Dirichlet boundary value problem for differential equation with ϕ-Laplacian and state-dependent impulses

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Abstract

The paper deals with the boundary value problem for differential equation with ϕ-Laplacian and state-dependent impulses of the form

$$\begin{gathered}\big(\phi(z'(t))\big)' = f(t, z(t), z'(t)), \quad \text{for a.e. } t \in [0,T] \subset \mathbb{R}, \\ \triangle z'(t) = M(z(t), z'(t-)), \quad t = \gamma(z(t)) \\ z(0) = z(T) = 0,\end{gathered}$$

Here, $T > 0$, $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $\phi(\mathbb{R}) = \mathbb{R}$, $\phi(0) = 0$, $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Carathéodory conditions, $M : \mathbb{R} \to \mathbb{R}$ is continuous and $\gamma : \mathbb{R} \to (0,T)$ is continuous, $\triangle z'(t) = z'(t+) - z'(t-)$. Sufficient conditions for the existence of at least one solution to this problem having no pulsation behaviour are provided.

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1 Introduction

In this paper we are interested in the ordinary differential equation of the second order with ϕ-Laplacian, state-dependent impulses and homogenous Dirichlet boundary conditions on the interval $[0,T] \subset \mathbb{R}$, $T > 0$. We consider the differential equation

$$\big(\phi(z'(t))\big)' = f(t, z(t), z'(t)) \quad \text{for a.e. } t \in [0,T], \tag{1}$$

subject to the state-dependent impulse conditions

$$\triangle z'(t) = M(z(t), z'(t-)), \quad t = \gamma(z(t)), \tag{2}$$

and homogeneous Dirichlet boundary conditions

$$z(0) = z(T) = 0, \tag{3}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $\phi(\mathbb{R}) = \mathbb{R}$, $\phi(0) = 0$, $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Carathéodory conditions, $M : \mathbb{R} \to \mathbb{R}$ is continuous, $\gamma : \mathbb{R} \to (0,T)$ is continuous, $\triangle z'(t) = z'(t+) - z'(t-)$ for $t \in (0,T)$, $z'(t\pm) = \lim_{s \to t\pm} z'(s)$.

During the last twenty years a great amount of papers dealt with the impulsive differential equations, including the boundary value problems. Various kinds of differential equations and boundary conditions were considered. Among others, some of the works were devoted to the boundary value problems for impulsive differential equations with *impulses at fixed times* with p-Laplacian, see e.g. [4, 14, 15, 20, 33, 34, 35, 36, 37] or more generally ϕ-Laplacian, see e.g. [11, 12, 21, 24, 31]. Impulsive problems with Dirichlet boundary conditions and impulses at fixed times were studied e.g. in [1, 3, 18, 23, 38]. Till now, not so much attention was paid to boundary value problems with state-dependent impulses. There can be traced works with periodic boundary conditions [5, 6, 7, 8, 16, 32], Dirichlet [17, 27, 30], Sturm–Liouville [17, 26], general linear boundary conditions [28, 29] and nonlinear boundary conditions [9].

On the other hand, to the author's knowledge, nothing is known about impulsive boundary value problems for differential equation with ϕ-Laplacian and state-dependent impulses. The aim of this paper is to fill in this gap. The investigation relies on the method originally used by Rachůnková and Tomeček in [27]. The approach exploits the fixed point theory, which is extensively used in many works for problems with impulses at fixed times, e.g. [2, 10, 13, 19, 22, 30]. The main idea of [27] lies in the investigation of a fixed point in the product space of continuous functions instead of the space of scalar valued discontinuous functions. This technique was also extended to relatively high generality [25, 26, 28, 29, 30].

Here we work with the linear space $\mathbb{C}(E)$ of all continuous functions on a set $E \subset \mathbb{R}^n$, $n \in \mathbb{N}$. Particularly, for a compact interval $J = [0,T] \subset \mathbb{R}$, where $T \in \mathbb{R}$, $T > 0$, $\mathbb{C}([0,T])$ will be equipped with the norm

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)|, \quad u \in \mathbb{C}([0,T]).$$

Further, the space $\mathbb{C}^1([0,T])$ of all continuously differentiable functions on the interval $[0,T]$ will be equipped with the norm

$$\|u\|_{1,\infty} = \|u\|_\infty + \|u'\|_\infty, \quad u \in \mathbb{C}^1([0,T]),$$

the space $\mathbb{L}^1([0,T])$ of all Lebesgue integrable functions with the norm

$$\|u\|_1 = \int_0^T |u(t)|\,\mathrm{d}t, \quad u \in \mathbb{L}^1([0,T]).$$

Also, we need to work with several product spaces. If $\mathbb{X}_i$, $i = 1,\dots,n$, $n \in \mathbb{N}$ are normed linear spaces with norms $\|\cdot\|_{\mathbb{X}_i}$, the product space $\mathbb{X}_1 \times \ldots \times \mathbb{X}_n$ is considered with the norm

$$\|(u_1,\dots,u_n)\| = \sum_{i=1}^n \|u_i\|_{\mathbb{X}_i}, \quad (u_1,\dots,u_n) \in \mathbb{X}_1 \times \ldots \times \mathbb{X}_n.$$

By $\mathrm{Car}([0,T]\times\mathbb{R}^2)$ we denote the set of all functions $f : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ that satisfy the so called *Carathéodory conditions*, i.e. they have the following properties:

(i) $f(\cdot,x,y)$ is measurable on $[0,T]$ for every $(x,y) \in \mathbb{R}^2$,

(ii) $f(t,\cdot,\cdot)$ is continuous on $\mathbb{R}^2$ for a.e. $t \in [0,T]$,

(iii) for every compact set $E \subset \mathbb{R}^2$ there exists $m \in \mathbb{L}^1([0,T])$ such that

$$|f(t,x,y)| \le m(t) \quad \text{for a.e. } t \in [0,T], \text{ all } (x,y) \in E.$$

The main result of this paper is the existence of at least one solution to the problem (1)–(3).

Definition 1 *A function $z \in \mathbb{C}([0,T])$ is called a* solution *of problem* (1)–(3)*, iff*

- *$\phi \circ z'$ is absolutely continuous on each interval $J \subset [0,T]$ satisfying condition*

$$t \ne \gamma(z(t)) \quad \textit{for all } t \in J,$$

- *z satisfies the differential equation* (1)*, fulfills impulse conditions* (2) *and boundary conditions* (3)*.*

The existence results are obtained under following assumptions:

(H_1) $T > 0$, $f \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$, $M \in \mathbb{C}(\mathbb{R}^2)$;

(H_2) $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $\phi(0) = 0$, $\phi(\mathbb{R}) = \mathbb{R}$;

(H_3) there exists $h : [0,T] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ increasing in its second and third variable such that
$$|f(t,x,y)| \le h(t,|x|,|y|) \qquad \text{for a.e. } t \in [0,T], \text{ all } (x,y) \in \mathbb{R}^2$$
and there exists $K > 0$ such that
$$2\int_0^T h(s,K,K/T)\,\mathrm{d}s + \phi\left(\phi^{-1}\left(\int_0^T h(s,K,K/T)\,\mathrm{d}s\right) + \overline{M}\right) < \phi(K/T),$$
$$-2\int_0^T h(s,K,K/T)\,\mathrm{d}s + \phi\left(\phi^{-1}\left(-\int_0^T h(s,K,K/T)\,\mathrm{d}s\right) - \overline{M}\right) > \phi(-K/T),$$
where
$$\overline{M} = \max_{[-K,K]\times[-K/T,K/T]} |M|;$$

(H_4) $\gamma \in \mathbb{C}(\mathbb{R})$, $0 < \gamma(x) < T$ for $x \in \mathbb{R}$ and there exists $L \ge 0$ such that
$$|\gamma(x) - \gamma(y)| \le L|x-y| \quad \text{for all } x,y \in [-K,K]$$
and
$$LK/T < 1.$$

These assumptions will guarantee the existence of at least one solution of (1)–(3) with exactly one cross through the barrier, i.e. such solution z satisfies the condition $t = \gamma(z(t))$ for exactly one $t \in (0,T)$.

