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DIRICHLET FINITE SOLUTIONS OF $\Delta u=Pu$ ON OPEN RIEMANN SURFACES*

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Consider a second order differential $P(z)dxdy$ $(z=x+iy)$ on an open Riemann surface R such that $P(z)$ is a nonnegative continuously differentiable function of a local parameter z, and the corresponding second order self-adjoint elliptic partial differential equation

$$\Delta u(z)=P(z)u(z) \qquad (P(z)\geqq 0) \tag{1}$$

on R where $\Delta u(z)dxdy=d{*}du(z)$. We are interested in solutions u of (1) with finite Dirichlet integrals $\int_R du\wedge{*}du$. The main result of this paper is the following

THEOREM. *If there exists a nonconstant Dirichlet finite solution of* (1) *on* R, *then there exists a nonconstant bounded Dirichlet finite solution of* (1) *on* R.

In no. 1 we will give an account of the background of the theorem. After establishing several auxiliary results in nos. 2-5, a general theorem will be proved in no. 6; from this the main result will follow. In no. 7 a sufficient condition is given for the space of bounded Dirichlet finite solutions to be isomorphic to the space of bounded Dirichlet finite harmonic functions. This isomorphism, together with auxiliary results in nos. 8 and 9, is used in no. 10 to deduce a criterion for a Riemann surface not to carry any Dirichlet finite solution of (1).

1. Background of the theorem. By a solution $u(z)$ of the equation (1) on an open subset ω of R we mean a real-valued C^2 function satisfying (1) on ω. We denote by $P(R)$ the space of solutions of (1) on R and we also consider its subspace $PX(R)$ with a certain property X. For $P\equiv 0$ we use the traditional notation $H(R)$ and $HX(R)$ instead of $O(R)$ and $OX(R)$. Let $\mathcal{O}_{PX}$ be the set of pairs (R, P) such that $PX(R)$ reduces to constants. Instead of $(R, P)\in\mathcal{O}_{PX}$ we simply write $R\in\mathcal{O}_{PX}$ if P is well understood. As for X we let B stand for boundedness, D for the finiteness of the Dirichlet integral $D_R(u)=\int_R du\wedge{*}du$, and E for the finiteness of the energy integral $E_R(u)=D_R(u)+\int_R Pu^2dxdy$; we also consider combinations of these properties. It has been known that

$$\mathcal{O}_G\subsetneqq\mathcal{O}_{PB}\subsetneqq\mathcal{O}_{PD}\subset\mathcal{O}_{PBD}\subset\mathcal{O}_{PE}=\mathcal{O}_{PBE}, \tag{2}$$

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where $\mathcal{O}_G$ is class of pairs (R, P) such that there exists no harmonic Green's function on R.

This type of classification theory of Riemann surfaces was initiated by Ozawa [9], who proved that $\mathcal{O}_G \subsetneqq \mathcal{O}_{PB} \subset \mathcal{O}_{PE} = \mathcal{O}_{PBE}$. The most interesting result of Ozawa is $\mathcal{O}_{PB} = \mathcal{O}_{PE}$ for $P \not\equiv 0$ with $\int_R P(z)\,dxdy < \infty$. Unrestricted existence of the Green's function of (1) for every (R, P) with $P \not\equiv 0$, established by Myrberg [4], eliminated the need to consider the nonexistence of positive solutions for $P \not\equiv 0$. The relations $\mathcal{O}_{PB} \subset \mathcal{O}_{PD} \subset \mathcal{O}_{PBD} \subset \mathcal{O}_{PE}$ were obtained partly by Royden [10] and partly by [7]. The strictness of the inclusion $\mathcal{O}_{PB} \subset \mathcal{O}_{PD}$ $(P \not\equiv 0)$ was only recently obtained by Glasner-Katz-Naikai [2].

The open problems in this context are thus, to prove or disprove the strictness of the inclusions $\mathcal{O}_{PD} \subset \mathcal{O}_{PBD}$ and $\mathcal{O}_{PBD} \subset \mathcal{O}_{PE}$. The theorem stated in the introduction settles one of these:[1)]

$$\mathcal{O}_{PD} = \mathcal{O}_{PBD}. \tag{3}$$

The relation (3) may also be viewed as a generalization of the Virtanen identity [12] $\mathcal{O}_{HD} = \mathcal{O}_{HBD}$. A straightforward extension of this identity is of course $\mathcal{O}_{PE} = \mathcal{O}_{PBE}$ since the energy integral $E(u)$ for $\Delta u = pu$ plays the same role as the Dirichlet integral $D(u)$ for the harmonic case $\Delta u = 0$. In this sense the proof for (3) requires a new approach and therefore (3) may be considered as a nontrivial extension of the Virtanen identity.

