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Abstract

We use viability techniques for solving Dirichlet problems with inequality constraints (obstacles) for a class of Hamilton-Jacobi equations. The hypograph of the "solution" is defined as the "capture basin" under an auxiliary control system of a target associated with the initial and boundary conditions, viable in an environment associated with the inequality constraint. From the tangential condition characterizing capture basins, we prove that this solution is the unique "upper semicontinuous" solution to the Hamilton-Jacobi-Bellman partial differential equation in the Barron/Jensen-Frankowska sense. We show how this framework allows us to translate properties of capture basins into corresponding properties of the solutions to this problem. For instance, this approach provides a representation formula of the solution which boils down to the Lax-Hopf formula in the absence of constraints.

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1 Introduction

1.1 Motivation

This article is motivated by macroscopic fluid models of highway traffic, following the pioneering work of Lighthill, Whitham and Richards [61, 75]. In their original work, the authors modelled highway traffic flow with a first order hyperbolic *partial differential equation* with concave flux function, called the Lighthill-Whitham-Richards (partial differential) equation. This model is the seminal model for numerous highway traffic flow studies available in the literature today [2, 42, 43, 60, 30, 82, 28]. It models the evolution of the density of vehicles on a highway by a conservation law, in which the mathematical model of the flux function inside the conservation law results from empirical measurements [57].

Solutions to such equations may have shocks (they are set-valued maps), which model abrupt changes in vehicle density on the highway [2], and only model physical phenomena to a certain degree. Hence discontinuous selections of these solutions are looked after, for instance, the *entropy solution* [2] of Oleinik [70], which is acknowledged to be the proper weak solution of this problem. There has been an extensive literature on this problem, of which we single out the work of Bardos, Leroux and Nedelec [23, 80].

Very few results applicable to highway traffic are available for control of first order hyperbolic conservation laws. Differential flatness [47] has been successfully applied to Burgers equation (and therefore to the Lighthill-Whitham-Richards equation) in [72] order to avoid the formation of such shockwaves. This analysis does not so far extend to the presence of shocks. Lyapunov based techniques have also been applied to the Burgers equation [59]. Adjoint based methods have been successfully applied to networks of Lighthill-Whitham-Richards equations in [54]; these results seem so far the most promising, but they do not have guarantees to provide an optimal control policy. Questions of interest in controlling first-order partial differential equations, and in particular, Lighthill-Whitham-Richards equations, are still open and difficult to solve due to the presence of shocks occurring in the solutions of these partial differential equations.

In order to alleviate the technical difficulties resulting from shocks present in solution of the Lighthill-Whitham-Richards equation, an alternate formulation consists in considering the *cumulated number of vehicles*, widely used in the transportation literature as well [67, 68, 69]. The cumulative number of vehicles can be thought of as a primitive of the density over space. Formally, the evolution of the cumulated number $\mathbf{N}(t,x)$ of vehicles is the solution of an *Hamilton-Jacobi (partial differential) equation* of the form

$$\frac{\partial \mathbf{N}(t,x)}{\partial t} + \psi\left(\frac{\partial \mathbf{N}(t,x)}{\partial x}\right) \;=\; \psi(v(t))$$

where the flux function ψ appearing in this Hamilton-Jacobi equation is in fact concave as shown by the empirically measured flux function of the Lighthill-Whitham-Richards equation [61, 75, 23, 80]. The function $v(\cdot)$ will be regarded as a control of the Hamilton-Jacobi equation in forthcoming studies. It could for example model the inflow of vehicles at the entrance of a stretch of highway. It is a given datum in this paper.

The solution of this Hamilton-Jacobi equation has no shocks, but is not necessarily differentiable. It is only upper semicontinuous. Actually, the non differentiability of the cumulated number of vehicles is closely related to the presence of the shocks of the solution to the Lighthill-Whitham-Richards equation (see for instance [36, 37, 38]).

Since the Lighthill-Whitham-Richards equation and the Hamilton-Jacobi equation model the same physical phenomenon and since both formulations are equivalently used in the highway transportation literature, we single out in this paper the study of the evolution of the cumulated number of vehicles for benefiting of the extensive knowledge of Hamilton-Jacobi equations for which control and viability techniques can be applied.

1.2 Contributions of the paper

We shall revisit this Hamilton-Jacobi equation by answering new questions:

- introducing a nontrivial right hand side,
- involve Dirichlet conditions,
- and above all, impose *inequality constraints* on the solution, for instance, upper bounds on the cumulated number of vehicles, depending on time and space variables.

For this purpose, we suggest to use a novel point of view based on the concept of capture basin of a target viable in an environment extensively studied in the framework of viability theory: Given a closed subset of a finite dimensional vector space regarded as an environment, a closed subset of this environment considered as a target and a control system, the viable capture basin is the subset of initial states of the environment from which starts at least one evolution governed by the control system viable in the environment until the finite time when it reaches the target (see Definition 9.3, p.25). It happens that the *hypograph* of the solution to the Hamilton-Jacobi equation satisfying initial and Dirichlet conditions as well as inequality constraints is the capture basin of an auxiliary target (involving initial and boundary conditions) viable in an auxiliary environment (involving inequality constraints) under an auxiliary control system (involving the flux function of the Hamilton-Jacobi equation).
Hence, anticipating on this property, we define the *viability hyposolution* of the Dirichlet problem for such an Hamilton-Jacobi equation with constraints from this property of being a viable capture basin (see Definition 3.1, p.6). Then we proceed by translating properties of viable capture basins (see [7] for instance) in the language of partial differential equations for this particular case. We shall prove that the viability hyposolution

1. is the *unique* generalized solution in the Barron/Jensen-Frankowska sense [5] (a weaker concept of viscosity solution introduced by Crandall, Evans and Lions in [41, 40] for continuous solutions adapted to the case when solution is only semicontinuous): Theorem 8.1, p.22,
2. is equivalently the *unique* upper semicontinuous solution in the contingent Frankowska sense: Theorem 7.1, p.19,
3. satisfies the sup-linearity property and depends "hypo-continuously" of the initial and Dirichlet conditions,
4. is represented by the Lax-Hopf formula (see Theorem 4.1, p.8) in the absence of inequality constraints, a more involved representation formula (see Theorem 4.5, p.11) in the presence of inequality constraints, upper estimates (maximum principe, see Proposition 4.3, p.10) and lower estimates (see Proposition 4.4, p.10).

[5] Hélène Frankowska proved that the epigraph of the value function of an optimal control problem—assumed to be only lower semicontinuous—is semipermeable (i.e., invariant and backward viable) under a (natural) auxiliary system. Furthermore, when it is continuous, she proved that its epigraph is viable and its hypograph invariant [50, 51, 53]. By duality, she proved that the latter property is equivalent to the fact that the value function is a viscosity solution of the associated Hamilton–Jacobi equation in the sense of Crandall and Lions. See also [25, 21, 8] for more details. Such concepts have been extended to solutions of systems of first-order partial differential equations without boundary conditions by Hélène Frankowska and one of the authors (see [13, 14, 15, 16, 17, 18] and chapter 8 of [5]). See also [10, 11].

1.3 Outline of the paper

In order to make the paper more readable by postponing the technical difficulties, we chose to begin by stating the problem and the main assumptions which will not be repeated all along the paper. We next define the viability hyposolution to the non homogeneous Dirichlet/initial value problem for our class of Hamilton-Jacobi equations under inequality constraint as the capture basin of a target summarizing the Dirichlet/initial data viable in a target associated with inequality constraints. Then, we translate the properties of capture basins to the viability hyposolution, starting with a general representation formula providing Lax-Hopf formulas in the absence of inequality constraints. We next check that the viability hyposolution satisfies the Dirichlet and initial conditions as well as the inequality constraints. The last three sections are devoted to the proof that the viability hyposolution is a solution to the Hamilton-Jacobi partial differential equation in two equivalent dual generalized sense by translating both the Viability Theorem and the Invariance Theorem characterizing the capture basin in terms of either tangential conditions or normal conditions, as it was done in a long series of papers by Hélène Frankowska. Using tangential conditions, we express the viability hyposolution as a solution to the Hamilton-Jacobi partial differential equation couched in terms of contingent hypoderivatives, whereas using normal conditions, we characterize it in terms of superdifferentials, as it was done independently by Barron/Jensen and Frankowska, in the spirit of nonsmooth analysis and viscosity solutions. The presence of inequality constraints complicates the technical formulation of the concept of solution at points where the solution touches the constraint, above all in the superdifferentials formulation, justifying the reason why we conclude this paper by this dual characterization. An appendix gathers some definitions, notations and basic prerequisites of viability theory and convex analysis for the convenience of readers who are not familiar with these topics.

