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Dirichlet curves, convex order and Cauchy distribution

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To the memory of Jean-François Chamayou (1942–2016)

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If α is a probability on $\mathbb{R}^d$ and $t > 0$, the Dirichlet random probability $P_t \sim \mathcal{D}(t\alpha)$ is such that for any measurable partition $(A_0, \dots, A_k)$ of $\mathbb{R}^d$ the random variable $(P_t(A_0), \dots, P_t(A_k))$ is Dirichlet distributed with parameters $(t\alpha(A_0), \dots, t\alpha(A_k))$. If $\int_{\mathbb{R}^d} \log(1 + \|x\|)\alpha(dx) < \infty$ the random variable $\int_{\mathbb{R}^d} x P_t(dx)$ of $\mathbb{R}^d$ does exist: let $\mu(t\alpha)$ be its distribution. The Dirichlet curve associated to the probability α is the map $t \mapsto \mu(t\alpha)$. It has simple properties like $\lim_{t\searrow 0} \mu(t\alpha) = \alpha$ and $\lim_{t\to\infty} \mu(t\alpha) = \delta_m$ when $m = \int_{\mathbb{R}^d} x\alpha(dx)$ exists. The present paper shows that if m exists and if ψ is a convex function on $\mathbb{R}^d$ then $t \mapsto \int_{\mathbb{R}^d} \psi(x)\mu(t\alpha)(dx)$ is a decreasing function, which means that $t \mapsto \mu(t\alpha)$ is decreasing according to the Strassen convex order of probabilities. The second aim of the paper is to prove a group of results around the following question: if $\mu(t\alpha) = \mu(s\alpha)$ for some $0 \le s < t$, can we claim that μ is Cauchy distributed in $\mathbb{R}^d$?

Keywords: Cauchy distribution; Dirichlet random probability; Strassen convex order

1. Introduction

If $a_0, \dots, a_k > 0$ and $t = a_0 + \dots + a_k$ recall that the Dirichlet distribution $\mathcal{D}(a_0, \dots, a_k)$ (as named by Wilks [28]) is the law of the random variable $(X_0, \dots, X_k)$ of $\mathbb{R}^{k+1}$ such that $X_i \ge 0$ for all $i = 0, \dots, k$ and $X_0 + \dots + X_k = 1$, with the density of $(X_1, \dots, X_k)$ equal to

$$\frac{\Gamma(t)}{\Gamma(a_0)\cdots\Gamma(a_k)}(1 - x_1 - \dots - x_k)^{a_0-1} x_1^{a_1-1} \cdots x_k^{a_k-1}.$$

For $f_0, \dots, f_k > 0$ it satisfies

$$\mathbb{E}\left(\frac{1}{(f_0X_0 + \dots + f_kX_k)^t}\right) = \frac{1}{f_0^{a_0} \cdots f_k^{a_k}}. \tag{1}$$

See, for instance [2], Proposition 2.1. By considering moments, we can prove the following weak limits:

$$\lim_{r\to\infty} \mathcal{D}(ra_0, \dots, ra_k) = \delta_{(a_0/t,\dots,a_k/t)}, \tag{2}$$

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$$\lim_{\varepsilon\to 0}\mathcal{D}(\varepsilon a_0,\ldots,\varepsilon a_k)=\sum_{i=0}^{k}\frac{a_i}{t}\delta_{e_i}, \tag{3}$$

where $(e_0,\ldots,e_k)$ is the canonical basis of $\mathbb{R}^{k+1}$.

More generally, consider a measured space $(\Omega,\mathfrak{A},t\alpha)$ where α is a probability on Ω and $t>0$. A quick way to introduce the Dirichlet random probability P_t on Ω associated to the bounded measure $t\alpha$ follows Sethuraman's stick breaking method: select independent random variables $B_1,Y_1,\ldots,B_n,Y_n,\ldots$ such that for all $n\geq 1$ we have $B_n\sim\alpha$ and $Y_n\sim\beta(1,t)(dy)=t(1-y)^{t-1}\mathbf{1}_{(0,1)}(y)\,dy$. Then define $W_1=Y_1$ and for $n>1$

$$W_n=Y_n(1-Y_{n-1})\cdots(1-Y_1).$$

It is an easy consequence of the strong law of large numbers that with probability 1, as $N\to\infty$ one has $\sum_{n=1}^{N}W_n=1-(1-Y_1)\cdots(1-Y_N)\to 1$. Sethuraman [24] has proved that the random purely atomic probability P_t on Ω defined by

$$P_t(dw)=\sum_{n=1}^{\infty}W_n\delta_{B_n}(dw), \tag{4}$$

satisfies for any measurable partition $(A_0,\ldots,A_k)$ of Ω

$$\big(P_t(A_0),\ldots,P_t(A_k)\big)\sim\mathcal{D}\big(t\alpha(A_0),\ldots,t\alpha(A_k)\big). \tag{5}$$

For this reason, the random probability P_t is said to be a Dirichlet random probability and its distribution is denoted by $\mathcal{D}(t\alpha)$. One says also that α is the governing probability of P_t and that t is its intensity. Of course, $(P_t)_{t\geq 0}$ has a venerable story and the papers by [4,6,9] and [17] are among the important papers to read on the subject.

Some simple considerations about $\{\mathcal{D}(t\alpha),t>0\}$ are in order. If f is a real bounded measurable function defined on Ω and if $P_t\sim\mathcal{D}(t\alpha)$, then the Fourier transform of the real random variable

$$X_t(f)=\int_\Omega f(w)P_t(dw)=\sum_{n=1}^{\infty}W_n f(B_n)$$

will satisfy for real s:

$$\lim_{t\to\infty}\mathbb{E}\big(e^{is\int_\Omega f(w)P_t(dw)}\big)=e^{is\int_\Omega f(w)\alpha(dw)}, \tag{6}$$

$$\lim_{t\searrow 0}\mathbb{E}\big(e^{is\int_\Omega f(w)P_t(dw)}\big)=\int_\Omega e^{isf(w)}\alpha(dw). \tag{7}$$

If f is taking a finite number of values, this is a reformulation of the statements (2) and (3). To show (6) when f is bounded denote $\alpha(f)=\int_\Omega f\,d\alpha$ for simplicity. Consider a sequence g_N of functions on Ω taking a finite number of values such that $\varepsilon_N=\sup|g_N-f|\to_{N\to\infty}0$. Then

$$\big|\mathbb{E}\big(e^{isX_t(f)}\big)-e^{is\alpha(f)}\big|\leq A+B+C,$$

where

$$A = \left|\mathbb{E}\left(e^{isX_t(f)}\right) - \mathbb{E}\left(e^{isX_t(g_N)}\right)\right|, \qquad B = \left|\mathbb{E}\left(e^{isX_t(g_N)}\right) - e^{is\alpha(g_N)}\right|,$$
$$C = \left|e^{is\alpha(g_N)} - e^{is\alpha(f)}\right|.$$

From $|e^{ia} - e^{ib}| \leq |a-b|$, we get that A and C are less that $2|s|\varepsilon_N$. Furthermore, $\lim_{t\searrow 0} B = 0$ since g_N takes a finite number of values. As a consequence $\limsup_{t\searrow 0}(A+B+C) \leq 2|s|\varepsilon_N$ for all N and this proves (6). Formula (7) is intuitively clear from the representation (4). A detailed proof is similar to the previous one.

Notice that, if we assume that Ω is a locally compact separable space, then equality (6) says that $\lim_{t\to\infty} \mathcal{D}(t\alpha) = \delta_\alpha$ whereas, if we denote by Q_α the distribution of the random probability on Ω defined by δ_X with $X \sim \alpha$, equality (7) says that $\lim_{t\searrow 0} \mathcal{D}(t\alpha) = Q_\alpha$ both in the sense of weak convergence.

The present paper focuses on the distribution of the random variable $X_t(f)$ when f is neither necessarily non-negative nor bounded, and it can be even valued in $\mathbb{R}^d$ rather than in $\mathbb{R}$. It is easily seen that if $f : \Omega \to \mathbb{R}^d$ and α' and P'_t are the respective images by f on $\mathbb{R}^d$ of the probabilities α and P_t on Ω, then $P'_t \sim \mathcal{D}(t\alpha')$. Therefore, in order to study the distribution of $X_t(f) = \int_\Omega f(w)P_t(dw) = \int_{\mathbb{R}^d} x P'_t(dx)$, there is no loss of generality in choosing $\Omega = \mathbb{R}^d$ and f equal to the identity.

The problem of the existence of

$$X_t = \int_{\mathbb{R}^d} x P_t(dx) = \sum_{n=1}^{\infty} W_n B_n \tag{8}$$

(where now the B_n's are i.i.d., α distributed in $\mathbb{R}^d$) has been solved by a crucial paper of [7] where they prove that for a fixed $t > 0$, then $\int_{\mathbb{R}^d} \|x\| P_t(dx) < \infty$ almost surely if and only if

$$\int_{\mathbb{R}^d} \log\left(1 + \|x\|\right)\alpha(dx) < \infty \tag{9}$$

(actually they did this for $d = 1$; the case $d > 1$ is easily deduced from it). Let us denote by $L^{\log}_d$ the set of probabilities α on $\mathbb{R}^d$ such that (9) holds. If $\alpha \in L^{\log}_d$ denote by $\mu(t\alpha)$ the distribution in $\mathbb{R}^d$ of X_t defined by (8). We anticipate that $\mu(t\alpha) \notin L^{\log}_d$ in general (see Proposition 5.6 below).

The main character of this paper is the map $t \mapsto \mu(t\alpha)$ from $(0,\infty)$ to the set of probabilities on $\mathbb{R}^d$. We call this map the Dirichlet curve associated to the probability $\alpha \in L^{\log}_d$ on $\mathbb{R}^d$. Here is a useful and classical characterization of $\mu(t\alpha)$ (see, e.g., [12], Proposition 2 and the subsequent discussion):

Proposition 1.1. *Let $\alpha \in L^{\log}_d$ and consider three independent random variables X (valued in $\mathbb{R}^d$), $B \sim \alpha$ and $Y \sim \beta(1,t)$. Then*

$$X \sim (1-Y)X + YB \tag{10}$$

if and only if $X \sim \mu(t\alpha)$.

The if part is obvious from the definition (8). The converse follows from a general result described in [2], (Proposition 1).

In Proposition 3.4, we will see that $t \mapsto \mu(t\alpha)$ is weakly continuous and that

$$\lim_{t \searrow 0} \mu(t\alpha) = \alpha. \tag{11}$$

Furthermore, if

$$\int_{\mathbb{R}^d} \|x\| \alpha(dx) < \infty \tag{12}$$

then $m = \int_{\mathbb{R}^d} x\alpha(dx)$ is well defined and Theorem 3.5 below in particular shows

$$\lim_{t \to \infty} \mu(t\alpha) = \delta_m. \tag{13}$$

If α has compact support, these two facts are immediate consequences of (6) and (7). Observe also that (12) implies through (8) that $\mathbb{E}(X_t)$ exists and is equal to m, for any $t > 0$. Comparing the behavior of $\mu(t\alpha)$ in the neighbourhood of 0 and ∞, one can make the vague observation that the concentration of $\mu(t\alpha)$ is increasing with t. In order to give a meaning to this statement, namely that for $0 \leq s \leq t$ the probability $\mu(t\alpha)$ is more concentrated than $\mu(s\alpha)$, we use the Strassen convex order. Before stating its definition, observe that if μ is a probability in $\mathbb{R}^d$ having a mean and if ψ is a convex function on $\mathbb{R}^d$ then $\int_{\mathbb{R}^d} \max(0, -\psi(x))\mu(dx) < +\infty$. This comes from the fact that there exists $a \in \mathbb{R}^d$ and $b \in R$ such that $\psi(x) \geq \langle a, x\rangle + b$ together with the fact that μ has a mean. As a consequence $\int_{\mathbb{R}^d} \psi(x)\mu(dx)$ makes sense, although it can be possibly $+\infty$.

