N : x^N = x\},$$

and 0 otherwise. It is well known (e.g., by Weyl’s dimension formula) that the $N(N - 1)/2$ -dimensional Euclidean volume of $GT_N(x)$ is

$$\left(\prod_{k=1}^{N-1} k!\right)^{-1} h(x),$$

where h is given by (2). It follows that, for $x \in C_N$,

$$(15) \quad \lim_{\beta \rightarrow \infty} \beta^{-N(N-1)/2} \psi_0(\beta x) = \left(\prod_{k=1}^{N-1} k!\right)^{-1} h(x).$$

Similarly, the probability measure $\sigma_0^{\beta x}(\beta \cdot)$ converges as $\beta \rightarrow \infty$ to the uniform probability measure on $GT_N(x)$. Putting all of this together, letting $\beta \rightarrow \infty$ in the statement of Corollary 6.1, we immediately recover parts (1) and (2) of the following theorem.

THEOREM 6.1 ([5–7, 40, 43]).

(1) *If W is a standard Brownian motion in \mathbb{R}^N , then $X^N = \mathcal{P}X$ is a Brownian motion conditioned (in the sense of Doob) never to exit C_N .*

(2) *The conditional law of $\mathbb{X}(t)$, given $\{X^N(s), s \leq t; X^N(t) = x\}$, is uniform on $GT_N(x)$.*

(3) *The conditional law of $W(t)$, given $\{X^N(s), s \leq t; X^N(t) = x\}$, is given by the probability measure κ^x which is characterized by*

$$\int_{\mathbb{R}^N} e^{(\lambda,y)} \kappa^x(dy) = \left(\prod_{k=1}^{N-1} k!\right) \frac{\sum_{\sigma \in \mathfrak{S}_N} (-1)^\sigma e^{(\sigma \lambda, x)}}{h(x)h(\lambda)}.$$

Part (3) of the above theorem can be deduced from part (2), noting that $\sum_{i=1}^k X_i^k = \sum_{i=1}^k W_i$, for each $1 \leq k \leq N$. Comparing this with Corollary 6.1(3) yields the asymptotic formula: for $x \in C_N$,

$$(16) \quad \lim_{\beta \rightarrow \infty} \beta^{-N(N-1)/2} \psi_{\lambda/\beta}(\beta x) = \frac{\sum_{\sigma \in \mathfrak{S}_N} (-1)^\sigma e^{(\sigma \lambda, x)}}{h(\lambda)}.$$

This formula can also be seen as a consequence of an alternative representation of ψ_λ as an alternating sum of *fundamental* Whittaker functions [3, 25, 31].

The mapping

$$\eta_{[0,t]} \mapsto (\{(\Gamma_k \eta)_i(t), 1 \leq i \leq k \leq N\}, \{\mathcal{P}\eta(s), s \leq t\})$$

is a continuous version of the RSK correspondence [5, 6, 40]. The mapping

$$\eta_{[0,t]} \mapsto (\{(\Pi_k \eta)_i(t), 1 \leq i \leq k \leq N\}, \{\mathcal{T}\eta(s), s \leq t\})$$

is a continuous version of the geometric (or “tropical”) RSK introduced by Kirillov [34] (see also [4, 39]). The probability measure κ^x is the (normalized) Duistermaat–Heckman measure associated with the point x . In this setting it can be interpreted, via the Harish–Chandra formula, as the conditional distribution of the diagonal of a $N \times N$ GUE random matrix given its eigenvalues x . The probability measure γ_0^x of Theorem 3.1 can thus be interpreted as a geometric analogue of the Duistermaat–Heckman measure. In keeping with this analogy, it is natural to record the following analogue of the Littlewood–Richardson rule, which follows from Theorem 3.1(3) (cf. [6], Theorem 5.16(ii)). For $s, t > 0$, define $\tau_s W(\cdot) = W(s + \cdot) - W(s)$ and

$$\mathcal{G}_{s,t} = \sigma \{ \mathcal{T}W(r), 0 < r \leq s; (\mathcal{T}\tau_s W)(u), 0 < u \leq t \}.$$

COROLLARY 6.2. For each $x, y \in \mathbb{R}^N$,

$$(17) \quad \frac{\psi_\lambda(x) \psi_\lambda(y)}{\psi_0(x) \psi_0(y)} = \int_{\mathbb{R}^N} \frac{\psi_\lambda(z)}{\psi_0(z)} \gamma^{x,y}(dz),$$

where $\gamma^{x,y}$ is a probability measure on \mathbb{R}^N which can be interpreted, for $s, t > 0$, as the conditional law of $\mathcal{T}W(s + t)$ given $\mathcal{G}_{s,t}$, $\mathcal{T}W(s) = x$ and $(\mathcal{T}\tau_s W)(t) = y$.

