Dirichlet forms: Some infinite dimensional examples

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Introduction

The theory of Dirichlet forms deserves to be better known. It is an area of Markov process theory that uses the energy of functionals to study a Markov process from a quantitative point of view. For instance, the recent notes of Saloff-Coste [S-C] use Dirichlet forms to analyze Markov chains with finite state space, by making energy comparisons. In this way, information about a simple chain is parlayed into information about another, more complicated chain. The upcoming book [AF] by Aldous and Fill will use Dirichlet forms for similar purposes.

Dirichlet form theory does not use the tools of partial differential equations, as in standard diffusion theory, and therefore is not as closely tied to analysis on Euclidean space. For example, Dirichlet forms can be used to study Markov processes taking values in spaces of fractional dimension, i.e. fractals (see [F3], [Ku2], [KuY]). This paper applies Dirichlet form techniques to study Markov processes taking values in infinite dimensional spaces. Such processes are used to describe a complex natural phenomenon, such as the diffusion of gas molecules or the genetic evolution of a population. Each such system is made up of an effectively infinite number of individuals whose evolution in time is governed by a combination of random chance and interactions with the other individuals in the system. The complexity of such a system makes this a forbidding mathematical problem.

This paper is not an introduction to Dirichlet form theory. We are not interested here in all the details and generalities of the theory; there are several good sources for that ([MR1] [BH] [FOT]). In fact, we do not even define Dirichlet forms, we simply motivate them. This paper is about calculations, and how you use energy estimates to give concrete results on the sample path properties of Markov processes. The four processes that we consider are:

1. Brownian motion on $\mathbb{R}^d$.
2. Ornstein-Uhlenbeck process on Hilbert space.
3. Fleming-Viot process on a space of probability measures.
4. A particle process on configuration space.

We hope that you find these examples interesting, and consider this paper as an invitation to further exploration of Dirichlet forms.

Some historical perspective

Dirichlet forms have their origin in the energy method used by Dirichlet to address the problem in classical electrostatics that now bears his name: given a continuous function f on the boundary ∂D of an open set D, find a continuous extension to the closure $\overline{D}$ that is harmonic on D. Recall that in electrostatics, two point charges in $\mathbb{R}^3$ feel a force whose magnitude is inversely proportional to the squared distance between them. Thus, placing a point charge at $x \in \mathbb{R}^3$ and integrating the force as you bring a charge of the same sign from infinity to $z \in \mathbb{R}^3$, you obtain $(2\pi|x - z|)^{-1}$. Here the constant factor 2π is simply chosen for convenience. We say that the charge at x induces a potential whose value at z is

$$G(x,z) = \frac{1}{2\pi|x-z|},$$

and by analogy we define the potential induced by a distribution μ of charge by

$$G\mu(z) = \int_{\mathbb{R}^3} G(x,z)\,\mu(dx).$$

The mutual energy between two distributions μ and ν is given by

$$(\mu,\nu) = \int_{\mathbb{R}^3} G\mu(z)\,\nu(dz),$$

and represents the potential energy stored in the system consisting of μ and ν that can be recovered by letting either μ or ν go to infinity.

Before pursuing Dirichlet's solution, let's rewrite the energy integral in a more convenient manner. A little calculus will convince you that, as a Schwartz distribution, the function $z \mapsto G(x,z)$ satisfies $\Delta G(x,\cdot) = -2\varepsilon_x$, where ε_x is the point mass at x. Therefore, provided the potentials $G\mu$ and $G\nu$ are sufficiently smooth, we can use integration by parts to rewrite the energy as

$$(\mu,\nu) = -\frac{1}{2}\int_{\mathbb{R}^3} G\mu(z)\Delta G\nu(z)\,dz = \frac{1}{2}\int_{\mathbb{R}^3} \langle \nabla G\mu(z), \nabla G\nu(z)\rangle\,dz. \tag{0.1}$$

In Dirichlet form theory, we choose to work directly with the potential functions, and avoid the underlying distributions. Accordingly, we adopt the expression in the right hand side of (0.1) as a starting point and denote it as $\mathcal{E}(G\mu, G\nu)$. But once this step is taken, it is natural to eliminate the function G also, and at the same time removing any qualms about smoothness of potentials, by defining the energy, only for smooth functions, by

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}^3} \langle \nabla u(z), \nabla v(z)\rangle\,dz, \quad u,v \in C_0^\infty(\mathbb{R}^3). \tag{0.2}$$

In his deliberations Dirichlet used a local version $\mathcal{E}_D$ of this integral, that is, where the range of integration is the set D. He then argued that the function u that minimizes the energy $\mathcal{E}_D(u,u)$, subject to the boundary condition $u|_{\partial D} = f$, will be harmonic on D, and hence solve the extension problem. Although his approach was not entirely rigorous, the spirit was essentially correct and Dirichlet's principle led to important developments in potential theory as a mathematical discipline [M].

A radically different solution to Dirichlet's problem, first published by Kakutani in 1944 [K], is based on a deep and unexpected connection to probability. Working in $\mathbb{R}^2$, though the result is true in all dimensions, Kakutani showed that if $(X_t)_{t\geq 0}$ is Brownian motion and $\tau(\omega) = \inf\{t > 0 : X_t(\omega) \in D^c\}$ is the time it leaves D, then $z \mapsto E^z(f(X_\tau))$ is harmonic on D. If the boundary ∂D is suitably regular, then this function also takes the right boundary values, and hence is the (unique) solution to Dirichlet's problem.

In the years following Kakutani's discovery, further research showed that the connection between probability and potential theory was no accident. Other notions in potential theory (balayage, capacity, equilibrium measure) also proved to have interpretations using Brownian motion paths (Chung's entertaining [C], or the popular article [GH] provide a gentle introduction to the topic. More mathematical treatments can be found in [B] and [PS]).

Let's see if we can find a direct relationship between Brownian motion $(X_t)_{t\geq 0}$ on $\mathbb{R}^3$ and the energy integral in (0.1). Since Brownian motion solves the martingale problem for the operator $\Delta/2$, for every $u \in C_0^\infty(\mathbb{R}^3)$ we have that $u(X_t) - u(X_0) - \int_0^t ((1/2)\Delta u)(X_s)\, ds$ is a mean-zero martingale. Now let's look at the average squared increment of $u(X_t)$ at $t = 0$. First of all, applying the martingale property for the function u and u^2, and taking expectations with respect to P_z (which starts the process at the point z), we get $E_z(u(X_t)) = u(z) + E_z(\int_0^t (\Delta u/2)(X_s)\, ds)$ and $E_z(u^2(X_t)) = u^2(z) + E_z(\int_0^t (\Delta u^2/2)(X_s)\, ds)$. Therefore we have

$$\begin{aligned}
&E_z([u(X_t) - u(X_0)]^2) \\
&\quad = E_z(u^2(X_t)) - 2E_z(u^2(X_t))u(z) + u^2(z) \\
&\quad = u^2(z) + E_z(\int_0^t (\Delta u^2/2)(X_s)\, ds) - 2u(z)\left(u(z) + E_z(\int_0^t (\Delta u/2)(X_s)\, ds)\right) + u^2(z) \\
&\quad = E_z\left(\int_0^t [(\Delta u^2/2)(X_s) - 2u(z)(\Delta u/2)(X_s)]\, ds\right).
\end{aligned}$$

Dividing by t and letting $t \to 0$ gives

$$\lim_{t\downarrow 0} E_z\left(\frac{[u(X_t) - u(X_0)]^2}{t}\right) = (\Delta u^2/2)(z) - 2u(z)(\Delta u/2)(z).$$

Using polarization and integrating with respect to Lebesgue measure gives

$$\begin{aligned}
\mathcal{E}(u,v) &= \frac{1}{2}\int_{\mathbb{R}^3} \lim_{t\downarrow 0} E_z\left(\frac{[u(X_t) - u(X_0)][v(X_t) - v(X_0)]}{t}\right) dz \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \big((\Delta(uv)/2)(z) - u(z)(\Delta v)(z) - v(z)(\Delta u)(z)\big)\, dz \qquad (0.3) \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \langle \nabla u(z), \nabla v(z)\rangle\, dz.
\end{aligned}$$

So we've found two ways to obtain the integral in (0.2), one using analysis and the other using probability. The theory of Dirichlet forms is a grand elaboration on this theme; that certain bilinear forms can serve as a bridge between analysis and probability.