Definition (Strassen convex order). *If μ and ν are probabilities on $\mathbb{R}^d$ having means, we write $\nu \prec \mu$ if $\int_{\mathbb{R}^d} \psi(x)\nu(dx) \leq \int_{\mathbb{R}^d} \psi(x)\mu(dx)$ for all convex functions ψ on $\mathbb{R}^d$.*

Needless to say, $\nu \prec \mu$ implies that μ and ν have the same mean. For general references on stochastic orders, see [25] or [19].

Our main theorem is the following:

Theorem 1.2. *Let $\alpha \in L_d^{\log}$. If $\int_{\mathbb{R}^d} \|x\|\alpha(dx) < \infty$ then for $0 \leq s \leq t$ we have $\mu(t\alpha) \prec \mu(s\alpha)$. In other terms, $t \mapsto \mu(t\alpha)$ is decreasing for the Strassen convex order on $(0, +\infty)$.*

We shall comment on this result and we will give examples in Section 2. We will prove it in Section 4, after gathering several properties of $\mu(t\alpha)$ in Section 3.

Next, we suppose that (9) is fulfilled but not (12). In the asymptotic behavior of $\mu(t\alpha)$ when $t \to \infty$, Cauchy laws play a crucial role. For $b > 0$ and $a \in \mathbb{R}$ denote $w = a + ib$ and consider the strict Cauchy distribution on $\mathbb{R}$

$$c_w(dx) = \frac{1}{\pi} \frac{b\,dx}{(x-a)^2 + b^2}. \tag{14}$$

This notation is borrowed from [15]; it enables us to write the Fourier transform of c_w in the following way. For $s > 0$

$$\int_{-\infty}^{\infty} e^{isx} c_w(dx) = e^{isw}$$

and obviously $\int_{-\infty}^{\infty} e^{isx} c_w(dx) = e^{is\overline{w}}$ if $s < 0$. Moreover, this formula has a sense for $b = 0$, in which case c_w is defined as the Dirac mass δ_a. Both strict Cauchy and Dirac distributions will be called Cauchy distributions in the sequel.

It is a well-known fact due to [29] that $\mu(tc_w) = c_w$ for all $t > 0$ when $\Im w > 0$. In other terms, the Dirichlet curve governed by c_w is reduced to a point. If (12) is not fulfilled, the asymptotic behavior of $\mu(t\alpha)$ is not yet well understood: Theorem 3.5 below shows if the limit of $\mu(t\alpha)$ as $t \to \infty$ exists, it is a Cauchy distribution in $\mathbb{R}^d$ (in $\mathbb{R}^d$, what we call a Cauchy distribution is a probability law such that all linear forms are one dimensional Cauchy). In Section 5, we shall study the α's such that $\mu(t\alpha) = \mu(s\alpha)$ for some $0 \leq s < t$. In many particular cases for (s, t), we will prove that these α's are Cauchy distributions in $\mathbb{R}^d$. On the basis of these partial results, we conjecture that this is true for any $0 \leq s < t$.

2. Comments and examples

Comments on Strassen convex order:

1. The Strassen convex order between probabilities on $\mathbb{R}^d$ has an important characterization due to [26]:

Theorem 2.1. *Let μ and ν be probabilities on $\mathbb{R}^d$ with the same finite mean. Then $\mu \prec \nu$ if and only if there exists a probability kernel $K(y, dx)$ from $\mathbb{R}^d$ to itself such that $\mu(dx) = \int_{\mathbb{R}^d} \nu(dy)K(y, dx)$, the integral $\int_{\mathbb{R}^d} \|x\| K(y, dx)$ exists and $\int_{\mathbb{R}^d} xK(y, dx)$ is equal to y, ν almost everywhere. In other terms if $X \sim \mu$ and $Y \sim \nu$, one can find a joint distribution $\nu(dy)K(y, dx)$ for (X, Y) such that $\mathbb{E}(X|Y) = Y$.*

2. More generally if I is an real interval, a family $(\nu_s)_{s \in I}$ of probabilities on $\mathbb{R}^d$ is sometimes called a "peacock" if $s_1 < s_2$ implies $\nu_{s_1} \prec \nu_{s_2}$. For $d = 1$, Kellerer [14] has shown that in this case there exists a Markovian martingale $(M_s)_{s \in I}$ such that $M_s \sim \nu_s$ for all $s \in I$. Hirsch and Roynette [10] have extended this result to any d. Therefore, our Theorem 1.2 says that if we denote for $s \in (0, 1)$

$$\nu_s = \mu\left(\frac{1-s}{s}\alpha\right) \tag{15}$$

then $(\nu_s)_{s \in (0,1)}$ is a peacock. In practical circumstances, it is difficult to make the kernels K and the martingale (M_s) explicit, in agreement with the fact that Kellerer's proof is not constructive (see also the comment after Proposition 2.2 below).

3. It is useful to know that if $\nu_n \prec \mu_n$ and if μ_n and ν_n converge weakly to μ and ν respectively, and if the means of μ_n and ν_n converge to the means of μ and ν, then $\nu \prec \mu$. This is

Theorem 3.4.6 of [19]. Here is an application of this fact: with the hypotheses and notations of Theorem 1.2, we have $\mu(t\alpha) \prec \alpha$ for any $t > 0$, because of (11).

4. If $\mu \prec \nu$ and $\nu \prec \mu$, we have $\mu = \nu$. To see this in dimension one, use the convex function $\psi_a(x) = (x-a)_+$, getting $\int_{[a,\infty)}(x-a)\mu(dx) = \int_{[a,\infty)}(x-a)\nu(dx)$. Thus,

$$\int_a^\infty \left(\int_{[t,\infty)} \mu(dx)\right) dt = \int_a^\infty \left(\int_{[t,\infty)} \nu(dx)\right) dt$$

for any a and $\mu = \nu$ follows. It is easy to pass to higher dimensions by taking linear forms.

Examples of Strassen convex order:

1. A classical example is offered by a sequence $X_1, \ldots, X_n, \ldots$ of i.i.d. random variables of $\mathbb{R}^d$ having a mean. If μ_n is the distribution of $\overline{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$, then $\mu_n \prec \mu_m$ if $1 \le m \le n$ since $\mathbb{E}(\overline{X}_m|\overline{X}_n) = \overline{X}_n$. To see this, observe that $j \mapsto \mathbb{E}(X_j|\overline{X}_n)$ does not depend on j. This sequence $(\mu_n)_{n\ge1}$ presents an analogy with the Dirichlet curve. Indeed, by the weak law of large numbers μ_n converges weakly to $\delta_{\mathbb{E}(X_1)}$. Moreover, if $X_1 \sim c_w$ is Cauchy distributed on $\mathbb{R}$ then $\mu_n \sim c_w$, for any for any integer n. Furthermore, if $\mu_1 = \mu_n = \mu_m$ where m is not a rational power of n, then X_1 is strictly one-stable, *that is*, is Cauchy (see [21] and [30], page 14 or [22], page 4). This does not hold under the assumption $\mu_1 = \mu_m$ for some $m > 1$, as proved by Lévy's counderexampled reported in [8], page 538. This behaviour differs from what we conjecture to be true for the Dirichlet curve at the end of the Introduction. In all the examples known so far, the asymptotic behaviour of $\mu(t\mu_1)$ as $t \to \infty$ is the same as that of μ_n as $n \to \infty$.

2. Suppose that $X \sim \mu$, $Y \sim \nu$ and $0 < U < 1$ are independent random variables such that $X \sim (1-U)X + UY$ where μ and ν are probabilities on $\mathbb{R}^d$ having a mean. Then $\mu \prec \nu$, since for any convex function ψ, with $m = \mathbb{E}(U) \in (0,1)$, we obtain

$$\begin{aligned}\mathbb{E}\big(\psi(X)\big) &= \mathbb{E}\big(\psi\big((1-U)X + UY\big)\big) \le (1-m)\mathbb{E}\big(\psi(X)\big) + m\mathbb{E}\big(\psi(Y)\big)\\ &\Rightarrow \mathbb{E}\big(\psi(X)\big) \le \mathbb{E}\big(\psi(Y)\big).\end{aligned}$$

3. To give an explicit example of application of 2 let us use the following result due to [1] (with a different proof).

Proposition 2.2. *Let* $0 < a < b$. *Let* $X_b \sim \beta(b,b)$, $X_a \sim \beta(a,a)$ *and* $U \sim \beta(2a, b-a)$ *be mutually independent. Then* $X_b \sim (1-U)X_b + UX_a$.

Proof. For $|z| < 1$ apply (1) to the Dirichlet distribution $(1-U, U) \sim D(b-a, 2a)$ and to $f_1 = 1 - zX_b$, $f_2 = 1 - zX_a$. We get

$$\mathbb{E}\left(\frac{1}{(1 - z((1-U)X_b + UX_a))^{b+a}}\right) = \mathbb{E}\left(\frac{1}{(1-zX_b)^{b-a}}\right) \times \mathbb{E}\left(\frac{1}{(1-zX_a)^{2a}}\right).$$

Now we use Euler's formula (see [20], page 47): for $V \sim \beta(B, C-B)$ then

$${}_2F_1(A,B;C;t) = \mathbb{E}\left(\frac{1}{(1-zV)^A}\right).$$

We apply it to $V = X_a$, with $B = a$ and $A = C = 2a$, then to $V = X_b$, with $A = b - a$, $B = b$ and $C = 2b$:

$$\mathbb{E}\left(\frac{1}{(1 - zX_a)^{2a}}\right) = \frac{1}{(1-z)^a}, \qquad \mathbb{E}\left(\frac{1}{(1 - zX_b)^{b\pm a}}\right) = {}_2F_1(b \pm a, b; 2b; z).$$

Now we use the other Euler formula (Rainville [20], page 60)

$${}_2F_1(A, B; C, z) = (1 - t)^{C-A-B}{}_2F_1(C - A, C - B; C; z).$$

for $A = b - a$, $B = b$ and $C = 2b$, obtaining

$$\mathbb{E}\left(\frac{1}{(1 - z((1 - U)X_b + UX_a))^{b+a}}\right) = \mathbb{E}\left(\frac{1}{(1 - zX_b)^{b+a}}\right)$$

which implies the result. □

As a consequence $\beta(b, b) \prec \beta(a, a)$ if $0 < a < b$. We shall use this fact in the proof of Theorem 1.2. No explicit probability kernel $K(x, dy)$ satisfying the characterization in Theorem 2.1 for this pair $(\beta(b, b), \beta(a, a))$ is known to us.

4. Suppose that α is concentrated on $[0, \infty)$ and has a moment of order n. Then $G_n(t) = \int_0^\infty x^n \mu(t\alpha)(dx)$ exists (see [12] and Section 3 below). Theorem 1.2 implies that $t \mapsto G_n(t)$ is decreasing. Proving directly this fact for small values of $n \geq 2$ is a painful process using classical inequalities for the moments of α, as exemplified by Proposition 3.3 below.

Examples of Dirichlet curves:

1. Bernoulli case: If $\Omega = \mathbb{R}^{d+1}$ and $\alpha = p_0\delta_{e_0} + \cdots + p_d\delta_{e_d}$ where $(e_0, \dots, e_d)$ is the canonical basis of $\mathbb{R}^{d+1}$ then from (5) we have $P_t = X_0\delta_{e_0} + \cdots + X_d\delta_{e_d}$ where $(X_0, \dots, X_d) \sim \mathcal{D}(tp_0, \dots, tp_d)$. This implies that $\mu(t\alpha) = \mathcal{D}(tp_0, \dots, tp_d)$. The fact that in this example we have $\mu(t\alpha) \prec \mu(s\alpha)$ for $0 \leq s < t$ is by no means obvious and is a consequence of Theorem 1.2. A particular example is obtained for $d = 1$: the ordinary Bernoulli distribution $\alpha(dx) = q\delta_0 + p\delta_1$ with $p = 1 - q \in (0, 1)$ governs the Dirichlet curve $\mu(t\alpha) = \beta(tp, tq)$, for $t > 0$. It should be mentioned that another proof of $\beta(tp, tq) \prec \beta(sp, sq)$ for $s < t$ is obtained from Theorem 2.A.7 of [25]. To apply this theorem, it is necessary to study the ratio between their densities. For the particular case $p = q = 1/2$, Theorem 1.2 is directly obtained by using Proposition 2.2, since

$$\mu\left(t\left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right)\right) = \beta\left(\frac{t}{2}, \frac{t}{2}\right).$$

2. If $\Omega = \mathbb{R}$ and $\alpha(dx) = \beta^{(2)}(\frac{1}{2}, \frac{1}{2})(dx) = \frac{1}{\pi}\frac{x^{-1/2}}{(1+x)}\mathbf{1}_{(0,+\infty)}(x)\,dx$, then

$$\mu(t\alpha)(dx) = \beta^{(2)}\left(t + \frac{1}{2}, \frac{1}{2}\right)(dx) = \frac{1}{B(t + 1/2, 1/2)}\frac{x^{t-1/2}}{(1 + x)^{1+t}}\mathbf{1}_{(0,\infty)}(x)\,dx. \qquad (16)$$

This fact has been observed by [3], Example 7, page 1394. It has been actually completed by [12], Remark 7, page 234. This example has no first moments so Theorem 1.2 cannot be applied to it.