When $N = 2$, (17) is equivalent to the formula

$$K_\nu(z)K_\nu(w) = \frac{1}{2} \int_0^\infty e^{(-1/2)[t+(z^2+w^2)/t]} K_\nu\left(\frac{zw}{t}\right) \frac{dt}{t}.$$

Theorem 6.1, in the case $N = 2$, is equivalent to Pitman’s celebrated “ $2M - X$ ” theorem [45], which states that, if $X_t, t \geq 0$, is a standard one-dimensional Brownian motion, then $2 \max_{0 \leq s \leq t} X_s - X_t, t \geq 0$, is a three-dimensional Bessel process. Setting $W = (B_N, \dots, B_1)$ as before, the random variable M_1^N defined by

(1) can be written as $M_1^N = X_1^N(1)$. Thus, we also recover the fact [2, 23] that M_1^N has the same law as the largest eigenvalue of a $N \times N$ GUE random matrix.

Theorem 6.1 has been generalized to arbitrary finite Coxeter groups in the papers [5, 6]. The definition of the operator \mathcal{T} also extends naturally to other Lie algebras, with \mathfrak{S}_N replaced by the corresponding Weyl group. This is described in [5, 6], where various Lie-theoretic interpretations are given. It is natural to expect the analogue of Theorem 3.1 to hold in this more general setting.

7. Intertwining relations. Consider the following extension of the operator $\mathcal{Q}_\theta^{(N)}$, defined on a suitable class of functions $f : \mathbb{R}^N \times \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ by

$$\mathcal{R}_\theta^{(N)} f(x) = \int_{\mathbb{R}^{N-1}} \mathcal{Q}_\theta^{(N)}(x, y) f(x, y) dy.$$

By a straightforward calculation, we obtain

$$(18) \quad (H^{(N)} - \theta^2) \circ \mathcal{R}_\theta^{(N)} = \mathcal{R}_\theta^{(N)} \circ U_\theta^{(N)},$$

where

$$\begin{aligned} U_\theta^{(N)} &= \sum_{i=1}^{N-1} \frac{\partial^2}{\partial y_i^2} - 2 \sum_{i=1}^{N-2} e^{y_{i+1}-y_i} + \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} \\ &\quad + 2(\theta + e^{y_1-x_1}) \frac{\partial}{\partial x_1} \\ &\quad + 2(\theta + e^{y_2-x_2} - e^{x_2-y_1}) \frac{\partial}{\partial x_2} \\ &\quad \vdots \\ &\quad + 2(\theta + e^{y_{N-1}-x_{N-1}} - e^{x_{N-1}-y_{N-2}}) \frac{\partial}{\partial x_{N-1}} \\ &\quad + 2(\theta - e^{x_N-y_{N-1}}) \frac{\partial}{\partial x_N}. \end{aligned}$$

Further integration by parts yields

$$(19) \quad (H^{(N)} - \theta^2) \circ \mathcal{R}_\theta^{(N)} = \mathcal{R}_\theta^{(N)} \circ V_\theta^{(N)},$$

where

$$\begin{aligned} V_\theta^{(N)} &= \sum_{i=1}^{N-1} \frac{\partial^2}{\partial y_i^2} - 2 \sum_{i=1}^{N-2} e^{y_{i+1}-y_i} + \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} \\ &\quad + 2 \left(\frac{\partial}{\partial y_1} + e^{x_2-y_1} \right) \frac{\partial}{\partial x_1} \end{aligned}$$

$$\begin{aligned}
 &+ 2\left(\frac{\partial}{\partial y_2} + e^{x_3-y_2} - e^{x_2-y_1}\right) \frac{\partial}{\partial x_2} \\
 &\vdots \\
 &+ 2\left(\frac{\partial}{\partial y_{N-1}} + e^{x_N-y_{N-1}} - e^{x_{N-1}-y_{N-2}}\right) \frac{\partial}{\partial x_{N-1}} \\
 &+ 2(\theta - e^{x_N-y_{N-1}}) \frac{\partial}{\partial x_N}.
 \end{aligned}$$

The intertwining relation (19) lies at the heart of this paper.