In pioneering work in 1959, Beurling and Deny [BD1, BD2] initiated the study of Dirichlet forms by identifying a crucial contraction property, now called the Markov property, possessed by the form (0.2). They explored many important consequences of the Markov property, but the explicit connection to probability had to wait until the fundamental work of Silverstein and Fukushima during the 1970s. In particular, Fukushima showed that if a Dirichlet form on a locally compact state space is *regular* one can construct an associated

Markov process with right continuous sample paths. By the 1980s, the demand for tools to study Markov processes on infinite dimensional (not locally compact) spaces led to various extensions of Fukushima's result, e.g. [AH1, AH2, AM, AR1, AR2, Ku1, RS]. Finally, in 1992, the general question was settled by the impressive characterization of Ma and Röckner [MR1] to the effect that a Dirichlet form on a separable metric space is associated with a Markov process with decent sample paths if and only if the form is *quasi-regular*. This result provides a method to construct Markov processes on metric spaces, and guarantees the existence of the processes considered in section ?.

Dirichlet forms

We begin with a complete separable metric space E and a σ-finite measure m on the Borel sets of E. In our examples we will always use a special type of Dirichlet form that has an associated square field 田*. We begin with a dense linear subspace $\mathcal{F}C_b^\infty$ of $L^2(E;m)$ that is closed under pointwise multiplication and assume that 田 : $\mathcal{F}C_b^\infty \times \mathcal{F}C_b^\infty \to L^1(E;m)$ is symmetric, is positive definite 田$(u,u) \geq 0$, and satisfies the product rule 田$(uv,w) = u$田$(v,w) + v$田(u,w). We define the pre-Dirichlet form for $u, v \in \mathcal{F}C_b^\infty$ by

$$\mathcal{E}(u,v) = \frac{1}{2}\int_E \text{田}(u,v)(z)\, m(dz). \tag{0.4}$$

Standard Dirichlet form theory shows that if $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is closed, then the closure $(\mathcal{E}, D(\mathcal{E}))$ is a local Dirichlet form. Then $(D(\mathcal{E}), \|\ \|_1)$ is a Hilbert space with the norm $\|u\|_1 = (\mathcal{E}(u,u) + (u,u)_{L^2(E;m)})^{1/2}$. The map $(u,v) \to$ 田(u,v) is continuous from $\mathcal{F}C_b^\infty \times \mathcal{F}C_b^\infty$ into $L^1(E;m)$ when $\mathcal{F}C_b^\infty$ is equipped with $\|\ \|_1$, and so the square field 田 extends to the full domain $D(\mathcal{E})$ in such a way that formula (0.4) continues to hold.

Adopting the shorthand 田$(u) =$ 田(u,u), the usual functional calculus for Dirichlet forms ensures (eg. [RS; Lemma 3.2]) that if $u, v \in D(\mathcal{E})$ and ψ is a smooth function on $\mathbb{R}$ that vanishes at the origin and has bounded derivative, then $\psi(u)$ belongs to $D(\mathcal{E})$ and

$$\text{田}(\psi(u)) = (\psi'(u))^2\, \text{田}(u).$$

In the same vein, you can show that $u \vee v$ belongs to $D(\mathcal{E})$ and 田$(u \vee v) \leq$ 田$(u) \vee$ 田(v). These bounds will be used repeatedly in our calculations in the next section.

We can think of a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ as a recipe for a Markov process $(X_t)_{t\geq 0}$, in the sense that $(\mathcal{E}, D(\mathcal{E}))$ describes the behaviour of the composed process $(u(X_t))_{t\geq 0}$ for every $u \in D(\mathcal{E})$. This association is given by the equation

$$\mathcal{E}(u,u) = \lim_{t\downarrow 0} \frac{1}{2t}\int_E E_z\left([u(X_t) - u(X_0)]^2\right)\, m(dz). \tag{0.5}$$

However, the existence of the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ does not guarantee the existence of an associated Markov process $(X_t)_{t\geq 0}$. There may be no way that the "coordinates"

* This notation is based on the Chinese character Tián, which means 'field'.

$(u(X_t))_{u\in D(\mathcal{E})}$ can be put together in a consistent way to form an E-valued process $(X_t)_{t\geq 0}$ with reasonable sample paths.

Ma and Röckner have proved the fundamental existence theorem for Markov processes coming from Dirichlet forms. Their result [MR1; Chapter IV, Theorem 6.7 and Chapter V, Theorem 1.5] shows that every quasi-regular, local Dirichlet form is associated with a Markov process $(X_t)_{t\geq 0}$ with continuous sample paths.

The hitting of sets

Suppose now that we have a quasi-regular, local, symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ and an associated Markov process $(X_t)_{t\geq 0}$. What can the form $(\mathcal{E}, D(\mathcal{E}))$ tell us about the path properties of the process?

Exceptional sets and quasi-continuous functions are important tools for understanding the diffusion process corresponding to a Dirichlet form. Exceptional sets are "almost empty", and quasi-continuous functions are "almost continuous" in a sense appropriate for Dirichlet forms.

We know that $(X_t)_{t\geq 0}$ has continuous sample paths so that if u is a continuous function, the composed process $(u(X_t))_{t\geq 0}$ obviously has continuous sample paths as well. A remarkably useful extension is to define a function u to be $\mathcal{E}$-quasi-continuous if, for m-almost every $z \in E$,

$$P_z\,(t \to u(X_t) \text{ is continuous }) = 1. \tag{0.6}$$

This can only give us new information if u is $\mathcal{E}$-quasi-continuous and not continuous. An extreme case is when $u = 1_N$ is the indicator function of some Borel set. If $m(N) = 0$ and 1_N is $\mathcal{E}$-quasi-continuous we say that N is $\mathcal{E}$-exceptional. Combined with the previous equation this shows that a set $N \in \mathcal{B}(E)$ is $\mathcal{E}$-exceptional if and only if, for m-almost every $z \in E$,

$$P_z\,(X_t \in N \text{ for some } 0 \leq t < \infty) = 0.$$

This method to find $\mathcal{E}$-exceptional sets relies on our ability to identify $\mathcal{E}$-quasi-continuous functions. To that end, we use the following lemma (see [MR1; Chapter III, Proposition 3.5] and [S3]).

Lemma 0.1. *Let $u_n \in D(\mathcal{E})$ be $\mathcal{E}$-quasi-continuous functions such that $\sup_n \mathcal{E}(u_n, u_n) < \infty$ and $u_n \to u$ pointwise everywhere, where $u \in L^2(E; m)$. Then u is $\mathcal{E}$-quasi-continuous member of $D(\mathcal{E})$ and 田$(u) \leq \limsup_n$ 田(u_n). If, in addition, $u_n \to u$ in $L^2(E; m)$ then 田$(u_n) \to$ 田(u) in $L^1(E; m)$.*

We now turn to the opposite problem: How can we prove that the process $(X_t)_{t\geq 0}$ must hit a particular set N? The answer lies in the following result.