However, notice that $\lim_{t\to\infty}\mu(t\alpha)$ does not exist (it goes to infinity, see also Corollary 3.6 for a generalization). More specifically, if $X_t \sim \beta^{(2)}(t+\frac{1}{2},\frac{1}{2})$ then the distribution of t/X_t converges to a gamma distribution $\gamma_{1/2}$ (in the sequel we write $\gamma_a(dx) = e^{-x}x^{a-1}\mathbf{1}_{(0,\infty)}(x)\,dx/\Gamma(a)$ for $a>0$). To see this, use the Mellin transform or the representation of X_t as a quotient of two independent gamma variables. Using this representation one easily sees that $t\mapsto \Pr(X_t>x)$ is increasing. More generally, one can conjecture that if α is supported by $(0,\infty)$ and has no mean, then $t\mapsto \Pr(X_t>x)$ is increasing for all x when $X_t\sim\mu(t\alpha)$. If α has a finite expectation m, this is impossible since in this case

$$m = \mathbb{E}(X_t) = \int_0^\infty \Pr(X_t>x)\,dx.$$

3. If $\Omega=\mathbb{R}$ and $\alpha=\beta(\frac{1}{2},\frac{1}{2})$ then $\mu(t\alpha)=\beta(t+\frac{1}{2},t+\frac{1}{2})$. To see this apply Proposition 2.2 to the particular case $a=\frac{1}{2}$ and $b=t+\frac{1}{2}$: the proposition says that if $X\sim\beta(t+\frac{1}{2},t+\frac{1}{2})$, $Y\sim\beta(1,t)$ and $B\sim\beta(\frac{1}{2},\frac{1}{2})$ are independent, then $X\sim(1-Y)X+YB$. From the characterization (10) of Proposition 1.1, we get the result. Comparing example 1 with $d=1$ with the present example 3, we notice the formula: for $t\geq 1/2$

$$\mu(t\alpha_1)=\beta\left(\frac{t}{2},\frac{t}{2}\right)=\mu\left(\frac{t-1}{2}\alpha\right),$$

with $\alpha_1=(\delta_0+\delta_1)/2$ and $\alpha=\beta(1/2,1/2)$: the curve of α_1 contains the curve of α. This is the only example we know in which this happens.

4. If $\Omega=\mathbb{R}^2$ and α is the uniform distribution on the circle $\mathbb{U}=\{(x,y);\,x^2+y^2=1\}$, then $\mu(t\alpha)$ is the distribution of $R_t\Theta$ where $R_t^2\sim\beta(1,t)$ is independent of $\Theta\sim\alpha$. To see this, observe from (8) that $\mu(t\alpha)$ must be invariant by rotation since α has this property. Furthermore, the image of α by the projection $(x,y)\mapsto x$ is also the image of $\beta(\frac{1}{2},\frac{1}{2})$ by $x\mapsto x'=2x-1$. Using the preceding example, the image of $\mu(t\alpha)$ by the projection $(x,y)\mapsto x$ is also the image of $\beta(t+\frac{1}{2},t+\frac{1}{2})$ by $x\mapsto x'=2x-1$. A slightly tedious calculation leads to the result: for this observe that $X_t'=R_t\cos\Theta$ where Θ is uniform on $(0,2\pi]$ and is independent of R_t. Therefore if $s>0$ we write $\mathbb{E}(R_t^{2s})=\mathbb{E}(((X_t')^2)^s)/\mathbb{E}((\cos^2\Theta)^s)$. Similar examples when α is the uniform distribution on the unit sphere of $\mathbb{R}^d$ with $d>2$ are manageable but they lead to untractable formulas for the distribution of R_t.

Already for $d=3$ we are led to deal with the Dirichlet curve of the uniform distribution α_1 on $(0,1)$. Diaconis and Kemperman [6] (Example 4, page 99) seem to be the first to have written that

$$\mu(\alpha_1)=\frac{e}{\pi}\frac{\sin\pi x}{x^{x-1}(1-x)^{-x}}\alpha_1(dx),$$

but $\mu(t\alpha_1)$ for $t\neq 1$ is notoriously complicated, as it can be seen in [17]. Explicit calculations about this problem appear in [16], in the comments following Theorem 16.

5. If $\alpha\in L_d^{\log}$, if $X\sim\mu(t\alpha)$ is independent of $U\sim\beta(t,t_0)$, then $XU\sim\mu(t_0\delta_0+t\alpha)$. This remark can be found in [13], Theorem 2.1. More generally, suppose that $X_0,\dots,X_n$ and $Y=(Y_0,\dots,Y_n)\sim\mathcal{D}(t_0,\dots,t_n)$ are independent, with $X_j\sim\mu(t_j\alpha_j)$ with $\alpha_j\in L_d^{\log}$, for

$j=0,\dots,n$. Then

$$Y_0X_0+\cdots+Y_nX_n\sim\mu(t_0\alpha_0+\cdots+t_n\alpha_n).$$

In particular, for $\alpha_j=\alpha\in L_d^{\log}$ for all $j=0,\dots,n$, the distribution of $Y_0X_0+\cdots+Y_nX_n$ still lies on the Dirichlet curve of α.

Comments on the Cauchy distribution in $\mathbb{R}^d$:

1. Recall that a Cauchy distribution c in $\mathbb{R}^d$ is a distribution such that if $X\sim c$ then $\langle f,X\rangle$ is Cauchy in $\mathbb{R}$ for any linear form f on $\mathbb{R}^d$. This means that $\int_{\mathbb{R}^d}e^{is\langle f,x\rangle}c(dx)=e^{isw(f)}$, with $f\mapsto w(f)$ positively homogeneous (i.e., $w(\lambda f)=\lambda w(f)$ for $\lambda\ge 0$): the admissible w's are described in the following proposition (see [23], Theorem 4.10, or [22], Chapter 2, Theorem 2.3.1).

Proposition 2.3. *The random variable X in $\mathbb{R}^d$ is Cauchy distributed if and only if there exists $a\in\mathbb{R}^d$ and a positive measure $b(ds)$ on the unit sphere S of $\mathbb{R}^d$ such that $\int_S sb(ds)=0$ and such that for all $f\in\mathbb{R}^d$ we have $\langle f,X\rangle\sim c_{w(f)}$ with*

$$w(f)=\langle a,f\rangle-\frac{2}{\pi}\int_S\langle f,s\rangle\log\big|\langle f,s\rangle\big|b(ds)+i\int_S\big|\langle f,s\rangle\big|b(ds). \tag{17}$$

There are several other definitions of the Cauchy distribution in a Euclidean space in the literature, generally more restrictive that the present one. The most popular is the distribution of X such that $\mathbb{E}(e^{\langle t,X\rangle})=e^{-\|t_t\|}$ and its affine deformations. For such an X, we have $w(f)=i\|f\|$ and $b(ds)=CU(ds)$ where $U(ds)$ is the uniform probability on the unit sphere S and $C=\sqrt{\pi}\Gamma((d+1)/2)/\Gamma(d/2)$.

2. A remarkable fact about the distribution of $\langle f,X\rangle$ is that its median

$$\langle a,f\rangle-\frac{2}{\pi}\int_S\langle f,s\rangle\log\big|\langle f,s\rangle\big|b(ds)$$

is not a linear form in f, which means that the distribution of X has not necessarily a center of symmetry. If $b(ds)$ is invariant by $s\mapsto -s$ of course $\int_S\langle f,s\rangle\log|\langle f,s\rangle|b(ds)=0$ and a is the center of symmetry. If $d=1$, b is necessary symmetric.

For an example of a Cauchy distribution in $\mathbb{R}^2$ without center of symmetry, one can consider $b=\delta_1+\delta_j+\delta_{j^2}$ where S is identified with the unit circle of the complex plane and where j and j^2 are the complex cubic roots of the unity. It satisfies $\int_S sb(ds)=0$. If $f=e^{i\theta}$ and if $g(\theta)=-\frac{2}{\pi}\cos\theta\log|\cos\theta|$ then the median of $\langle f,X\rangle$ is

$$r(\theta)=g(\theta)+g\left(\theta-\frac{2\pi}{3}\right)+g\left(\theta+\frac{2\pi}{3}\right).$$

and $\theta\mapsto r(\theta)e^{i\theta}$ is the equation of a nice trefoil curve.

3. If α is a probability on $[0,\infty)$ and if ρ is a probability in $\mathbb{R}^d$ we denote by $\rho\odot\alpha$ the distribution of XY when $X\sim\rho$ and $Y\sim\alpha$ are independent. For $d=1$, the following invariance principle was obtained by [29] in the particular case $\alpha=\delta_1$ and in general by [12]:

Proposition 2.4. *If c is Cauchy in $\mathbb{R}^d$ and if α is a probability on $[0,\infty)$ belonging to $L_1^{\log}$, then*

$$\mu(tc \odot \alpha) = c \odot \mu(t\alpha). \tag{18}$$

Proof. The proof is quite easy: since c is Cauchy, then $c \in L_d^{\log}$. Furthermore, if $\alpha \in L_1^{\log}$, then $c \odot \alpha \in L_d^{\log}$ and $\mu(tc \odot \alpha)$ makes sense. Let $X = (X_n)$, $A = (A_n)$ and $Y = (Y_n)$ be three independent i.i.d. sequences such that $X_n \sim c$, $A_n \sim \alpha$ and $Y_n \sim \beta(1,t)$ then

$$\mu(tc \odot \alpha) \sim \sum_{n=1}^{\infty} X_n A_n W_n,$$

where $W_1 = Y_1$, and W_n denotes $Y_n \prod_{j=1}^{n-1}(1 - Y_j)$ as usual. So we have to prove that the latter has the same law as $X_0 \sum_{n=1}^{\infty} A_n W_n$, where $X_0 \sim c$ is independent of everything else. Recall that the Fourier transform of c is $e^{isw(f)}$, with w positively homogeneous, from which the Fourier transform of $\mu(tc \odot \alpha)$ is obtained as follows:

$$\begin{aligned}
\int_{\mathbb{R}^d} e^{is\langle f,x\rangle} \mu(tc \odot \alpha)(dx) &= \mathbb{E}\big(\mathbb{E}\big(e^{is\sum_{n=1}^{\infty}\langle f,X_n\rangle A_n W_n} | A, W\big)\big) = \mathbb{E}\big(e^{\sum_{n=1}^{\infty} isw_f A_n W_n}\big) \\
&= \int_0^{\infty} e^{isw_f a} \mu(t\alpha)(da) = \mathbb{E}\big(e^{is\langle f,X\rangle \sum_{n=1}^{\infty} A_n W_n}\big) \\
&= \int_{\mathbb{R}^d} e^{is\langle f,x\rangle} c \odot \mu(t\alpha)(dx).
\end{aligned}$$
□

Corollary 2.5. *If c is Cauchy in $\mathbb{R}^d$, then $\mu(tc) = c$ for all $t > 0$.*

Proof. Choose $\alpha = \delta_1$ in Proposition 2.4. □

3. Moments and asymptotic properties of the Dirichlet curve

The basic link between $\mu(t\alpha)$ and α is the Proposition 3.1 below, generally attributed to [4]. Other proofs are given in Theorem 1 of [12] or in Theorem 2 of [27] (where it is called Markov–Krein identity). It is a considerable extension of (1). For convenience, we give two versions. For a real number t and a nonzero complex number z such that its argument $\arg z$ is in $(-\pi,\pi)$, symbols $\log z$ and z^t mean $\log|z| + i\arg(z)$ and $e^{t\log z}$, respectively.