8. Proof of Theorem 3.1. We begin by using the intertwining relation (19) to prove a Markov functions result. We will then proceed by induction to prove a version of Theorem 3.1 for general starting position. The final step will be to let the starting position $x_0 \rightarrow -\infty$ (in a sense that will be made precise later). Let $v \in \mathbb{R}^N$, and define

$$\mathcal{L}_v^{(N)} = \frac{1}{2}(\psi_v^{(N)})^{-1} \left(H^{(N)} - \sum_{i=1}^N v_i^2 \right) \psi_v^{(N)}.$$

We consider a Markov process $((X(t), Y(t)), t \geq 0)$ taking values in $\mathbb{R}^N \times \mathbb{R}^{(N-1)}$, defined as follows. The process Y evolves as an autonomous Markov process with infinitesimal generator $\mathcal{L}_{v_1, \dots, v_{N-1}}^{(N-1)}$. Let W be standard one-dimensional Brownian motion, independent of Y , and define the evolution of the process X via the stochastic differential equations

$$\begin{aligned}
 dX_1 &= dY_1 + e^{X_2-Y_1} dt, \\
 dX_2 &= dY_2 + (e^{X_3-Y_2} - e^{X_2-Y_1}) dt, \\
 &\vdots \\
 dX_{N-1} &= dY_{N-1} + (e^{X_N-Y_{N-1}} - e^{X_{N-1}-Y_{N-2}}) dt, \\
 dX_N &= dW + (v_N - e^{X_N-Y_{N-1}}) dt.
 \end{aligned}$$

Then (X, Y) is a Markov process taking values in $\mathbb{R}^N \times \mathbb{R}^{(N-1)}$ with generator

$$\mathcal{G}_v^{(N)} = \psi_{v_1, \dots, v_{N-1}}^{(N-1)}(y)^{-1} \left(V_{v_N}^{(N)} - \sum_{i=1}^{N-1} v_i^2 \right) \psi_{v_1, \dots, v_{N-1}}^{(N-1)}(y).$$

Consider the Markov operator $\Lambda_v^{(N)}$ defined, for bounded measurable functions on $\mathbb{R}^N \times \mathbb{R}^{(N-1)}$, by

$$\Lambda_v^{(N)} f(x) = \psi_v^{(N)}(x)^{-1} \int_{\mathbb{R}^{N-1}} Q_\theta^{(N)}(x, y) \psi_{v_1, \dots, v_{N-1}}^{(N-1)}(y) f(x, y) dy.$$

For $x \in \mathbb{R}^N$, define a probability measure λ_v^x on $\mathbb{R}^N \times \mathbb{R}^{(N-1)}$ by

$$\int f d\lambda_v^x = \Lambda_v^{(N)} f(x).$$

By (19), we have the intertwining relation

$$\mathcal{L}_v^{(N)} \circ \Lambda_v^{(N)} = \Lambda_v^{(N)} \circ \mathcal{G}_v^{(N)}.$$

From the theory of Markov functions [48], we conclude the following proposition:

PROPOSITION 8.1. *Fix $x_0, v \in \mathbb{R}^N$ and let (X, Y) be a Markov process with infinitesimal generator $\mathcal{G}_v^{(N)}$, started with initial law $\lambda_v^{x_0}$. Then X is a Markov process with infinitesimal generator $\mathcal{L}_v^{(N)}$, started at x_0 . Moreover, for each $t \geq 0$, the conditional law of $Y(t)$, given $\{X(s), s \leq t; X(t) = x\}$, is given by*

$$\psi_v^{(N)}(x)^{-1} Q_{v_N}^{(N)}(x, y) \psi_{v_1, \dots, v_{N-1}}^{(N-1)}(y) dy.$$

The next step is to deduce, by induction, an analogue of Theorem 3.1 for general starting position. We construct a Markov process Z taking values in Γ as follows. Let W be a standard Brownian motion in \mathbb{R}^N with drift v . The evolution of Z is defined recursively by $dZ_{1,1} = dW_1$ and, for $k = 2, \dots, N$,

$$\begin{aligned} dZ_{k,1} &= dZ_{k-1,1} + e^{Z_{k,2} - Z_{k-1,1}} dt, \\ dZ_{k,2} &= dZ_{k-1,2} + (e^{Z_{k,3} - Z_{k-1,2}} - e^{Z_{k,2} - Z_{k-1,1}}) dt, \\ &\vdots \\ dZ_{k,k-1} &= dZ_{k-1,k-1} + (e^{Z_{k,k} - Z_{k-1,k-1}} - e^{Z_{k,k-1} - Z_{k-1,k-2}}) dt, \\ dZ_{k,k} &= dW_k - e^{Z_{k,k} - Z_{k-1,k-1}} dt. \end{aligned}$$