2 Statement of the Problem

We set $X := \mathbb{R}^n$. Let us consider

1. A concave function $\psi : X \mapsto \mathbb{R}$ satisfying growth conditions

$$\forall\, v \in X,\;\; \beta - \sigma_A(v) \;\leq\; \psi(v) \;\leq\; \delta - \sigma_A(v)$$

   for some compact convex subset $A \subset X$, where $\sigma_A(v) := \sup_{u\in A} \langle u, v\rangle$ is the support function of A and where $\beta \leq \delta$,

2. A bounded continuous function $v : \mathbb{R}_+ \mapsto \mathrm{Dom}(\psi)$,

3. An upper semicontinuous initial datum $\mathbf{N}_0 : X \mapsto \mathbb{R}_+$. We set $\mathbf{N}_0(0,x) := \mathbf{N}_0(x)$ and $\mathbf{N}_0(t,x) := -\infty$ if $t > 0$.

4. A closed subset $K \subset X$ with nonempty interior $\mathrm{Int}(K) =: \Omega$ and boundary $\partial K =: \Gamma$,

5. An upper semicontinuous boundary datum $\gamma : \mathbb{R}_+ \times X \mapsto \mathbb{R}$, satisfying[6]

$$\forall\, x \in \partial K,\;\; \mathbf{N}_0(x) \;=\; \gamma(0,x) \;\text{ and }\; \forall\, t \geq 0,\;\; \forall\, x \in \mathrm{Int}(K),\;\; \gamma(t,x) \;=\; -\infty$$

6. A Lipschitz function $\mathbf{b} : \mathbb{R}_+ \times X \mapsto \mathbb{R} \cup \{-\infty\}$ setting the upper constraint.

[6] This is not mandatory. We can take any function such that $\mathrm{Dom}(\gamma) \subset K$ is strictly contained in K, an instance which may be useful for defining "guards" in impulse or hybrid systems, for instance. Boundary conditions are obtained when $\mathrm{Dom}(\gamma) = \partial K$.

We shall also assume in this paper that the data satisfy the following consistency conditions

$$\left\{\begin{array}{lll} (i) & \forall\, x \in \partial K, \;\; \mathbf{N}_0(x) \;=\; \gamma(0,x) \\ (ii) & \forall\, t \geq 0,\, \forall\, x \in K, \;\; \max\left(\mathbf{N}_0(t,x), \gamma(t,x)\right) \;\leq\; \mathbf{b}(t,x) \\ (iii) & \forall\, 0 \leq r \leq s,\, \forall\, x \in \partial K, \forall\, y \in \partial K, \;\; \gamma(r,x) - \gamma(s,y) \;\leq\; \left\langle \frac{1}{s-r}\int_r^s v(\tau)d\tau, x-y \right\rangle \\ (iv) & \forall\, x \in K, \forall\, y \in \partial K, \;\; \mathbf{N}_0(x) \;\leq\; \inf_{s\geq 0}\left(\gamma(s,y) + \left\langle \frac{1}{s}\int_0^s v(\tau)d\tau, x-y \right\rangle\right) \end{array}\right. \quad (1)$$

which are needed only to prove that the Dirichlet/initial conditions are satisfied (see Theorem 5.1, p.13 below). When the function $v(\cdot) \equiv v$ is constant, they boil down to

$$\left\{\begin{array}{lll} (i) & \forall\, x \in \partial K, \;\; \mathbf{N}_0(x) \;=\; \gamma(0,x) \\ (ii) & \forall\, t \geq 0,\, \forall\, x \in K, \;\; \max\left(\mathbf{N}_0(t,x), \gamma(t,x)\right) \;\leq\; \mathbf{b}(t,x) \\ (iii) & \forall\, 0 \leq r \leq s,\, \forall\, x \in \partial K,\; y \in \partial K, \;\; \gamma(r,x) - \gamma(s,y) \;\leq\; \langle v, x-y \rangle \\ (iv) & \forall\, x \in K,\; y \in \partial K, \;\; \mathbf{N}_0(x) \;\leq\; \inf_{s\geq 0} \gamma(s,y) + \langle v, x-y \rangle \end{array}\right.$$

Under the above mentioned assumptions, that are assumed all along this paper, we shall solve the existence of a solution to the *non-homogenous Hamilton-Jacobi equation*

$$\forall\, t > 0, \;\; x \in \mathrm{Int}(K), \;\; \frac{\partial \mathbf{N}(t,x)}{\partial t} + \psi\left(\frac{\partial \mathbf{N}(t,x)}{\partial x}\right) \;=\; \psi(v(t)) \quad (2)$$

satisfying the *initial and Dirichlet conditions*

$$\left\{\begin{array}{ll} (i) & \forall\, x \in K, \;\; \mathbf{N}(0,x) = \mathbf{N}_0(x) \text{ (initial condition)} \\ (ii) & \forall\, t \geq 0,\, \forall\, x \in \partial K, \;\; \mathbf{N}(t,x) = \gamma(t,x) \text{ (Dirichlet boundary condition)} \end{array}\right. \quad (3)$$

and the *viability constraints*

$$\forall\, t \geq 0,\; x \in K, \;\; \mathbf{N}(t,x) \;\leq\; \mathbf{b}(t,x) \text{ (upper inequality constraint)} \quad (4)$$

Example: — This equation is motivated by a commonly used first order model equation in highway traffic (*Lighthill-Whitham-Richards* equation) when $X := \mathbb{R}$ and $K := [\xi, +\infty[$, ψ a concave flux function vanishing at density 0 and at a jam density $\omega > 0$ and $\mathbf{N}(t,x)$ is the cumulated number of vehicles at time t and at location $x \in K$. Consistency conditions (1), p.5 read in this case: $\mathbf{N}_0(\xi) = \gamma(0,\xi)$ and

$$\left\{\begin{array}{lll} (i) & \forall\, t \geq 0,\, \forall\, x \in K, \;\; \max\left(\mathbf{N}_0(t,x), \gamma(t,x)\right) \;\leq\; \mathbf{b}(t,x) \\ (ii) & \forall\, 0 \leq r \leq s, \;\; \gamma(r,\xi) - \gamma(s,\xi) \;\leq\; 0 \text{ (monotonocity)} \\ (iii) & \forall x \in K, \;\; \mathbf{N}_0(x) \;\leq\; \inf_{s\geq 0}\left(\gamma(s,\xi) + \left\langle \frac{1}{s}\int_0^s v(\tau)d\tau, x-\xi \right\rangle\right) \end{array}\right. \quad (5)$$

Then the trapezoidal flux function (such as the one proposed by Daganzo [42, 43]) defined by

$$\psi(v) \;=\; \left\{\begin{array}{lll} \nu^\flat v & \text{if} & v \leq \gamma^\flat \\ \delta & \text{if} & v \in [\gamma^\flat, \gamma^\sharp] \\ \nu^\sharp(\omega - v) & \text{if} & v \geq \gamma^\sharp \end{array}\right.$$

and the Greenshield flux function

$$\psi(v) \;=\; \left\{\begin{array}{lll} \nu v & \text{if} & v \leq 0 \\ \frac{\nu}{\omega} v(\omega - v) & \text{if} & v \in [0,\omega] \\ \nu(\omega - v) & \text{if} & v \geq \omega \end{array}\right.$$

satisfy the assumptions on the function ψ with $A := [-\nu^\flat, +\nu^\sharp]$ and $A := [-\nu, +\nu]$ respectively (see Lemma 6.2, p.16). ■

We characterize the solution to this non-homogenous Dirichlet/Initial Value problem with inequality constraints through the capture basin of a target defined by the Dirichlet/initial conditions viable in an environment defined by inequality constraints under an adequate control system.