2 Problem with impulses at fixed times

In order to obtain the operator equation corresponding to problem (1)–(3) we consider its special case for right-hand side independent of the solution and with one impulse at fixed time, i.e.
$$\gamma(x) = \tau, \quad x \in \mathbb{R},$$
where $\tau \in (0,T)$. More precisely, we investigate the impulsive boundary value problem
$$(\phi(u'(t)))' = \ell(t) \quad \text{for a.e. } t \in [0,T], \tag{4}$$
$$u'(\tau+) - u'(\tau-) = \mu, \tag{5}$$
$$u(0) = u(T) = 0, \tag{6}$$
where
$$\ell \in \mathbb{L}^1([0,T]),\ \tau \in (0,T),\ \mu \in \mathbb{R} \text{ and } (\mathrm{H}_2) \text{ is satisfied.} \tag{7}$$

Definition 2 *A function $u \in \mathbb{C}([0,T])$ is called a* solution *of problem* (4)–(6)*, iff*

- *$\phi \circ u'$ is absolutely continuous on $[0,\tau)$ and $(\tau,T]$,*
- *u satisfies differential equation* (4)*, fulfills impulse conditions* (5) *and boundary conditions* (6)*.*

Lemma 3 *Assume* (7). *Let* v, $w \in \mathbb{C}([0,T])$ *be defined by*

$$\left.\begin{aligned} v(t) &= \int_0^t \phi^{-1}\left(A + \int_0^s \ell(\omega)\,\mathrm{d}\omega\right)\mathrm{d}s, \\ w(t) &= \int_T^t \phi^{-1}\left(B + \int_T^s \ell(\omega)\,\mathrm{d}\omega\right)\mathrm{d}s, \end{aligned}\right\} \quad t \in [0,T], \tag{8}$$

where $A, B \in \mathbb{R}$ *satisfy*

$$\int_0^\tau \phi^{-1}\left(A + \int_0^s \ell(\omega)\,\mathrm{d}\omega\right)\mathrm{d}s + \int_\tau^T \phi^{-1}\left(B + \int_T^s \ell(\omega)\,\mathrm{d}\omega\right)\mathrm{d}s = 0, \tag{9a}$$

$$\phi^{-1}\left(B + \int_T^\tau \ell(s)\,\mathrm{d}s\right) - \phi^{-1}\left(A + \int_0^\tau \ell(s)\,\mathrm{d}s\right) = \mu. \tag{9b}$$

Then the function

$$u(t) = \begin{cases} v(t), & t \in [0,\tau], \\ w(t), & t \in (\tau,T] \end{cases} \tag{10}$$

is a solution of problem (4)–(6).

Proof. Let us consider v, w, u defined by (8) and (10). After differentiation of (8) and application of ϕ we get

$$\phi(v'(t)) = A + \int_0^t \ell(s)\,\mathrm{d}s, \quad \phi(w'(t)) = B + \int_T^t \ell(s)\,\mathrm{d}s, \quad t \in [0,T].$$

Clearly, $\phi \circ v'$ and $\phi \circ w'$ are absolutely continuous on $[0,T]$ and v, w satisfy the differential equation (4). Hence $\phi \circ u'$ is absolutely continuous on $[0,\tau)$ and $(\tau,T]$ and u satisfies (4), as well. According to (10), (8) and (9) we have

$$\begin{aligned} u(\tau+) - u(\tau-) &= w(\tau) - v(\tau) \\ &= \int_T^\tau \phi^{-1}\left(B + \int_T^s \ell(\omega)\,\mathrm{d}\omega\right)\mathrm{d}s - \int_0^\tau \phi^{-1}\left(A + \int_0^s \ell(\omega)\,\mathrm{d}\omega\right)\mathrm{d}s = 0 \end{aligned}$$

and

$$\begin{aligned} u'(\tau+) - u'(\tau-) &= w'(\tau) - v'(\tau) \\ &= \phi^{-1}\left(B + \int_T^\tau \ell(s)\,\mathrm{d}s\right) - \phi^{-1}\left(A + \int_0^\tau \ell(s)\,\mathrm{d}s\right) = \mu. \end{aligned}$$

Therefore u is continuous on $[0,T]$ and satisfies (5). Finally,

$$u(0) = v(0) = 0 \quad \text{and} \quad u(T) = w(T) = 0,$$

which implies that u is a solution of problem (4)–(6). □

Lemma 4 *Assume* (7). *A couple* $(A,B) \in \mathbb{R}^2$ *is a solution of system* (9), *iff* $A \in \mathbb{R}$ *is a root of the function* $F(\cdot,\ell,\tau,\mu) : \mathbb{R} \to \mathbb{R}$, *where*

$$\begin{aligned} F(x,\ell,\tau,\mu) = &\int_0^\tau \phi^{-1}\left(x + \int_0^s \ell(\omega)\,\mathrm{d}\omega\right)\mathrm{d}s \\ &+ \int_\tau^T \phi^{-1}\left(\int_\tau^s \ell(\omega)\,\mathrm{d}\omega + \phi\left(\phi^{-1}\left(x + \int_0^\tau \ell(\omega)\,\mathrm{d}\omega\right) + \mu\right)\right)\mathrm{d}s \end{aligned} \tag{11}$$

and $B \in \mathbb{R}$ *is a root of the function* $G(\cdot,\ell,\tau,\mu) : \mathbb{R} \to \mathbb{R}$, *where*

$$\begin{aligned} G(x,\ell,\tau,\mu) = &\int_0^\tau \phi^{-1}\left(\int_\tau^s \ell(\omega)\,\mathrm{d}\omega + \phi\left(\phi^{-1}\left(x + \int_T^\tau \ell(\omega)\,\mathrm{d}\omega\right) - \mu\right)\right)\mathrm{d}s \\ &+ \int_\tau^T \phi^{-1}\left(x + \int_T^s \ell(\omega)\,\mathrm{d}\omega\right)\mathrm{d}s. \end{aligned} \tag{12}$$

Proof. From (9b) we derive

$$\phi^{-1}\left(B+\int_T^\tau \ell(s)\,\mathrm{d}s\right)=\phi^{-1}\left(A+\int_0^\tau \ell(s)\,\mathrm{d}s\right)+\mu$$

and after applying ϕ on this equality and subtracting $\int_T^\tau \ell(s)\,\mathrm{d}s$ from both its sides, we get

$$B=-\int_T^\tau \ell(s)\,\mathrm{d}s+\phi\left(\phi^{-1}\left(A+\int_0^\tau \ell(s)\,\mathrm{d}s\right)+\mu\right).$$

After substituting B into (9a) we get the equality $F(A,\ell,\tau,\mu)=0$. Similarly we get $G(B,\ell,\tau,\mu)=0$. □

Under the assumption (H_2) we define the mappings

$$F,G:\mathbb{R}\times\mathbb{L}^1([0,T])\times(0,T)\times\mathbb{R}\to\mathbb{R}$$

by (11), (12), respectively. In the first place, let us explore their properties.

Lemma 5 *Assume* (H_2)*. The mappings F and G are continuous.*

Proof. Let us prove the continuity of F. Similar argument holds for G. Let $(x_n,\ell_n,\tau_n,\mu_n),(x,\ell,\tau,\mu)\in\mathbb{R}\times\mathbb{L}^1([0,T])\times(0,T)\times\mathbb{R}$ for $n\in\mathbb{N}$ such that

$$\lim_{n\to\infty}(x_n,\ell_n,\tau_n,\mu_n)=(x,\ell,\tau,\mu)\quad\text{in }\mathbb{R}\times\mathbb{L}^1([0,T])\times\mathbb{R}^2.$$

