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# Directional output distance functions: endogenous directions based on exogenous normalization constraints

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**Abstract** In response to a question raised by Knox Lovell, we develop a method for estimating directional output distance functions with endogenously determined direction vectors based on exogenous normalization constraints. This is reminiscent of the Russell measure proposed by Färe and Lovell (J Econ Theory 19:150–162, 1978). Moreover it is related to the slacks-based directional distance function introduced by Färe and Grosskopf (Eur J Oper Res 200:320–322, 2010a, Eur J Oper Res 206:702, 2010b). Here we show how to use the slacks-based function to estimate the optimal directions.

**Keywords** DEA · Directional distance functions · Slack-based measures

**JEL Classification** C61 · D24 · D92 · O33

## 1 Introduction

This paper is inspired by a question raised many years ago by Knox Lovell when we were presenting work on directional distance functions: namely, how should the researcher choose the direction vectors when estimating the directional distance function other than in some ad hoc way? We have struggled with this issue for a number of years and here suggest that one might determine the direction vectors endogenously. The approach is reminiscent of the structure of the Russell measure proposed in 1978 by Färe and Lovell. More recently we show how this model is related to the slacks-based directional distance function introduced by Färe and Grosskopf (2010a, b) and show how to use the slacks-based function to estimate the optimal directions.

In the standard case in which the researcher chooses the directional vector, the resulting efficiency scores depend on that vector.[1] By endogenizing the direction vector, i.e., optimizing over them, the efficiency scores are in some sense 'optimal' rather than ad hoc choices of the researcher. Directional distance functions, see Chambers et al. (1998), are defined on a technology and were introduced by Luenberger (1992, 1995) as shortage functions; our model also applies to these functions.

In a recent paper (Zofio et al. 2012), the authors state: 'When market prices are observed and firms have a profit maximizing behavior, it seems natural to choose as the directional vector that projecting inefficient firms towards profit maximizing benchmarks.' This leads the authors to optimize over the directional vector and hence endogenize it (see expression 9). Here we address the case of the directional distance function without appeal to profit maximization, and therefore without requiring price data.

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[1] See Briec (1997, 1998), Chung et al (1997) and Färe and Grosskopf (2004) for alternative choices for the directional vectors.

## 2 Main results

In order to keep our exposition as simple as possible we assume that we have one input $x \geqq 0$ which is used to produce two outputs $(y_1, y_2) \geqq 0$. Moreover we assume that there are two decision making units (DMUs) or firms. The output set may then be formalized using Activity Analysis or Data Envelopment Analysis as

$$\begin{aligned} P(x) = \{(y_1, y_2) : & z_1 y_{11} + z_2 y_{21} \geqq y_1 \\ & z_1 y_{12} + z_2 y_{22} \geqq y_2 \\ & z_1 x_1 + z_2 x_2 \leqq x \\ & z_1, z_2 \geqq 0\}, \end{aligned} \tag{1}$$

where $z = (z_1, z_2)$ are the intensity variables.

Let $g = (g_1, g_2) \geqq 0$ be a nonnegative directional output vector and assume that the components belong to the unit simplex,[2] i.e.,

$$g_1 + g_2 = 1, \tag{2}$$

which ensures compactness, and guarantees that the problem we specify in (4) below has a solution. Among all possible normalizations of the directional vector we have followed Luenberger (1995, p. 78), which is also a standard approach in economics. This normalization also serves our purpose in our proof, see expression (8) which follows.

We may now formulate a directional output distance function with $(g_1, g_2)$ as *variables*. Their values will be endogenously determined through the following optimization problem,

$$\max_{z,g,\beta} \{\beta : (y_{11} + \beta g_1, y_{21} + \beta g_2) \in P(x), \quad g_1 + g_2 = 1, \\ g_1, g_2 \geqq 0\}, \tag{3}$$

which we formulate for DMU 1 as

$$\begin{aligned} \max_{z,g,\beta} \beta \quad \text{s.t.} \quad & z_1 y_{11} + z_2 y_{21} \geqq y_{11} + \beta g_1 \\ & z_1 y_{12} + z_2 y_{22} \geqq y_{12} + \beta g_2 \\ & z_1 x_1 + z_2 x_2 \leqq x_1 \\ & g_1 + g_2 = 1 \\ & z_1, z_2 \geqq 0, \\ & \beta \geqq 0, g_1, g_2 \geqq 0. \end{aligned} \tag{4}$$