3 The Viability Hyposolution

The assumption that the flux function ψ is concave and upper semicontinuous plays a crucial role for defining the viability hyposolution. Indeed, the Fenchel theorem allows us to characterize it by

$$\psi(p) \; = \; \inf_{u \in \mathrm{Dom}(\varphi^\star)} [\varphi^\star(u) - \langle p, u\rangle] \tag{6}$$

where $\varphi^\star$ is the Fenchel conjugate function, which is the convex lower semicontinuous function defined by

$$\varphi^\star(u) \; := \; \sup_{p \in \mathrm{Dom}(\psi)} [\langle p, u\rangle + \psi(p)] \tag{7}$$

We introduce the auxiliary characteristic control system:

$$\left\{ \begin{array}{lcl} \tau'(t) & = & -1 \\ x'(t) & = & u(t) \\ y'(t) & = & \varphi^\star(u(t)) - \psi(v(\tau(t))) \text{ where } u(t) \; \in \; \mathrm{Dom}(\varphi^\star) \end{array} \right. \tag{8}$$

To be rigorous, we have to mention *once and for all* that the controls $u(\cdot)$ are measurable integrable functions with values in $\mathrm{Dom}(\varphi^\star)$, and thus, ranging $L^1(0,\infty;\mathrm{Dom}(\varphi^\star))$, and that the above system of differential equations is valid for almost all $t \geq 0$.
We set $\mathbf{c}(t,x) := \max(\mathbf{N}_0(t,x), \gamma(t,x))$, defined by

$$\mathbf{c}(t,x) \; := \left\{ \begin{array}{lll} -\infty & \text{if} & t > 0 \text{ and } x \in \Omega := \mathrm{Int}(K) \\ \mathbf{N}_0(x) & \text{if} & t = 0 \text{ and } x \in K \\ \gamma(t,x) & \text{if} & t \geq 0 \text{ and } x \in \Gamma := \partial K \end{array} \right.$$

We introduce the environment $\mathcal{K} := \mathcal{H}yp(\mathbf{b})$ is the subset of triples $(T,x,y) \subset \mathbb{R}_+ \times X \times \mathbb{R}$ such that $y \leq \mathbf{b}(T,x)$ (this is the *hypograph* of the function $\mathbf{b}$) and the target $\mathcal{C} := \mathcal{H}yp(\mathbf{c})$ the subset of triples $(T,x,y) \subset \mathbb{R}_+ \times X \times \mathbb{R}$ such that $y \leq \mathbf{c}(T,x)$ (which is the *hypograph* of the function $\mathbf{c}$).

Definition 3.1 ***The Viability Hyposolution.*** *The capture basin* $\mathrm{Capt}_{(8)}(\mathcal{K},\mathcal{C})$ *of a target* $\mathcal{C}$ *viable in the environment* $\mathcal{K}$ *under control system (8) is the subset of initial states* (t,x,y) *such that there exists a measurable control* $u(\cdot)$ *such that the associated solution*

$$s \mapsto \left(t - s, x + \int_0^s u(\tau) d\tau, y + \int_0^s (\varphi^\star(u(\tau)) - \psi(v(t-\tau))) d\tau \right)$$

is viable in $\mathcal{K}$ *until it reaches the target* $\mathcal{C}$.
The viability hyposolution $\mathbf{N}$ *is defined by*

$$\mathbf{N}(t,x) \; := \; \sup_{(t,x,y) \in \mathrm{Capt}_{(8)}(\mathcal{K},\mathcal{C})} y \tag{9}$$

We shall prove the following:

Theorem 3.2 Non-homogenous Dirichlet/Initialvalue Problem with inequality contraints. *The viability hyposolution* $\mathbf{N}$ *defined by (9) is the* largest *upper semicontinuous solution to Hamilton-Jacobi equation (2) satisfying initial and Dirichlet conditions (3) and inequality constraints (4) in both the contingent solution sense (see (22), p.19) and in the contingent normal sense (see (25), p.22). If the functions* ψ, $\varphi^\star$ *and* v *are furthermore Lipschitz, then the viability hyposolution* $\mathbf{N}$ *is its* unique *upper semicontinuous solution in both the contingent Frankowska sense (see (23) and (24)), p.19) and in the Barron-Jensen/Frankowska sense (see (28) and (27), p.22 and Theorems 7.1, p.19 and 8.1, p.22 for the precise statement).*

We shall derive this theorem and other results from the properties of capture basins gathered in [7], [9]. Since the capture basin of a union of targets is the union of the capture basins of these targets, we infer that whenever $\mathbf{c} := \sup_i \mathbf{c}_i$ is the upper enveloppe of a family of functions $\mathbf{c}_i$, then the viability hyposolution is the upper enveloppe

$$\forall\, t \geq 0,\ x \in X,\ \ \mathbf{N}(t,x) \ = \ \sup_i \mathbf{N}_{\mathbf{c}_i}(t,x)$$

of the solutions $\mathbf{N}_{\mathbf{c}_i}$ (sup-linearity property).
In particular, since $\mathbf{c}(t,x) := \max(\mathbf{N}_0(t,x), \gamma(t,x))$ (extended to $-\infty$ when $t > 0$ or $x \in \mathrm{Int}(K)$), we obtain the decomposition formula

$$\mathbf{N}(t,x) \ = \ \max\left(\mathbf{N}_{N_0}(t,x), \mathbf{N}_\gamma(t,x)\right) \tag{10}$$

in terms of initial condition component $\mathbf{N}_{\mathbf{N}_0}$ and the Dirichlet component $\mathbf{N}_\gamma$ of the viability hyposolution $\mathbf{N}$ defined by

$$\left\{ \begin{array}{lll} \mathbf{N}_{\mathbf{N}_0}(t,x) & := & \sup_{(t,x,y) \in \mathrm{Capt}_{(8)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{N}_0))} y \\ \mathbf{N}_\gamma(t,x) & := & \sup_{(t,x,y) \in \mathrm{Capt}_{(8)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\gamma))} y \end{array} \right.$$

The viability hyposolution depends continuously on the data in the following sense: If the hypographs of a sequence of initial data $\mathbf{c}_j$ converge in the upper Painlevé-Kuratowski sense (see for instance [12]) to the hypographs of data $\mathbf{c}$, then the upper Painlevé-Kuratowski limit of the hypographs of the solutions $\mathbf{N}_j$ associated with data $\mathbf{c}_j$ is contained in the hypograph of the hyposolution $\mathbf{N}$ associated with data $\mathbf{c}$ (upper hypocontinuity property). If the functions ψ, $\varphi^\star$ and v are furthermore Lipschitz, the hypograph of the hyposolution $\mathbf{N}$ associated with data $\mathbf{c}$ is contained in the lower Painlevé-Kuratowski limit of the hypographs of the solutions $\mathbf{N}_j$ associated with data $\mathbf{c}_j$ (lower hypocontinuity property), so that both the upper and lower limits coincide with the hypograph of the hyposolution $\mathbf{N}$ (hypoconvergence of the solutions, defined in [12] or [76] for instance). These statements follow from Theorem 6.6 of [7] stating that if the system is both Marchaud and Lipschitz, the capture basin of a Painlevé-Kuratowski limit of targets is the Painlevé-Kuratowski limit of the capture basins of the targets.

4 The Lax-Hopf formula

When there is no inequality constraint, we prove that the viability hyposolution can be represented explicitly as a simple maximization problem involving the Fenchel conjugate $\varphi^\star$ defined by (7).

Theorem 4.1 ***The Lax-Hopf Formula.*** *Let us consider the case without inequality constraints and set*

$$\tau(x,u) \;:=\; \inf_{x+tu\notin K} t \quad \text{and} \quad \sigma(t,x,u) \;:=\; \min(t,\tau(x,u))$$

Then the viability hyposolution (12), p.8 can be written

$$\left\{ \begin{array}{l} \mathbf{N}(t,x) \\ = \displaystyle\sup_{\{u\in \mathrm{Dom}(\varphi^\star)\}} \left(\mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \sigma(t,x,u)\varphi^\star(u)\right) + \int_{t-\sigma(t,x,u)}^{t} (\psi(v(\tau)))d\tau \end{array} \right. \tag{11}$$

Using the decomposition $\mathbf{N}(t,x) = \max\left(\mathbf{N}_{N_0}(t,x), \mathbf{N}_\gamma(t,x)\right)$, we derive the more explicit formula

$$\left\{ \begin{array}{lcl} \mathbf{N}_{N_0}(t,x) &=& \displaystyle\sup_{u\in\mathrm{Dom}(\varphi^\star)} \left(\mathbf{N}_0(x+tu) - t\varphi^\star(u)\right) + \int_0^t \psi(v\,(\tau))d\tau \\ \mathbf{N}_\gamma(t,x) &=& \displaystyle\sup_{\{u\in\mathrm{Dom}(\varphi^\star)\mid \tau(x,u)\le t\}} \left[\gamma\left(t-\tau(x,u), x+\tau(x,u)u\right) - \tau(x,u)\varphi^\star(u)\right] \\ \displaystyle + \int_{t-\tau(x,u)}^{t} \psi(v\,(\tau))d\tau && \end{array} \right. \tag{12}$$

involving the initial and Dirichlet condition.