Proposition 3.1. *If $\alpha \in L_1^{\log}$, then for any real s we have*

$$\int_{-\infty}^{+\infty} \frac{\mu(t\alpha)(dx)}{(1 - isx)^t} = e^{-t\int_{-\infty}^{+\infty}\log(1-isx)\alpha(dx)}$$

and, for $\Im z \neq 0$:

$$\int_{-\infty}^{+\infty} \frac{\mu(t\alpha)(dx)}{(x - z)^t} = e^{-t\int_{-\infty}^{+\infty}\log(x-z)\alpha(dx)}.$$

With the methods of [12] the next proposition gives information on the Mellin transform of $\|X\|$ when $X \sim \mu(t\alpha)$. We use the Pochhammer symbol $(t)_k = t(t+1)\cdots(t+k-1)$ if k is a positive integer and $(t)_0 = 1$.

Proposition 3.2. *Let $\alpha \in L_d^{\log}$. Let $X_t \sim \int_{\mathbb{R}^d} x P_t(dx)$, where $P_t \sim \mathcal{D}(t\alpha)$, and let $B \sim \alpha$. Then for any number $s > 0$ we have*

$$\mathbb{E}\big(\|X_t\|^s\big) < \infty \quad \Leftrightarrow \quad \mathbb{E}\big(\|B\|^s\big) < \infty.$$

Under these circumstances, for $d = 1$ and if $s = n$ is a positive integer we have the Hjort–Ongaro formula

$$\mathbb{E}\big(X_t^n\big) = \frac{(n-1)!}{(t+1)_{n-1}} \sum_{k=0}^{n-1} (t)_k \frac{\mathbb{E}(X_t^k)}{k!} \mathbb{E}\big(B^{n-k}\big). \tag{19}$$

Furthermore if $s \geq 1$, we have $\mathbb{E}(\|X_t\|^s) \leq \mathbb{E}(\|B\|^s)$ and if $0 < s < 1$ we have

$$\frac{\mathbb{E}(\|X_t\|^s)}{\mathbb{E}(\|B\|^s)} \leq t \frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)}, \qquad \mathbb{E}\big(\|B\|^s\big) \leq \mathbb{E}\bigg(\bigg(\int_{-\infty}^{+\infty} \|x\| P_t(dx)\bigg)^s\bigg). \tag{20}$$

Proof. For simplicity, write $X = X_t$. We prove first the equivalence for $s \geq 1$. If X, Y, B are independent and $Y \sim \beta_{1,t}$, we have $X \sim (1-Y)X + YB$ from (10). Introduce a random variable $G \sim \gamma_{1+t}$ independent of X, Y, B and observe that $G' = G(1-Y) \sim \gamma_t$ and $G'' = GY \sim \gamma_1$ are independent. Therefore,

$$GX \sim G'X + G''B \tag{21}$$

with X, G', G'' and B mutually independent. For proving part $\Leftarrow$, we use (4). Since $s \geq 1$, one has

$$\|X\|^s \leq \bigg(\int_{\mathbb{R}^d} \|x\| P_t(dx)\bigg)^s \leq \int_{\mathbb{R}^d} \|x\|^s P_t(dx) = \sum_{i=1}^{\infty} \|B_i\|^s Y_i \prod_{k=1}^{i-1} (1-Y_k),$$

$$\mathbb{E}\big(\|X\|^s\big) \leq \mathbb{E}\bigg(\int_{\mathbb{R}^d} \|x\|^s P_t(dx)\bigg) = \mathbb{E}\bigg(\sum_{i=1}^{\infty} \|B_i\|^s Y_i \prod_{k=1}^{i-1} (1-Y_k)\bigg) = \mathbb{E}\big(\|B\|^s\big) < \infty.$$

For proving part $\Rightarrow$, let us denote $U = G'X$ and $V = G''B$. If $\mathbb{E}(\|X\|^s) < \infty$, then $\mathbb{E}(\|U + V\|^s) = \mathbb{E}(G^s)\mathbb{E}(\|X\|^s) < \infty$. Denote $C_s(u) = \mathbb{E}(\|u + V\|^s) \leq \infty$. Since $\mathbb{E}(C_s(U)) < \infty$, by Fubini's theorem there exists u_0 such that $C_s(u_0) < \infty$. We get from Minkowski's inequality

$$\mathbb{E}\big(\|V\|^s\big) \leq \big(\|u_0\| + \big(\mathbb{E}\big(\|V + u_0\|^s\big)\big)^{1/s}\big)^s < \infty$$

since V is the sum of $V + u_0$ and the constant $-u_0$. Since $\mathbb{E}(\|V\|^s) = \mathbb{E}((G'')^s)\mathbb{E}(\|B\|^s)$ we get $\mathbb{E}(\|B\|^s) < \infty$ and part $\Rightarrow$ is proved. Suppose now that $d = 1$ and that $\mathbb{E}(\|B\|^n) < \infty)$. Then

(19) is easily seen from (21):

$$(t+1)_n\frac{\mathbb{E}(X^n)}{n!}=\frac{\mathbb{E}(G^nX^n)}{n!}=\sum_{k=0}^{n}\frac{\mathbb{E}((G')^kX^k)}{k!}\frac{\mathbb{E}((G'')^{n-k}B^{n-k})}{(n-k)!}=\sum_{k=0}^{n}(t)_k\frac{\mathbb{E}(X^k)}{k!}\mathbb{E}\big(B^{n-k}\big).$$

Subtracting from both sides the nth term of the sum and simplifying one gets the desired expression. Finally, assume $0<s<1$ and observe that for all $t>0$ we have $(1+t)^s\leq 1+t^s$ (just show that $t\mapsto 1+t^s-(1+t)^s$ is increasing). Together with the triangle inequality, this implies that $\|U+V\|^s\leq\|U\|^s+\|V\|^s$ and therefore by taking expectations

$$\big(\mathbb{E}\big(G^s\big)-\mathbb{E}\big(\big(G'\big)^s\big)\big)\mathbb{E}\big(\|X\|^s\big)\leq\mathbb{E}\big(\big(G''\big)^s\big)\mathbb{E}\big(\|B\|^s\big)$$

which is (20) since $t\frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)}=\mathbb{E}((G'')^s)/(\mathbb{E}(G^s)-\mathbb{E}((G')^s))$. For (20), integrate $x\mapsto\|x\|^s$ with $P_t(dx)$ defined by (4), use the equality inside (22) and the following inequality (correct for $0<s<1$)

$$\int_{\mathbb{R}^d}\|x\|^sP_t(dx)\leq\left(\int_{\mathbb{R}^d}\|x\|P_t(dx)\right)^s. \qquad\square$$

Comment. About the first inequality in (20) note that $t\frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)}\geq 1$ for $0<s<1$: just observe that since $\log\Gamma$ is convex, then $s\mapsto\log t\frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)}$ is decreasing and zero for $s=1$.

Next, the proposition shows that if α is concentrated on $[0,\infty)$, then the first moments of $X_t\sim\mu(t\alpha)$ have certain delicate properties (which are probably true for any moment). These properties imply that $t\mapsto\mathbb{E}(X_t^n)$ is decreasing. This fact has been an incentive for guessing the statement of Theorem 1.2.

Proposition 3.3. *Let α be a probability on $[0,\infty)$, let $X_t\sim\mu(t\alpha)$ and let k be a fixed positive integer. Suppose that $m_k=\int_0^\infty x^k\alpha(dx)<\infty$. Consider the function*

$$c_k(t)=\frac{\mathbb{E}(X_t^k)}{k!}.$$

Then $P_k(t)=(t+1)_kc_{k+1}(t)$ is a polynomial of degree k. In particular

$$P_0(t)=m_1,\qquad P_1(t)=\frac{m_2}{2}+\frac{m_1^2}{2}t,\qquad P_2(t)=\frac{m_3}{3}+\frac{m_1m_2}{2}t+\frac{m_1^3}{6}t^2,$$
$$P_3(t)=\frac{m_4}{4}+\left(\frac{m_1m_3}{3}+\frac{m_2^2}{8}\right)t+\frac{m_1^2m_2}{4}t^2+\frac{m_1^4}{24}t^3.$$

Finally, the polynomial $t\mapsto Q_k(t)=-[(t+1)_k]^2c'_{k+1}(t)$ of degree $2k-1$ has nonnegative coefficients for $k=1,2,3$. As a consequence, the functions $t\mapsto\frac{\mathbb{E}(X_t^n)}{n!}$ are decreasing for $n=2,3,4$.

Proof. From (19), one easily gets $P_0(t) = m_1$ and

$$P_n(t) = \frac{1}{n+1} m_{n+1} + \frac{t}{n+1} \sum_{k=0}^{n-1} P_k(t) m_{n-k}$$

from which P_1, P_2, P_3 are deduced. One also gets

$$-\big[(t+1)_k\big]^2 c'_{k+1}(t) = Q_k(t) = P_k(t) \frac{d}{dt}(t+1)_k - (t+1)_k P'_k(t).$$

The first Q_k's are

$$\begin{aligned}
Q_1(t) &= \frac{1}{2}\big(m_2 - m_1^2\big), \qquad Q_2(t) = (m_3 - m_1 m_2) + \frac{2}{3}\big(m_3 - m_1^2\big)t + \frac{m_1}{2}\big(m_2 - m_1^2\big)t^2, \\
Q_3(t) &= \left(2(m_4 - m_1 m_3) + \frac{3}{4}\big(m_4 - m_2^2\big)\right) + 3\big(m_4 - m_1^2 m_2\big)t \\
&\quad + \left(\frac{3}{4}\big(m_4 - m_1^2 m_2\big) + \frac{3m_2}{4}\big(m_2 - m_1^2\big) + 2m_1(m_3 - m_1 m_2)\right)t^2 \\
&\quad + \left(\frac{2m_1}{3}\big(m_3 - m_1^3\big) + \frac{1}{4}\big(m_2^2 - m_1^4\big)\right)t^3 + \frac{m_1^4}{4}\big(m_2 - m_1^2\big)t^4.
\end{aligned}$$

If $B \sim \alpha$, then $m_2 - m_1^2 = \mathbb{E}((B - m_1)^2) \geq 0$, $m_4 - m_2^2 = \mathbb{E}((B^2 - m_2)^2) \geq 0$ and

$$\begin{aligned}
m_3 - m_2 m_1 &= \mathbb{E}\big((B - m_1)^2 (B + 2m_1)\big) \geq 0, \\
m_4 - m_3 m_1 &= \mathbb{E}\big((B - m_1)^2 \big(B^2 + m_1 B + 2m_1^2\big)\big) \geq 0, \\
m_3 - m_1^3 &= (m_3 - m_2 m_1) + m_1\big(m_2 - m_1^2\big) \geq 0, \\
m_4 - m_1^2 m_2 &= (m_4 - m_3 m_1) + m_1(m_3 - m_1 m_2) \geq 0.
\end{aligned}$$

This shows the nonnegativity of the coefficients of Q_1, Q_2 and Q_3. □

Proposition 3.4. *If $\alpha \in L_d^{\log}$ then $t \mapsto \mu(t\alpha)$ is weakly continuous on $(0, \infty)$. Furthermore, we have $\lim_{t \searrow 0} \mu(t\alpha) = \alpha$.*