The theorem stated in the introduction was announced in [8] in which the use of the compactification theory for its proof was suggested. It has the advantage of giving a clearer geometric insight to the result. However, to avoid rather heavy machinery, we will give a direct analytic proof.

Every result in this paper is obviously valid if R is replaced by a noncompact Riemannian manifold of arbitrary dimension $\geqq 2$.

2. Weak Dirichlet principle. As already mentioned the Dirichlet principle is the basic tool for the study of classes HD and PE. It appears that the class PD suffers from the lack of such a tool. However the following weaker version of the Dirichlet principle proves to be useful. This and also the result in no. 3 were already obtained in our earlier paper [7] but for the sake of completeness we include them here.

Let Ω be a regular subregion of R, and $\mathcal{F}_\varphi$ the class of nonnegative Dirichlet finite subsolutions v of (1) on Ω with continuous boundary values φ at $\partial\Omega$. Here an upper (resp. lower) semicontinuous function v on Ω is a subsolution (resp. supersolution) of (1) on Ω if there exists a parametric disk $|z|<1$ for each point in Ω such that $u \geqq v$ (resp. $u \leqq v$) on $|z|=r$ $(0<r<1)$ implies $u \geqq v$ (resp. $u \leqq v$) on $|z|<r$ for every r and every solution u of (1) on $|z|<r$ with a continuous exten-

1) After the completion of the present work the author found that $(\{|z|<1\}, (1-|z|)^{-1}) \in \mathcal{O}_{PE} - \mathcal{O}_{PBD}$. (The proof is not too simple as the example appears. See Bull. Amer. Math. Soc. **77** (1971), 527-530.) Thus the classification problem in this context is completely settled.

sion to $|z|\leqq r$ (for fundamental properties of sub- and supersolutions, we refer the reader to e.g. Myrberg [5]).

The weak Dirichlet principle reads: *There exists a unique function u in the class $\mathfrak{F}_\varphi\cap PBD(\Omega)$ such that*

$$D_\Omega(u)=\min_{v\in\mathfrak{F}_\varphi} D_\Omega(v). \tag{4}$$

The proof is trivial. In fact, there is one and only one function u in $\mathfrak{F}_\varphi\cap PBD(\Omega)$. By the Dirichlet (or energy) principle,

$$E_\Omega(u)\leqq E_\Omega(v)$$

for every $v\in\mathfrak{F}_\varphi$. Therefore

$$D_\Omega(u)\leqq D_\Omega(v)+\int_\Omega P(v^2-u^2)\,dxdy.$$

Since $u\geqq v\geqq 0$, we deduce (4).

3. The Riesz decomposition. Let $u\in PD(R)$ (resp. $PBD(R)$). We shall show that there exist solutions u^+ and u^- in $PD(R)$ (resp. $PBD(R)$) such that

$$u=u^+-u^-,\quad u^+\geqq 0,\quad u^-\geqq 0 \tag{5}$$

on R, i.e. *$PD(R)$ (resp. $PBD(R)$) admits the Riesz decomposition.*

For the proof, set $v(z)=\max(u(z), 0)$. It is a Dirichlet finite subsolution on R. Let Ω be a regular subregion of R. Denote by P_v^Ω (resp. H_v^Ω) the solution of (1) (resp. $\Delta u=0$) on Ω with continuous boundary values v at $\partial\Omega$. Clearly

$$v\leqq P_v^\Omega\leqq H_v^\Omega$$

on Ω. It is also obvious that

$$P_v^\Omega\leqq P_v^{\Omega'}$$

on Ω for $\Omega\subset\Omega'$. Since $\lim_{\Omega\to R} H_v^\Omega$ exists (cf. e.g. Sario-Nakai [11]), we conclude that $P_v^R=\lim_{\Omega\to R} P_v^\Omega$ exists and is a solution of (1) on R. By the weak Dirichlet principle, $D_\Omega(P_v^\Omega)\leqq D_R(v)$. The Fatou lemma yields $D_R(P_v^R)\leqq D_R(v)<\infty$. Therefore the relations $u^+=P_v^R\geqq 0$ and $u^-=P_v^R-u\geqq 0$ establish the desired decomposition.

4. An integral representation. Let Ω be a regular subregion of R and $u\in P(\Omega)\cap C^1(\bar{\Omega})$. Then the following identity is valid:

$$u(z)=H_u^\Omega(z)-\frac{1}{2\pi}\int_\Omega G_\Omega(z,\zeta)P(\zeta)u(\zeta)\,d\xi d\eta \tag{6}$$

for $z\in\Omega$, where $\zeta=\xi+i\eta$ and G_Ω is the harmonic Green's function on Ω. If $u\in P(R)$, then the transition from this to the limit

$$u(z)=H_u^R(z)-\frac{1}{2\pi}\int_R G_R(z,\zeta)P(\zeta)u(\zeta)\,d\xi d\eta \tag{7}$$

is permissible provided $H_u^R=\lim_{\Omega\to R}H_u^\Omega$ exists, the harmonic Green's function G_R on R exists, and $G_R(z,\zeta)u(\zeta)$ is $P(\zeta)\,d\xi d\eta$-integrable. This is the case, for example, when $u\in PD(R)$ for $R\notin\mathcal{O}_{PD}$.