We have

$$\begin{aligned}\left|\int_{\tau_n}^s \ell_n(\omega)\,\mathrm{d}\omega-\int_\tau^s \ell(\omega)\,\mathrm{d}\omega\right| &\le \left|\int_{\tau_n}^s |\ell_n(\omega)-\ell(\omega)|\,\mathrm{d}\omega\right|\\ &\quad+\left|\int_{\tau_n}^s \ell(\omega)\,\mathrm{d}\omega-\int_\tau^s \ell(\omega)\,\mathrm{d}\omega\right|\\ &\le \|\ell_n-\ell\|_1+\left|\int_{\tau_n}^\tau \ell(\omega)\,\mathrm{d}\omega\right|\to 0\end{aligned}$$

uniformly w.r.t. $s\in[0,T]$ as $n\to\infty$. Therefore

$$\int_{\tau_n}^s \ell_n(\omega)\,\mathrm{d}\omega+\phi\left(\phi^{-1}\left(x_n+\int_0^{\tau_n}\ell_n(\omega)\,\mathrm{d}\omega\right)+\mu_n\right)$$

tends to

$$\int_{\tau}^s \ell(\omega)\,\mathrm{d}\omega+\phi\left(\phi^{-1}\left(x+\int_0^{\tau}\ell(\omega)\,\mathrm{d}\omega\right)+\mu\right)$$

w.r.t. $s\in[0,T]$ as $n\to\infty$. This implies

$$\begin{aligned}&\lim_{n\to\infty}\int_0^{\tau_n}\phi^{-1}\left(\int_{\tau_n}^s \ell_n(\omega)\,\mathrm{d}\omega+\phi\left(\phi^{-1}\left(x_n+\int_0^{\tau_n}\ell_n(\omega)\,\mathrm{d}\omega\right)+\mu_n\right)\right)\mathrm{d}s\\ &\quad=\int_0^{\tau}\phi^{-1}\left(\int_{\tau}^s \ell(\omega)\,\mathrm{d}\omega+\phi\left(\phi^{-1}\left(x+\int_0^{\tau}\ell(\omega)\,\mathrm{d}\omega\right)+\mu\right)\right)\mathrm{d}s.\end{aligned}$$

Similarly

$$\lim_{n\to\infty}\int_0^{\tau_n}\phi^{-1}\left(x_n+\int_0^s \ell_n(\omega)\,\mathrm{d}\omega\right)\mathrm{d}s=\int_0^{\tau}\phi^{-1}\left(x+\int_0^s \ell(\omega)\,\mathrm{d}\omega\right)\mathrm{d}s.$$

We get $\lim_{n\to\infty}F(x_n,\ell_n,\tau_n,\mu_n)=F(x,\ell,\tau,\mu)$. □

Lemma 6 *Assume* (7)*. Then the function $F(\cdot,\ell,\tau,\mu):\mathbb{R}\to\mathbb{R}$ has the following properties:*

(a) it is strictly increasing,

(b) it is continuous,

(c) for $x \in \mathbb{R}$

$$F(x,\ell,\tau,\mu) \geq \tau\phi^{-1}\left(x - \|\ell\|_1\right) + (T-\tau)\phi^{-1}\left(-\|\ell\|_1 + \phi\left(\phi^{-1}\left(x - \|\ell\|_1\right) + \mu\right)\right)$$

and

$$F(x,\ell,\tau,\mu) \leq \tau\phi^{-1}\left(x + \|\ell\|_1\right) + (T-\tau)\phi^{-1}\left(\|\ell\|_1 + \phi\left(\phi^{-1}\left(x + \|\ell\|_1\right) + \mu\right)\right),$$

(d) $\lim_{x\to\pm\infty} F(x,\ell,\tau,\mu) = \pm\infty$,

(e) it has a unique root $A(\ell,\tau,\mu) \in \mathbb{R}$, for which

$$\min\{-\|\ell\|_1, -\|\ell\|_1 + \phi\left(\phi^{-1}\left(-\|\ell\|_1\right) - \mu\right)\} \leq A(\ell,\tau,\mu)$$

and

$$A(\ell,\tau,\mu) \leq \max\{\|\ell\|_1, \|\ell\|_1 + \phi\left(\phi^{-1}\left(\|\ell\|_1\right) - \mu\right)\}.$$

Proof. (a) The monotony follows directly from monotony of ϕ and ϕ^{-1}.
(b) The continuity follows directly from Lemma 5.
(c) Let us prove the first inequality only. The second one can be obtained by similar manners. Let $x \in \mathbb{R}$ be arbitrary. We have

$$x + \int_0^s \ell(\omega)\,\mathrm{d}\omega \geq x - \|\ell\|_1$$

and in view of monotony of ϕ and ϕ^{-1} we get

$$\int_0^\tau \phi^{-1}\left(x + \int_0^s \ell(\omega)\,\mathrm{d}\omega\right)\mathrm{d}s \geq \int_0^\tau \phi^{-1}\left(x - \|\ell\|_1\right)\mathrm{d}s = \tau\phi^{-1}\left(x - \|\ell\|_1\right).$$

Similarly

$$x + \int_0^\tau \ell(\omega)\,\mathrm{d}\omega \geq x - \|\ell\|_1$$

and in view of monotony of ϕ and ϕ^{-1} we get

$$\phi\left(\phi^{-1}\left(x + \int_0^\tau \ell(s)\,\mathrm{d}s\right) + \mu\right) \geq \phi\left(\phi^{-1}\left(x - \|\ell\|_1\right) + \mu\right).$$

Further

$$\begin{aligned}
\int_\tau^T &\phi^{-1}\left(\int_\tau^s \ell(\omega)\,\mathrm{d}\omega + \phi\left(\phi^{-1}\left(x + \int_0^\tau \ell(\omega)\,\mathrm{d}\omega\right) + \mu\right)\right)\mathrm{d}s \\
&\geq \int_\tau^T \phi^{-1}\left(-\|\ell\|_1 + \phi\left(\phi^{-1}\left(x - \|\ell\|_1\right) + \mu\right)\right)\mathrm{d}s \\
&= (T-\tau)\phi^{-1}\left(-\|\ell\|_1 + \phi\left(\phi^{-1}\left(x - \|\ell\|_1\right) + \mu\right)\right),
\end{aligned}$$

which gives

$$F(x,\ell,\tau,\mu) \geq \tau\phi^{-1}\left(x - \|\ell\|_1\right) + (T-\tau)\phi^{-1}\left(-\|\ell\|_1 + \phi\left(\phi^{-1}\left(x - \|\ell\|_1\right) + \mu\right)\right).$$

(d) The values of limits directly follow from (c) and

$$\lim_{x\to\pm\infty} \phi(x) = \pm\infty, \quad \lim_{x\to\pm\infty} \phi^{-1}(x) = \pm\infty.$$

(e) The existence and uniqueness of the root $A(\ell,\tau,\mu)$ of $F(\cdot,\ell,\tau,\mu)$ follow from (a), (b) and (d). Let us prove its upper bound. For the simplicity of notation we put

$$\bar{A}(\ell,\mu) = \max\{\|\ell\|_1, \|\ell\|_1 + \phi\left(\phi^{-1}\left(\|\ell\|_1\right) - \mu\right)\}.$$

From the first inequality in (e) and the monotony of ϕ and ϕ^{-1} we have

$$\begin{aligned} F(\bar{A}(\ell,\mu),\ell,\tau,\mu) &\geq \tau\phi^{-1}\left(\bar{A}(\ell,\mu)-\|\ell\|_1\right) \\ &\quad + (T-\tau)\phi^{-1}\left(-\|\ell\|_1+\phi\left(\phi^{-1}\left(\bar{A}(\ell,\mu)-\|\ell\|_1\right)+\mu\right)\right) \\ &\geq \tau\phi^{-1}\left(\|\ell\|_1-\|\ell\|_1\right) \\ &\quad + (T-\tau)\phi^{-1}\big(-\|\ell\|_1 \\ &\qquad + \phi\left(\phi^{-1}\left(\|\ell\|_1+\phi\left(\phi^{-1}\left(\|\ell\|_1\right)-\mu\right)-\|\ell\|_1\right)+\mu\right)\big) \\ &= 0 = F(A(\ell,\tau,\mu),\ell,\tau,\mu). \end{aligned}$$

The concluded inequality together with (a) implies that $A(\ell,\tau,\mu) \leq \bar{A}(\ell,\mu)$. □

Similarly, we can prove the following lemma for the function G.