Proof. We fix $t_0 > 0$. We consider a sequence $(U_n)_{n \geq 1}$ of i.i.d. random variables which are uniform on $(0, 1)$. Then $1 - U_n^{1/t} \sim \beta(1, t)$. If the B_n's are independent with the same distribution α, we consider for $t > 0$ and N integer

$$X_{N,t} = \sum_{n=N}^{\infty} (U_1 \cdots U_{n-1})^{1/t} \big(1 - U_n^{1/t}\big) B_n$$

with the convention $U_1 \cdots U_{n-1} = 1$ for $n = 1$. We have $X_t = X_{1,t} \sim \mu(t\alpha)$. Consider $M_{N,t} = \sum_{n=N}^{\infty} (U_1 \cdots U_{n-1})^{1/t} \|B_n\|$. Having $\mathbb{E}(\log(1 + \|B_n\|))$ finite we get $\lim_n \|B_n\|^{1/n} = 1$ almost surely. This comes from

$$\sum_{n=1}^{\infty} \Pr\left(\frac{1}{n} \log\big(1 + \|B_n\|\big) > \varepsilon\right) < \infty$$

and the Borel Cantelli lemma. From the law of large numbers and the fact that $U_n \sim \gamma_1$ we have that $\lim_n \frac{1}{n} \sum_{k=1}^{n} \log U_k = -1$. By Cauchy criterion for real series, these two remarks imply that $M_{N,t}$ converges almost surely to zero for $N \to \infty$. Since $t \mapsto M_{N,t}$ is increasing we conclude that for $0 < t \le t_0$ we have

$$\|X_{N,t}\| \le M_{N,t} \le M_{N,t_0}.$$

This implies the almost sure normal convergence of the series X_t on $(0, t_0]$. This implies that $t \mapsto X_t$ is almost surely continuous on $(0, \infty)$. Finally, let us extend the definition of X_t to $t = 0$ by $X_0 = B_1$. The above uniform convergence extends to $[0, t_0]$ and $\lim_{t \searrow 0} X_t = B_1$ almost surely. Since almost sure convergence implies weak convergence the proof is complete. $\square$

Theorem 3.5. *If $\int_{\mathbb{R}^d} \|x\| \alpha(dx) < \infty$ and $m = \int_{\mathbb{R}^d} x \alpha(dx)$, then $\mu(t\alpha) \underset{t \to \infty}{\to} \delta_m$.*

In case $\int_{\mathbb{R}^d} \|x\| \alpha(dx) = \infty$, with $\alpha \in L_d^{\log}$, and $\mu(t\alpha) \underset{t \to \infty}{\to} \mu$ exists and it is a probability, then μ is a Cauchy distribution.

In particular for $d = 1$, $\mu(t\alpha)$ converges to $\mu = c_w$, with $w = a + ib$, if and only if

$$\begin{aligned} b &= \lim_{t \to \infty} \int_0^{\infty} \Pr\big(|B| > x\big) \frac{xt}{x^2 + t^2}\, dx, \\ a &= \lim_{t \to \infty} \int_0^{\infty} \big(\Pr(B > x) - \Pr(B < -x)\big) \frac{t^2}{x^2 + t^2}\, dx, \end{aligned} \tag{22}$$

where $B \sim \alpha$.

Comments and examples:

1. For $d = 1$, consider the case where B is symmetric and $|B|$ has a Pareto distribution of the form $\Pr(|B| > x) = 1/x^r$ for $x > 1$ where r is a positive parameter. If $r > 1$, then B is integrable, so $\mathbb{E}(B) = 0$ and $b = 0$. Elementary calculations from (22) show that $b = \pi$ if $r = 1$, meaning that $\mu(t\alpha)$ converges to the strict Cauchy distribution $c_{i\pi}$. If $r \in (0, 1)$ $b = \infty$, so $\mu(t\alpha)$ has no limit.
2. For $d = 1$, consider B symmetric with $\Pr(|B| > x) = \frac{e}{x \log x}$ if $x > e$. The change of variables $x = e^u$ and $t = e^s$ shows that

$$\int_e^{\infty} \frac{1}{x \log x} \frac{xt}{x^2 + t^2}\, dx = \int_1^{\infty} \frac{du}{u \cosh(u - s)}\, du$$

and it is easy to see that the limit b of this expression when $s \to \infty$ is zero. This proves that in this example $\lim_{t\to\infty} \mu(t\alpha) = \delta_0$ while $\mathbb{E}(|B|) = \infty$. This shows that the Cauchy distribution in the second statement of Theorem 3.5 can be a Dirac mass.

3. If m exists, with the notation coined in (15) let us define $\nu_0 = \delta_m$ and $\nu_1 = \alpha$. Therefore $(\nu_s)_{s\in[0,1]}$ is a peacock. From Proposition 3.4, it is weakly continuous. In dimension 1, from Theorem 4.4. of [11], the Kellerer martingale $(M_s)_{0\leq s\leq 1}$ is continuous and uniformly integrable.

4. In case $\int_{\mathbb{R}^d} \|x\|\alpha(dx) = \infty$, we have seen in (16) that $\lim_{t\to\infty} \mu(t\alpha)$ may fail to exist. Proposition 2.4 has shown that if α is the distribution of $M > 0$, if $C \sim c$ is Cauchy in $\mathbb{R}^d$ and is independent of $M > 0$, and if α_1 is the distribution in $\mathbb{R}^d$ of MC, then $\mu(t\alpha_1)$ is the distribution of X_tC where $X_t \sim \mu(t\alpha)$ is independent of C. Now if $\mathbb{E}(M) = m$, Proposition 2.4 shows that the limit distribution of X_tC when $t \to \infty$ is the Cauchy distribution of mC. This example helped us to guess the second statement of Theorem 3.5. The Dirichlet curve $(\mu(t\alpha))_{t\geq 0}$ is not always tight, as shown by the example (16). But even if the Dirichlet curve is tight, it is not clear that a limit $\mu(t\alpha) \underset{t\to\infty}{\to} \mu$ always exists.

Proof of Theorem 3.5. We assume first that $\int_{\mathbb{R}^d} \|x\|\alpha(dx) < \infty$. It is enough to prove the result for $d = 1$. The idea of the proof is to use Proposition 3.1. For real s, we have

$$\int_{-\infty}^{+\infty} \frac{\mu(t\alpha)(dx)}{(1-\frac{isx}{t})^t} = e^{-t\int_{-\infty}^{+\infty} \log(1-(isx)/t)\alpha(dx)}. \tag{23}$$

We will show that the left-hand side converges to some $\int_{-\infty}^{+\infty} e^{isx}\mu(dx)$ and we will show that the right-hand side to converges to e^{ism}.

For the left-hand side of (23), we first establish the tightness of the family $\{\mu(t\alpha), t > 0\}$. To see this, we consider let $X_t \sim \mu(t\alpha)$ and observe that from Markov inequality and Proposition 3.2 we have for all $t > 0$:

$$\Pr\big(|X_t| > a\big) \leq \frac{1}{a}\mathbb{E}\big(|X_t|\big) \leq \frac{\mathbb{E}(|B|)}{a}.$$

Next, suppose that for some increasing sequence (t_n), the sequence $\mu(t_n\alpha)$ converges weakly to a probability μ as $n \to \infty$. Now we consider

$$A(t) = \int_{-\infty}^{+\infty} \left(\frac{1}{(1-(isx)/t)^t} - e^{isx}\right)\mu(t\alpha)(dx),$$
$$B(t) = \int_{-\infty}^{+\infty} e^{isx}\big(\mu(t\alpha)(dx) - \mu(dx)\big).$$

The left-hand side of (23) is $A(t) + B(t) + \int_{-\infty}^{+\infty} e^{isx}\mu(dx)$. By Paul Lévy's theorem, the sequence $B(t_n)$ goes to zero when $n \to \infty$.

We now show that $\lim_{t\to\infty} A(t) = 0$. We assume $s \neq 0$. Let us fix $\varepsilon > 0$ and $a = \mathbb{E}(|B|)/\varepsilon$, and define

$$A_0(t) = \int_{|x|\geq a} \left(\frac{1}{(1-(isx)/t)^t} - e^{isx}\right)\mu(t\alpha)(dx), \qquad A_1(t) = A(t) - A_0(t).$$

Since $\int_{|x|\geq a}\mu(t\alpha)(dx)\leq\varepsilon$ and since $|(1-\frac{isx}{t})^{-t}|=(1+\frac{s^2x^2}{t^2})^{-t/2}\leq 1$ we can claim that $A_0(t)\leq 2\varepsilon$ for all t.

Next for $0\leq y<t$ introduce the function

$$f(t,y)=\frac{1}{(1-y/t)^t}-e^y.$$

This is a nonnegative function since $\frac{(t)_n}{t^n}-1\geq 0$ shows $f(t,y)=\sum_{n=0}^{\infty}\frac{y^n}{n!}(\frac{(t)_n}{t^n}-1)\geq 0$. Furthermore, $y\mapsto f(t,y)$ is non-decreasing on $(0,t)$ since $\frac{\partial}{\partial y}f(t,y)=\frac{t}{t-y}f(t,y)+\frac{y}{t-y}e^y\geq 0$. For $-t<sx<t$, we have

$$\left|\frac{1}{(1-(isx)/t)^t}-e^{isx}\right|=\left|\sum_{n=0}^{\infty}\frac{(isx)^n}{n!}\left(\frac{(t)_n}{t^n}-1\right)\right|\leq f\big(t,|sx|\big).$$

As a consequence, for $t>|sa|$

$$\big|A_1(t)\big|\leq\int_{-a}^{a}f\big(t,|sx|\big)\mu(t\alpha)(dx)\leq f\big(t,|sa|\big)\underset{t\to\infty}{\to}=0.$$

This finally proves that $\lim_{t\to\infty}A(t)=0$.

For the right-hand side of (23), we introduce the function $g(t,y)=\frac{t}{2}\log(1+\frac{y^2}{t^2})$. Now we consider

$$-t\int_{-\infty}^{+\infty}\log\left(1-\frac{isx}{t}\right)\alpha(dx)=R(t)+iI(t),$$

where $R(t)=-\int_{-\infty}^{+\infty}g(t,sx)\alpha(dx)$ and where

$$\begin{aligned}I(t)&=-t\int_{-\infty}^{+\infty}\mathrm{Arg}\left(1-\frac{isx}{t}\right)\alpha(dx)=t\int_{-\infty}^{+\infty}\arctan\left(\frac{sx}{t}\right)\alpha(dx)\\&=\int_{-\infty}^{+\infty}\left(\int_0^{sx}\frac{t^2\,dv}{t^2+v^2}\right)\alpha(dx)\underset{t\to\infty}{\to}\int_{-\infty}^{+\infty}sx\alpha(dx)=sm\end{aligned}$$

(here we have used dominated convergence). In order to show $\lim_{t\to\infty}R(t)=0$, we fix $\varepsilon>0$; we introduce $a>0$ such that $\int_{|sx|>a}|x|\alpha(dx)\leq\varepsilon$ and such that $\frac{1}{2}\log(1+y^2)\leq|y|$ if $|y|\geq a$. Since $y\mapsto g(t,y)$ is increasing we get

$$\big|R(t)\big|=\int_{|sx|\leq a}+\int_{|sx|\geq a}g(t,sx)\alpha(dx)\leq g(t,a)+t\int_{|sx|\geq a}\frac{|sx|}{t}\alpha(dx)\leq g(t,a)+|s|\varepsilon$$

leading to the result since $\lim_{t\to\infty}g(t,a)=0$.

Finally, we have proved that for all probability μ such that there exists an increasing sequence (t_n) satisfying $\lim_{n\to\infty}\mu(t_n\alpha)=\mu$ we have $\int_{-\infty}^{+\infty}e^{isx}\mu(dx)=e^{ism}$, that is $\mu=\delta_m$. This is enough to claim that $\lim_{t\to\infty}\mu(t\alpha)=\delta_m$.