To prove (6), take a parametric disk B about z with radius $\varepsilon>0$ such that $\bar{B}\subset\Omega$. By Green's formula,

$$\int_{\Omega-\bar{B}}[(u(\zeta)-H_u^\Omega(\zeta))\Delta_\zeta G_\Omega(z,\zeta)-G_\Omega(z,\zeta)\Delta_\zeta(u(\zeta)-H_u^\Omega(\zeta))]\,d\xi d\eta$$

$$=\int_{\partial\Omega-\partial B}[(u(\zeta)-H_u^\Omega(\zeta))*d_\zeta G_\Omega(z,\zeta)-G_\Omega(z,\zeta)*d_\zeta(u(\zeta)-H_u^\Omega(\zeta))].$$

On letting $\varepsilon\to 0$, we deduce

$$\int_\Omega G_\Omega(z,\zeta)\Delta_\zeta u(\zeta)\,d\xi d\eta=-2\pi(u(z)-H_u^\Omega(z)).$$

Since $\Delta_\zeta u(\zeta)=P(\zeta)u(\zeta)$, we obtain (6).

We next prove that (7) is valid for $u\in PD(R)$ for $R\notin\mathcal{O}_{PD}$. In this case, (2) assures that $R\notin\mathcal{O}_G$. By the Riesz decomposition, we may assume that $u\geqq 0$ on R. By $D_R(u)<\infty$, we have the convergence $H_u^R(z)=\lim_{\Omega\to R}H_u^\Omega(z)$ (cf. e.g. [11]). Since the integrand of the integral in (6) is nonnegative for every Ω, by the Lebesgue-Fatou theorem we deduce (7).

5. The Green energy. Again let $u\in P(\Omega)\cap C^1(\bar{\Omega})$ for a regular subregion Ω of R. Then

$$(8)\qquad D_\Omega(u)=D_\Omega(H_u^\Omega)+\frac{1}{2\pi}\int_{\Omega\times\Omega}G_\Omega(z,\zeta)u(z)u(\zeta)P(z)P(\zeta)\,dxdyd\xi d\eta.$$

If $u\in P(R)$, then

$$(9)\qquad D_R(u)=D_R(H_u^R)+\frac{1}{2\pi}\int_{R\times R}G_R(z,\zeta)u(z)u(\zeta)P(z)P(\zeta)\,dxdyd\xi d\eta\leqq\infty$$

provided the integral is definite. This is the case, for example, when $u\in PD(R)$. For brevity we will write

$$\langle u_1,u_2\rangle_R=\frac{1}{2\pi}\int_R G_R(z,\zeta)u_1(z)u_2(\zeta)P(z)P(\zeta)\,dxdyd\xi d\eta$$

whenever the integral is meaningful. This quantity is referred to as the (Green) mutual energy of u_1 and u_2 with respect to the density P. The (Green) energy $\|u\|_R$ of u is then given by $\|u\|_R^2=\langle u,u\rangle_R$. The fact that $\langle u,u\rangle_R\geqq 0$ is equivalent to the Dirichlet principle $D_R(u)\geqq D_R(H_u^R)$.

To prove (8), let

$$g_\Omega(z)=\frac{1}{2\pi}\int_\Omega G(z,\zeta)P(\zeta)u(\zeta)\,d\xi d\eta.$$

By the Stokes formula, $D_\Omega(H_u^\Omega,g_\Omega)=\int_\Omega dH_u^\Omega\wedge *dg_\Omega=0$. Therefore

$$D_\Omega(u)=D_\Omega(H_u^\Omega)+D_\Omega(g_\Omega).$$

What remains to be shown is $D_\Omega(g_\Omega)=\langle u, u\rangle_\Omega$. From (6) it follows that $\Delta g_\Omega =\Delta(H_u^\Omega-u)=-\Delta u=-Pu$. By Green's formula and the Fubini Theorem, we infer that

$$\begin{aligned}D_\Omega(g_\Omega)&=-\int_\Omega g_\Omega(z)\Delta_z g_\Omega(z)\,dxdy\\&=\int_\Omega\left[\frac{1}{2\pi}\int_\Omega G(z,\zeta)P(\zeta)u(\zeta)\,d\xi d\eta\right]P(z)u(z)\,dxdy\\&=\langle u, u\rangle_R.\end{aligned}$$