Lemma 7 *Assume* (7). *Then the function $G(\cdot,\ell,\tau,\mu):\mathbb{R}\to\mathbb{R}$ has the following properties:*

(a) it is strictly increasing,

(b) it is continuous,

(c) for $x\in\mathbb{R}$

$$G(x,\ell,\tau,\mu) \geq \tau\phi^{-1}\left(-\|\ell\|_1+\phi\left(\phi^{-1}\left(x-\|\ell\|_1\right)-\mu\right)\right)+(T-\tau)\phi^{-1}\left(x-\|\ell\|_1\right)$$

and

$$G(x,\ell,\tau,\mu) \leq \tau\phi^{-1}\left(\|\ell\|_1+\phi\left(\phi^{-1}\left(x+\|\ell\|_1\right)-\mu\right)\right)+(T-\tau)\phi^{-1}\left(x+\|\ell\|_1\right),$$

(d) $\lim_{x\to\pm\infty} G(x,\ell,\tau,\mu)=\pm\infty$,

(e) it has a unique root $B(\ell,\tau,\mu)\in\mathbb{R}$, for which

$$\min\{-\|\ell\|_1,-\|\ell\|_1+\phi\left(\phi^{-1}\left(-\|\ell\|_1\right)+\mu\right)\} \leq B(\ell,\tau,\mu)$$

and

$$B(\ell,\tau,\mu) \leq \max\{\|\ell\|_1,\|\ell\|_1+\phi\left(\phi^{-1}\left(\|\ell\|_1\right)+\mu\right)\}.$$

Due to Lemma 6(e) and Lemma 7(e) we can define the mappings

$$A,B:\mathbb{L}^1([0,T])\times(0,T)\times\mathbb{R}\to\mathbb{R}$$

assigning for every triple (ℓ,τ,μ) a unique root of the functions F, G, respectively. Now, let us turn our attention to the properties of these mappings.

Lemma 8 *Let* (H_2) *be satisfied and $\ell^*,\mu^*\in\mathbb{R}$, $\ell^*,\mu^*>0$. Then for each $(\ell,\tau,\mu)\in\mathbb{L}^1([0,T])\times(0,T)\times\mathbb{R}$ satisfying*

$$\|\ell\|_1\leq\ell^*,\quad |\mu|\leq\mu^*, \tag{13}$$

the bounds

$$-\ell^*+\phi(\phi^{-1}(-\ell^*)-\mu^*)\leq A(\ell,\tau,\mu)\leq \ell^*+\phi(\phi^{-1}(\ell^*)+\mu^*)$$

and

$$-\ell^*+\phi(\phi^{-1}(-\ell^*)-\mu^*)\leq B(\ell,\tau,\mu)\leq \ell^*+\phi(\phi^{-1}(\ell^*)+\mu^*)$$

hold.

Proof. Let (13) be satisfied for some $(\ell,\tau,\mu)\in\mathbb{L}^1([0,T])\times(0,T)\times\mathbb{R}$. Then from (H_2) and Lemma 6(e) it follows that

$$\begin{aligned} A(\ell,\tau,\mu) &\geq \min\{-\|\ell\|_1,-\|\ell\|_1+\phi\left(\phi^{-1}\left(-\|\ell\|_1\right)-\mu\right)\} \\ &\geq \min\{-\ell^*,-\ell^*+\phi(\phi^{-1}(-\ell^*)-\mu^*)\} \\ &= -\ell^*+\phi(\phi^{-1}(-\ell^*)-\mu^*). \end{aligned}$$

Similarly we get the upper bound of $A(\ell,\tau,\mu)$. The bounds for $B(\ell,\tau,\mu)$ can be obtained using Lemma 7.
□

Lemma 9 *Let* (H_2) *be satisfied. The mappings* A, B *are continuous.*

Proof. Let $(\ell_n, \tau_n, \mu_n), (\ell, \tau, \mu) \in \mathbb{L}^1([0,T]) \times (0,T) \times \mathbb{R}$, $n \in \mathbb{N}$ be such that

$$\lim_{n\to\infty} (\ell_n, \tau_n, \mu_n) = (\ell, \tau, \mu) \quad \text{in } \mathbb{L}^1([0,T]) \times \mathbb{R}^2.$$

The convergence implies boundedness, i.e. there exist ℓ^*, $\mu^* \in \mathbb{R}$, $\ell^*, \mu^* > 0$ such that

$$\|\ell_n\|_1 \le \ell^*, \quad |\mu_n| \le \mu^*, \quad n \in \mathbb{N}.$$

From Lemma 8 we get

$$-\ell^* + \phi(\phi^{-1}(-\ell^*) - \mu^*) \le A(\ell_n, \tau_n, \mu_n) \le \ell^* + \phi(\phi^{-1}(\ell^*) + \mu^*), \quad n \in \mathbb{N},$$

i.e. $\{A(\ell_n, \tau_n, \mu_n)\}$ is bounded too. Therefore it has a convergent subsequence – let us denote it by $\{A_{k_n}\}$ and its limit as $\overline{A}$. From Lemma 5 it follows that

$$0 = F(A_{k_n}, \ell_{k_n}, \tau_{k_n}, \mu_{k_n}) \to F(\overline{A}, \ell, \tau, \mu) \quad \text{as } n \to \infty,$$

which implies $F(\overline{A}, \ell, \tau, \mu) = 0$. From Lemma 6(e) it follows that $\overline{A} = A(\ell, \tau, \mu)$. We have proved that each convergent subsequence of the sequence $\{A(\ell_n, \tau_n, \mu_n)\}$ has the same limit equal to $A(\ell, \tau, \mu)$ and therefore the sequence itself is convergent having the same value of limit. □

The following lemma will be needed to prove the compactness of the operator corresponding to the investigated boundary value problem.

Lemma 10 *Let* $S \subset \mathbb{C}([0,T])$ *be a set of uniformly bounded and equicontinuous functions. Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be continuous and increasing on* $\mathbb{R}$. *Then the set*

$$\phi(S) = \{\phi \circ u \ : \ u \in S\}$$

is also a set of uniformly bounded and equicontinuous functions.

Proof. From the boundedness of S it follows the existence of a constant $c > 0$ such that

$$-c \le u(t) \le c, \quad t \in [0,T], \ u \in S.$$

Then

$$\phi(-c) \le \phi(u(t)) \le \phi(c), \quad t \in [0,T], \ u \in S,$$

which proves the boundedness of $\phi(S)$.
Now, let us choose an arbitrary $\epsilon \in \mathbb{R}$, $\epsilon > 0$. According to the uniform continuity of ϕ on $[-c, c]$ there exists $\delta_1 > 0$ such that for each $x_1, x_2 \in [-c, c]$

$$|x_1 - x_2| < \delta_1 \quad \Longrightarrow \quad |\phi(x_1) - \phi(x_2)| < \epsilon. \tag{14}$$

From equicontinuity of functions from the set S there exists $\delta > 0$ such that for each $t_1, t_2 \in [0,T]$ and all $u \in S$

$$|t_1 - t_2| < \delta \quad \Longrightarrow \quad |u(t_1) - u(t_2)| < \delta_1. \tag{15}$$

Since for all $u \in S$ we have $\|u\|_\infty \le c$, from (14), (15) we get for each $t_1, t_2 \in [0,T]$, $u \in S$ the implication

$$|t_1 - t_2| < \delta \quad \Longrightarrow \quad |\phi(u(t_1)) - \phi(u(t_2))| < \epsilon.$$

□

3 Transversality conditions

In this section we assume some constant $K > 0$ and validity of the assumption (H_4). We define the set $\mathcal{B} \subset \mathbb{C}^1([0,T])$ as

$$\mathcal{B} = \{u \in \mathbb{C}^1([0,T]) \ : \ \|u\|_\infty < K, \ \|u'\|_\infty < K/T\}.$$

The following two lemmas were already presented in the past in various forms, see [30, 26, 28, 29, 25]. Their meaning lies in the fact that each function from $\overline{\mathcal{B}}$ crosses the barrier $t = \gamma(x)$ once and the time of the hit has continuous dependence on the choice of such function.

Lemma 11 *Assume a constant $K > 0$ and* (H_4)*. For each $u \in \overline{\mathcal{B}}$, there exists exactly one solution of the equation*

$$t = \gamma(u(t))$$

in $(0,T)$.