Let us now assume that $\int_{\mathbb{R}^d} \|x\|\alpha(dx) = \infty$ and that $\mu(t\alpha) \underset{t\to\infty}{\to} \mu$ exists and is a probability. We imitate much of the preceding proof, by starting from (23) and proving that $A(t)$ and $B(t)$ both converge to 0: the tightness of $(\mu(t\alpha))_{t>0}$ is guaranteed by the existence of μ. Therefore, the right-hand side of (23) has a limit when $t \to \infty$. As a consequence, the limit iw of $-\frac{t}{s}\int_{-\infty}^{+\infty} \log(1 - \frac{isx}{t})\alpha(dx)$ exists but does not depend on $s > 0$. This implies that the limit of the right-hand side of (23) is e^{iws}, which means that μ is the one dimensional Cauchy distribution c_w. For checking (22), observe that the real part of $\int_{-\infty}^{\infty} \log(1 - i\frac{x}{t})\alpha(dx)$ is

$$\frac{t}{2}\mathbb{E}\left(\log\left(1+\frac{B^2}{t^2}\right)\right) = \frac{t}{2}\int_0^\infty \Pr\left(\log\left(1+\frac{|B|^2}{t^2}\right) > z\right)dz = \int_0^\infty \Pr\big(|B| > x\big)\frac{xt}{x^2+t^2}.$$

The proof for a in (22) is similar. $\square$

Corollary 3.6. *If $\alpha \in L_1^{\log}$ suppose that $\int_{-\infty}^0 |x|\alpha(dx) < \infty$ and that $\int_0^\infty x\alpha(dx) = \infty$. If $X_t \sim \mu(t\alpha)$, then for any x we have $\Pr(X_t > x) \underset{t\to\infty}{\to} 1$.*

Proof. If $N > 0$, denote $p_N = \alpha([N,\infty))$ and $\alpha_N(dx) = \alpha(dx)1_{(-\infty,N)}(x) + p_N\delta_N(dx)$. Denote $m_N = \int_{-\infty}^{\infty} x\alpha_N(dx)$. Theorem 3.5 implies that $\mu(t\alpha_N)$ tends to δ_{m_N} and the assumption implies that $\lim_N m_N = +\infty$. If $Y_1,\dots,Y_n,\dots,B_1,\dots,B_n,\dots$ are independent with $Y_n \sim \beta_{1,t}$ and $B_n \sim \alpha$, then

$$X_t^N = \sum_{n=1}^{\infty}(1-Y_1)\cdots(1-Y_{n-1})Y_n \max(B_n, N) \sim \mu(t\alpha_N)$$

and this shows that $N \mapsto \Pr(X_t^N > x)$ is an increasing sequence with limit $\Pr(X_t > x)$. Furthermore if $x < m_N$ from weak convergence $\lim_{t\to\infty}\Pr(X_t^N > x) = 1$. Therefore, for fixed ε and N_0 such that $m_{N_0} > x$ there exists T_ε such that $t > T_\varepsilon$ implies

$$1 - \varepsilon \le \Pr\big(X_t^{N_0} > x\big) \le \Pr(X_t > x)$$

which ends the proof. $\square$

4. Proof of Theorem 1.2

First step. The following proposition belongs to folklore (see [12], Theorem 1). We give below a self-contained proof. In the particular case where α is uniform on the unit sphere of $\mathbb{R}^d$, additional details are given in Section 6 of [16].

Proposition 4.1. *If $(W_1,\dots,W_n) \sim \mathcal{D}(t/n,\dots,t/n)$ and $B_1,\dots,B_n$ are independent, with $B_j \sim \alpha \in L_d^{\log}$ then the limit distribution of $M_n = W_1B_1 + \cdots + W_nB_n$ for $n \to \infty$ is $\mu(t\alpha)$.*

Proof. Let $f \in \mathbb{R}^d$ and z complex with $\Im z > 0$. Then if $W_t \sim \mu(t\alpha)$ we have

$$\mathbb{E}\left(\frac{1}{(\langle f, M_n\rangle - z)^t}\right) = \mathbb{E}\left(\frac{1}{(\langle f, W_1 B_1 + \cdots + W_n B_n\rangle - z(W_1 + \cdots + W_n))^t}\right)$$
$$= \mathbb{E}\left(\frac{1}{(\langle f, B_1\rangle - z)^{t/n}} \cdots \frac{1}{(\langle f, B_n\rangle - z)^{t/n}}\right) = \left(\mathbb{E}\left(\frac{1}{(\langle f, B_1\rangle - z)^{t/n}}\right)\right)^n.$$

We compute the limit of the last expression as follows. If $z = a + ib$ with $b > 0$, write

$$e^{U+iV} = \frac{1}{(\langle f, B_1\rangle - a - ib)^t},$$

where U and V are real. We have $U \leq -t \log b$ and $V \in (0, \pi)$. Therefore, $\mathbb{E}(U)$ makes sense, by allowing $-\infty \leq \mathbb{E}(U)$. Consider now i.i.d. random variables $(U_1, V_1), \ldots, (U_n, V_n)$ with the distribution of (U, V). Then the law of large numbers applies and $\frac{1}{n}(U_1 + iV_1 + \cdots + U_n + iV_n)$ converges almost surely to $\mathbb{E}(U) + i\mathbb{E}(V)$. Also from $U \leq -t \log b$ we are able to claim that by dominated convergence:

$$\begin{aligned}\left(\mathbb{E}\left(\frac{1}{(\langle f, B_1\rangle - z)^{t/n}}\right)\right)^n &= \mathbb{E}\left(\exp \frac{1}{n}(U_1 + iV_1 + \cdots + U_n + iV_n)\right) \\ &\underset{n\to\infty}{\to} \exp\big(\mathbb{E}(U) + i\mathbb{E}(V)\big) = e^{-t\mathbb{E}(\log(\langle f, B_1\rangle - z))} \\ &= \mathbb{E}\left(\frac{1}{(\langle f, W_t\rangle - z)^t}\right)\end{aligned}$$

by Proposition 3.1. ☐

Second step. We want to use Proposition 4.1 in the particular case $n = 2^k$. The reason is that we can realise $\mathcal{D}(t/2^k, \ldots, t/2^k)$ by using products of beta random variables as follows. If $k = 1$ and $Z^t \sim \beta(\frac{t}{2}, \frac{t}{2})$, then $(W_1^t, W_2^t) = (1 - Z^t, Z^t) \sim \mathcal{D}(\frac{t}{2}, \frac{t}{2})$. If $k = 2$ and if Z^t, Z_0^t and Z_1^t are independent and if Z_i^t are $\beta(\frac{t}{4}, \frac{t}{4})$ distributed, then

$$\big(W_1^t, W_2^t, W_3^t, W_4^t\big) = \big(\big(1 - Z^t\big)\big(1 - Z_0^t\big), \big(1 - Z^t\big)Z_0^t, Z^t\big(1 - Z_1^t\big), Z^t Z_1^t\big) \sim D\left(\frac{t}{4}, \frac{t}{4}, \frac{t}{4}, \frac{t}{4}\right).$$

It is worth giving the details of the proof; taking $f_1, f_2, f_3, f_4 > 0$ we write

$$\begin{aligned}&\mathbb{E}\big[\big(f_1 W_1^t + f_2 W_2^t + f_3 W_3^t + f_4 W_4^t\big)^{-t}\big] \\ &\quad = \mathbb{E}\big[\big(\big(1 - Z^t\big)\big(f_1\big(1 - Z_0^t\big) + f_2 Z_0^t\big) + Z^t\big(f_3\big(1 - Z_1^t\big) + f_4 Z_1^t\big)\big)^{-t}\big] \\ &\quad = \mathbb{E}\big[\big(\big(f_1\big(1 - Z_0^t\big) + f_2 Z_0^t\big)\big)^{-t/2}\big] \times \mathbb{E}\big[\big(\big(f_3\big(1 - Z_1^t\big) + f_4 Z_1^t\big)\big)^{-t/2}\big] = (f_1 f_2 f_3 f_4)^{-t/4}.\end{aligned}$$

More generally, the set $\{1,\dots,2^k\}$ is put in a one to one correspondence $j \mapsto (i_1(j),\dots,i_k(j))$ with $\{0,1\}^k$ by

$$j = 1 + \sum_{h=1}^{k} i_h(j)2^{h-1},$$

we introduce for each $h = 1,\dots,k-1$ and each $(i_1,\dots,i_h) \in \{0,1\}^h$ the random variable

$$Z^t_{(i_1,\dots,i_h)} \sim \beta\left(\frac{t}{2^{h+1}}, \frac{t}{2^{h+1}}\right)$$

in such a way that these random variables are all independent (and are independent of Z^t). We define for $h = 1,\dots,k$

$$\begin{aligned} T^t_{(i_1,\dots,i_h)} &= Z^t_{(i_1,\dots,i_{h-1})} \qquad \text{if } i_h = 1, \\ &= 1 - Z^t_{(i_1,\dots,i_{h-1})} \qquad \text{if } i_h = 0, \\ W^t_j &= \prod_{h=1}^{k} T^t_{(i_1(j),\dots,i_h(j))}. \end{aligned}$$

One can now prove by induction on k along lines similar to the case $k = 2$ that $(W^t_j)_{j=1}^{2^k} \sim \mathcal{D}(t/2^k,\dots,t/2^k)$. We skip the details.

Third step. We have seen in the comment following Proposition 2.2 that $0 < s < t$ implies that $\beta(t,t) \prec \beta(s,s)$. From Theorem 2.1, this implies the existence of a probability kernel $K_{s,t}(x,dy)$ on $(0,1)^2$ such that

$$K_{s,t}(x,dy)\beta(t,t)(dx)$$

is a joint distribution of (X,Y) with $X \sim \beta(t,t)$, $Y \sim \beta(s,s)$ and $\mathbb{E}(Y|X) = X$.

Next, for fixed $0 < s < t$ and each $(i_1,\dots,i_h)$ with $h = 1,\dots,k-1$ we consider a pair $(Z^s_{(i_1,\dots,i_h)}, Z^t_{(i_1,\dots,i_h)})$ with respective margins $\beta(\frac{s}{2^{h+1}}, \frac{s}{2^{h+1}})$ and $\beta(\frac{t}{2^{h+1}}, \frac{t}{2^{h+1}})$ and such that the conditional distribution of the former given the latter is $K_{s/2^{h+1},t/2^{h+1}}$. Finally, all these pairs are mutually independent. Now we create also the W^s_j's from the Z^s's as done in the second step. The important point is now

$$\begin{aligned} &\mathbb{E}\big(W^s_j | Z^t_{(i_1,\dots,i_h)}, (i_1,\dots,i_h) \in \{0,1\}^h, h = 0,1,\dots,k-1\big) \\ &\quad = \prod_{h=1}^{k} \mathbb{E}\big(T^s_{(i_1(j),\dots,i_h(j))} | Z^t_{(i_1(j),\dots,i_{h-1}(j))}\big) = \prod_{h=1}^{k} T^t_{(i_1(j),\dots,i_h(j))} = W^t_j. \end{aligned} \tag{24}$$

Essentially we are using that if (X_i, Y_i), $i = 1,\dots,n$ are mutually independent pairs of random variables with X_i integrable and $\mathbb{E}(X_i|Y_i) = Y_i$ for $i = 1,\dots,n$, then $\mathbb{E}(\prod_{i=1}^n X_i|Y_1,\dots,Y_n) = \prod_{i=1}^n \mathbb{E}(X_i|Y_i)$. From (24) we get

$$\mathbb{E}\big(W^s_j | W^t_j\big) = W^t_j \tag{25}$$

by using the tower property of conditional expectations: if $\mathbb{E}(X|\mathcal{F}) = Y$ then $\mathbb{E}(X|\mathcal{G}) = Y$ if $\mathcal{G} \subset \mathcal{F}$ and if Y is $\mathcal{G}$-measurable.

Fourth step. For simplicity, we continue to omit in the notations W_j^s and W_j^t the fact that these random variables depend on k. Defining like in Proposition 4.1

$$X_k^t = \sum_{j=1}^{2^k} B_j W_j^t, \qquad X_k^s = \sum_{j=1}^{2^k} B_j W_j^s$$

we can now claim that from (25) that

$$\mathbb{E}\big(X_k^s | W_j^t, B_j, \forall j = 1, \dots, 2^k\big) = X_k^t.$$

Again by the tower property we get $\mathbb{E}(X_k^s|X_k^t) = X_k^t$. By Theorem 2.1, this implies that $X_k^t \prec X_k^s$. Furthermore $\mathbb{E}(X_k^t) = \mathbb{E}(X_k^s) = \mathbb{E}(B_1)$ for any integer k. By Proposition 4.1, X_k^t and X_k^s converge in law to $\mu(t\alpha)$ and $\mu(s\alpha)$, respectively, as $k \to \infty$. Moreover, these limit distributions keep the same mean vector $\mathbb{E}(B_1)$. The proof of Theorem 1.2 is completed by an application of comment 3 on the Strassen convex order in Section 2.