The identity (9) is a consequence of (8) for all admissible u. We prove, in particular, that (9) is valid for $u\in PD(R)$. First suppose $u\geqq 0$. Then since the integrand is nonnegative, the Lebesgue-Fatou theorem yields (9). If u changes sign on R, then let $u=u^+-u^-$ be the Riesz decomposition of u. Since $u^+ +u^-\in PD(R)$,

$$D_R(u^++u^-)=D_R(H^R_{u^++u^-})+\langle u^++u^-, u^++u^-\rangle_R<\infty.$$

Therefore $\langle |u|, |u|\rangle_R\leqq\langle u^++u^-, u^++u^-\rangle$. The Lebesgue convergence theorem permits the transition from (8) to (9) as $\Omega\to R$.

6. The main theorem. We shall study the relation between the class $PD(R)$ and its subclass $PBD(R)$. The class $PBD(R)$ is dense in $PD(R)$:

THEOREM 1. *For any u in $PD(R)$ there exists a sequence $\{v_n\}$ in $PBD(R)$ such that* $\sup_R|v_n|=\min(n, \sup_R|u|)$, *$\{v_n\}$ converges to u uniformly on each compact subset of R, and* $\lim_n D_R(u-v_n)=0$. *If, moreover, u is nonnegative, then $\{v_n\}$ can be chosen nondecreasing.*

Proof. Suppose there exists a nonconstant u in $PD(R)$. First we assume that $u>0$ on R. For an arbitrary fixed positive integer n, the function

$$u_n(z)=(u\cap n)(z)=\min(u(z), n)$$

is a supersolution of (1) on R. Clearly

$$D_R(u_n)<\infty. \tag{10}$$

By nos. 4 and 5, we have

$$u(z)=H_u^R(z)-\frac{1}{2\pi}\int_R G_R(z,\zeta)u(\zeta)P(\zeta)\,d\xi d\eta \tag{11}$$

for every $z\in R$ and also

$$D_R(u)=D_R(H_u^R)+\frac{1}{2\pi}\int_{R\times R}G_R(z,\zeta)u(z)u(\zeta)P(z)P(\zeta)\,dxdyd\xi d\eta<\infty. \tag{12}$$

Let Ω be a regular subregion of R and $v_{n\Omega}=P^{\Omega}_{u_n}$. Since u_n is a supersolution of (1), we see that

$$u_n \geqq v_{n\Omega} \geqq v_{n\Omega'} \geqq 0$$

on Ω for $\Omega \subset \Omega'$. Therefore

$$v_n(z)=\lim_{\Omega\to R} v_{n\Omega}(z)$$

exists, $v_n \in P(R)$, and $0 \leqq v_n \leqq u_n$ on R.

From (7) it follows that

$$v_{n\Omega}(z)=H^{\Omega}_{u_n}(z)-\frac{1}{2\pi}\int_{\Omega} G_{\Omega}(z, \zeta)v_{n\Omega}(\zeta)P(\zeta)\,d\xi d\eta \tag{13}$$

for every $z \in R$. Here $H^{\Omega}_{u_n}(z)=\lim_{\Omega'\to\Omega} H^{\Omega'}_{v_{n\Omega}}(z)$ is used. Since

$$0 \leqq G_{\Omega}(z, \zeta)v_{n\Omega}(\zeta) \leqq G_R(z, \zeta)u_n(\zeta) \leqq G_R(z, \zeta)u(\zeta)$$

and by (11) the function $G_R(z, \zeta)u(\zeta)$ is $P(\zeta)\,d\xi d\eta$-integrable, we can apply the Lebesgue convergence theorem to (13) to conclude that

$$v_n(z)=h_n(z)-\frac{1}{2\pi}\int_{R} G_R(z, \zeta)v_n(\zeta)P(\zeta)\,d\xi d\eta \tag{14}$$

for $z \in R$. Here

$$h_n=H^R_u \wedge n \in HBD(R). \tag{15}$$

This follows from the fact that, because of (10), (11) is nothing but the harmonic decomposition of u with the harmonic projection $\pi_R u=H^R_u$, and that $\pi_R(u \cap n) =(\pi_R u)\wedge n$ (see [11]). The symbol $\wedge$ stands for the lattice meet in the vector lattice $HD(R)$. We only have to observe that $\pi_R(u \cap n)=\pi_R u_n=\lim_{\Omega\to R} H^{\Omega}_{u_n}$.