Proof. Let $u \in \overline{\mathcal{B}}$. We define the function $\sigma : [0,T] \to \mathbb{R}$ by

$$\sigma(t) = \gamma(u(t)) - t, \quad t \in [0,T].$$

From the properties of γ and u it follows that σ is continuous on $[0,T]$. By (H_4) we have that

$$\sigma(0) = \gamma(u(0)) > 0 \quad \text{and} \quad \sigma(T) = \gamma(u(T)) - T < 0.$$

Therefore there exists a solution of the equation. Let us prove that it is unique. If not, there exist $t, s \in (0,T)$, $t \neq s$ such that $t = \gamma(u(t))$, $s = \gamma(u(s))$. By (H_4) and the inclusion $u \in \overline{\mathcal{B}}$ we have

$$\begin{aligned}|t-s| &= |\gamma(u(t)) - \gamma(u(s))| \le L|u(t) - u(s)| \le L\|u'\|_\infty |t-s| \\ &\le LK/T|t-s| < |t-s|,\end{aligned}$$

which is a contradiction. □

Let us define the operator $\mathcal{P} : \overline{\mathcal{B}} \to (0,T)$ by

$$\mathcal{P} : u \mapsto \tau_u,$$

where τ_u is given by Lemma 11 as a unique solution of the equation $t = \gamma(u(t))$.

Lemma 12 *Assume a constant $K > 0$ and* (H_4)*. The operator $\mathcal{P}$ is continuous on $\overline{\mathcal{B}}$ in $\mathbb{C}([0,T])$.*

Proof. Let $u, v \in \overline{\mathcal{B}}$. Then $\mathcal{P}u = \gamma(u(\mathcal{P}u))$, $\mathcal{P}v = \gamma(v(\mathcal{P}v))$ and by (H_4) we get

$$\begin{aligned}|\mathcal{P}u - \mathcal{P}v| &\le |\gamma(u(\mathcal{P}u)) - \gamma(v(\mathcal{P}v))| \le L|u(\mathcal{P}u) - v(\mathcal{P}v)| \\ &\le L|u(\mathcal{P}u) - u(\mathcal{P}v) + u(\mathcal{P}v) - v(\mathcal{P}v)| \\ &\le L\|u'\|_\infty |\mathcal{P}u - \mathcal{P}v| + L\|u-v\|_\infty \le LK/T|\mathcal{P}u - \mathcal{P}v| + L\|u-v\|_\infty.\end{aligned}$$

Therefore

$$|\mathcal{P}u - \mathcal{P}v| \le \frac{L}{1 - LK/T}\|u-v\|_\infty.$$

□

4 Fixed point problem

In this section we construct the operator, corresponding to the boundary value problem (1)–(3), defined on a certain subset of some Banach space.

Let us denote

$$\mathbb{X} = \mathbb{C}^1([0,T]) \times \mathbb{C}^1([0,T])$$

and the set

$$\Omega = \mathcal{B} \times \mathcal{B} \subset \mathbb{X},$$

with $\mathcal{B}$ defined in the previous section for arbitrary $K > 0$. Let us define the mapping $\widetilde{f} : (0,T) \times \overline{\Omega} \to \mathbb{L}^1([0,T])$ by

$$\widetilde{f}(\tau, v, w)(t) = \begin{cases} f(t, v(t), v'(t)), & t \in [0, \tau], \\ f(t, w(t), w'(t)), & t \in (\tau, T], \end{cases}$$

for $(\tau, v, w) \in (0,T) \times \overline{\Omega}$.

Lemma 13 *Assume some constant $K > 0$ and* (H_1). *Then the mapping $\widetilde{f}$ is continuous.*

Proof. Let $(\tau_n, v_n, w_n), (\tau, v, w) \in (0,T) \times \overline{\Omega}$, $n \in \mathbb{N}$ be such that

$$\lim_{n\to\infty} (\tau_n, v_n, w_n) = (\tau, v, w) \quad \text{in } \mathbb{R} \times \mathbb{X}.$$

Let $n \in \mathbb{N}$ be such that $\tau \le \tau_n$. Then

$$\begin{aligned}
&\int_0^T \left| \widetilde{f}(\tau_n, v_n, w_n)(s) - \widetilde{f}(\tau, v, w)(s) \right| \mathrm{d}s \\
&\quad = \int_0^\tau \left| \widetilde{f}(\tau_n, v_n, w_n)(s) - \widetilde{f}(\tau, v, w)(s) \right| \mathrm{d}s \\
&\qquad + \int_\tau^{\tau_n} \left| \widetilde{f}(\tau_n, v_n, w_n)(s) - \widetilde{f}(\tau, v, w)(s) \right| \mathrm{d}s \\
&\qquad + \int_{\tau_n}^T \left| \widetilde{f}(\tau_n, v_n, w_n)(s) - \widetilde{f}(\tau, v, w)(s) \right| \mathrm{d}s \\
&\quad = \int_0^\tau |f(s, v_n(s), v_n'(s)) - f(s, v(s), v'(s))| \,\mathrm{d}s \\
&\qquad + \int_\tau^{\tau_n} |f(s, v_n(s), v_n'(s)) - f(s, w(s), w'(s))| \,\mathrm{d}s \\
&\qquad + \int_{\tau_n}^T |f(s, w_n(s), w_n'(s)) - f(s, w(s), w'(s))| \,\mathrm{d}s \\
&\quad \le \int_0^T |f(s, v_n(s), v_n'(s)) - f(s, v(s), v'(s))| \,\mathrm{d}s \\
&\qquad + \int_0^T |f(s, w_n(s), w_n'(s)) - f(s, w(s), w'(s))| \,\mathrm{d}s \\
&\qquad + 2 \left| \int_\tau^{\tau_n} m(s) \,\mathrm{d}s \right|,
\end{aligned}$$

where $m \in \mathbb{L}^1([0,T])$ is the function from the third Carathéodory property for f and a compact set

$$E = [-K, K] \times [-K/T, K/T].$$

The last inequality is true for the case $\tau_n \le \tau$, as well. Since $\lim_{n\to\infty} \tau_n = \tau$, $v_n \to v$, $w_n \to w$ in $\mathbb{C}^1([0,T])$ and from the Carathéodory property of f, we conclude that $\widetilde{f}(\tau_n, v_n, w_n) \to \widetilde{f}(\tau, v, w)$ in $\mathbb{L}^1([0,T])$. $\square$

Let us note that in Lemma 13 the constant K was arbitrary positive real number. From now on, this constant will have the meaning from (H_3). As a consequence, the set $\mathcal{B}$ will be considered with the constant K from (H_3).

Lemma 14 *Assume* (H_1)–(H_4). *Let* $v, w \in \overline{\mathcal{B}}$ *be such that*

$$
\begin{aligned}
v(t) = \int_0^t \phi^{-1}\Big(& A\left(\widetilde{f}(\mathcal{P}v, v, w), \mathcal{P}v, M(v(\mathcal{P}v), v'(\mathcal{P}v))\right) \\
& + \int_0^s \widetilde{f}(\mathcal{P}v, v, w)(\omega)\,\mathrm{d}\omega\Big)\,\mathrm{d}s, \quad t \in [0,T], \\
w(t) = \int_T^t \phi^{-1}\Big(& B\left(\widetilde{f}(\mathcal{P}v, v, w), \mathcal{P}v, M(v(\mathcal{P}v), v'(\mathcal{P}v))\right) \\
& + \int_T^s \widetilde{f}(\mathcal{P}v, v, w)(\omega)\,\mathrm{d}\omega\Big)\,\mathrm{d}s, \quad t \in [0,T].
\end{aligned}
\tag{16}
$$

Then the function

$$
z(t) = \begin{cases} v(t), & t \in [0, \mathcal{P}v], \\ w(t), & t \in (\mathcal{P}v, T] \end{cases}
\tag{17}
$$

is a solution of problem (1)–(3) *hitting the barrier* γ *exactly once.*

Proof. Let $v, w \in \overline{\mathcal{B}}$ satisfy (16) and z be defined by (17). By Lemma 3, 4 and the definition of mappings A and B, z is a solution of problem (4)–(6) with

$$
\ell(t) = \widetilde{f}(\mathcal{P}v, v, w)(t) \text{ for a.e. } t \in [0,T],\ \tau = \mathcal{P}v,\ \mu = M(v(\mathcal{P}v), v'(\mathcal{P}v)).
\tag{18}
$$