Proof — Let us associate with $u(\cdot)$

$$\tau(x,u(\cdot)) \;:=\; \inf_{x+\int_0^t u(\tau)d\tau\notin K} t \quad \text{and} \quad \sigma(t,x,u(\cdot)) \;=\; \min(t,\tau(x,u(\cdot)))$$

The formula is derived from the general representation formula

$$\left\{ \begin{array}{l} \mathbf{N}(t,x) = \displaystyle\sup_{u(\cdot)} \left(\mathbf{c}\left(t-\sigma(t,x,u(\cdot)), x+\int_0^{\sigma(t,x,u(\cdot))} u(\tau)d\tau\right) - \int_0^{\sigma(t,x,u(\cdot))} \varphi^\star(u(\tau))d\tau\right) \\ \displaystyle + \int_{t-\sigma(t,x,u(\cdot))}^{t} \psi(v(\tau))d\tau \end{array} \right.$$

of the viability hyposolution without constraints given by Corollary 4.6, p.13.

1. Taking constant controls $u(\cdot)\equiv u$ and observing that $\tau(x,u) = \tau(x,u(\cdot))$, we infer that

$$\sup_{u\in\mathrm{Dom}(\varphi^\star)} \left(\mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \sigma(t,x,u)\varphi^\star(u)\right) + \int_{t-\sigma(t,x,u)}^{t} \psi(v(\tau))d\tau \;\le\; \mathbf{N}(t,x)$$

2. Let us associate with $u(\cdot)$ the function $\widehat{u}$ defined by $\widehat{u}(s) := \frac{1}{s}\int_0^s u(\tau)d\tau$. We first observe that

$$\tau(x,u(\cdot)) \;=\; \tau(x,\widehat{u}(\tau(x,u(\cdot))))$$

Since $\varphi^\star$ is convex and lower semicontinuous and ψ is concave and upper semicontinuous, Jensen inequality implies

$$\varphi^{\star}\left(\frac{1}{s}\int_0^s u(\tau)d\tau\right) \;\leq\; \frac{1}{s}\int_0^s \varphi^{\star}(u(\tau))d\tau \quad \text{and} \quad \frac{1}{s}\int_0^s \psi(v(t-\tau))d\tau \;\leq\; \psi\left(\frac{1}{s}\int_{t-s}^t v(\tau)d\tau\right)$$

and thus

$$\int_0^s \psi(v(t-\tau))d\tau - \int_0^s \varphi^{\star}(u(\tau))d\tau \;\leq\; s\left(\psi\left(\frac{1}{s}\int_{t-s}^t v(\tau)d\tau\right) - \varphi^{\star}\left(\widehat{u}(s)\right)\right) \tag{13}$$

Consequently, setting $t^{\sharp} := \sigma(t,x,u(\cdot)) = \tau(x,\widehat{u}(\sigma(t,x,u(\cdot))))$ and $u^{\sharp} := \widehat{u}(t^{\sharp})$, we obtain inequalities

$$\left\{\begin{array}{l} \mathbf{c}\left(t-t^{\sharp}, x+\int_0^{t^{\sharp}} u(\tau)d\tau\right) - \int_0^{t^{\sharp}} \varphi^{\star}(u(\tau))d\tau + \int_{t-t^{\sharp}}^t \psi(v(\tau))d\tau \\ \leq\; \mathbf{c}\left(t-t^{\sharp}, x+t^{\sharp}u^{\sharp}\right) - t^{\sharp}\varphi^{\star}(u^{\sharp}) + \int_{t-t^{\sharp}}^t \psi(v(\tau))d\tau \\ \leq \sup_{\{u\in\mathrm{Dom}(\varphi^{\star})\}} \left(\mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \sigma(t,x,u)\varphi^{\star}(u)\right) + \int_{t-\sigma(t,x,u)}^t \psi(v(\tau))d\tau \end{array}\right.$$

Therefore, by taking the supremum, we obtain

$$\mathbf{N}(t,x) \;\leq\; \sup_{u\in\mathrm{Dom}(\varphi^{\star})} \left(\mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \sigma(t,x,u)\varphi^{\star}(u)\right) + \int_{t-\sigma(t,x,u)}^t \psi(v(\tau))d\tau$$

This completes the proof of the Lax-Hopf inequality. ■

Corollary 4.2 Case of the Traffic Model. *When $X := \mathbb{R}$, $K := [\xi, +\infty[$, ψ is a concave flux function vanishing at density 0 and at a jam density $\omega > 0$ and $\mathbf{N}(t,x)$ is the cumulated number of vehicles at time t and at location $x \in K$, consistency conditions (5), p.5 imply the existence of a unique upper semicontinuous solution $\mathbf{N}(t,x) = \max\left(\mathbf{N}_{N_0}(t,x), \mathbf{N}_{\gamma}(t,x)\right)$ to this problem in the Barron-Jensen/Frankowska sense satisfying the Lax-Hopf formula:*

$$\left\{\begin{array}{lll} \mathbf{N}_{N_0}(t,x) &=& \sup_{u\in\mathrm{Dom}(\varphi^{\star})} \left(\mathbf{N}_0(x+tu) - t\varphi^{\star}(u) + \int_0^t \psi(v\left(t-\tau\right))d\tau\right) \\ \mathbf{N}_{\gamma}(t,x) &=& \sup_{\{u\in\mathrm{Dom}(\varphi^{\star}) \mid u \leq \frac{\xi-x}{t}\}} \left(\gamma\left(t-\frac{\xi-x}{u}, \xi\right) - \frac{\xi-x}{u}\varphi^{\star}(u) + \int_0^{\frac{\xi-x}{u}} \psi(v\left(t-\tau\right))d\tau\right) \end{array}\right. \tag{14}$$

We provide an upper estimate:

Proposition 4.3 ***Upper Estimate of the Viability Hyposolution*** *The viability hyposolution satisfies*

$$\mathbf{N}(t,x) \;\leq\; \sup_{u\in \mathrm{Dom}(\varphi^\star)} \left(\mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \left\langle u, \int_{t-\sigma(t,x,u)}^{t} v(\tau)d\tau \right\rangle \right)$$

Consequently, the viability hyposolution satisfies the following (a posteriori *instead of* a priori*) estimate*

$$\mathbf{N}(t,x) \;\leq\; \sup_{t\geq 0,\; x\in K} \mathbf{c}(t,x) + t\mathrm{Diam}(\mathrm{Dom}(\varphi^\star)) \sup_{t\geq 0} \|v(t)\|$$

(maximum principle)

Proof — Fix $u \in \mathrm{Dom}(\varphi^\star)$ and set $\sigma(t,x,u) =: s$. Definition (7), p.6 of the conjugate function implies

$$\psi\left(\frac{1}{s}\int_{t-s}^{t} v(\tau)d\tau\right) - \varphi^\star(u) \;\leq\; -\left\langle \frac{1}{s}\int_{t-s}^{t} v(\tau)d\tau, u\right\rangle \tag{15}$$

Consequently

$$\begin{cases} \mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \sigma(t,x,u)\varphi^\star(u) + \displaystyle\int_{t-\sigma(t,x,u)}^{t} (\psi(v(\tau)))d\tau \\ \leq\; \mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \left\langle u, \displaystyle\int_{t-\sigma(t,x,u)}^{t} v(\tau)d\tau \right\rangle \\ \leq\; \displaystyle\sup_{w\in \mathrm{Dom}(\varphi^\star)} \left(\mathbf{c}\left(t-\sigma(t,x,w), x+\sigma(t,x,w)w\right) - \left\langle w, \int_{t-\sigma(t,x,w)}^{t} v(\tau)d\tau \right\rangle \right) \end{cases}$$