By $G_R(z, \zeta)u(z)u(\zeta) \geqq G_R(z, \zeta)v_n(z)v_n(\zeta)$, and relations (12), (9), we see that

$$D_R(v_n)=D_R(H^R_u \wedge n)+\frac{1}{2\pi}\int_{R\times R} G_R(z, \zeta)v_n(z)v_n(\zeta)P(z)P(\zeta)\,dxdyd\xi d\eta<\infty. \tag{16}$$

Thus we have shown that $v_n \in PBD(R)$.

Since $0 \leqq v_{n\Omega} \leqq v_{n+1\,\Omega} \leqq u$, we see that $0 \leqq v_n \leqq v_{n+1} \leqq u$ on R and consequently $v(z)=\lim_{n\to\infty} v_n(z)$ exists on R, $v \in P(R)$, and $0 \leqq v \leqq u$ on R. In view of

$$\lim_{n\to\infty} [(H^R_u(z)-(H^R_u \wedge n)(z))+D_R(H^R_u-H^R_u \wedge n)]=0 \tag{17}$$

(see [11]), (11), and $0 \leqq v \leqq u$, we can apply the Lebesgue convergence theorem to (14) to deduce

$$v(z)=H^R_u(z)-\frac{1}{2\pi}\int_{R} G_R(z, \zeta)v(\zeta)P(\zeta)\,d\xi d\eta \tag{18}$$

for $z \in R$. The subtraction of (18) from (11) gives

(19) $$u(z)-v(z)=-\frac{1}{2\pi}\int_R G_R(z,\zeta)(u(\zeta)-v(\zeta))P(\zeta)\,d\xi d\eta$$

for $z\in R$. The left-hand side of (19) is nonnegative while the right-hand side is nonpositive. Therefore we obtain $u=v$ on R and a fortiori

(20) $$u(z)=\lim_{n\to\infty} v_n(z)$$

on R increasingly. The convergence is uniform on each compact subset on R. By no. 5, (11), and (14), we have

$$D_R(u-v_n)=D_n(H_u^R-H_u^R\wedge n)+\langle u-v_n,\, u-v_n\rangle_R.$$

Since $0<u-v_n<u$ on R, the Lebesgue convergence theorem yields

(21) $$\lim_{n\to\infty} D_R(u-v_n)=0.$$

Next suppose $u\in PD(R)$ changes sign on R. By the Riesz decomposition, there exists $u_j\in PD(R)$ ($j=1,2$) such that $u_j>0$ and $u=u_1-u_2$ on R. Let $\{v_{jn}\}_{n=1}^\infty$ be the sequence in $PBD(R)$ obtained as above for u_j. Then $v_n=v_{1n}-v_{2n}\in PBD(R)$ satisfies (20). Since $D_R(u-v_n)^{1/2}\leqq D_R(u_1-v_{1n})^{1/2}+D_R(u_2-v_{2n})^{1/2}$, (21) is also satisfied.

The proof is herewith complete.

The theorem stated in the introduction is an immediate consequence of Theorem 1. It is also clear that

$$\mathcal{O}_{PD}=\mathcal{O}_{PBD}.$$

7. Isomorphisms. We shall next study the relation between $PBD(R)$ and $HBD(R)$. A vector space isomorphism T of $PBD(R)$ onto $HBD(R)$ will be referred to as the *canonical isomorphism* if $Tu-u$ is a potential, i.e. a superharmonic function on R whose greatest harmonic minorant is zero, for every u in $PBD(R)$ with $u\geqq 0$ on R. This means that "$u=Tu$" on the ideal boundary of R in the intuitive sense. By virtue of the Riesz representation theorem of superharmonic functions and the Riesz decomposition of $PBD(R)$, the canonical isomorphism T must have the form

(22) $$Tu=u+\frac{1}{2\pi}\int_R G_R(\cdot,\zeta)u(\zeta)P(\zeta)\,d\xi d\eta$$

for every $u\in PBD(R)$ (see (7)). We shall prove:

THEOREM 2. *If the pair (R, P) satisfies*

(23) $$\int_{R\times R} G_R(z,\zeta)P(z)P(\zeta)\,dxdyd\xi d\eta<\infty,$$

then there exists a canonical isomorphism of $PBD(R)$ onto $HBD(R)$.

Proof. Under the assumption (23), the function Tu in (22) can be always defined for every u in $PBD(R)$. By (7) and (9) we infer that $Tu \in HBD(R)$. Clearly T defines a vector space homomorphism of $PBD(R)$ into $HBD(R)$. We shall see that T is injective. Suppose $Tu=0$ for some u in $PBD(R)$. Observe that, since $|u|$ is a nonnegative subsolution of (1), $|u|$ is subharmonic. From $Tu=0$ it follows that

$$0 \leqq |u| \leqq \frac{1}{2\pi}\int_R G_R(\cdot, \zeta)|u(\zeta)|P(\zeta)\,d\xi d\eta.$$

Since the subharmonic function $|u|$ is dominated by the potential

$$\frac{1}{2\pi}\int_R G_R(\cdot, \zeta)|u(\zeta)|P(\zeta)\,d\xi d\eta$$

on R, we deduce that $|u| \leqq 0$, i.e. $u=0$. Therefore $Tu=0$ implies that $u=0$.