PROPOSITION 8.2. *Fix $x_0, v \in \mathbb{R}^N$ and let Z be the process defined as above with initial law $\sigma_v^{x_0}$. Then $Z_{N,\cdot}$ is a Markov process with infinitesimal generator $\mathcal{L}_v^{(N)}$, started at x_0 . Moreover, for each $t \geq 0$, the conditional law of $Z(t)$, given $\{Z_{N,\cdot}(s), s \leq t; Z_{N,\cdot}(t) = x\}$, is given by σ_v^x , and the intertwining relation (12) holds.*

Next we give a formula for the process Z started at $Z(0) = z$ in terms of the driving Brownian motion W . For $i = 1, \dots, N - 1$, and continuous $\eta : (0, \infty) \rightarrow \mathbb{R}^N$, define

$$(\mathcal{I}_i^\xi \eta)(t) = \eta(t) + \log\left(e^\xi + \int_0^t e^{\eta_{i+1}(s) - \eta_i(s)} ds\right)(e_i - e_{i+1}).$$

Fix $z \in \Gamma$ and, for $1 \leq i \leq k \leq N - 1$, define

$$\xi_{k,i} = z_{k,i} - z_{k+1,i+1}.$$

Let Π_1^z be the identity map and, for $2 \leq k \leq N$,

$$\Pi_k^z = (\mathcal{T}_1^{\xi_{k-1,1}} \circ \dots \circ \mathcal{T}_{k-1}^{\xi_{k-1,k-1}}) \circ \Pi_{k-1}^z.$$

Then, for $1 \leq i \leq k \leq N$, we can write

$$Z_{k,i}(t) = z_{1,1} + (\Pi_k^z W)_i(t).$$

For convenience we will write $\mathcal{T}^z = \Pi_N^z$ and note that $Z_{N,\cdot} = z_{1,1}\mathbf{1} + \mathcal{T}^z W$, where $\mathbf{1} = (1, 1, \dots, 1)$. Proposition 8.2 can now be restated as follows.

PROPOSITION 8.3. *Fix $x_0, \nu \in \mathbb{R}^N$. Let W be a standard Brownian motion in \mathbb{R}^N with drift ν and ζ a random element of Γ chosen according to the distribution $\sigma_\nu^{x_0}$, independent of W . Then $Z_{N,\cdot} = \zeta_{1,1}\mathbf{1} + \mathcal{T}^\zeta W$ is a Markov process with infinitesimal generator $\mathcal{L}_\nu^{(N)}$, started at x_0 . Moreover, for each $t \geq 0$, the conditional law of $Z(t)$, given $\{Z_{N,\cdot}(s), s \leq t; Z_{N,\cdot}(t) = x\}$, is given by σ_ν^x .*

For $k = 1, \dots, N$, define

$$\rho^k = \left(\frac{k-1}{2}, \frac{k-1}{2} - 1, \dots, 1 - \frac{k-1}{2}, -\frac{k-1}{2} \right).$$

We remark that the vector ρ^k is half the sum of the positive roots associated with the Lie algebra \mathfrak{gl}_k . To complete the proof of Theorem 3.1, we will consider the starting position $x_0 = -M\rho^N$, and let $M \rightarrow \infty$. For this we need to understand the asymptotic behavior of $\psi_\nu(-M\rho^N)$ and the probability measures $\sigma_\nu^{-M\rho^N}$ as $M \rightarrow \infty$. It was shown by Rietsch ([47], Theorem 10.2) that the function $-\mathcal{F}_0(T)$ on $\Gamma(x)$ has a unique critical point T^x , which is a minimum, and that the Hessian is everywhere totally positive. It is straightforward to verify from the critical point equations that

$$\frac{1}{k} \sum_{i=1}^k T_{k,i}^x = \frac{1}{N} \sum_{i=1}^N x_i, \quad 1 \leq k \leq N-1.$$

Define $\mathcal{S}_\nu(T) = \mathcal{F}_\nu(T) - \mathcal{F}_0(T)$ and consider the change of variables

$$T'_{k,i} = T_{k,i} + M\rho_i^k, \quad 1 \leq i \leq k \leq N.$$

Then we can write

$$\psi_\nu(-M\rho^N) = \int_{\Gamma(0)} e^{\mathcal{S}_\nu(T') + e^{M/2}\mathcal{F}_0(T')} \prod_{k=1}^{N-1} \prod_{i=1}^k dT'_{k,i}.$$