By Lemma 3 the definition of $\widetilde{f}$ and (17), z satisfies

$$
(\phi(z'(t)))' = \widetilde{f}(\mathcal{P}v, v, w)(t) = f(t, v(t), v'(t)) = f(t, z(t), z'(t))
$$

for a.e. $t \in [0, \mathcal{P}v]$ and

$$
(\phi(z'(t)))' = \widetilde{f}(\mathcal{P}v, v, w)(t) = f(t, w(t), w'(t)) = f(t, z(t), z'(t))
$$

for a.e. $t \in (\mathcal{P}v, T]$, i.e. z is a solution of differential equation (1). Further, by Lemma 3, z is continuous on $[0,T]$ and satisfies

$$
\triangle z'(\mathcal{P}v) = M(v(\mathcal{P}v), v'(\mathcal{P}v)) = M(z(\mathcal{P}v), z'(\mathcal{P}v-)),
$$

where $\mathcal{P}v \in (0,T)$ is such that $\mathcal{P}v = \gamma(v(\mathcal{P}v)) = \gamma(z(\mathcal{P}v))$. This is the condition (2). Since z satisfies (6), it satisfies (3) as well. The only thing that left to prove is that z hits the barrier exactly once. It suffices to prove that

$$
t \neq \gamma(z(t)) \quad \text{for all } t \in [0, \mathcal{P}v) \cup (\mathcal{P}v, T].
$$

From the fact that $\mathcal{P}v$ is the only solution of equation $t = \gamma(v(t))$ on $[0,T]$ and from (17) it follows

$$
t \neq \gamma(v(t)) = \gamma(z(t)) \quad t \in [0, \mathcal{P}v).
$$

Moreover, from the continuity of z at $\mathcal{P}v$ it follows that $v(\mathcal{P}v) = z(\mathcal{P}v) = w(\mathcal{P}v)$ and hence

$$
\mathcal{P}v = \gamma(v(\mathcal{P}v)) = \gamma(w(\mathcal{P}v)),
$$

i.e. $\mathcal{P}v$ is the only solution of $t = \gamma(w(t))$, as well. Hence

$$
t \neq \gamma(w(t)) = \gamma(z(t)) \quad t \in (\mathcal{P}v, T].
$$

$\square$

Now, we are motivated enough to introduce the operator corresponding to problem (1)–(3). We define the operator $\mathcal{F} : \overline{\Omega} \to \mathbb{X}$ by $\mathcal{F}(v, w) = (x, y) \in \mathbb{X}$, where

$$\begin{aligned} x(t) = \int_0^t \phi^{-1}\Big(A\left(\widetilde{f}(\mathcal{P}v, v, w), \mathcal{P}v, M(v(\mathcal{P}v), v'(\mathcal{P}v))\right) \\ + \int_0^s \widetilde{f}(\mathcal{P}v, v, w)(\omega)\,\mathrm{d}\omega\Big)\,\mathrm{d}s, \quad t \in [0, T], \\ y(t) = \int_T^t \phi^{-1}\Big(B\left(\widetilde{f}(\mathcal{P}v, v, w), \mathcal{P}v, M(v(\mathcal{P}v), v'(\mathcal{P}v))\right) \\ + \int_T^s \widetilde{f}(\mathcal{P}v, v, w)(\omega)\,\mathrm{d}\omega\Big)\,\mathrm{d}s, \quad t \in [0, T]. \end{aligned} \tag{19}$$

Remark 15 *Obviously, a fixed point* (v, w) *of operator* $\mathcal{F}$ *is a solution of system* (16)*, which by Lemma 14 gives a solution* z *of problem* (1)*–*(3) *defined by* (17)*.*

5 Main results

Here we investigate the properties of the operator $\mathcal{F}$ and prove that it has at least one fixed point. From this fixed point we construct a solution of problem (1)–(3).

Lemma 16 *Assume* (H_1)*–*(H_4)*. The operator* $\mathcal{F}$ *is compact.*

Proof. Let us prove the continuity of $\mathcal{F}$. Let $(v_n, w_n) \in \overline{\Omega}$, $n \in \mathbb{N}$, $(v, w) \in \overline{\Omega}$ and

$$(v_n, w_n) \to (v, w) \quad \text{in } \mathbb{X}. \tag{20}$$

Let us denote

$$\tau_n = \mathcal{P}v_n, (x_n, y_n) = \mathcal{F}(v_n, w_n),\ n \in \mathbb{N},\ \tau = \mathcal{P}v,\ (x, y) = \mathcal{F}(v, w).$$

By Lemma 12 we get $\lim_{n\to\infty} \tau_n = \tau$. Together with (20) we have

$$\lim_{n\to\infty} v_n(\tau_n) = v(\tau) \quad \text{and} \quad \lim_{n\to\infty} v_n'(\tau_n) = v'(\tau),$$

and by (H_1)

$$\lim_{n\to\infty} M(v_n(\tau_n), v_n'(\tau_n)) = M(v(\tau), v'(\tau)).$$

From Lemma 13 it follows that

$$\lim_{n\to\infty} \int_0^T |\widetilde{f}(\tau_n, v_n, w_n)(s) - \widetilde{f}(\tau, v, w)(s)|\,\mathrm{d}s = 0.$$

These facts together with Lemma 9 give

$$\lim_{n\to\infty} A(\widetilde{f}(\tau_n, v_n, w_n), \tau_n, M(v_n(\tau_n), v_n'(\tau_n))) = A(\widetilde{f}(\tau, v, w), \tau, M(v(\tau), v'(\tau)))$$

and

$$\lim_{n\to\infty} B(\widetilde{f}(\tau_n, v_n, w_n), \tau_n, M(v_n(\tau_n), v_n'(\tau_n))) = B(\widetilde{f}(\tau, v, w), \tau, M(v(\tau), v'(\tau))).$$

Further, using the continuity of ϕ^{-1} we see that $x_n' \to x'$, $y_n' \to y'$ in $\mathbb{C}([0, T])$ as $n \to \infty$. Since $x_n(0) = 0 \to x(0) = 0$ and $y_n(T) = 0 \to y(T) = 0$ as $n \to \infty$, we conclude that $x_n \to x$, $y_n \to y$ in $\mathbb{C}^1([0, T])$ as $n \to \infty$.

Let us turn our attention to the relative compactness of the set $\mathcal{F}(\overline{\Omega})$. Let $(v, w) \in \overline{\Omega}$. Then

$$|\widetilde{f}(\mathcal{P}v, v, w)(t)| \le \max\{|f(t, v(t), v'(t))|, |f(t, w(t), w'(t))|\} \le m(t) \tag{21}$$

for a.e. $t \in [0,T]$, where $m \in \mathbb{L}^1([0,T])$ is the function from the third Carathéodory property of f for compact set $E = [-K,K] \times [-K/T, K/T]$. Let $(x,y) \in \mathcal{F}(\overline{\Omega})$, i.e. there exists $(v,w) \in \overline{\Omega}$ such that $(x,y) = \mathcal{F}(v,w)$. From the definition of $\mathcal{F}$ we have

$$x'(t) = \phi^{-1}\left(A\left(\widetilde{f}(\mathcal{P}v,v,w),\mathcal{P}v,M(v(\mathcal{P}v),v'(\mathcal{P}v))\right) + \int_0^t \widetilde{f}(\mathcal{P}v,v,w)(s)\,\mathrm{d}s\right)$$

for all $t \in [0,T]$. Since $v,w \in \overline{\mathcal{B}}$, Lemma 9 gives us

$$\begin{aligned}-\|m\|_1 + \phi(\phi^{-1}(-\|m\|_1) - \overline{M}) \le A\left(\widetilde{f}(\mathcal{P}v,v,w),\mathcal{P}v,M(v(\mathcal{P}v),v'(\mathcal{P}v))\right)\\ \le \|m\|_1 + \phi(\phi^{-1}(\|m\|_1) + \overline{M}).\end{aligned}$$

These inequalities and (21) give

$$-2\|m\|_1 + \phi(\phi^{-1}(-\|m\|_1) - \overline{M}) \le \phi(x'(t)) \le 2\|m\|_1 + \phi(\phi^{-1}(\|m\|_1) + \overline{M})$$

and the monotony of ϕ^{-1} that

$$\begin{aligned}\phi^{-1}\left(-2\|m\|_1 + \phi(\phi^{-1}(-\|m\|_1) - \overline{M})\right) \le x'(t)\\ \le \phi^{-1}\left(2\|m\|_1 + \phi(\phi^{-1}(\|m\|_1) + \overline{M})\right)\end{aligned}$$

for all $t \in [0,T]$. The same bounds hold also for $y'(t)$. Since $x(0) = 0$ and $y(T) = 0$, by integrating of obtained inequalities, we get the bounds also for x and y depending only on m and $\overline{M}$. Therefore $\mathcal{F}(\overline{\Omega})$ is bounded in $\mathbb{X}$.