Taking the supremum over $u \in \mathrm{Dom}(\varphi^\star)$, Lax-Hopf formula (11), p.8 implies the upper estimate

$$\mathbf{N}(t,x) \;\leq\; \sup_{u\in \mathrm{Dom}(\varphi^\star)} \left(\mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \left\langle u, \int_{t-\sigma(t,x,u)}^{t} v(\tau)d\tau \right\rangle \right)$$

This completes the proof. ■

Proposition 4.4 ***Lower Estimate.*** *Assume that $v(t) := v$ is constant and, for simplicity, that the function ψ is differentiable. Then*

$$\mathbf{c}(t-\sigma(t,x,-\psi'(v)), x-\sigma(t,x,-\psi'(v))\psi'(v)) + \sigma(t,x,\psi'(v))\left\langle v, \psi'(v)\right\rangle \;\leq\; \mathbf{N}(t,x)$$

Consequently, the hyposolution is nonnegative on its positivity domain $\mathrm{Dom}_+(\mathbf{N})$*, defined as the subset of pairs $(t,x) \in \mathbb{R}_+ \times K$ such that*

$$\mathbf{c}(t-\sigma(t,x,-\psi'(v)), x-\sigma(t,x,-\psi'(v))\psi'(v)) + \sigma(t,x,-\psi'(v))\left\langle v, \psi'(v)\right\rangle \;\geq\; 0$$

Proof — By definition 9.9, p.28 of the superdifferential,

$$\forall\, u \in \partial_+\psi(v),\;\; \psi(v) - \varphi^\star(-u) \;=\; \langle v, u\rangle$$

Therefore, if ψ is differentiable, taking $u := -\psi'(v)$ the unique element of $-\partial_+\psi(v) = \partial_-\varphi(v)$, Legendre equality $\psi(v) - \varphi^\star(-\psi'(v)) \;=\; \langle v, \psi'(v)\rangle$ yields

$$\left\{\begin{array}{l}\mathbf{c}(t-\sigma(t,x,-\psi'(v)),x-\sigma(t,x,-\psi'(v))\psi'(v))+\sigma(t,x,-\psi'(v))\left\langle v,\psi'(v)\right\rangle\\ =\ \mathbf{c}(t-\sigma(t,x,u),x+\sigma(t,x,u)u)-\sigma(t,x,u)\left\langle v,u\right\rangle\\ =\ \mathbf{c}(t-\sigma(t,x,u),x+\sigma(t,x,u)u)+\sigma(t,x,u)(\psi(v)-\varphi^\star(u))\leq\mathbf{N}(t,x)\end{array}\right.$$

thanks to the Lax-Hopf formula. ■

Theorem 4.5 ***Representation Formula of the viability solution (case with constraints).*** *We already set*

$$\tau(x,u(\cdot))\ :=\ \inf_{x+\int_0^t u(\tau)d\tau\notin K} t \text{ and } \sigma(t,x,u(\cdot))\ =\ \min(t,\tau(x,u(\cdot)))$$

The viability hyposolution can be represented in the form:

$$\left\{\begin{array}{l}\mathbf{N}(t,x)\ =\ \sup_{u(\cdot)}\Bigg[\max\Bigg(\\ \mathbf{c}\left(t-\sigma(t,x,u(\cdot)),x+\int_0^{\sigma(t,x,u(\cdot))}u(\tau)d\tau\right)-\int_0^{\sigma(t,x,u(\cdot))}\varphi^\star(u(\tau))d\tau+\int_{t-\sigma(t,x,u(\cdot))}^t\psi(v(\tau))d\tau,\\ \inf_{s\in[0,\sigma(t,x,u(\cdot))]}\left(\mathbf{b}\left(t-s,x+\int_0^s\right)u(\tau)d\tau\right)-\int_0^s\varphi^\star(u(\tau))d\tau+\int_{t-s}^t\psi(v(\tau))d\tau\Bigg)\Bigg)\Bigg]\end{array}\right. \tag{16}$$

Using the decomposition $\mathbf{N}(t,x)=\max\left(\mathbf{N}_{N_0}(t,x),\mathbf{N}_\gamma(t,x)\right)$, this formula boils down to

$$\left\{\begin{array}{l}\mathbf{N}_{N_0}(t,x)\ =\ \sup_{u(\cdot)}\left[\max\left(\mathbf{N}_0\left(x+\int_0^t u(\tau)d\tau\right)-\int_0^t\varphi^\star(u(\tau))d\tau+\int_0^t\psi(v(\tau))d\tau,\right.\right.\\ \left.\left.\inf_{s\in[0,t]}\left(\mathbf{b}\left(t-s,x+\int_0^s u(\tau)d\tau\right)-\int_0^s\varphi^\star(u(\tau))d\tau+\int_{t-s}^t\psi(v(\tau))d\tau\right)\right]\end{array}\right.$$

and

$$\left\{\begin{array}{l}\mathbf{N}_\gamma(t,x)\ =\ \sup_{\{u(\cdot)|\tau(x,u(\cdot))\leq t\}}\Bigg[\max\Bigg(\\ \gamma\left(t-\tau(x,u(\cdot)),x+\int_0^{\tau(x,u(\cdot))}u(\tau)d\tau\right)-\int_0^{\tau(x,u(\cdot))}\varphi^\star(u(\tau))d\tau+\int_{t-\tau(x,u(\cdot))}^t\psi(v(\tau))d\tau,\\ \inf_{s\in[0,\tau(x,u(\cdot))]}\left(\mathbf{b}\left(t-s,x+\int_0^s u(\tau)d\tau\right)-\int_0^s\varphi^\star(u(\tau))d\tau+\int_{t-s}^t\psi(v(\tau))d\tau\right)\Bigg)\Bigg]\end{array}\right.$$

Proof — We begin by observing that a solution $(\tau(\cdot),x(\cdot),y(\cdot))$ to control system (8) starting from (t,x,y) is given by $\tau(s)=t-s$, $x(s)=x+\int_0^s u(r)dr$ and

$$y(s)\ =\ y+\int_0^s(\varphi^\star(u(r))-\psi(v(t-r)))dr$$

for some $u(\cdot)$.
Therefore, to say that (t,x,y) belongs to the capture basin $\text{Capt}_{(8)}(\mathcal{K},\mathcal{C})$ amounts to saying that there exist a solution $(\tau(\cdot),x(\cdot),y(\cdot))$ to the characteristic control system (8) starting from (t,x,y) and $t^\star\in[0,t]$ such that

1. $(t - t^\star, x(t^\star), y(t^\star))$ belongs to the target $\mathcal{C}$, i.e., such that

$$y(t^\star) := y + \int_0^{t^\star} (\varphi^\star(u(\tau)) - \psi(v(t-\tau)))d\tau \ \le \ \mathbf{c}\,(t - t^\star, x(t^\star)) \ = \ \mathbf{c}\left(t - t^\star, x + \int_0^{t^\star} u(\tau)d\tau\right)$$

2. $\forall\, s \in [0, t^\star]$, $(t - s, x(s), y(s))$ belongs to the environment $\mathcal{K}$, i.e., such that

$$y(s) = y + \int_0^{s} (\varphi^\star(u(\tau)) - \psi(v(t-\tau)))d\tau \ \le \ \mathbf{b}(t - s, x(s)) \ = \ \mathbf{b}\left(t - s, x + \int_0^{s} u(\tau)d\tau\right)$$

This implies that

$$y \ \le \min \left(\begin{array}{l} \mathbf{c}\left(t - t^\star, x + \int_0^{t^\star} u(\tau)d\tau\right) - \int_0^{t^\star} (\varphi^\star(u(\tau)) - \psi(v(t-\tau)))d\tau, \\ \inf_{s\in[0,t^\star]} \mathbf{b}\left(t - s, x + \int_0^{s} u(\tau)d\tau\right) - \int_0^{s} (\varphi^\star(u(\tau)) - \psi(v(t-\tau)))d\tau \end{array} \right)$$

Since y is finite, this implies that $\mathbf{c}\left(t - t^\star, x + \int_0^{t^\star} u(\tau)d\tau\right)$ must be finite, and thus, that

1. either $t - t^\star = 0$, in which case $\mathbf{c}\left(t - t^\star, x + \int_0^{t^\star} u(\tau)d\tau\right) = \mathbf{N}_0\left(x + \int_0^{t} u(\tau)d\tau\right)$
2. or $x(t^\star) \in \partial K$, which means that $t^\star = \tau(x, u(\cdot)) = \sigma(t, x, u(\cdot)) \le t$, in which case $\mathbf{c}\left(t - t^\star, x + \int_0^{t^\star} u(\tau)d\tau\right) = \gamma\left(t - \sigma(t, x, u(\cdot)), x + \int_0^{\sigma(t,x,u(\cdot))} u(\tau)d\tau\right)$.