Lemma 0.2. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular Dirichlet form on $L^2(E; m)$, $\mathcal{F}$ a dense linear subspace of $D(\mathcal{E})$ consisting of $\mathcal{E}$-quasi-continuous functions, and ν a σ-finite measure on $\mathcal{B}(E)$. If $u \mapsto \int u\, d\nu$ is continuous on $(\mathcal{F}, \|\ \ \|_1)$, then $\nu(N) = 0$ for any $\mathcal{E}$-exceptional N.*

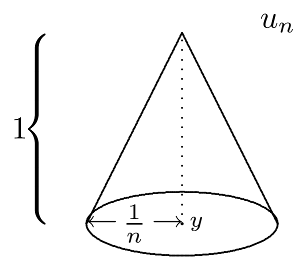
A measure ν that satisfies the conditions of Lemma 0.2 is called a measure of finite energy. The lemma tells us that if we can find a measure ν of finite energy so that $\nu(N) > 0$, then N cannot be $\mathcal{E}$-exceptional and hence the process $(X_t)_{t\geq 0}$ must hit the set N.

Examples

In this section we will study four different processes $(X_t)_{t\geq 0}$ via their associated Dirichlet forms $(\mathcal{E}, D(\mathcal{E}))$. The state space E is a complete separable metric space, and m, the invariant measure for the process, is a σ-finite Borel measure on E. In each case we begin with a core $\mathcal{F}C_b^\infty$ of functions, a square field 田, and define a pre-Dirichlet form $(\mathcal{E}, \mathcal{F}C_b^\infty)$ as in (0.4). In each of our examples $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is closable, and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a local, quasi-regular Dirichlet form. Proofs may be found in the references provided. We will then use Lemmas 0.1 and 0.2 to study the sample paths of the corresponding process $(X_t)_{t\geq 0}$ by determining whether certain sets are hit or not.

In all of our computations, we use c to denote a constant whose value may change from line to line, and which does not depend on n.

1. Brownian motion. Will Brownian motion ever hit the point y? Although the answer to this question is well known, this example is a useful illustration of the kind of calculations needed to apply Dirichlet form techniques to the question of the hitting of sets. Here $E = \mathbb{R}^d$, m is Lebesgue measure, $\mathcal{F}C_b^\infty = C_0^\infty(\mathbb{R}^d)$, and 田$(u, v) = \langle \nabla u, \nabla v\rangle$. The closure $(\mathcal{E}, D(\mathcal{E}))$ is the Dirichlet form associated with Brownian motion on $\mathbb{R}^d$. Take the set N to be the singleton $\{y\}$ and for $n \geq 1$, define the function $u_n(z) = (1 - n\|z - y\|)_+$.



These functions are continuous, belong to $D(\mathcal{E})$, and converge pointwise to 1_N as $n \to \infty$. Let us calculate the Dirichlet norm of u_n. Taking the gradient of u_n gives

$$\nabla u_n(z) = n\,\frac{z-y}{\|z-y\|}\,1_{\{0<\|z-y\|<1/n\}},$$

so that

$$田(u_n)(z) = n^2\,1_{\{0<\|z-y\|<1/n\}}.$$

Integrating over E gives

$$\mathcal{E}(u_n, u_n) = \frac{n^2}{2}\,m(0 < \|z-y\| < 1/n) = c\,n^{2-d}.$$

Since $\mathcal{E}(u_n, u_n)$ is bounded for $d \geq 2$, we apply Lemma 0.1 to find that 1_N is $\mathcal{E}$-quasi-continuous. From (0.6) we conclude that the singleton $N = \{y\}$ is not hit by Brownian motion in two or more dimensions.

What happens when $d = 1$? Define the function $v(z) = \exp(z - y)$, and for any

$u \in C_0^\infty(\mathbb{R})$ apply the product rule to uv to obtain

$$
\begin{aligned}
|u(y)| = |u(y)v(y)| &= \left| \int_{-\infty}^{y} (uv)'(z)\, dz \right| \\
&= \left| \int_{-\infty}^{y} v(z)\,(u(z) + u'(z))\, dz \right| \\
&\leq \left(\int_{-\infty}^{y} v(z)^2\, dz \right)^{1/2} \left(\int_{-\infty}^{y} (u(z) + u'(z))^2\, dz \right)^{1/2} \\
&\leq c \left(\int_{-\infty}^{y} u(z)^2 + u'(z)^2\, dz \right)^{1/2} \\
&\leq c\, \mathcal{E}_1(u,u)^{1/2}.
\end{aligned} \tag{1.1}
$$

This inequality shows that the measure $\nu = \varepsilon_y$ has finite energy, and hence by Lemma 0.2 we know that the singleton $N = \{y\}$ is hit by one dimensional Brownian motion with positive probability.

2. Walsh's stochastic model of neural response. In his 1981 paper [W], Walsh proposed a model for a nerve cylinder undergoing random stimulus along its length. The interval $[0, L]$ represents the cylinder and $\{X(x,t,\omega) : 0 \leq x \leq L,\ 0 \leq t,\ \omega \in \Omega\}$ denotes the value of the nerve membrane potential at time t at a location x along the axis. He found that this potential could be approximated by the solution of the stochastic differential equation

$$dX = (\Delta - I)X\, dt + dW,$$

where the Laplacian Δ is given reflecting boundary conditions at 0 and L. Here W is a white noise on $\mathbb{R}_+ \times [0, L]$ based on the measure $\eta(dx)dt$, where η models the intensity of the random stimulation acting along the nerve cylinder. In order to avoid non-symmetric Dirichlet forms we shall take $\eta(dx)$ equal to Lebesgue measure, but a treatment that includes the non-symmetric case can be found in [S2].

Let $E = L^2([0, L]; dx)$ be our state space and let $\langle \cdot\, , \cdot \rangle$ denote the inner product in E. On the space E, the operator $A = I - \Delta$ has the eigensystem

$$
\begin{aligned}
&e_0(x) \equiv L^{-1/2} \quad \text{and} \quad e_j(x) = 2^{1/2} L^{-1/2} \cos(\pi j x L^{-1}), \quad j \geq 1 \\
&A e_j = \lambda_j e_j = (1 + \pi^2 j^2 L^{-2}) e_j.
\end{aligned}
$$

Let m be the mean zero Gaussian measure on E with covariance $(2A)^{-1}$, that is, such that for $f, g \in E$ we have

$$\int_E \langle f, z\rangle \langle g, z\rangle\, m(dz) = \frac{1}{2} \langle A^{-1} f, g\rangle.$$

We define a linear subspace $\mathcal{F}C_b^\infty$ of $L^2(E; m)$ by

$$\mathcal{F}C_b^\infty = \{u : u(z) = \psi(\langle f_1, z\rangle, \dots, \langle f_n, z\rangle),\ f_i \in E, \psi \in C_b^\infty(\mathbb{R}^n)\}.$$

For $u \in \mathcal{F}C_b^\infty$ we define the gradient $\nabla u(z) = \sum_{i=1}^n (\partial_i \psi)(\langle f_1, z\rangle, \ldots, \langle f_n, z\rangle)\, f_i$, and the square field $\boxplus(u,v)(z) = \langle \nabla u(z), \nabla v(z)\rangle$. The pre-Dirichlet form $(\mathcal{E}, \mathcal{F}C_b^\infty)$ defined as in (0.4) is closable, and its closure $(\mathcal{E}, D(\mathcal{E}))$ is the local, quasi-regular Dirichlet form associated with Walsh's process $(X_t)_{t\geq 0}$.

Since the state space E of the process $(X_t)_{t\geq 0}$ is $L^2([0,L];dx)$ we know that the functions $x \mapsto X(x,t,\omega)$ are square integrable. Can we use the form $(\mathcal{E}, D(\mathcal{E}))$ to obtain further information on these functions?