Further, from the definition of $\mathcal{F}$ and by (21) we have

$$|(\phi(x'(t)))'| = |(\phi(y'(t)))'| = |\widetilde{f}(\mathcal{P}v,v,w)(t)| \le m(t) \quad \text{for a.e. } t \in [0,T].$$

It follows that

$$\{\phi \circ x' \;:\; (x,y) \in \mathcal{F}(\overline{\Omega})\} \quad \text{and} \quad \{\phi \circ y' \;:\; (x,y) \in \mathcal{F}(\overline{\Omega})\}$$

are sets of equicontinuous functions. From Lemma 10 we have that

$$\{x' \;:\; (x,y) \in \mathcal{F}(\overline{\Omega})\} \quad \text{and} \quad \{y' \;:\; (x,y) \in \mathcal{F}(\overline{\Omega})\}$$

are sets of equicontinuous functions, too. Therefore

$$\{(x',y') \;:\; (x,y) \in \mathcal{F}(\overline{\Omega})\}$$

is a set of equicontinuous (vector valued) functions, as well. Consequently, by Arzelà–Ascoli Theorem, we have proved that $\mathcal{F}(\overline{\Omega})$ is a relatively compact set in $\mathbb{X}$. □

Lemma 17 *Let* (H_1)–(H_4) *be satisfied. Then for each* $u \in \overline{\Omega}$ *and* $\lambda \in [0,1]$ *satisfying* $u = \lambda\mathcal{F}u$*, the inclusion* $u \in \Omega$ *holds.*

Proof. Let $u = (v,w) \in \overline{\Omega}$, $\lambda \in [0,1]$ be such that

$$u = \lambda\mathcal{F}u.$$

Obviously, for $\lambda = 0$ we have $u = 0 \in \Omega$. Let $\lambda > 0$. Then

$$v'(t) = \lambda\phi^{-1}\left(A\left(\widetilde{f}(\mathcal{P}v,v,w),\mathcal{P}v,M(v(\mathcal{P}v),v'(\mathcal{P}v))\right) + \int_0^t \widetilde{f}(\mathcal{P}v,v,w)(s)\,\mathrm{d}s\right),$$

for all $t \in [0,T]$. Hence

$$\phi(\lambda^{-1}v'(t)) = A\left(\widetilde{f}(\mathcal{P}v,v,w),\mathcal{P}v,M(v(\mathcal{P}v),v'(\mathcal{P}v))\right) + \int_0^t \widetilde{f}(\mathcal{P}v,v,w)(s)\,\mathrm{d}s \tag{22}$$

for all $t \in [0,T]$. From the definition of $\widetilde{f}$ and the facts $v, w \in \overline{\mathcal{B}}$, we get

$$|\widetilde{f}(\mathcal{P}v, v, w)(t)| \leq \max\{|f(t, v(t), v'(t))|, |f(t, w(t), w'(t))|\} \leq h(t, K, K/T)$$

for a.e. $t \in [0,T]$, and consequently

$$\int_0^T |\widetilde{f}(\mathcal{P}v, v, w)(s)| \,\mathrm{d}s \leq \int_0^T h(s, K, K/T) \,\mathrm{d}s. \tag{23}$$

From Lemma 8 and (23) we have

$$\begin{aligned} &A\left(\widetilde{f}(\mathcal{P}v, v, w), \mathcal{P}v, M(v(\mathcal{P}v), v'(\mathcal{P}v))\right) + \int_0^t \widetilde{f}(\mathcal{P}v, v, w)(s) \,\mathrm{d}s \\ &\leq 2\int_0^T h(s, K, K/T) \,\mathrm{d}s + \phi\left(\phi^{-1}\left(\int_0^T h(s, K, K/T) \,\mathrm{d}s\right) + \overline{M}\right) \end{aligned}$$

and

$$\begin{aligned} &A\left(\widetilde{f}(\mathcal{P}v, v, w), \mathcal{P}v, M(v(\mathcal{P}v), v'(\mathcal{P}v))\right) + \int_0^t \widetilde{f}(\mathcal{P}v, v, w)(s) \,\mathrm{d}s \\ &\geq -2\int_0^T h(s, K, K/T) \,\mathrm{d}s + \phi\left(\phi^{-1}\left(-\int_0^T h(s, K, K/T) \,\mathrm{d}s\right) - \overline{M}\right). \end{aligned}$$

These inequalities, (22) and (H$_3$) imply

$$\phi(-K/T) < \phi(\lambda^{-1}v'(t)) < \phi(K/T), \quad t \in [0,T].$$

Using the monotony of ϕ^{-1} we obtain

$$-K/T \leq -\lambda K/T < v'(t) < \lambda K/T \leq K/T, \quad t \in [0,T].$$

The continuity of v' on $[0,T]$ gives

$$\|v'\|_\infty < K/T.$$

This inequality together with $v(0) = 0$ implies

$$|v(t)| = \left|v(0) + \int_0^t v'(s) \,\mathrm{d}s\right| \leq T\|v'\|_\infty < TK/T = K, \quad t \in [0,T],$$

which, again according to the continuity of v on $[0,T]$, proves the inclusion $v \in \mathcal{B}$. Similarly, we get $w \in \mathcal{B}$ and therefore $u = (v, w) \in \Omega$. □

Theorem 18 *Let* (H$_1$)–(H$_4$) *be satisfied. Then there exists at least one solution z of* (1)–(3) *hitting the barrier at exactly one instant and*

$$\|z\|_\infty < K.$$

Proof. Under the assumptions (H$_1$)–(H$_4$) we can define the set $\Omega \subset \mathbb{X}$, and the operator $\mathcal{F} : \overline{\Omega} \to \mathbb{X}$ by (19). By Lemma 16 and 17 the mapping $\mathcal{I} - \mathcal{F}$, where $\mathcal{I} : \mathbb{X} \to \mathbb{X}$ stands for identity mapping, has nonzero Leray–Schauder degree on Ω and hence it has a fixed point $(v, w) \in \Omega$. By Remark 15, the function z defined by (17) is a solution of (1)–(3). The bound of its norm follows from

$$\|z\|_\infty \leq \max\{\|v\|_\infty, \|w\|_\infty\}$$

and the inclusions $u, v \in \overline{\mathcal{B}}$. □

Remark 19 *If the function ϕ is odd, then the two inequalities in* (H$_3$) *are the same, i.e. in this case it suffices to check just one of them.*

To show that Theorem 18 is applicable, let us present the following example.

Example 20 *Let us consider problem* (1)–(3) *with*

$$\phi(x) = x^3, \quad f(t,x,y) = t^\alpha + |x|^\beta \operatorname{sgn} x, \quad \alpha \geq 0, \ \beta \in [0,1], \tag{24}$$

$$M(x,y) = -\sqrt[3]{y}, \quad \gamma(x) = \frac{T}{2} + L \sin x, \quad x, y \in \mathbb{R}, \ \textit{where} \ |L| < \frac{T}{2}. \tag{25}$$

Obviously, the assumptions (H_1) *and* (H_2) *are satisfied. Let us check the validity of condition* (H_3). *We put*

$$h(t,x,y) = t^\alpha + x^\beta, \quad (t,x,y) \in [0,T] \times [0,\infty)^2.$$

According to Remark 19, it suffices to find such $K > 0$ *for which*

$$2\left(\frac{T^{\alpha+1}}{\alpha+1} + TK^\beta\right) + \left(\sqrt[3]{\frac{T^{\alpha+1}}{\alpha+1} + TK^\beta} + \sqrt[3]{\frac{K}{T}}\right)^3 < \left(\frac{K}{T}\right)^3.$$

This is equivalent to the inequality

$$2\left(\frac{T^{\alpha+4}}{(\alpha+1)K^3} + T^4 K^{\beta-3}\right) + \left(\frac{T}{K}\right)^3 \left(\sqrt[3]{\frac{T^{\alpha+1}}{\alpha+1} + TK^\beta} + \sqrt[3]{\frac{K}{T}}\right)^3 < 1.$$

For sufficiently large K *the left-hand side of the last inequality can be made arbitrarily small. For instance, if*

$$\alpha = 2, \quad \beta = 1, \quad T = 1,$$

the inequality from (H_3) *reduces to*

$$\frac{2}{3K^3} + \frac{2}{K^2} + \frac{1}{K^3}\left(\sqrt[3]{\frac{1}{3} + K} + \sqrt[3]{K}\right)^3 < 1,$$

which is satisfied for $K = 4$. *The assumption* (H_4) *holds for each* $L \in \mathbb{R}$ *satisfying*

$$|L| < T \min\left\{\frac{1}{2}, \frac{1}{K}\right\}.$$

Due to Theorem 18 there exists at least one solution z *of the problem having exactly one jump.*

References

[1] R.P. Agarwal and D. O'Regan. Multiple nonnegative solutions for second order impulsive differential equations. *Appl. Math. Comput.*, 114(1):51–59, 2000.