It follows, by Laplace’s method (see, e.g., [17], Theorem 4.14), that the following asymptotic equivalence holds:

$$(20) \quad \psi_\nu(-M\rho^N) \sim C e^{-N(N-1)M/8} \exp(e^{M/2}\mathcal{F}_0(T^0))$$

as $M \rightarrow \infty$, where C is a constant which is independent of ν . Moreover, recalling the above change of variables, we see that, in probability, $\zeta_{k,i} - \zeta_{k+1,i+1} \rightarrow -\infty$ for each $1 \leq i \leq k \leq N - 1$ and $\zeta_{1,1} \rightarrow 0$. It follows by the continuous mapping theorem that that $\zeta_{1,1}\mathbf{1} + \mathcal{T}^\zeta W$ converges in law to $\mathcal{T}W$, and, for each $t > 0$, $\{(\Pi_k^\zeta W)_i(t), 1 \leq i \leq k \leq N\}$ converges in law to $\{(\Pi_k W)_i(t), 1 \leq i \leq k \leq N\}$. We conclude that $\mathcal{T}W$ is a diffusion with generator $\mathcal{L}_\nu^{(N)}$, and that the conditional law of $\{(\Pi_k W)_i(t), 1 \leq i \leq k \leq N\}$, given $\{\mathcal{T}W(s), s \leq t; \mathcal{T}W(t) = x\}$, is σ_ν^x . This proves parts (1) and (2) of the theorem. Part (3) of the theorem follows from part (2), noting that for each $k \leq N$,

$$W_k = \sum_{i=1}^k (\Pi_k W)_i - \sum_{i=1}^{k-1} (\Pi_{k-1} W)_i.$$

Part (4) follows from part (3) by the spectral decomposition theorem.

REMARK 8.1. The asymptotic equivalence (20) is well known in the case $N = 2$ and can be compared to the full asymptotic expansion obtained in [11] in the case $N = 3$, where it was remarked that the leading term in the expansion is independent of the parameter ν .

9. A symmetric version of Proposition 8.2. Proposition 8.2 has a ‘‘symmetric’’ analogue which can be regarded as a geometric version of a result of Dubedat [16] in the case $N = 2$, and Warren [62] in the general case. It is obtained by applying the intertwining relation (18) rather than (19). In this case, we construct a Markov process S on Γ as follows. Let $\{W_{k,i}, 1 \leq i \leq k \leq N\}$ be a collection of independent standard one-dimensional Brownian motions. The evolution of S is defined recursively by $dS_{1,1} = dW_{1,1}$ and, for $k = 2, \dots, N$,

$$\begin{aligned} dS_{k,1} &= dW_{k,1} + (\nu_k + e^{S_{k-1,1} - S_{k,1}}) dt, \\ dS_{k,2} &= dW_{k,2} + (\nu_k + e^{S_{k-1,2} - S_{k,2}} - e^{S_{k,2} - S_{k-1,1}}) dt, \\ &\vdots \\ dS_{k,k-1} &= dW_{k,k-1} + (\nu_k + e^{S_{k-1,k-1} - S_{k,k-1}} - e^{S_{k,k-1} - S_{k-1,k-2}}) dt, \\ dS_{k,k} &= dW_{k,k} + (\nu_k - e^{S_{k,k} - S_{k-1,k-1}}) dt. \end{aligned}$$

PROPOSITION 9.1. Fix $x_0, \nu \in \mathbb{R}^N$ and let S be the process defined as above with initial law $\sigma_\nu^{x_0}$. Then $S_{N,\cdot}$ is a Markov process with infinitesimal generator $\mathcal{L}_\nu^{(N)}$, started at x_0 . Moreover, for each $t \geq 0$, the conditional law of $S(t)$, given $\{S_{N,\cdot}(s), s \leq t; S_{N,\cdot}(t) = x\}$, is σ_ν^x .

In the case $N = 2$, with zero drift, we deduce the following corollary:

COROLLARY 9.2. *Let B_1, B_2 and B_3 be independent standard one-dimensional Brownian motions. Define*

$$X(t) = B_1(t) + \log \int_0^t e^{B_2(s) - B_1(s)} ds,$$

$$Y(t) = B_3(t) - \log \int_0^t e^{B_3(s) - B_2(s)} ds.$$

Then $(X + Y)/\sqrt{2}$ is a standard Brownian motion and $(X - Y)/\sqrt{2}$ is a diffusion process [independent of $(X + Y)/\sqrt{2}$] with infinitesimal generator

$$\frac{1}{2} \frac{d^2}{dx^2} + \left(\frac{d}{dx} \log K_0(e^{-x}) \right) \frac{d}{dx}.$$

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