It is easiest to consider the properties of the coefficients in the cosine expansion of $x \to X_t(x)$. Before we continue then, let us look more closely at the cosine expansion of a randomly selected element $z \in E$. To simplify the notation we define $z_j = \langle e_j, z\rangle$ for $j \geq 0$, and consider the sequence of random variables $(z_j)_{j\in\mathbb{N}}$ on (E,m). This sequence is mean zero Gaussian with covariance

$$\mathrm{Cov}\,(z_i, z_j) = \int_E \langle e_i, z\rangle \langle e_j, z\rangle\, m(dz) = \frac{1}{2}\langle A^{-1} e_i, e_j\rangle = \frac{1}{2\lambda_j}\delta_{ij}.$$

In other words, $(z_j)_{j\geq 0}$ are independent, and the standard deviation of z_j is $\sigma_j = (2\lambda_j)^{-1}$. It is useful to note that the standard deviation σ_j is of the order $1/j$. In particular, the sequence $j\sigma_j$ is increasing so that for $j \geq 1$ we have $j\sigma_j \geq \sigma_1 = [2(1+\pi^2 L^{-2})]^{-1/2}$.

Now, two properties of a function that are related to its cosine expansion are Hölder continuity and bounded variation. In fact, a slight modification of [Kat; Chapter 1, Theorem 4.5] and [Kat; Chapter 1, Section 6.3], where the topology of the unit circle is used, proves the following lemma.

Lemma 2.1.
(i) If z is of bounded variation, then $|z_j| = O(1/j)$.
(ii) If z is α-Hölder continuous with $\alpha > 1/2$, then $\sum_j |z_j| < \infty$.

In the next two propositions we show that with probability one and at all times t, the function $x \mapsto X(x,t,\omega)$ is of unbounded variation and is not α-Hölder continuous for $\alpha > 1/2$. These complement Walsh's fixed time result [W; Proposition 6.1] which says that, as a function of x, $X(t,x)$ looks like a Brownian motion path plus a C^2-function.

Proposition 2.2. For m-almost every $z \in E$,

$$P_z(x \mapsto X(x,t) \text{ is of unbounded variation for all } t) = 1.$$

Proof. From Lemma 2.1 (i) it suffices to show that the set $N = \{z : \sup_{j\geq 1} |jz_j| < \infty\}$ is $\mathcal{E}$-exceptional. The fact that $m(N) = 0$ follows from letting $n \to \infty$ in (2.3) below, it remains to show that 1_N is $\mathcal{E}$-quasi-continuous.

For $M \geq 1$, let ψ_M be a smooth function on $\mathbb{R}$ so that $1_{[M,\infty)} \leq \psi_M \leq 1_{[M-1,\infty)}$ and $|\psi_M'| \leq 2 \times 1_{[M-1,M]}$. For $n, M \geq 1$ define a continuous function on E by

$$u_{n,M}(z) = \psi_M\Big(\sup_{j=1}^n |jz_j|\Big).$$

Then $u_{n,M} \in D(\mathcal{E})$ and, almost surely on E we have

$$\nabla u_{n,M}(z) = \psi'_M\Big(\sup_{j=1}^{n} |jz_j|\Big)\nabla\Big(\sup_{j=1}^{n} |jz_j|\Big)$$
$$= \psi'_M\Big(\sup_{j=1}^{n} |jz_j|\Big)\sum_{i=1}^{n} ie_i \ \mathrm{sign}(z_i)\, 1_{(\sup_{j=1}^{n} |jz_j|=|iz_i|)},$$

and hence

$$\boxplus(u_{n,M})(z) \le c\,n^2 1_{(M-1<\sup_{j=1}^{n} |jz_j|<M)}. \tag{2.1}$$

Therefore integrating over E we get

$$\mathcal{E}(u_{n,M}, u_{n,M}) \le cn^2 m\Big(\sup_{j=1}^{n} |jz_j| \le M\Big). \tag{2.2}$$

Since the random variables jz_j are independent mean zero Gaussian with standard deviation bigger than σ_1,

$$m\Big(\sup_{j=1}^{n} |jz_j| \le M\Big) = \prod_{j=1}^{n} m(|jz_j| \le M) \le \Phi(M/\sigma_1)^n, \tag{2.3}$$

where Φ is the standard normal cumulative distribution function. For fixed M, the sequence $(u_{n,M})_{n\in\mathbb{N}}$ of continuous functions increases pointwise to

$$u_M(z) = \psi_M\Big(\sup_{j=1}^{\infty} |jz_j|\Big).$$

Since by (2.2) and (2.3) we know that $\mathcal{E}(u_{n,M}, u_{n,M})$ is bounded, Lemma 0.1 tells us that u_M is an $\mathcal{E}$-quasi-continuous member of $D(\mathcal{E})$. In addition, combining (2.1) and (2.3) shows that $\boxplus(u_{n,M}) \to 0$ as $n \to \infty$ and so $\boxplus(u_M) = 0$. Now the sequence u_M converges pointwise to 1_N as $M \to \infty$, and since $\mathcal{E}(u_M, u_M) = 0$, a second application of Lemma 0.1 shows that 1_N is $\mathcal{E}$-quasi-continuous, and this gives the result. □

Proposition 2.3. For m-almost every $z \in E$,

$$P_z\Big(x \mapsto X(x,t) \text{ is not } \alpha\text{-Hölder continuous with } \alpha > 1/2 \text{ for all } t \ge 0\Big) = 1.$$

Proof. From Lemma 2.1 (ii) it suffices to show that the set $N = \{z : \sum_j |z_j| < \infty\}$ is $\mathcal{E}$-exceptional. The fact that $m(N) = 0$ follows from letting $n \to \infty$ in (2.5) below, so it remains to show that 1_N is $\mathcal{E}$-quasi-continuous.

For $M \ge 1$, let ψ_M be a smooth function on $\mathbb{R}$ so that $1_{[M,\infty)} \le \psi_M \le 1_{[M-1,\infty)}$ and $|\psi'_M| \le 2 \times 1_{[M-1,M]}$. For $n, M \ge 1$ define the continuous function on E by

$$u_{n,M}(z) = \psi_M\Big(\sum_{j=1}^{n} |z_j|\Big).$$

Then $u_{n,M} \in D(\mathcal{E})$ and, almost surely we have

$$\begin{aligned}\nabla u_{n,M}(z) &= \psi_M'\Big(\sum_{j=1}^n |z_j|\Big)\nabla\Big(\sum_{j=1}^n |z_j|\Big)\\ &= \psi_M'\Big(\sum_{j=1}^n |z_j|\Big)\sum_{j=1}^n e_j \text{ sign } (z_j),\end{aligned}$$

so

$$\boxplus(u_{n,M})(z) \leq cn^2 1_{(\sum_{j=1}^n |z_j| \leq M)}. \tag{2.4}$$

Therefore

$$\mathcal{E}(u_{n,M}, u_{n,M}) \leq cn^2 m\Big(\sum_{j=1}^n |z_j| \leq M\Big).$$

The random variables z_j are independent mean zero Gaussian with standard deviation $\sigma_j \geq \sigma_1/j$. From Chebyshev's inequality we find that for any $t > 0$,

$$m\Big(\sum_{j=1}^n |z_j| \leq M\Big) \leq e^{tM} E(e^{-t\sum_{j=1}^n |z_j|}) = e^{tM}\prod_{j=1}^n E(e^{-t\sigma_j|Z|}), \tag{2.5}$$

where Z is a standard normal random variable. Using the bound $\sum_{j=1}^n \sigma_j \geq \sum_{j=1}^n \sigma_1/j \geq \sigma_1 \ln(n)$ and the fact that $E(e^{-a|Z|}) \leq e^{-a/2}$ for $0 \leq a \leq 1$, we find

$$\prod_{j=1}^n E(e^{-t\sigma_j|Z|}) \leq \left[\prod_{\{j:t\sigma_j>1\}} e^{(t/2)\sigma_j}\right] e^{(-t/2)\sum_{j=1}^n \sigma_j} \leq c\, n^{-\sigma_1 t/2}.$$