What remains to be shown is the surjectiveness of T. Let h be in $HBD(R)$. We wish to find a u in $PBD(R)$ such that $Tu=h$. Since $HBD(R)$ also admits the Riesz decomposition, it suffices to consider the case $h>0$ on R. For any regular subregion Ω of R, (6) implies that

$$P_h^{\Omega}(z) = h(z) - \frac{1}{2\pi}\int_{\Omega} G_{\Omega}(z, \zeta) P_h^{\Omega}(\zeta) P(\zeta)\,d\xi d\eta. \tag{24}$$

Since $0 \leqq P_h^{\Omega'} \leqq P_h^{\Omega} \leqq h$ on Ω for $\Omega' \supset \Omega$, $u(z) = \lim_{\Omega \to R} P_h^{\Omega}(z)$ exists and belongs to $PB(R)$. By (23) we can apply the Lebesgue convergence theorem to (24) to conclude

$$u(z) = h(z) - \frac{1}{2\pi}\int_R G_R(z, \zeta) u(\zeta) P(\zeta)\,d\xi d\eta. \tag{25}$$

Let $c = \sup_R h$. From (9) it follows that

$$D_R(u) = D_R(h) + \langle u, u\rangle_R \leqq D_R(h) + c^2\langle 1, 1\rangle_R.$$

Since (23) is nothing but $\langle 1, 1\rangle_R < \infty$, we conclude that $u \in PBD(R)$. By (25), we obtain $Tu=h$.

The proof of Theorem 2 is herewith complete.

We already know (cf. [6], also Maeda [3]) that the weaker condition

$$\int_R G_R(z, \zeta) P(\zeta)\,d\xi d\eta < \infty \tag{26}$$

for one and hence for all $z \in R$ assures the existence of the canonical isomorphism of $PB(R)$ onto $HB(R)$.

It is also known (Royden [10], Glasner-Katz [1]) that the condition

$$\int_R P(\zeta)\,d\xi d\eta < \infty \tag{27}$$

asssures the existence of the canonical isomorphism of $PBE(R)$ onto $HBD(R)$. In this case, $PBD(R)=PBE(R)$.

8. Relative classes. Let S be a subregion of R whose relative boundary ∂S consists of regular points for the harmonic Dirichlet problem on S. We denote by $PX(S, \partial S)$ the subclass of $PX(S)$ consisting of functions u vanishing continuously on ∂S. As a counterpart of Theorem 2 we obtain the follwing

THEOREM 3. *If the pair* (S, P) *satisfies*

$$\int_{S\times S} G_S(z, \zeta)P(z)P(\zeta)\,dxdyd\xi d\eta<\infty, \tag{28}$$

then the canonical isomorphism

$$T_S u=u+\frac{1}{2\pi}\int_S G_S(\cdot, \zeta)u(\zeta)P(\zeta)\,d\xi d\eta \tag{29}$$

of $PBD(S)$ *onto* $HBD(S)$ *maps* $PBD(S, \partial S)$ *onto* $HBD(S, \partial S)$.

Proof. Let Ω be a regular subregion of R such that $\Omega\cap S$ is connected. By (7) we obtain

$$H_u^{S\cap\Omega}=u+\frac{1}{2\pi}\int_{S\cap\Omega} G_{S\cap\Omega}(\cdot, \zeta)u(\zeta)P(\zeta)\,d\xi d\eta \tag{30}$$

on $\Omega\cap S$ for $u\in PBD(S)\cap C(\bar{S})$. In view of (28)

$$\lim_{\Omega\to R}\int_{S\cap\Omega} G_{S\cap\Omega}(\cdot, \zeta)u(\zeta)P(\zeta)\,d\xi d\eta=\int_S G_S(\cdot, \zeta)u(\zeta)P(\zeta)\,d\xi d\eta.$$

Thus (29) and (30) imply

$$T_S u=\lim_{\Omega\to R} H_u^{S\cap\Omega}$$

on S. Since ∂S consists of regular points for S, $H_u^{S\cap\Omega}$ has continuous boundary values u on $(\partial S)\cap\Omega$ and a fortiori $T_S u$ has continuous boundary values u on ∂S. Therefore we deduce that

$$T_S(PBD(S, \partial S))\subset HBD(S, \partial S).$$

Conversely let $h\in HBD(S, \partial S)$. There is a unique $u\in PBD(S)$ with $Tu=h$. We assert that $u\in PBD(S, \partial S)$. Since $HBD(S, \partial S)$ admits the Riesz decomposition ([11]), we may assume that $h>0$ on S. Again by (7)

$$h=P_h^{S\cap\Omega}+\frac{1}{2\pi}\int_{S\cap\Omega} G_{S\cap\Omega}(\cdot, \zeta)P_h^{S\cap\Omega}(\zeta)P(\zeta)\,d\xi d\eta.$$