This implies that $\mathbf{N}(t, x) \le \mathbf{V}(t, x)$ where

$$\left\{ \begin{array}{l} \mathbf{V}(t,x) \ = \ \sup_{u(\cdot)} (\max (\\ \mathbf{c}\left(t - \sigma(t,x,u(\cdot)), x + \int_0^{\sigma(t,x,u(\cdot))} u(\tau)d\tau\right) - \int_0^{\sigma(t,x,u(\cdot))} \varphi^\star(u(\tau))d\tau + \int_{t-\sigma(t,x,u(\cdot))}^{t} \psi(v(\tau))d\tau, \\ \inf_{s\in[0,\sigma(t,x,u(\cdot))]} \left(\mathbf{b}\left(t - s, x + \int_0^{s} u(\tau)d\tau\right) - \int_0^{s} \varphi^\star(u(\tau))d\tau + \int_{t-s}^{t} \psi(v(\tau))d\tau \right) \Big) \end{array} \right.$$

For proving the converse inequality, we associate with every $\varepsilon > 0$ a control $t \mapsto u_\varepsilon(t) \in \mathrm{Dom}(\varphi^\star)$ such that

$$\left\{ \begin{array}{l} \mathbf{V}(t,x) - \varepsilon \ \le \ \max (\\ \mathbf{c}\left(t - \sigma(t,x,u_\varepsilon(\cdot)), x + \int_0^{\sigma(t,x,u_\varepsilon(\cdot))} u_\varepsilon(\tau)d\tau\right) - \int_0^{\sigma(t,x,u_\varepsilon(\cdot))} \varphi^\star(u_\varepsilon(\tau))d\tau + \int_{t-\sigma(t,x,u_\varepsilon(\cdot))}^{t} \psi(v(\tau))d\tau, \\ \inf_{s\in[0,\sigma(t,x,u_\varepsilon(\cdot))]} \left(\mathbf{b}\left(t - s, x + \int_0^{s} u(\tau)d\tau\right) - \int_0^{s} \varphi^\star(u_\varepsilon(\tau))d\tau + \int_{t-s}^{t} \psi(v(\tau))d\tau \right) \Big) \end{array} \right.$$

Therefore, setting $x_\varepsilon(t) := x + \int_0^t u_\varepsilon(s)ds$ and

$$y_\varepsilon(t) \;:=\; \mathbf{V}(t,x) - \varepsilon + \int_0^t (\varphi^\star(u_\varepsilon(r)) - \psi(v(t-r)))\,dr$$

we observe that the function $s \mapsto (t-s, x_\varepsilon(s), y_\varepsilon(s))$ starts from $(t, x, \mathbf{V}(t,x) - \varepsilon)$, is a solution to characteristic control system (8), viable in $\mathcal{K}$ for $s \leq \sigma(t, x, u_\varepsilon(\cdot))$ because

$$y_\varepsilon(s) \;=\; \mathbf{V}(t,x) - \varepsilon + \int_0^s (\varphi^\star(u_\varepsilon(r)) - \psi(v(t-r))dr) \;\leq\; \mathbf{b}\,(t-s, x_\varepsilon(s))$$

and reaching the target $\mathcal{C} := \mathcal{H}yp(\mathbf{c})$ at time $t_\varepsilon := \sigma(t, x, u_\varepsilon(\cdot))$,

$$y_\varepsilon(t_\varepsilon) \;=\; \mathbf{V}(t,x) - \varepsilon + \int_0^{t_\varepsilon} (\varphi^\star(u_\varepsilon(r)) - \psi(v(t-r)))\,dr \;\leq\; \mathbf{c}\,(t - t_\varepsilon, x_\varepsilon(t_\varepsilon))$$

This implies that $(t, x, \mathbf{V}(t,x) - \varepsilon)$ belongs to the capture basin $\mathrm{Capt}_{(8)}(\mathcal{K}, \mathcal{C})$, and thus, that $\mathbf{V}(t,x) - \varepsilon \leq \mathbf{N}(t,x)$. Letting ε converge to 0 provides the converse inequality, and thus, the representation formula we were looking for. ■

Corollary 4.6 ***Representation Formula of the viability solution (case without constraints).*** *Without inequality constraints, the viability hyposolution can be represented in the form:*

$$\begin{cases} \mathbf{N}(t,x) \;=\; \sup\limits_{u(\cdot)} \left(\displaystyle\int_{t-\sigma(t,x,u(\cdot))}^{t} \psi(v(\tau))d\tau \right. \\ \left. +\mathbf{c}\left(t - \sigma(t,x,u(\cdot)), x + \displaystyle\int_0^{\sigma(t,x,u(\cdot))} u(\tau)d\tau\right) - \displaystyle\int_0^{\sigma(t,x,u(\cdot))} \varphi^\star(u(\tau))d\tau \right) \end{cases}$$

5 Dirichlet/Initial Conditions and Inequality Constraints

We begin by checking that the viability hyposolution satisfies the initial condition, the Dirichlet condition and the inequality constraints:

Theorem 5.1 Dirichlet/Initial Conditions and Inequality Constraints. *Consistency conditions (1), p.5 imply that the viability hyposolution satisfies the initial and Dirichlet conditions (3), p.5 and inequality contraints (4), p.5.*

Proof — Inclusions

$$\mathcal{C} \;:=\; \mathcal{H}yp(\mathbf{c}) \;\subset\; \mathrm{Capt}_{(8)}(\mathcal{K}, \mathcal{C}) \;\subset\; \mathcal{K} \;:=\; \mathcal{H}yp(\mathbf{b})$$

imply that

$$\forall\, t \geq 0,\; \forall\, x \in K,\;\; \mathbf{c}(t,x) \;\leq\; \mathbf{N}(t,x) \;\leq\; \mathbf{b}(t,x)$$

and thus inequality constraint $\mathbf{N}(t,x) \leq \mathbf{b}(t,x)$ and inequalities $\mathbf{N}_0(x) \leq \mathbf{N}(0,x)$ for all $x \in K$ and $\gamma(t,x) \leq \mathbf{N}(t,x)$ for all $t \geq 0$ and $x \in \partial K$. We now prove by contradiction that consistency conditions (1), p.5 imply converse inequalities $\mathbf{N}_0(x) \geq \mathbf{N}(0,x)$ for all $x \in K$ and $\gamma(t,x) \geq \mathbf{N}(t,x)$ for all $t \geq 0$ and $x \in \partial K$ that we summarize in

$$\forall\ (t,x) \in \mathrm{Dom}(\mathbf{c}),\ \ \mathbf{N}(t,x) \leq \mathbf{c}(t,x)$$

Assume that there exist $(t,\xi) \in \mathrm{Dom}(\mathbf{c})$ and $\varepsilon > 0$ such that

$$\mathbf{N}(t,\xi)\ =\ \mathbf{c}(t,\xi) + \varepsilon$$

Since $(t,\xi,\mathbf{N}(t,\xi))$ belongs to the capture basin $\mathrm{Capt}_{(8)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c}))$, there exist a solution $(\tau(\cdot),x(\cdot),y(\cdot))$ to the characteristic control system (8) starting from $(t,\xi,\mathbf{N}(t,\xi))$ and $t^\star > 0$ such that $(t-t^\star, x(t^\star), y(t^\star))$ belongs to the hypograph $\mathcal{H}yp(\mathbf{c})$: Setting $x(t^\star) = \xi + \int_0^{t^\star} u(\tau)d\tau = \eta$, we obtain

$$y(t^\star)\ =\ \mathbf{N}(t,\xi) + \int_0^{t^\star} \varphi^\star(u(\tau))d\tau - \int_0^{t^\star} \psi(v(t-\tau))d\tau\ \leq\ \mathbf{c}(t-t^\star,\eta)$$