5. Cauchy distributions in $\mathbb{R}^d$ and Dirichlet curves

The next problem to deal with is the study of the Dirichlet curve $t \mapsto \mu(t\alpha)$ when $\int_{\mathbb{R}^d} \|x\|\alpha(dx) = \infty$. Theorem 3.5 has shown that if the probability $\mu(\infty) = \lim_{t\to\infty} \mu(t\alpha)$ exists then $\mu(\infty)$ is Cauchy in $\mathbb{R}^d$. In this section, we will prove various characterizations of the Cauchy distributions related to the Dirichlet curve. These characterizations are linked with the general conjecture $\mu(t\alpha) = \mu(s\alpha)$ for $t \neq s$ implies that α is Cauchy. Propositions 5.1 and 5.2 consider the cases $(s,t) = (n, n+1)$ and $(n, n+2)$ for $n = 0, 1, 2, \dots$. Propositions 5.3 and 5.4 and Corollary 5.5 consider the case where $\mu(t\alpha) = \mu(s\alpha)$ holds for some infinite sets of t's. Propositions 5.6 and 5.7 concern the iteration of the map $\alpha \mapsto \mu(\alpha)$. The proofs use the properties of analytic functions and differential equations in the complex plane.

All along this section we exploit the properties of the Stieltjes transform of a probability α on $\mathbb{R}$, namely the function, defined for all complex numbers z with $\Im z > 0$ by $y(z) = \int_{-\infty}^{+\infty} \frac{\alpha(dw)}{w-z}$. Recall that the Stieltjes transform of the Cauchy distribution c_ζ with $\zeta = a + ib \in H_+$ and $\overline{\zeta} = a - ib$ is

$$\int_{-\infty}^{+\infty} \frac{c_\zeta(dt)}{t-z} = \frac{1}{\overline{\zeta} - z}.$$

Note that for any positive integer k we have $y^{(k)}(z) = k! \int_{-\infty}^{+\infty} \frac{\alpha(dw)}{(w-z)^{k+1}}$.

Proposition 5.1. *Let $\alpha \in L_1^{\log}$ and let y be its Stieltjes transform. Then $\mu(n\alpha) = \alpha$ if and only if*

$$n y(z) y^{(n-1)}(z) = y^{(n)}(z). \tag{26}$$

In particular for $n=1$ and $n=2$ this implies that α is Cauchy. If $\alpha \in L_d^{\log}$ again $\mu(\alpha)=\alpha$ or $\mu(2\alpha)=\alpha$ if and only if α is Cauchy in $\mathbb{R}^d$.

Proof. Suppose $d=1$ and use Proposition 3.1. If $\mu(n\alpha)=\alpha \in L_1^{\log}$ we can write with $g(z) = -\int_{-\infty}^{+\infty} \log(w-z)\alpha(dw)$:

$$\int_{-\infty}^{+\infty} \frac{\alpha(dw)}{(w-z)^n} = e^{ng(z)}.$$

Both sides are analytic functions on the half plane $H^+ = \{z \in \mathbb{C} : \Im z > 0\}$. Deriving in z and using $y=g'$ we get

$$n\int_{-\infty}^{+\infty} \frac{\alpha(dw)}{(w-z)^{n+1}} = ne^{ng(z)}g'(z) = ny(z)\int_{-\infty}^{+\infty} \frac{\alpha(dw)}{(w-z)^n},$$

from which (26) is immediate. Conversely, from (26) we write

$$ny(z) = ng'(z) = \frac{y^{(n)}(z)}{y^{(n-1)}(z)}$$

and we get that $y^{(n-1)}$ is proportional to e^{ng}. Since, up to a muliplicative constant, the left-hand side is equal to $\int_{-\infty}^{+\infty} \frac{\alpha(dw)}{(w-z)^n}$, we get for some constant C

$$\int_{-\infty}^{+\infty} \frac{\alpha(dw)}{(w-z)^n} = Ce^{ng(z)}.$$

To see that $C=1$ we use the fact that α has mass 1 and we replace z by ri with $r>0$ in the equality. We get

$$\int_{-\infty}^{+\infty} r^n \frac{\alpha(dw)}{(w-ri)^n} = Ce^{n(g(ri)+\log r)}.$$

Now $\lim_{r\to\infty} \int_{-\infty}^{+\infty} r^n \frac{\alpha(dw)}{(w-ri)^n} = i^n$. Also

$$g(ri) + \log r = -\int_{-\infty}^{+\infty} \log\left(\frac{w}{r} - i\right)\alpha(dw) \underset{r\to\infty}{\to} -\log(-i) = \frac{\pi}{2}i$$

and therefore $\lim_{r\to\infty} e^{n(g(ri)+\log r)} = e^{n\frac{\pi}{2}i} = i^n$ which implies $C=1$.

As far as the second statement is concerned, for $n=1$ this is a result due to [18]. Our proof is shorter, since the general solution of the differential equation $y'(z) = y^2(z)$, corresponding to (26) for $n=1$ is $y(z) = \frac{1}{a-ib-z}$ where $a-ib$ is an arbitrary complex constant. However, since $z \mapsto y(z)$ is analytic in H^+ we have necessarily $b \geq 0$. If $b>0$ one gets the Stieltjes transform of the Cauchy distribution c_{a+ib}, if $b=0$, then $\alpha = \delta_a$.

For $n=2$ things are more involved. Any solution of the differential equation $y'' = 2yy'$, corresponding to (26) for $n=2$, which is analytic in H^+ satisfies $y' = y^2 - C^2$ where C is some complex constant. If $C = 0$, we get that $y(z) = \frac{1}{a-ib-z}$ as in the case $n = 1$. In this case, α is Cauchy. Let us show now that taking $C \neq 0$ does not lead to an acceptable solution. We write first $1 = \frac{y'}{y^2-C^2}$ leading with an arbitrary constant z_0 to $y(z) = C \operatorname{cotanh} C(z_0 - z)$. If $\Re C \neq 0$ the meromorphic function $z \mapsto \operatorname{cotanh} C(z_0 - z)$ has poles in H^+ and y would not be holomorphic in H^+. If $C = ir$ is purely imaginary, we observe that $y(z) = C \operatorname{cotanh} C(z_0 - z)$ cannot be a Stieltjes transform since the condition $\lim_{t\to\pm\infty} y(z+t) = 0$ is not fulfilled, the function $t \mapsto y(z+t)$ being periodic.

Finally we consider the d-dimensional case. If $\alpha \in L_d^{\log}$ and if $\mu(n\alpha) = \alpha$, let $f \in \mathbb{R}^d$ and denote by α_f the image of α by $x \mapsto \langle f, x\rangle$. Then $\mu(n\alpha_f) = \alpha_f$. If $n = 1$ or $n = 2$, we have seen that α_f is Cauchy: the definition of a Cauchy distribution in $\mathbb{R}^d$ implies the result. $\square$

In the sequel, all the characterizations of the Cauchy distribution in $\mathbb{R}$ are extendable to $\mathbb{R}^d$ as done in Proposition 5.1, so we shall not mention it anymore and set $d = 1$ from now on.

Proposition 5.2. *Let $\alpha \in L_1^{\log}$. Let $n < m$ any positive integers. Suppose that $\mu(n\alpha) = \mu(m\alpha)$ and let $y(z) = \int_{-\infty}^{+\infty} \frac{\mu(n\alpha)(dw)}{w-z}$. Then*

$$\left(\frac{y^{(n-1)}}{(n-1)!}\right)^m = \left(\frac{y^{(m-1)}}{(m-1)!}\right)^n. \tag{27}$$

In particular if $m = n+1$ or if $m = n+2$ then α is Cauchy.

Proof. As usual we write $g(z) = -\int_{-\infty}^{+\infty} \log(w-z)\alpha(dw)$. From Proposition 3.1, we have

$$e^{ng(z)} = \int_{-\infty}^{+\infty} \frac{\mu(n\alpha)(dw)}{(w-z)^n} = \frac{y^{(n-1)}(z)}{(n-1)!}.$$

From this (27) is plain.

Suppose now that $m = n+1$ and denote $Y = y^{(n-1)}/(n-1)!$. From (27), we get $(\frac{Y'}{n})^n = Y^{n+1}$. Clearly Y is not identically zero, since the Stieltjes transform of a probability cannot be a polynomial. Select an open ball $U \subset H^+$ where $Y(z) \neq 0$ for all $z \in U$. Therefore there exists a nth root of unity ω such that $Y' = n\omega Y^{1+\frac{1}{n}}$. Integrating this differential equation we get that there exists a complex number $a - ib$ such that $Y^{-1/n} = \omega(a - ib - z)$ leading to $\frac{y^{(n-1)}}{(n-1)!} = \frac{1}{(a-ib-z)^n}$. Integrating $n-1$ times we get $y(z) = P(z) + \frac{1}{a-ib-z}$ where P is a polynomial with degree $< n$. This is correct for $z \in U$, but by analytic continuation it extends to the whole H^+. Since y is a Stieltjes transform $P = 0$ and one concludes as the usual way that $b \geq 0$ and that $\mu(n\alpha)$ is either strict Cauchy c_{a+ib} or Dirac δ_a (from the Stieltjes transform of the Cauchy distribution). Since, again by Proposition 3.1, the map $\alpha \mapsto \mu(n\alpha)$ is injective and from Corollary 2.5 $\mu(nc_{a+ib}) = c_{a+ib}$ and $\mu(n\delta_a) = \delta_a$ we conclude that $\mu(n\alpha) = \alpha$, so α is Cauchy.

Consider now the case $m = n+2$. From (27) we get

$$\left(\frac{y^{(n-1)}}{(n-1)!}\right)^{n+2} = \left(\frac{y^{(n+1)}}{(n+1)!}\right)^n.$$

Again taking $Y = y^{(n-1)}/(n-1)!$ we get $Y'' = n(n+1)y^{(n+1)}/(n+1)!$ and finally

$$\left(\frac{Y''}{n(n+1)}\right)^n = Y^{n+2}.$$

Using again a ball $U \subset H^+$ on which $Y(z) \neq 0$ there exists a nth root of unity ω such that

$$Y'' = n(n+1)\omega Y^{1+2/n}.$$

We now use a classical trick for ordinary differential equations of the form $Y'' = f(Y', Y)$. From the implicit function theorem in the analytic case, there exists an open set $V \subset U$ such that $z \mapsto Y(z)$ is injective while restricted to V and such that $Y(V)$ is open. As a consequence, there exists an analytic function p on $Y(V)$ such that $Y'(z) = p(Y(z))$ for $z \in V$. Deriving we get $Y''(z) = p'(Y(z))p(Y(z))$ leading to

$$2p'\big(Y(z)\big)p\big(Y(z)\big) = 2n(n+1)\omega Y^{1+2/n}(z).$$

Thus integrating this differential equation in p there exists a complex constant C such that

$$p\big(Y(z)\big)^2 = \big(Y'(z)\big)^2 = n^2\omega\big(Y^{(2n+2)/n}(z) - C^{(2n+2)/n}\big).$$

Now $Y(z) = \int_{-\infty}^{+\infty} \frac{\alpha(dw)}{(w-z)^n}$ and $Y'(z) = n\int_{-\infty}^{+\infty} \frac{\alpha(dw)}{(w-z)^{n+1}}$ imply that $C = 0$ and that for some $2n$th root of unity ω_1 we have, for z in some non-empty open subset V_1 of V

$$Y'(z) = n\omega_1 Y^{(n+1)/n}(z)$$

leading to the existence of a complex number $a - ib$ such that $Y^{-1/n} = \omega_1(z - a + ib)$. Since $\omega_1^{2n} = 1$ we get $\omega_1^n = \pm 1$ and

$$Y(z) = \pm\frac{1}{(a-ib-z)^n}.$$

Finally we get that $y(z) = P(z) \pm \frac{1}{a-ib-z}$ where P is a polynomial. The fact that y is a Stieltjes transform leads easily to $P = 0$ and to $y(z) = \frac{1}{a-ib-z}$ where $b \geq 0$: this implies again that α is Cauchy. □

Proposition 5.3. *Let $\alpha \in L_1^{\log}$. Let N be an integer and suppose that $\mu(n\alpha) = \alpha$ for all $n \geq N$. Then α is Cauchy.*