Since $\{P_h^{S\cap\Omega}\}_\Omega$ is decreasing, it converges to a solution $v\in PBD(R)$. We infer that

$$h=v+\frac{1}{2\pi}\int_S G_S(\cdot, \zeta)v(\zeta)P(\zeta)\,d\xi d\eta,$$

because of (28), (31), the Lebesgue convergence theorem, and (9). Therefore $Tu = Tv$ implies that $u=v\geqq 0$. A fortiori $0\leqq u\leqq Tu$ on S which in turn implies that $u\in PBD(S, \partial S)$.

The proof is herewith complete.

As a counterpart of (26) we conclude that the condition

$$\int_S G_S(z, \zeta)P(\zeta)\,d\xi d\eta<\infty \tag{31}$$

for one and hence for all $z\in S$ is sufficient for the existence of the canonical isomorphism of $PB(S, \partial S)$ onto $HB(S, \partial S)$ (cf. [6], also Maeda [3]).

As a counterpart of (27) it is known (Royden [10], Glasner-Katz [1]) that the condition

$$\int_S P(\zeta)\,d\xi d\eta<\infty \tag{32}$$

assures the existence of the canonical isomorphism of $PBE(S, \partial S)$ onto $HBD(S, \partial S)$. Obviously $PBD(S, \partial S)=PBE(S, \partial S)$ under the condition (32).

9. Canonical extension. Let Ω be a regular subregion of R. We extend $u\in PB(S, \partial S)$ to R by $u=0$ on $R-S$, and maintain:

$$\lambda_P u=\lim_{\Omega\leftarrow R} P_u^{\Omega} \tag{33}$$

exists and $\lambda_P u\in PB(R)$. This is clear for $u>0$, and the Riesz decomposition of $PB(S, \partial S)$ implies it for every $u\in PB(S, \partial S)$. Clearly λ_P is a linear mapping of $PB(S, \partial S)$ into $PB(R)$ with

$$\lambda_P u\geqq u \tag{34}$$

for $u\geqq 0$. We call λ_P the *canonical extension.* For $P\equiv 0$ we denote it by λ_H instead of λ_0.

THEOREM 4. *The canonical extension of a Dirichlet finite function is again Dirichlet finite*:

$$\lambda_P(PBD(S, \partial S))\subset PBD(R). \tag{35}$$

Proof. Let Ω be a regular subregion. Set $v_1=\max(u, 0)$ and $v_2=\max(-u, 0)$ pointwise on R for $u\in PBD(S, \partial S)$, where u is extended to R by $u=0$ on $R-S$. Observe that v_1 and v_2 are nonnegative subsolutions of (1) on R. The sequence $\{P_{v_i}^{\Omega}\}_{\Omega}$ is increasing and bounded. Therefore

$$u_i=\lim_{\Omega\to R} P_{v_i}^{\Omega}$$

exists on R and belongs to $PB(R)$. By the weak Dirichlet principle,

$$D_{\Omega}(P_{v_i}^{\Omega})\leqq D_{\Omega}(v_i)\leqq D_{\Omega}(u)<D_R(u)=D_S(u).$$

By the Fatou theorem, we conclude that $D_R(u_i)\leqq D_S(u)$, *i.e.* $u_i\in PBD(R)$ $(i=1, 2)$. Since

$$P_u^{\Omega}=P_{v_1}^{\Omega}-P_{v_2}^{\Omega},$$

we deduce that $\lambda_P u=u_1-u_2\in PBD(R)$.

This completes the proof.

By using the energy principle instead of the weak Dirichlet principle, the same proof is valid for

$$\lambda_P(PBE(S, \partial S))\subset PBE(R). \tag{36}$$

In passing we remark that

$$T\lambda_P u=\lambda_H Tu \tag{37}$$

for every $u\in PX(S, \partial S)$ $(X=B, BD, BE)$. We shall, however, not make use of this relation.

10. One-domain criterions. We denote by SO_{HX} the class of bordered Riemann surfaces $(S, \partial S)$ for which $HX(S, \partial S)=\{0\}$ (cf. [11]). We shall prove the following one-domain criterion for $O_{PD}=O_{PBD}$:

THEOREM 5. *A pair (R, P) does not belong to O_{PD} if and only if there exists a subregion S of R with regular relative boundary ∂S such that $(S, \partial S)\notin SO_{HD}$ and (S, P)* satisfies (28).