Combining the previous three inequalities, we see that

$$\mathcal{E}(u_{n,M}, u_{n,M}) \leq c\, n^{2-\sigma_1 t/2}.$$

Now choosing t so that $\sigma_1 t/2 \geq 2$ makes $\mathcal{E}(u_{n,M}, u_{n,M})$ bounded in n, and so by Lemma 0.1 the pointwise limit $u_M(z) = \psi_M(\sum_{j\geq 1} |z_j|)$ is $\mathcal{E}$-quasi-continuous. In addition, combining (2.4) and (2.5) shows that $\boxplus(u_{n,M}) \to 0$ as $n \to \infty$ and so $\boxplus(u_M) = 0$. Now the sequence u_M converges pointwise to 1_N as $M \to \infty$, and since $\mathcal{E}(u_M, u_M) = 0$, a second application of Lemma 0.1 shows that 1_N is $\mathcal{E}$-quasi-continuous, and this gives the result. □

3. The Fleming-Viot process. The Fleming-Viot process models the evolution of the genetic profile of a population. Each individual in the population has a genetic type belonging to the type space S, and X_t denotes the empirical distribution of types at time t. The process $(X_t)_{t\geq 0}$ lives on the space of probability measures on S. The changes to the genetic makeup of this population come from two opposing sources; *genetic drift* which encourages conformity by preferring the offspring of individuals with dominant type and *mutation* which continually adds fresh variation. In our process, these forces are in perfect balance and the result is that $(X_t)_{t\geq 0}$ is stationary, i.e., in equilibrium.

Mathematically, we proceed as follows. Let (S, d) be a locally compact, separable metric space and let E denote the space of probability measures on the Borel σ-algebra in S. The topology of weak convergence turns E into a complete separable metric space. We let $\mathcal{B}_b(S)$ denote the space of bounded, Borel measurable functions on S. For $z \in E$ and $f \in \mathcal{B}_b(S)$ we define $\langle f, z\rangle$ to be the integral $\int_S f(x)\, z(dx)$. Consider the space of functions

$$\mathcal{F}C_b^\infty = \{u : u(z) = \psi(\langle f_1, z\rangle, \dots, \langle f_n, z\rangle),\ f_i \in \mathcal{B}_b(S), \psi \in C_b^\infty(\mathbb{R}^n)\}\,.$$

Even though the functions in $\mathcal{F}C_b^\infty$ are not continuous, a monotone class argument shows that they are $\mathcal{E}$-quasi-continuous where $(\mathcal{E}, D(\mathcal{E}))$ is defined in (3.4) below.

For every $x \in S$, let ε_x be the point mass at x, and for $u \in \mathcal{F}C_b^\infty$ define

$$\frac{\partial u}{\partial \varepsilon_x}(z) = \lim_{s\to 0} \frac{u(z + s\varepsilon_x) - u(z)}{s}.$$

We will write

$$\nabla u(z; x) = (\partial u/\partial \varepsilon_x)(z) = \sum_{i=1}^n (\partial_i \psi)(\langle f_1, z\rangle, \dots, \langle f_n, z\rangle) f_i(x).$$

For $f, g \in \mathcal{B}_b(S)$, we set $\langle f, g\rangle_z = \int fg\, dz - (\int f\, dz)(\int g\, dz)$ and define the square field on $\mathcal{F}C_b^\infty$ by $田(u, v)(z) = \langle \nabla u(z), \nabla v(z)\rangle_z$.

Before we can define the invariant measure m, we need to introduce the mutation operator and Fleming-Viot generator. The mutation operator A acts on functions $f : S \to \mathbb{R}$, and is given by

$$Af(x) = (\theta/2) \int_S \big(f(y) - f(x)\big)\, \mu(dy).$$

The interpretation of the parameters $\theta > 0$ and $\mu \in E$ is that the mutation intensity θ governs how rapidly mutation occurs, and when mutation occurs the new type is chosen according to the fixed measure μ. Consider the Fleming-Viot generator

$$Lu(z) = \frac{1}{2} \int_S \int_S \frac{\partial^2 u}{\partial \varepsilon_x \partial \varepsilon_y}(z) \left(\varepsilon_x(dy) - z(dy)\right) z(dx) + \int_S A\big(\nabla u(z)\big)(x)\, z(dx).$$

From Theorem 8.1 of [EK], there is a unique probability measure $m = m(\mu, \theta)$ on E such that

$$\mathcal{E}(u, v) = \frac{1}{2} \int_E 田(u, v)(z)\, m(dz) = \int_E (-Lu)(z) v(z)\, m(dz) \qquad u, v \in \mathcal{F}C_b^\infty.$$

The form $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is closable, and its closure $(\mathcal{E}, D(\mathcal{E}))$ is the local, quasi-regular Dirichlet form [ORS] associated with the Fleming-Viot process with parameters θ and μ.

Before we turn to the sample paths of $(X_t)_{t\geq 0}$ we first gather some information on the invariant measure m.

Definition 3.1. If $\theta_1, \dots, \theta_n > 0$, then the Dirichlet $(\theta_1, \dots, \theta_n)$ distribution is the measure ξ on $S_n = \{x \in \mathbb{R}^n \mid x_i \geq 0, \sum_{i=1}^n x_i = 1\}$ given by

$$\xi(dx) = \frac{\Gamma(\theta_1 + \cdots + \theta_n)}{\Gamma(\theta_1) \cdots \Gamma(\theta_n)} x_1^{\theta_1 - 1} \cdots x_n^{\theta_n - 1} \, dx_1 \dots dx_{n-1}.$$

For $n = 1$, the Dirichlet (θ) distribution is the point mass at 1.

Lemma 3.2. *If $(B_j)_{j=1}^n$ is a measurable partition of S, with $\mu(B_j) > 0$ for all $1 \leq j \leq n$, then the random vector $(z(B_1), \dots, z(B_n))$ on (E, m) has a Dirichlet $(\theta\mu(B_1), \dots, \theta\mu(B_n))$ distribution.*

Lemma 3.2 says that if F is a Borel set in S with $\mu(F) = 0$, then $z(F) = 0$ for m-almost every $z \in E$; while if $\mu(F) > 0$, then $z(F) > 0$ for m-almost every $z \in E$. Furthermore, if $\mu(F) = 0$, then since the map $z \mapsto z(F)$ is $\mathcal{E}$-quasi-continuous, $P_z(X_t(F) = 0 \text{ for all } t) = 1$ for m-almost every $z \in E$. That is, if the set F of types is not charged on average, then it is not charged ever by the process $(X_t)_{t \geq 0}$.

When $\mu(F) > 0$ the situation is more complicated. As you would expect, most of the time $X_t(F) > 0$, but it is possible that there exist exceptional times when $X_t(F) = 0$. Whether or not such exceptional times exist depends on the value of the mutation parameter θ. If θ is large enough, then there is a lot of mutation and this tends to keep the measure X_t spread out, so $X_t(F) > 0$ for all times t. But if the rate of mutation is small, the population will occasionally collapse to F^c and give $X_t(F) = 0$.