Proof. By Proposition 5.1, the hypothesis implies that for all $n \geq N$ we have

$$y\frac{y^{(n-1)}}{(n-1)!} = \frac{y^{(n)}}{n!},$$

where $y(z)=\int_{-\infty}^{+\infty}\frac{\alpha(dw)}{w-z}$ is the Stieltjes transform of α, which is analytic in $H^+=\{z\in\mathbb{C};\Im z>0\}$. Since the above equality is true for all $n\geq N$, we deduce from it that for all $n\geq N$ we have

$$y^{n-N+1}\frac{y^{(N-1)}}{(N-1)!}=\frac{y^{(n)}}{n!}. \tag{28}$$

Since y is analytic in H^+, when $z\in H^+$ the Taylor expansion of $t\mapsto y(z+t)$ converges for $|t|<\Im z$ and we can write for such (z,t)

$$\begin{aligned} y(z+t) &= \sum_{n=0}^{N-1}\frac{y^{(n)}(z)t^n}{n!}+\sum_{n=N}^{\infty}\frac{y^{(n)}(z)t^n}{n!} \\ &= \sum_{n=0}^{N-1}\frac{y^{(n)}(z)t^n}{n!}+\frac{y^{(N-1)}(z)}{(N-1)!}\sum_{n=N}^{\infty}y^{n-N+1}(z)t^n \qquad (29)\\ &= \sum_{n=0}^{N-1}\frac{y^{(n)}(z)t^n}{n!}+\frac{y^{(N-1)}(z)}{(N-1)!}\frac{y(z)t^N}{1-ty(z)}, \qquad (30)\end{aligned}$$

where (29) comes from (28). From (30), we get that $t\mapsto y(z+t)$ is a rational function. Since y is analytic on H^+ this implies that (30) holds for all $z\in H^+$ and all real t. We deduce from (30) by expanding the rational function $t\mapsto y(z+t)$ in partial fractions that there exists a polynomial $t\mapsto A_z(t)$ whose coefficients depend on z such that

$$y(z+t)=A_z(t)+\frac{B_z}{1-ty(z)}, \tag{31}$$

where $B_z=\frac{y^{(N-1)}(z)}{(N-1)!}y(z)^{1-N}$ if $y(z)\neq 0$ and $B_z=0$ if $y(z)=0$. The trick is now to observe that since y is the Stieltjes transform of the probability α we can write

$$\lim_{t\to\infty}ty(z+t)=\lim_{t\to\infty}t\int_{-\infty}^{+\infty}\frac{\alpha(dw)}{w-z-t}=-1.$$

Applying this remark to (31), we obtain that $A_z=0$, that $B_z=y(z)$ and finally that $y(z+t)=\frac{y(z)}{1-ty(z)}$. Deriving with respect to t and setting $t=0$ we get $y'(z)=y^2(z)$, from which one concludes as in Proposition 5.1. $\square$

Proposition 5.4. *Let $\alpha\in FT_1$ and $0\leq b<c$. Suppose that $\nu=\mu(a\alpha)$ for all $a\in(b,c)$. Then $\alpha=\nu$ is Cauchy.*

Proof. Again with $g(z)=-\int_{-\infty}^{+\infty}\log(w-z)\alpha(dw)$, with $z\in H^+$, we can differentiate n times with respect to $a\in(b,c)$ both sides of

$$\int_{-\infty}^{+\infty}\frac{\nu(dw)}{(w-z)^a}=e^{ag(z)}.$$

We get for all $a \in (b, c)$

$$\int_{-\infty}^{+\infty} \big[-\log(w-z)\big]^n \frac{\nu(dw)}{(w-z)^a} = e^{ag(z)} g(z)^n. \tag{32}$$

The idea of the proof is to multiply both sides of (32) by $t^n/n!$, to sum up in n, to invert sum and integral in order to get finally

$$\int_{-\infty}^{+\infty} \frac{\alpha(dw)}{(w-z)^{a+t}} = e^{(a+t)g(z)}.$$

However, the inversion of the sum and the integral needs some care. For this reason, denote $u_n(w) = |-\log(w-z)|^n \frac{1}{|w-z|^a}$ and observe that $F(w,t) = \sum_{n=0}^{\infty} u_n(w) \frac{t^n}{n!} < \infty$. If $0 < t < a$, let us observe that

$$\int_{-\infty}^{+\infty} F(w,t)\nu(dw) < \infty.$$

This obtained since $u_n(w) \le (|\log|w-z|| + \pi)^n \frac{1}{|w-z|^a}$ and therefore if $|w-z| > 1$

$$F(w,t) \le \frac{1}{|w-z|^a} e^{t|\log|w-z||+t\pi} = \frac{1}{|w-z|^{a-t}} e^{\pi t}.$$

We now write from (32) and the dominated convergence theorem

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{\nu(dw)}{(w-z)^{a+t}} &= e^{(a+t)g(z)} = \sum_{n=0}^{\infty} e^{ag(z)} g(z)^n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} \big[-\log(w-z)\big]^n \frac{t^n}{n!} \frac{\alpha(dw)}{(w-z)^a} \\
&= \int_{-\infty}^{+\infty} \frac{\alpha(dw)}{(w-z)^{a+t}}.
\end{aligned}$$

As a result $\alpha = \nu$ and furthermore $\mu((a+t)\alpha) = \alpha$ for all $t \in (0, a)$. By induction, we get easily that $\mu((a+t)\alpha) = \alpha$ for all $t > 0$. Now we apply Proposition 5.3 since $\mu(n\alpha) = \alpha$ for all integers n large enough and the proof is complete. □

Corollary 5.5. *If for a fixed b and c such that $0 \le b < c$ we have $\mu(b\alpha) = \mu(c\alpha)$ and if α has a mean, then α is Dirac.*

Proof. If $b < a < c$ from Theorem 1.2 we have $\mu(c\alpha) \prec \mu(a\alpha) \prec \mu(b\alpha)$. From comment 4 on the Strassen convex order in Section 2 and from the hypothesis of the present corollary we have $\mu(c\alpha) = \mu(a\alpha) = \mu(b\alpha)$. Therefore, the hypothesis of Proposition 5.4 is fulfilled and α is strict Cauchy or Dirac. By since α has a mean, the first possibility is ruled out. □

Comment. Suppose that $\alpha \in L_d^{\log}$ is invariant by rotation and consider $X_t \sim \mu(t\alpha)$. Suppose that $t \mapsto \Pr(\|X_t\| \leq x)$ is increasing on $[0, \infty)$ for any $x \geq 0$. In other terms, the laws of the $\|X_t\|$'s are decreasing in the stochastic order. In this case $\mu(b\alpha) = \mu(c\alpha)$ for some $0 \leq b < c$ implies that α is Cauchy. For seeing this, observe that $\|X_b\| \sim \|X_c\|$ and therefore $\|X_b\| \sim \|X_a\|$ for all $a \in (b, c)$. From the invariance by rotation, we get that $\mu(a\alpha) = \mu(b\alpha)$ for all $a \in (b, c)$ and Proposition 5.4 applies.

Proposition 5.6. *There exists a probability $\alpha \in L_1^{\log}$ such that $\mu(\alpha) \notin L_1^{\log}$.*

Proof. Let us fix $1 < a \leq 2$ and consider

$$\alpha(dw) = \frac{a}{(1 + \log(1 + w))^{a+1}} \mathbf{1}_{(0,\infty)}(w) \frac{dw}{1 + w}.$$

With this definition, if $B \sim \alpha$, then $\Pr(\log(1 + B) > t) = \frac{1}{(1+t)^a}$ for $t > 0$, so $\mathbb{E}(\log(1 + B)) < \infty$. Let us compute

$$\begin{aligned} g(x) &= -\int_0^\infty \log|x - w|\alpha(dw) = -\int_0^\infty \log|x - w| \frac{a}{(1 + \log(1 + w))^{a+1}} \frac{dw}{1 + w} \\ &= -a \int_0^\infty \log\left|x + 1 - e^y\right| \frac{dy}{(1 + y)^{a+1}}, \\ g\left(e^u - 1\right) &= -a \int_0^\infty \log\left|e^u - e^y\right| \frac{dy}{(1 + y)^{a+1}} = -u - a \int_0^\infty \log\left|1 - e^{y-u}\right| \frac{dy}{(1 + y)^{a+1}}. \end{aligned}$$

From Cifarelli and Regazzini [4] the density $f(x)$ of $X \sim \mu(\alpha)$ is, for $x > 0$,

$$f(x) = \frac{1}{\pi} \sin\left(\pi \int_x^\infty \alpha(dw)\right) e^{g(x)} \underset{x\to\infty}{\sim} \left(\int_x^\infty \alpha(dw)\right) e^{g(x)}.$$

From this remark, $\mathbb{E}(\log(1 + X)) = \infty$ if and only if the integral

$$I = \int_0^\infty \log(1 + x)\left(\int_x^\infty \alpha(dw)\right) e^{g(x)}\, dx$$

diverges. Doing in I the change of variable $x = e^u - 1$, we obtain

$$I = \int_0^\infty \frac{u}{(1 + u)^a} e^{g(e^u - 1) + u}\, du.$$

From dominated convergence, we have

$$g\left(e^u - 1\right) + u = -a \int_0^\infty \log\left|1 - e^{y-u}\right| \frac{dy}{(1 + y)^{a+1}} \underset{u\to\infty}{\to} 0.$$

Therefore, I diverges like the integral $J = \int_0^\infty \frac{u\, du}{(1+u)^a}$ since $1 < a \leq 2$. $\square$

Proposition 5.7. *For $\alpha \in L_1^{\log}$ let $\mu_1(\alpha) = \mu(\alpha)$, and define by induction $\mu_n(\alpha) = \mu(\mu_{n-1}(\alpha))$, if $\mu_{n-1}(\alpha) \in L_1^{\log}$. Let $n \geq 2$ be an integer, and suppose that $\alpha \in L_1^{\log}$ and $\mu_k(\alpha) \in L_1^{\log}$ for $k = 2, \dots, n-1$ and $\mu_n(\alpha) = \alpha$. Denote $y_j(z) = \int_{-\infty}^{+\infty} \frac{\mu_j(\alpha)(dw)}{w-z}$, for $j = 1, \dots, n$. Then*

$$\left(y_1', \dots, y_n'\right) = (y_n y_1, y_1 y_2, \dots, y_{n-1} y_n). \tag{33}$$

In particular, if $\mu(\mu(\alpha)) = \alpha$ then α is Cauchy.

Proof. With the convention $\mu_0(\alpha) = \alpha$ and the assumption $\mu_n(\alpha) = \alpha$, we can write for $j = 1, \dots, n$:

$$\int_{-\infty}^{+\infty} \frac{\mu_j(\alpha)(dw)}{w - z} = e^{g_{j-1}(z)}, \tag{34}$$

where $g_j(z) = -\int_{-\infty}^{+\infty} \log(w - z)\mu_j(\alpha)(dw)$, for $j = 0, \dots, n-1$. Since $g_j' = y_j$, taking derivatives in (34) we get $y_j' = e^{g_{j-1}} g_{j-1}' = y_j y_{j-1}$, which is (33). If $n = 2$, the differential system (33) gives $y_1' = y_1 y_2 = y_2'$. Therefore, there exists a complex constant C such that $y_2 = y_1 + C$. If $C = 0$ we get $y_1' = y_1^2$ leading to α being Cauchy as above. We are going to prove that $C \neq 0$ is impossible. Suppose the contrary: then, being $y_1' = y_1(y_1 + C)$ we get

$$\frac{1}{C}\left(\frac{y_1'}{y_1} - \frac{y_1'}{y_1 + C}\right) = 1$$

from which there exists a complex constant z_0 such that $y_1 = \frac{C}{e^{-C(z-z_0)} - 1}$. The constant z_0 cannot belong to H^+: otherwise it is a pole of y_1, which is impossible. Finally, if $\Re C \neq 0$ the function y_1 has poles in H^+, whereas if $C = ir$ is purely imaginary the function $t \mapsto y_1(z + t)$ is periodic and this contradicts the fact that y_1 is a Stieltjes transform. The proof is finished. $\square$

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