Proof. Suppose $(R, P)\notin O_{PD}$. There exists a nonconstant function v in $PD(R)$. We may assume that there exists a constant $\varepsilon>0$ such that $S_{2\varepsilon}=\{z\in R|v(z)>2\varepsilon\}\neq\emptyset$. Let S be a subregion of R such that $\bar{S}\subset S_\varepsilon$, S contains a component of $S_{2\varepsilon}$, and ∂S consists of a countable number of disjoint C^1 arcs. Take a regular subregion Ω of R such that $S\cap R$ is connected. Let $v_0=\min(v, 2\varepsilon)$. Clearly

$$v\leqq H_v^{S\cap\Omega}\leqq H_v^{S\cap\Omega'}\ (\Omega\subset\Omega'),\qquad D_{S\cap\Omega}(H_v^{S\cap\Omega})\leqq D_{S\cap\Omega}(v)\leqq D_R(v).$$

Consequently $h=\lim_{\Omega\to R}H_v^{S\cap\Omega}$ exists on S and belongs to $HD(S)$. Similarly

$$0\leqq H_{v_0}^{S\cap\Omega}\leqq 2\varepsilon,\qquad D_{S\cap\Omega}(H_{v_0}^{S\cap\Omega})\leqq D_{S\cap\Omega}(v_0)\leqq D_{S\cap\Omega}(v)\leqq D_R(v).$$

Therefore $h_0=\lim_{\Omega\to R}H_{v_0}^{S\cap\Omega}$ exists on S and belongs to $HD(S)$. Since h and h_0 have continuous boundary values v on ∂S, $u=h-h_0\in HD(S, \partial S)$. Observe that

$$u(z)=h(z)-h_0(z)\geqq v(z)-2\varepsilon>2\varepsilon-2\varepsilon=0$$

for $z\in S\cap S_{2\varepsilon}$. Therefore $(S, \partial S)\notin SO_{HD}$.

By (6) and (7) we obtain

$$v=h-\frac{1}{2\pi}\int_S G_S(\cdot, \zeta)v(\zeta)P(\zeta)\,d\xi d\eta$$

and then deduce by (9) that $D_S(v) \geqq \langle v, v\rangle_S$. Since $v > \varepsilon$ on S,

$$\langle 1, 1\rangle_S \leqq \varepsilon^{-2}\langle v, v\rangle_S \leqq \varepsilon^{-2} D_S(v) < \infty .$$

This shows that (S, P) satisfies (28).

Conversely suppose that there exists a subregion S of R with regular relative boundary ∂S such that $(S, \partial S) \notin SO_{HD}$ and (S, P) satisfies (28). By Theorem 3, $PBD(S, \partial S)$ is isomorphic to $HBD(S, \partial S)$. Since $HBD(S, \partial S)$ is dense in $HD(S, \partial S)$ with respect to $D_S(\cdot)$ (see e.g. [11]), $(S, \partial S) \notin SO_{HD}$ implies that $HBD(S, \partial S) \neq \{0\}$. A fortiori $PBD(S, \partial S) \neq \{0\}$.

Let $u \in PBD(S, \partial S)$. Set $v_1 = \max(u, 0)$ and $v_2 = \max(-u, 0)$ pointwise on S. Take a regular subregion Ω of R such that $S \cap \Omega$ is connected. Since v_i is a non-negative subsolution on S, we see that $\{P_{v_i}^{S\cap\Omega}\}_\Omega$ is bounded and increasing. Thus $u_i = \lim_{\Omega \to R} P_{v_i}^{S\cap\Omega}$ exists on S and belongs to $PB(S, \partial S)$. By the weak Dirichlet principle, $u_i \in PBD(S, \partial S)$ and $u = u_1 - u_2$. In view of this we can assume that $u > 0$ on S.

By (34) and (35), $\lambda_P u > 0$ and belongs to $PBD(R) \subset PD(R)$. We conclude that $(R, P) \notin O_{PD}$.

The proof of Theorem 5 is herewith complete.

By using (27), (32), (36), and the energy principle, we can prove by the same argument as above that $(R, P) \notin O_{PE}$ *if and only if there exists a subregion* S *of* R *with regular relative boundary* ∂S *such that* $(S, \partial S) \notin SO_{HD}$ *and* (S, P) *satisfies* (32). This is a theorem of Glasner and Katz [1], the one-domain criterion for $O_{PE} = O_{PBE}$.

The one-domain criterion for O_{PB} reads as follows: $(R, P) \in O_{PB}$ *if and only if there exists a subregion* S *of* R *with regular relative boundory* ∂S *such that* $(S, \partial S) \notin SO_{HB}$ *and* (S, P) *satisfies* (31). The proof is clear, in view of (26) and (31).

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