This is a nonlinear optimization problem, and we would like to transform it into a linear optimization problem, namely into

$$\begin{aligned} \max_{z,\beta_1,\beta_2} \beta_1 + \beta_2 \quad \text{s.t.} \quad & z_1 y_{11} + z_2 y_{21} \geqq y_{11} + \beta_1 \cdot 1 \\ & z_1 y_{12} + z_2 y_{22} \geqq y_{12} + \beta_2 \cdot 1 \\ & z_1 x_1 + z_2 x_2 \leqq x_1 \\ & z_1, z_2 \geqq 0, \beta_1, \beta_2 \geqq 0 \end{aligned} \tag{5}$$

Expression (5) is the output-oriented version of the slacks-based directional distance function introduced by Färe and Grosskopf (2010a, b), which bears some resemblance to the Russell measure proposed by Färe and Lovell (1978). In the Färe-Grosskopf formulation, $g_1 = g_2 = 1$ which implies that they are in the same units as the outputs. However, here the $\beta$s are unit free, and therefore may be added. This distance function takes value zero if and only if the output vector belongs to the efficient output set, also referred to as the Pareto-Koopmans efficient subset, see references.

In order to use (5) to estimate (4) we need to show that they are equivalent. We begin by showing that the model in (4) can be derived from (5). There are two possible cases to consider:

(1) both $\beta_1$ and $\beta_2 = 0$
(2) at least one $\beta_i$, $i = 1, 2$ is positive.

In the first case, which is when the DMU is efficient, the direction vector is not uniquely determined. Thus we may set the direction vector arbitrarily in a positive direction, for example, let $g_1 = g_2 = 1/2$. If at least one $\beta_i > 0$, $i = 1, 2$, then in (5) take $\beta_i = \beta g_i \geqq 0, i = 1, 2$ with $g_i \geqq 0$ and $g_1 + g_2 = 1$, then we can rewrite (5) as

$$\begin{aligned} \max_{z,g,\beta} \beta g_1 + \beta g_2 \quad \text{s.t.} \quad & z_1 y_{11} + z_2 y_{21} \geqq y_{11} + \beta g_1 \\ & z_1 y_{12} + z_2 y_{22} \geqq y_{12} + \beta g_2 \\ & z_1 x_1 + z_2 x_2 \leqq x_1 \\ & g_1 + g_2 = 1 \\ & z_1, z_2 \geqq 0, \beta \geqq 0 \\ & g_1, g_2 \geqq 0. \end{aligned} \tag{6}$$

and transform (6) into

$$\begin{aligned} \max_{z,g,\beta} \beta \quad \text{s.t.} \quad & z_1 y_{11} + z_2 y_{21} \geqq y_{11} + \beta g_1 \\ & z_1 y_{12} + z_2 y_{22} \geqq y_{12} + \beta g_2 \\ & z_1 x_1 + z_2 x_2 \leqq x_1 \\ & g_1 + g_2 = 1 \\ & z_1, z_2 \geqq 0 \\ & \beta \geqq 0, g_1, g_2 \geqq 0. \end{aligned} \tag{7}$$

i.e., expression (4).

Next we show the converse, i.e., that (5) can be derived from (4). Let $\beta \geqq 0$ and multiply $\beta$ in the objective function of (3) with $(g_1 + g_2) = 1$ then we have

[2] The unit of measurement problem that can occur is trivially corrected by introducing appropriate weights.

$$\max_{z,g,\beta} \beta \cdot (g_1 + g_2) \quad \text{s.t.} \quad \begin{aligned} & z_1 y_{11} + z_2 y_{21} \geqq y_{11} + \beta g_1 \\ & z_1 y_{12} + z_2 y_{22} \geqq y_{12} + \beta g_2 \\ & z_1 x_1 + z_2 x_2 \leqq x_1 \\ & z_1, z_2 \geqq 0, \beta \geqq 0. \\ & g_1, g_2 \geqq 0. \end{aligned} \tag{8}$$