Inequality (13), p.9 and the definition (15), p.10 imply

$$\int_0^s \psi(v(t-\tau))d\tau - \int_0^s \varphi^\star(u(\tau))d\tau \leq\ -\left\langle \frac{1}{s}\int_{t-s}^t v(\tau)d\tau, \widehat{u}(s)\right\rangle \tag{17}$$

Piecing these inequalities together and taking $s = t^\star$, we infer that

$$\left\{\begin{array}{l} \mathbf{c}(t,\xi) + \varepsilon + \left\langle \frac{1}{t^\star}\int_{t-t^\star}^t v(\tau)d\tau, \eta - \xi\right\rangle \\ \leq\ \mathbf{N}(t,\xi) + \int_0^{t^\star} \varphi^\star(u(\tau))d\tau - \int_0^{t^\star} \psi(v(t-\tau))d\tau \leq\ \mathbf{c}(t-t^\star,\eta) \end{array}\right.$$

from which we deduce that

$$\varepsilon\ \leq\ \mathbf{c}(t-t^\star,\eta) - \mathbf{c}(t,\xi) - \left\langle \frac{1}{t^\star}\int_{t-t^\star}^t v(\tau)d\tau, \eta - \xi\right\rangle$$

Consistency conditions (1), p.5 can be written in the form

$$\forall\ 0 \leq r \leq s,\ \forall\ x \in K,\ y \in \partial K,\ \ \mathbf{c}(r,x) - \mathbf{c}(s,y)\ \leq\ \left\langle \frac{1}{s-r}\int_r^s v(\tau)d\tau, x - y\right\rangle$$

Taking $r := t - t^\star$, $s := t$, $x := \eta$ and $y := \xi$, we obtain the contradiction $\varepsilon \leq 0$, and thus, we proved that for any $(t,\xi) \in \mathrm{Dom}(\mathbf{c})$, $\mathbf{N}(t,\xi) = \mathbf{c}(t,\xi)$. ■

6 Other Auxiliary Systems

For proving that the viability hyposolution is the solution in a generalized sense to the Hamilton-Jacobi partial differential equation derived from the tangential or normal conditions characterizing capture basins, we need assumptions that control system (8), p.6 does not satisfy. It happens that the capture basin of the hypograph of $\mathbf{c}$ viable in the hypograph of $\mathbf{b}$ under control system (8) is still the capture basin under other auxiliary systems which satisfy these assumptions, so that we shall be able to transfer the theorems concerning capture basins.

The function ψ being concave and finite, it is then continuous, so that, the function $v(\cdot)$ being bounded, the constant

$$\alpha \;:=\; \sup_{u\in\mathrm{Dom}(\varphi^\star)} \varphi^\star(u) - \inf_{\tau\geq 0}\psi(v(\tau))$$

is finite by Lemma 6.3, p.17 below. The new characteristic control systems are defined by

$$\left\{\begin{array}{lllll} \tau'(t) & = & -1 & & \\ x'(t) & = & u(t) & \text{where } u(t) & \in \; \mathrm{Dom}(\varphi^\star) \\ y'(t) & = & -\psi(v(\tau(t)))+\varphi^\star(u(t))+\pi(t) & \text{where } \pi(t) & \in \; [0,\alpha+\psi(v(\tau(t))-\varphi^\star(u(t))] \end{array}\right. \qquad (18)$$

and

$$\left\{\begin{array}{lllll} \tau'(t) & = & -1 & & \\ x'(t) & = & u(t) & \text{where } u(t) & \in \; \mathrm{Dom}(\varphi^\star) \\ y'(t) & = & -\psi(v(\tau(t)))+\varphi^\star(u(t))+\pi(t) & \text{where } \pi(t) & \geq \; 0 \end{array}\right. \qquad (19)$$

where we added a new control π ranging in different intervals.

Lemma 6.1 Equality between Capture Basins. *The capture basins of the hypograph of the function* **c** *by systems (8), (18) and (19) coincide:*

$$\mathrm{Capt}_{(19)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c})) \;=\; \mathrm{Capt}_{(18)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c})) \;=\; \mathrm{Capt}_{(8)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c}))$$

Furthermore,

$$\mathrm{Capt}_{(18)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c})) \;=\; \mathrm{Capt}_{(18)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c})) - \{0\}\times\{0\}\times\mathbb{R}_+$$

Proof — Inclusions

$$\mathrm{Capt}_{(8)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c})) \;\subset\; \mathrm{Capt}_{(18)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c})) \;\subset\; \mathrm{Capt}_{(19)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c}))$$

are obvious. For proving that

$$\mathrm{Capt}_{(19)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c})) \;\subset\; \mathrm{Capt}_{(8)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c}))$$

let us consider an element $(t,x,y) \in \mathrm{Capt}_{(19)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c}))$. Hence there exist $u(\cdot) \in L^1(0,+\infty;\mathrm{Dom}(\varphi^\star))$ and a corresponding solution $(\tau(\cdot),x(\cdot),y(\cdot))$ to the characteristic control system (19) starting from (t,x,y) is given by $\tau(s)=t-s$, $x(s)=x+\int_0^s u(r)dr$ and

$$y(s) \;\geq\; y - \int_0^s \left(\psi(v(t-r)) - \varphi^\star(u(r))\right)dr$$

and there exists $t^\star\in[0,t]$ such that $(t-t^\star,x(t^\star),y(t^\star))\in\mathcal{H}yp(\mathbf{c})$ and, for all $s\in[0,t^\star]$, $(t-s,x(s),y(s))\in\mathcal{H}yp(\mathbf{b})$. Setting

$$y_0(s) \;:=\; y + \int_0^s \left(\varphi^\star(u(r)) - \psi(v(t-r))\right)dr$$

we infer that $(\tau(\cdot),x(\cdot),y_0(\cdot))$ is a solution to the to the characteristic control system (8) starting from (t,x,y) viable in the environment $\mathcal{H}yp(\mathbf{b})$ because

$$\forall\, s \in [0, t^\star],\;\; y_0(s) \;\leq\; y(s) \;\leq\; \mathbf{b}(t-s, x(s))$$

until time $t^\star$ where it reaches the target $\mathcal{H}yp(\mathbf{c})$ because

$$y_0(t^\star) \;\leq\; y(t^\star) \;\leq\; \mathbf{c}(t-t^\star, x(t^\star))$$

This means that $(t,x,y) \in \mathrm{Capt}_{(8)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c}))$.

We also observe that whenever let us consider an element $(t,x,y) \in \mathrm{Capt}_{(19)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c}))$ and $z \leq y$, (t,x,z) also belongs to the capture basin, so that,

$$\mathrm{Capt}_{(18)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})) \;=\; \mathrm{Capt}_{(18)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})) - \{0\} \times \{0\} \times \mathbb{R}_+$$

The proof is completed. ■

We also need

Lemma 6.2 *Let $\psi : X \mapsto \mathbb{R}$ be an upper semicontinuous concave function.*
The domain of its Fenchel transform $\varphi^\star$ is contained in a closed convex subset A if and only if the function ψ satisfies inequality

$$\exists\, \beta \in \mathbb{R} \;\; \textit{such that} \; \forall\, v \in X,\;\; \beta - \sigma_A(v) \;\leq\; \psi(v)$$

Its Fenchel transform $\varphi^\star$ is bounded on a convex subset A if and only if the function ψ satisfies

$$\exists\, \delta \in \mathbb{R} \;\; \textit{such that} \; \forall\, v \in X^\star,\;\; \psi(v) \;\leq\; \delta - \sigma_A(v)$$