[2] N.V. Azbelev, V.P. Maksimov, and L.F. Rakhmatullina. *Introduction to the theory of functional differential equations.* Nauka, Moscow, 1991. (in Russian).

[3] L. Bai and B. Dai. An application of variational method to a class of Dirichlet boundary value problems with impulsive effects. *J. Franklin Inst.*, 348:2607–2624, 2011.

[4] L Bai and B. Dai. Three solutions for a p-Laplacian boundary value problem with impulsive effects. *Appl. Math. Comput.*, 217:9895–9904, 2011.

[5] D.D. Bainov and P.S. Simeonov. *Impulsive Differential Equations: Periodic Solutions and Applications*, volume 66 of *Pitman Monographs and Surveys in Pure and Applied Mathematics.* Longman Scientific and Technical, Essex, England, 1993.

[6] I. Bajo and E. Liz. Periodic boundary value problem for first order differential equations with impulses at variable times. *J. Math. Anal. Appl.*, 204(1):65–73, 1996.

[7] J.M. Belley and M. Virgilio. Periodic Duffing delay equations with state dependent impulses. *J. Math. Anal. Appl.*, 306(2):646–662, 2005.

[8] J.M. Belley and M. Virgilio. Periodic Liénard-type delay equations with state-dependent impulses. *Nonlinear Anal., Theory Methods Appl.*, 64(3):568–589, 2006.

[9] M. Benchohra, J.R. Graef, S.K. Ntouyas, and A. Ouahab. Upper and lower solutions method for impulsive differential inclusions with nonlinear boundary conditions and variable times. *Dynam. Contin. Discrete Impuls. Systems, Ser. A, Math. Anal.*, 12(3-4):383–396, 2005.

[10] A. Cabada, E. Liz, and S. Lois. Green's function and maximum principle for higher order ordinary differential equations with impulses. *Rocky Mountain Journal of Mathematics*, 30:435–444, 2000.

[11] A. Cabada and B. Thompson. Nonlinear second-order equations with functional implicit impulses and nonlinear functional boundary conditions. *Nonlinear Anal., Theory Methods Appl.*, 74:7198–7209, 2011.

[12] A. Cabada and J. Tomeček. Extremal solutions for nonlinear functional ϕ-Laplacian impulsive equations. *Nonlinear Anal.*, 67:827–841, 2007.

[13] J. Chu and J.J. Nieto. Impulsive periodic solutions of first-order singular differential equations. *Bull. London Math. Soc.*, 40:143–150, 2008.

[14] M. Feng, B. Du, and W. Ge. Impulsive boundary value problems with integral boundary conditions and one-dimensional p-Laplacian. *Nonlinear Anal., Theory Methods Appl.*, 70(9):3119–3126, 2009.

[15] M. Ferrara and Sh. Heidarkhani. Multiple solutions for perturbed p-Laplacian boundary-value problems with impulsive effects. *Electron. J. Differential Equations*, 2014(106):1–14, 2014.

[16] M. Frigon and D. O'Regan. First order impulsive initial and periodic problems with variable moments. *J. Math. Anal. Appl.*, 233(2):730–739, 1999.

[17] M. Frigon and D. O'Regan. Second order Sturm-Liouville BVP's with impulses at variable moments. *Dynam. Contin. Discrete Impuls. Systems, Ser. A, Math. Anal.*, 8(2):149–159, 2001.

[18] M. Galewski. On variational impulsive boundary value problems. *Cent. Eur. J. Math.*, 10(6):1969–1980, 2012.

[19] J. Li, J.J. Nieto, and J. Shen. Impulsive periodic boundary value problems of first-order differential equations. *J. Math. Anal. Appl.*, 325:226–236, 2007.

[20] P. Li and Y. Wu. Triple positive solutions for nth-order impulsive differential equations with integral boundary conditions and p-Laplacian. *Results. Math.*, 61:401–419, 2012.

[21] S. Liang and J. Zhang. The existence of countably many positive solutions for some nonlinear singular three-point impulsive boundary value problems. *Nonlinear Anal., Theory Methods Appl.*, 71(10):4588–4597, 2009.

[22] J.J. Nieto. Basic theory for nonresonance impulsive periodic problems of first order. *J. Math. Anal. Appl.*, 205:423–433, 1997.

[23] J.J. Nieto and D. O'Regan. Variational approach to impulsive differential equations. *Nonlinear Anal., Real World Appl.*, 10:680–690, 2009.

[24] V. Polášek. Periodic BVP with ϕ-Laplacian and impulses. *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica*, 44(1):131–150, 2005.

[25] I. Rachůnková and L. Rachůnek. First-order nonlinear differential equations with state-dependent impulses. *Bound. Value Probl.*, 2013:1–18, 2013. Article ID 195.

[26] I. Rachůnková and J. Tomeček. Existence principle for BVPs with state-dependent impulses. Topol. Methods Nonlinear Anal., to appear.

[27] I. Rachůnková and J. Tomeček. A new approach to BVPs with state-dependent impulses. *Bound. Value Probl.*, 2013, 2013. Article ID 22.

[28] I. Rachůnková and J. Tomeček. Existence principle for higher-order nonlinear differential equations with state-dependent impulses via fixed point theorem. *Bound. Value Probl.*, 2014(1):1–15, 2014. Article ID 118.

[29] I. Rachůnková and J. Tomeček. Fixed point problem associated with state-dependent impulsive boundary value problems. *Bound. Value Probl.*, 2014(1):1–17, 2014. Article ID 172.

[30] I. Rachůnková and J. Tomeček. Second order BVPs with state dependent impulses via lower and upper functions. *Cent. Eur. J. Math.*, 12(1):128–140, 2014.

[31] I. Rachůnková and M. Tvrdý. Second-order periodic problem with ϕ-Laplacian and impulses. *Nonlinear Anal.*, 63:e257–e266, 2005.

[32] A.M. Samoilenko and N.A. Perestyuk. *Impulsive Differential Equations.* World Scientific, Singapore, 1995.

[33] K. Teng and C. Zhang. Existence of solution to boundary value problem for impulsive differential equations. *Nonlinear Anal., Real World Appl.*, 11:4431–4441, 2010.

[34] Y. Tian and W. Ge. Multiple positive solutions of second-order Sturm-Liouville boundary value problems for impulsive differential equations. *Rocky Mt. J. Math.*, 40(2):643–672, 2010.

[35] L. Wang and W. Ge. Infinitely many many solutions of a second-order p-Laplacian problem with impulsive conditions. *Appl. Math.*, 55(5):405–418, 2010.

[36] L. Wang and M. Pei. Infinitely many solutions of a Sturm-Liouville system with impulses. *J. Appl. Math. Comput.*, 35:577–593, 2011.

[37] J. Xu, P. Kang, and Z. Wei. Singular multipoint impulsive boundary value problem with p-Laplacian operator. *J. Appl. Math. Comput.*, 30(1-2):105–120, 2009.

[38] D. Zhang and B. Dai. Existence of solutions for nonlinear impulsive differential equations with Dirichlet boundary conditions. *Math. Comput. Modelling*, 53:1154–1161, 2011.