Proposition 3.3. *Let $F \in \mathcal{B}(S)$ so that $0 < \mu(F) < 1$. Then $P_z(X_t(F) > 0 \text{ for all } t) = 1$ for m-almost every $z \in E$, if and only if $\theta\mu(F) \geq 1$.*

Proof. Define the set $N = \{z : z(F) = 0\}$; since $\mu(F) > 0$ we know that $m(N) = 0$. For $n \geq 1$, let ψ_n be a smooth function on $\mathbb{R}$ so that $1_{(-\infty,0]} \leq \psi_n \leq 1_{(-\infty,1/n]}$ and $|\psi_n'| \leq 2n \times 1_{[0,1/n]}$. If we let $u_n(z) = \psi_n(z(F))$, then

$$\boxplus(u_n)(z) = \big(\psi_n'(z(F))\big)^2 (z(F) - z(F)^2).$$

Lemma 3.2 says that the random variable $z(F)$ on $(E; m)$ has a Beta distribution with parameters $\theta\mu(F)$ and $\theta\mu(F^c)$. Hence integrating the square field over E gives

$$\begin{aligned} \mathcal{E}(u_n, u_n) &\leq c \int n^2 \big(z(F) - z(F)^2\big) 1_{(0 < z(F) < 1/n)} \, m(dz) \\ &= c\, n^2 \int_0^{1/n} (y - y^2)\, y^{\theta\mu(F) - 1} \, (1 - y)^{\theta\mu(S \setminus F) - 1} \, dy \\ &\leq c\, n^2 \, (1/n)^{\theta\mu(F) + 1} \\ &= c\, n^{1 - \theta\mu(F)}. \end{aligned}$$

Since $\theta\mu(F) \geq 1$, we see that $\mathcal{E}(u_n, u_n)$ is bounded. The functions u_n are $\mathcal{E}$-quasi-continuous and converge pointwise to 1_N as $n \to \infty$. By Lemma 0.1, the function 1_N is an $\mathcal{E}$-quasi-continuous member of $D(\mathcal{E})$ which gives the result.

To prove the converse statement we need a lemma on the Dirichlet distribution. Its proof is an exercise in calculus that is, in principle, the same as the calculation we did in (1.1) to find a measure of finite energy for Brownian motion.

Lemma 3.4. Suppose that $1 \le k \le n$ and let ξ_k be the Dirichlet $(\theta_1, \ldots, \theta_k)$ measure on S_k and ξ_n be the Dirichlet $(\theta_1, \ldots, \theta_n)$ measure on S_n. Then for every $\psi \in C^\infty(\mathbb{R}^n)$,

$$\frac{1}{2}\left(\frac{1-\tilde\theta_2}{2^{\tilde\theta_1}}\right)\frac{\Gamma(\tilde\theta_1+\tilde\theta_2)}{\Gamma(\tilde\theta_1)\Gamma(\tilde\theta_2)}\int_{S_k}\psi^2(u_1,\ldots,u_k,0,\ldots,0)\,\xi_k(du)$$
$$\le \int_{S_n}\left(\langle\nabla\psi(w), a(w)\nabla\psi(w)\rangle_{\mathbb{R}^n} + \psi^2(w)\right)\xi_n(dw).$$

Here $\tilde\theta_1 = \theta_1 + \cdots + \theta_k$, $\tilde\theta_2 = \theta_{k+1} + \cdots + \theta_n$, and for $w \in S_n$, $a(w)$ is the $n \times n$ matrix with entries $a(w)_{ij} = w_i\delta_{ij} - w_i w_j$.

Define a probability measure on S by $\tilde\mu(A) = \mu(A \cap F^c)/\mu(F^c)$ and then let $\nu = m(\theta\mu(F^c), \tilde\mu)$ be an invariant measure for a different Fleming-Viot process. Let $\mathcal{A} = \{(A_j)_{j=1}^n$ is a measurable partition of S for some n with $\mu(A_j) > 0$ for all j, $F^c = A_1 \cup \cdots \cup A_k$ for some $1 \le k \le n\}$, and define

$$D = \{u : u(z) = \psi(z(A_1), \ldots, z(A_n)); (A_j)_{j=1}^n \in \mathcal{A}, \psi \in C_b^\infty(\mathbb{R}^n)\}.$$

You can show that D is dense in $D(\mathcal{E})$. Under m, the random vector $(z(A_1), \ldots, z(A_n))$ has a Dirichlet $(\theta\mu(A_1), \ldots, \theta\mu(A_n))$ distribution, which has full support on S_n. On the other hand, under ν the random vector $(z(A_1), \ldots, z(A_k))$ has a Dirichlet $(\theta\mu(A_1), \ldots, \theta\mu(A_k))$ distribution, and $z(A_j) = 0$ ν-almost everywhere for $k+1 \le j \le n$. In particular, ν only charges the set N. By Lemma 3.4, we see that for $u \in D$,

$$\frac{1}{2}\left(\frac{1-\theta\mu(F)}{2^{\theta\mu(F^c)}}\right)\frac{\Gamma(\theta)}{\Gamma(\theta\mu(F))\Gamma(\theta\mu(F^c))}\int u(z)^2\nu(dz) \le \mathcal{E}_1(u,u).$$

If $\theta\mu(F) < 1$, this shows that ν has finite energy and Lemma 0.2 gives us the result. □

4. Particle systems in $\mathbb{R}^d$. In much recent work [AKR 1–4, O, Y] the theory of Dirichlet forms has been used to construct and study Markov processes that take values in the space of locally finite configurations on a Riemannian manifold (for a nice survey see [R]). For simplicity we take the manifold to be Euclidean space $\mathbb{R}^d$ so that the configuration space is defined by

$$\Gamma_{\mathbb{R}^d} = \{z \subset \mathbb{R}^d : |z \cap K| < \infty \text{ for every compact } K\}.$$

A configuration, then, is simply a collection of points in $\mathbb{R}^d$ with the property that only finitely many points inhabit any compact set. Every configuration z can be identified with the Radon measure $\sum_{x\in z}\varepsilon_x$, and we will make this identification without comment. For $f \in \mathcal{B}_b(\mathbb{R}^d)$ we let $\langle f, z\rangle$ be the integral of f with respect to the measure z, that is, $\langle f, z\rangle = \sum_{x\in z} f(x)$.

The space $\Gamma_{\mathbb{R}^d}$ will be given the topology of vague convergence of measures, and measures on $\Gamma_{\mathbb{R}^d}$ are defined on the corresponding Borel sets $\mathcal{B}(\Gamma_{\mathbb{R}^d})$. But since $\Gamma_{\mathbb{R}^d}$ is not complete with respect to the vague topology it is necessary to use the completed state space

$$E = \{\mathbb{Z}_+ \cup \{+\infty\}\text{-valued Radon measures on } \mathbb{R}^d\}.$$

Unlike the measures $\sum_{x\in z} \varepsilon_x$ in the space $\Gamma_{\mathbb{R}^d}$, the measures in E allow for the possibility that more than one particle could occupy the same position in $\mathbb{R}^d$, resulting in a point with mass of two or more. Proposition 4.1 gives conditions so that, with probability one, the process $(X_t)_{t\geq 0}$ will not hit the set of such measures, so the completion of the state space was unnecessary after all.

A probability measure on E models a randomly chosen configuration, that is, a point process on $\mathbb{R}^d$. In the free case, a Poisson measure is used to model random particles that act independently; while in the Gibbs case, a Gibbs measure is used to model random particles that interact via a potential function. Although the mathematically challenging Gibbs case is more interesting, analysis of the free case often serves as a useful guideline. In this paper, we will only consider Poisson measures, but more general treatments can be found in [R] and the references therein.

Let σ be a measure on $\mathbb{R}^d$ that has a strictly positive, continuously differentiable density ϱ with respect to Lebesgue measure. We let m be Poisson measure with intensity σ, that is, the probability measure on E characterized by the formula

$$\int_E \exp(\langle f, z\rangle)\, m(dz) = \exp\left(\int_{\mathbb{R}^d} (e^{f(x)} - 1)\, \sigma(dx)\right), \tag{4.1}$$

for $f \in C_0(\mathbb{R}^d)$. If A and B are disjoint Borel subsets of $\mathbb{R}^d$, then under the measure m, $z(A) = \langle 1_A, z\rangle$ and $z(B) = \langle 1_B, z\rangle$ are independent Poisson random variables with means $\sigma(A)$ and $\sigma(B)$.