If we take $\beta_1 = \beta g_1 \cdot 1$, $\beta_2 = \beta g_2 \cdot 1$, then (5) follows, namely

$$\max_{z,\beta_1,\beta_2} \beta \cdot (\beta_1 + \beta_2) \quad \text{s.t.} \quad \begin{aligned} & z_1 y_{11} + z_2 y_{21} \geqq y_{11} + \beta_1 \cdot 1 \\ & z_1 y_{12} + z_2 y_{22} \geqq y_{12} + \beta_2 \cdot 1 \\ & z_1 x_1 + z_2 x_2 \leqq x_1 \\ & z_1, z_2 \geqq 0, \beta_1, \beta_2 \geqq 0. \end{aligned} \tag{9}$$

Thus the two models can be derived from each other allowing us to use the 'linear' model in (5) to find the optimal $g_1$ and $g_2$. Solving (5) yields optimal $\beta_1^*$ and $\beta_2^*$, and with at least one $\beta_i^* > 0$ (if both are zero assign any positive value to $g_i$, $i = 1, 2$). To continue, let's assume that both $\beta_i$'s are greater than zero, then we have

$$\beta = \frac{\beta_1^*}{g_1} = \frac{\beta_2^*}{g_2} \tag{10}$$

This together with $g_1 + g_2 = 1$ can be used to solve for optimal $(g_1^*, g_2^*)$.

We have

$$\frac{g_1}{g_2} + 1 = \frac{1}{g_2}, \quad \text{and} \quad \frac{\beta_1^*}{\beta_2^*} = \frac{g_1}{g_2} \tag{11}$$

thus

$$g_2^* = \frac{1}{\beta_1^*/\beta_2^* + 1} = \frac{\beta_2^*}{\beta_1^* + \beta_2^*} \tag{12}$$

and $g_1^* = 1 - g_2^*$ or $g_1^* = \beta_1^*/(\beta_1^* + \beta_2^*)$ so $g_1^* + g_2^* = 1$. If one $\beta_i^* = 0$, our conclusion still applies.

Thus by solving model (5) we can find the optimal directional vector for each DMU or firm. It is straightforward to generalize the above results to the case of multiple inputs and outputs as well as alternative input/output orientations.

Finally, let us consider a simple numerical example with two DMUs:

| DMU | 1 | 2 |
|---|---|---|
| $y_1$ | 1 | 2 |
| $y_2$ | 1 | 2 |
| $x$ | 1 | 1 |

Based on these data, model (5) yields the following results for DMU 1:

$$\beta_1^* = \beta_2^* = 1.$$

which we can use to solve for the optimal direction vector, i.e.,

$$g_1^* = g_2^* = 1/2.$$

For DMU 2, which is efficient, model (5) yields $\beta_1^* = \beta_2^* = 0$, which implies that we may assign arbitrary positive values to $g_1$ and $g_2$.

## References


Briec W (1997) A graph-type extension of Farrell technical efficiency measures. J Prod Anal 8:95–111

Briec W (1997) Hölder distance function and the measurement of technical efficiency. J Prod Anal 11:111–132

Chambers RG, Chung Y, Färe R (1998) Profit, directional distance functions, and Nerlovian efficiency. J Optim Theory Appl 98:351–364

Chung Y, Färe R, Grosskopf S (1997) Productivity and undesirable outputs: a directional distance function approach. J Environ Manage 51:229–240

Färe R, Grosskopf S (2010a) Directional distance functions and slacks-based measures of efficiency. Eur J Oper Res 200: 320–322

Färe R, Grosskopf S (2010b) Directional distance functions and slacks-based measures of efficiency: some clarifications. Eur J Oper Res 206:702

Färe R, Grosskopf S (2004) New directions: efficiency and productivity. Kluwer, Boston

Färe R, Lovell CAK (1978) Measuring the technical efficiency of production. J Econ Theory 19:150–162

Luenberger DG (1992) Benefit functions and duality. J Math Econ 21:461–481

Luenberger DG (1995) Microeconomic theory. McGraw Hill, Boston

Zofio JL, Pastor JT, Aparicio J (2012) The directional profit efficiency measure: on why profit efficiency is either technical or allocative. J Prod Anal. doi:10.1007/s11123-012-0292-0