We define a subspace of $L^2(E; m)$ by

$$\mathcal{F}C_b^\infty = \left\{u : u(z) = \psi(\langle f_1, z\rangle, \ldots, \langle f_n, z\rangle),\ f_i \in C_0^\infty(\mathbb{R}^d), \psi \in C_b^\infty(\mathbb{R}^n)\right\}.$$

In contrast to the Fleming-Viot model, here we do not get $\mathcal{E}$-quasi-continuous functions if the f_i's above are bounded and measurable; they must be continuous. For $u \in \mathcal{F}C_b^\infty$ define the gradient

$$(\nabla^\Gamma u)(z; x) = \sum_{i=1}^n (\partial_i \psi)(\langle f_1, z\rangle, \langle f_2, z\rangle, \ldots, \langle f_n, z\rangle)(\nabla f_i)(x),$$

where ∇ is the usual gradient on $\mathbb{R}^d$. We now define the square field on $\mathcal{F}C_b^\infty$ by

$$\boxplus(u, v)(z) = \int_{\mathbb{R}^d} \langle (\nabla^\Gamma u)(z; x), (\nabla^\Gamma v)(z; x)\rangle_{\mathbb{R}^d}\, z(dx).$$

The form $(\mathcal{E}, \mathcal{F}C_b^\infty)$ defined by (0.6) is closable, and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a local, quasi-regular Dirichlet form associated with a diffusion on configuration space [MR2].

Our first result shows that the set of multiple configurations is $\mathcal{E}$-exceptional if the underlying Euclidean space is at least two dimensional.

Proposition 4.1. *If $d \geq 2$, then the set $E \setminus \Gamma_{\mathbb{R}^d}$ is $\mathcal{E}$-exceptional.*

Proof. Our goal is to show that the set of measures taking values greater than one is $\mathcal{E}$-exceptional. It clearly suffices to prove this locally, that is, to show that for every positive integer a, the function 1_N is $\mathcal{E}$-quasi-continuous, where

$$N = \{z : \sup(z(\{x\}) : x \in [-a, a]^d) \geq 2\}.$$

Since m is a Poisson measure, then $\sup_x z(\{x\}) = 1$ for m-almost every $z \in E$, and so $m(N) = 0$.

Our analysis begins with a smooth partition of $\mathbb{R}^d$ into small pieces. Let f be a $C_0^\infty(\mathbb{R})$ function satisfying $1_{[0,1]} \leq f \leq 1_{[-1/2,3/2)}$ and $|f'| \leq c\,1_{[-1/2,3/2)}$, and for any $n \in \mathbb{N}$ and $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$, define a $C_0^\infty(\mathbb{R}^d)$ function by

$$f_i(x) = \prod_{k=1}^{d} f(nx_k - i_k).$$

We also let $I_i(x) = \prod_{k=1}^d 1_{[-1/2,3/2)}(nx_k - i_k)$ and note that $f_i \leq I_i$. Taking the jth partial derivative of f_i gives

$$\partial_j f_i(x) = nf'(nx_j - i_j) \prod_{k \neq j} f(nx_k - i_k),$$

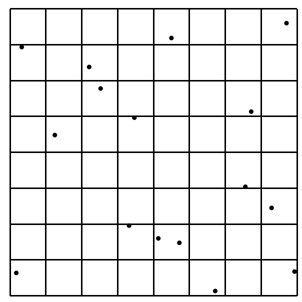
and so $(\partial_j f_i(x))^2 \leq c\,n^2 I_i(x)$. Adding over j from 1 to d gives us

$$|\nabla f_i(x)|^2 \leq c\,n^2 I_i(x).$$

Let ψ be a smooth function on $\mathbb{R}$ satisfying $1_{[2,\infty)} \leq \psi \leq 1_{[1,\infty)}$ and $|\psi'| \leq c\,1_{(1,\infty)}$. Choosing $A = \mathbb{Z}^d \cap [-na, na]^d$, define a continuous element of $D(\mathcal{E})$ by

$$u_n(z) = \psi\big(\sup_{i \in A} \langle f_i, z \rangle\big).$$

Then $u_n \to 1_N$ pointwise as $n \to \infty$, so to apply Lemma 0.1 we must prove that $(u_n)_{n \in \mathbb{N}}$ is $\mathcal{E}$-bounded.



The function u_n picks out configurations with points that are "too" close

First note that

$$\Big(\psi'\big(\sup_{i \in A} \langle f_i, z \rangle\big)\Big)^2 \leq c\,1_{(\sup_{i \in A} \langle f_i, z \rangle > 1)} \leq c\,1_{(\sup_{i \in A} \langle I_i, z \rangle \geq 2)}, \tag{4.2}$$

where for the final inequality we use the fact that $\langle I_i, z \rangle$ is an integer. Therefore, using the inequalities in (4.2) and (4.3), we get

$$
\begin{aligned}
\boxplus(u_n)(z) &= \Big(\psi'\big(\sup_{i\in A}\langle f_i, z\rangle\big)\Big)^2 \boxplus\big(\sup_{i\in A}\langle f_i,\cdot\rangle\big)(z) \\
&\le \Big(\psi'\big(\sup_{i\in A}\langle f_i, z\rangle\big)\Big)^2 \sup_{i\in A}\boxplus(\langle f_i,\cdot\rangle)(z) \\
&= \Big(\psi'\big(\sup_{i\in A}\langle f_i, z\rangle\big)\Big)^2 \sup_{i\in A}\int |\nabla f_i(x)|^2\, z(dx) \\
&\le c\, 1_{(\sup_{i\in A}\langle I_i,z\rangle\ge 2)}\; n^2 \sup_{i\in A}\langle I_i, z\rangle \\
&\le c\, n^2 \sum_{i\in A} 1_{(\langle I_i,z\rangle\ge 2)}\langle I_i, z\rangle.
\end{aligned}
\tag{4.3}
$$

Since $\langle I_i, z\rangle$ is a Poisson random variable on $(E;m)$, we have $\int_{(\langle I_i,z\rangle\ge 2)}\langle I_i, z\rangle\, m(dz) \le \langle I_i,\sigma\rangle^2$, and combined with (4.4) this gives $\mathcal{E}(u_n,u_n)\le c\,n^2\sum_{i\in A}\langle I_i,\sigma\rangle^2$.

Although the supports of the indicator functions I_i are not disjoint, each point belongs to at most 2^d of the sets $\{I_i = 1\}$ for $i\in A$. Therefore the Cauchy-Schwarz inequality gives us

$$
\begin{aligned}
\sum_{i\in A}\langle I_i,\sigma\rangle^2 &= \sum_{i\in A}\left(\int I_i(x)\,\varrho(x)\,dx\right)^2 \\
&\le \sum_{i\in A}\left(\int I_i(x)\,\varrho(x)^2\,dx\right)\left(\int I_i(x)\,dx\right) \\
&\le 2^d\int_{[-(a+1),a+1]^d}\varrho(x)^2\,dx\,(2/n)^d,
\end{aligned}
$$

and combining this with the previous bound we find that $\mathcal{E}(u_n,u_n)\le c\,n^{2-d}$. Since $d\ge 2$ we see that $\sup_n\mathcal{E}(u_n,u_n)<\infty$, and conclude that N is $\mathcal{E}$-exceptional. □

We will now show that the particles in the random configuration X_t satisfy the law of large numbers. Lemma 4.2 gives the fixed time result and the general result follows in Proposition 4.3. In the discussion below, S_r denotes the sphere in $\mathbb{R}^d$ with radius r, centered at the origin.

Lemma 4.2. If $\sigma(\mathbb{R}^d)=\infty$, then $\lim_{r\to\infty} z(S_r)/\sigma(S_r)=1$, for m-almost every $z\in E$.

Proof. Choose radii $(r_n)_{n\in\mathbb{N}}$ so that $\sigma(S_{r_n})=n$. Then under the measure m, the function $z(S_{r_n}) = z(S_1)+z(S_2\setminus S_1)+\cdots+z(S_{r_n}\setminus S_{r_n-1})$ is the sum of n independent Poisson(1) random variables. By the law of large numbers, $z(S_{r_n})/n\to 1$ m-almost surely. Now for $r_n\le r\le r_{n+1}$ we have

$$
\frac{n}{n+1}\,\frac{z(S_{r_n})}{n}\le\frac{z(S_r)}{\sigma(S_r)}\le\frac{z(S_{r_{n+1}})}{n+1}\,\frac{n+1}{n}
\tag{4.4}
$$

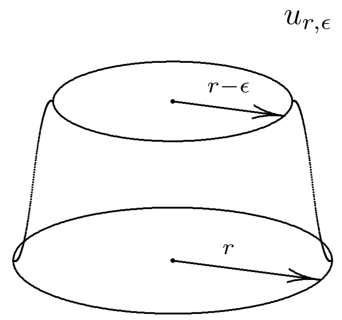
and so the limit is attained over the continuous index r as well. $\square$

Proposition 4.3. If $\sigma(\mathbb{R}^d) = \infty$ and $\lim_{\epsilon\to 0} \liminf_{r\to\infty} \sigma(S_{r-\epsilon})/\sigma(S_r) = 1$, then for m-almost every $z \in E$,

$$P_z\left(\lim_{r\to\infty} \frac{X_t(S_r)}{\sigma(S_r)} = 1 \text{ for all } t \geq 0\right) = 1.$$

Proof. For $\epsilon > 0$ and $r > \epsilon$, let $\psi_{r,\epsilon}$ be a smooth function satisfying $1_{(-\infty,r-\epsilon]} \leq \psi_{r,\epsilon} \leq 1_{(-\infty,r]}$, and $|\psi'_{r,\epsilon}| \leq (c/\epsilon)\, 1_{(-\infty,r]}$. Define a continuous element of $D(\mathcal{E})$ by

$$u_{r,\epsilon}(z) = \langle \psi_{r,\epsilon}(|\cdot|), z\rangle/\sigma(S_r).$$



Then we have

$$\frac{\sigma(S_{r-\epsilon})}{\sigma(S_r)} \frac{z(S_{r-\epsilon})}{\sigma(S_{r-\epsilon})} \leq u_{r,\epsilon}(z) \leq \frac{z(S_r)}{\sigma(S_r)},$$

so the regularity assumption on the measure σ gives

$$\lim_{\epsilon\to 0} \limsup_{r\to\infty} u_{r,\epsilon}(z) = \limsup_{r\to\infty} z(S_r)/\sigma(S_r).$$

Bounding the square field gives

$$\boxplus(u_{r,\epsilon})(z) \leq \frac{c^2}{\epsilon^2} \frac{z(S_r)}{\sigma(S_r)^2}. \tag{4.5}$$

As in the previous lemma, define radii $(r_n)_{n\in\mathbb{N}}$ so that $\sigma(S_{r_n}) = n$. Therefore,

$$\int_E \sup_{n\in\mathbb{N}} \frac{z(S_{r_n})}{n}\, m(dz) \leq c + c\int_E z(S_{r_1}) \log^+(z(S_{r_1}))\, m(dz) < \infty.$$

Defining the random variable $X^*(z) = \sup_{r\geq r_1} z(S_r)/\sigma(S_r)$, the bound above together with (4.5) gives $\int_E X^*(z)\, m(dz) < \infty$. For fixed $n \geq r_1$, let $(A_j)_{j\in\mathbb{N}}$ be an increasing sequence of finite subsets of $[n,\infty)$ so that $\cup_j A_j$ is dense in $[n,\infty)$. For fixed z and ϵ, the function $r \mapsto u_{r,\epsilon}(z)$ is continuous on $[r_1,\infty)$ and so

$$\sup_{r\geq n} u_{r,\epsilon}(z) = \sup_j \sup_{r\in A_j} u_{r,\epsilon}(z).$$

Now for each $j \in \mathbb{N}$, $\sup_{r\in A_j} u_{r,\epsilon}(z) \in D(\mathcal{E})$ and is $\mathcal{E}$-quasi-continuous. Repeated use of the inequality $\boxplus(u \vee v) \leq \boxplus(u) \vee \boxplus(v)$ combined with the bound (4.6) gives $\boxplus(\sup_{r\in A_j} u_{r,\epsilon}) \leq c^2 X^*/\epsilon^2\sigma(S_n)$, and so

$$\sup_j \mathcal{E}\Big(\sup_{r\in A_j} u_{r,\epsilon}, \sup_{r\in A_j} u_{r,\epsilon}\Big) \leq \int_E \frac{c^2 X^*(z)}{\epsilon^2\sigma(S_n)}\, m(dz) < \infty.$$

Applying Lemma 0.1, we see that the pointwise limit $\sup_{r\geq n} u_{r,\epsilon}$ belongs to $D(\mathcal{E})$ and is $\mathcal{E}$-quasi-continuous. Also the bound for the square field also carries over; $\boxplus(\sup_{r\geq n} u_{r,\epsilon}) \leq c^2 X^*/\epsilon^2\sigma(S_n)$. Applying the same argument to the decreasing sequence $(\sup_{r\geq n} u_{r,\epsilon})_{n\in\mathbb{N}}$, we find that the pointwise limit $u_\epsilon = \limsup_{r\to\infty} u_{r,\epsilon}$ belongs to $D(\mathcal{E})$, is $\mathcal{E}$-quasi-continuous, and has $\boxplus(u_\epsilon) = 0$. The extra factor of $\sigma(S_n)$ in the denominator accounts for the fact that the square field is zero in the limit. Since $\mathcal{E}(u_\epsilon, u_\epsilon) = 0$ is bounded in ϵ, we may apply Lemma 0.1 to conclude that $u = \lim_{\epsilon\to 0} u_\epsilon$ belongs to $D(\mathcal{E})$, is $\mathcal{E}$-quasi-continuous, and has $\boxplus(u) = 0$.

For any two rational numbers $0 < a < b$, we let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of smooth, compactly supported functions that vanish at the origin, decreasing pointwise to the indicator function $1_{[a,b]}$. Then $\psi_n(u)$ belongs to $D(\mathcal{E})$, is $\mathcal{E}$-quasi-continuous, and has $\boxplus(\psi_n(u)) = 0$. Letting $n \to \infty$ and applying Lemma 0.1 once more, we find that $1_{[a,b]}(u)$ belongs to $D(\mathcal{E})$ and is $\mathcal{E}$-quasi-continuous.

Applying the continuity result (0.6) simultaneously to the countable set of functions $\{1_{[a,b]}(u) : 0 < a < b \in \mathbb{Q}\}$, we conclude that the value of $\limsup_{r\to\infty} X_t(S_r)/\sigma(S_r)$ is almost surely constant in t. To be precise, for m-almost every $z \in E$, we have

$$P_z\left(\limsup_{r\to\infty} X_0(S_r)/\sigma(S_r) = \limsup_{r\to\infty} X_t(S_r)/\sigma(S_r) \text{ for all } t \geq 0\right) = 1.$$

A parallel argument shows that $\liminf_{r\to\infty} X_t(S_r)/\sigma(S_r)$ is almost surely constant in t.

Letting $A = \{z : \lim_{r\to\infty} z(S_r)/\sigma(S_r) = 1\}$, Lemma 4.2 says that $P_z(X_0 \in A) = 1_A(z) = 1$ for m-almost every $z \in E$. Therefore, for m-almost every $z \in E$, we have

$$P_z\left(\lim_{r\to\infty} X_t(S_r)/\sigma(S_r) \to 1 \text{ for all } t \geq 0\right) = 1.$$

□

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