Proof — Since $\psi(0) = \inf_{u \in \mathrm{Dom}(\varphi^\star)} \varphi^\star(u)$, we infer that

$$\forall\, v \in X,\;\; \forall\, u \in \mathrm{Dom}(\varphi^\star),\;\; \psi(0) - \sigma_{\mathrm{Dom}(\varphi^\star)}(v) \;\leq\; \varphi^\star(u) - \langle u, v\rangle$$

so that, by taking the infimum over u, we obtain inequality $\psi(0) - \sigma_{\mathrm{Dom}(\varphi^\star)}(v) \leq \psi(v)$. It is enough to set $\beta := \psi(0)$ and to take $A := \mathrm{Dom}(\varphi^\star)$. Conversely, assume that $\forall\, v \in X,\;\; \psi(v) \;\geq\; \beta - \sigma_A(v)$. We shall prove that $\mathrm{Dom}(\varphi^\star) \subset A$. If not, there would exist $u \in \mathrm{Dom}(\varphi^\star) \setminus A$. The Separation Theorem states there exist $p_0 \in X$ and $\varepsilon > 0$ such that $\varepsilon \leq \langle p_0, u\rangle - \sigma_A(p_0)$. Consequently, for every $\lambda > 0$,

$$\lambda\varepsilon \;\leq\; \langle \lambda p_0, u\rangle - \sigma_A(\lambda p_0) \leq\; \langle \lambda p_0, u\rangle + \psi(\lambda p_0) - \beta \;\leq\; \varphi^\star(u) - \beta$$

by assumption and by the definition of $\varphi^\star$. Letting $\lambda \mapsto +\infty$ implies that $\varphi^\star(u) = +\infty$, i.e., that $u \notin \mathrm{Dom}(\varphi^\star)$, a contradiction.

For proving the second statement, we observe that if $\delta := \sup_{u \in \mathrm{Dom}(\varphi^\star)} \varphi^\star(u) < +\infty$ is finite, then

$$\psi(v) \;\leq\; \delta + \inf_{u \in \mathrm{Dom}(\varphi^\star)} \langle v, -u\rangle \;=\; \delta - \sigma_{\mathrm{Dom}(\varphi^\star)}(v)$$

so that the inequality holds true with $A := \mathrm{Dom}(\varphi^\star)$. Conversely, inequality $\psi(v) \leq \delta - \sigma_A(v)$ implies that

$$\forall\, u \in A,\;\; \varphi^\star(u) \;\leq\; \sup_{v \in \mathrm{Dom}(\psi)} \left[\langle v, u\rangle + \delta - \sigma_A(v)\right] \;=\; c$$

is bounded on A, and thus, on $\mathrm{Dom}(\varphi^\star)$ whenever this domain is contained in A. ■

Control systems (18) and (19) p.15 are actually differential inclusions

$$(\tau'(t), x'(t), y'(t)) \;\in\; F(\tau(t), x(t), y(t))$$

where

$$F(\tau, x, y) \;:=\; \{(-1, u, -\psi(v(\tau)) + \varphi^\star(u) + \pi)\}_{u \in \mathrm{Dom}(\varphi^\star),\ \pi \in [0, \alpha + \psi(v(\tau)) - \varphi^\star(u)]} \tag{20}$$

$$(\tau'(t), x'(t), y'(t)) \;\in\; F_\infty(\tau(t), x(t), y(t))$$

and where

$$F_\infty(\tau, x, y) \;:=\; \{(-1, u, -\psi(v(\tau)) + \varphi^\star(u) + \pi)\}_{u \in \mathrm{Dom}(\varphi^\star),\ \pi \geq 0} \tag{21}$$

Lemma 6.3 *The set-valued map F is Marchaud and, if the functions ψ, $\varphi^\star$ and v are Lipschitz, the set-valued map F_∞ is Lipschitz with closed images.*

Proof — For proving that the set-valued map F is Marchaud, we shall check successively that:

1. *The values $F(\tau, x, y)$ of the set-valued map F are convex*: Indeed, for convex weight $\lambda_i \geq 0$ such that $\sum \lambda_i = 1$, we can write

$$\sum \lambda_i(-1, u_i, -\psi(v(\tau)) + \varphi^\star(u_i) + \pi_i) \;=\; (-1, \overline{u}, \varphi^\star(\overline{u}) - \psi(v(\tau)) + \overline{\pi})$$

where $\overline{u} := \sum \lambda_i u_i$ and

$$\overline{\pi} \;:=\; \sum \lambda_i \varphi^\star(u_i) - \varphi^\star\left(\sum \lambda_i u_i\right) + \sum \lambda_i \pi_i$$

Since the domain of $\varphi^\star$ is convex, $\overline{u} \in \mathrm{Dom}(\varphi^\star)$. We observe that $\overline{\pi}$ is non negative and smaller than or equal to $\alpha + \psi(v(\tau)) - \varphi^\star(\overline{u})$ because

$$\begin{cases} \overline{\pi} \;\leq\; \sum \lambda_i \varphi^\star(u_i) - \varphi^\star\left(\sum \lambda_i u_i\right) + \sum \lambda_i \left(\alpha + \psi(v(\tau)) - \varphi^\star(u_i)\right) \\ \;=\; \alpha + \psi(v(\tau)) - \varphi^\star\left(\sum \lambda_i u_i\right) \end{cases}$$

2. *The graph of the set-valued map F is closed:* Indeed, let us consider a sequence of elements $((\tau_n, x_n, y_n), (-1, u_n, \lambda_n))$ of the graph of F converging to $((\tau, x, y), (-1, u, \lambda))$ where $\lambda_n := -\psi(v(\tau_n)) + \varphi^\star(u_n) + \pi_n$ and where $\pi_n \in [0, \alpha + \psi(v(\tau_n)) - \varphi^\star(u_n)]$.

   Since the function $(\tau, x, y, u) \mapsto \varphi^\star(u) - \psi(v(\tau))$ is lower semicontinuous and since

$$(\tau_n, x_n, y_n, u_n, \lambda_n) \;=\; (\tau_n, x_n, y_n, u_n, -\psi(v(\tau_n)) + \varphi^\star(u_n) + \pi_n)$$

   belongs to the epigraph of this function (because π_n is positive by construction) which is closed, we deduce that the limit (τ, x, y, u, λ) also belongs to this epigraph, i.e., that $\lambda \geq \varphi^\star(u) - \psi(v(\tau))$. It is enough to set $\pi := \lambda - \varphi^\star(u) - \psi(v(\tau)) \geq 0$, which from now on defines π. Recall that $\pi_n = \lambda_n + \psi(v(\tau_n)) - \varphi^\star(u_n) \leq \alpha + \mathbf{l}(\tau_n, x_n) - \varphi^\star(u_n)$ by construction of π_n. Therefore, $\lambda_n \leq \alpha$. Therefore, taking the limit, $\lambda = \pi + \varphi^\star(u) - \psi(v(\tau)) \leq \alpha$. In summary, the limit $((\tau, x, y), (-1, u, \lambda))$ of elements $((\tau_n, x_n, y_n), (-1, u_n, \lambda_n))$ belongs to the graph of F since $\lambda = -\psi(v(\tau)) + \varphi^\star(u) + \pi$ where $\pi \in [0, \alpha + \psi(v(\tau)) - \varphi^\star(u)]$.

3. *The images $F(\tau, x, y)$ of F are bounded:* This follows from Lemma 6.2, p.16 because $\text{Dom}(\varphi^\star)$ is bounded and
$$\varphi^\star(u) - \psi(v(\tau)) + \pi \ \leq \ \alpha \ := \sup_{u \in \text{Dom}(\varphi^\star)} \varphi^\star(u) - \inf_{\tau \geq 0} \psi(v(\tau))$$
is finite since $\varphi^\star$ is bounded above. Therefore
$$\|(-1, u, \varphi^\star(u) - \psi(v(\tau)) + \pi)\| \leq \max(1, \|\text{Dom}(\varphi^\star)\|, \alpha)$$

Hence, we have proved that the set-valued map F is Marchaud. The Lipschitziannity of F_∞ is obvious. ■

We thus deduce that

Proposition 6.4 Upper semicontinuity of the Solution. *The viability hyposolution is upper semicontinuous and its hypograph satisfies:*
$$\mathcal{H}yp(\mathbf{N}) \ = \ \text{Capt}_{(18)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})) \ = \ \text{Capt}_{(19)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c}))$$
The viability hyposolution is concave whenever the functions $\mathbf{b}$ *and* $\mathbf{c}$ *are concave.*

Proof — The first statement follows from Proposition 4.3 of [5] stating that under a Marchaud control system, the capture basin of a target is closed whenever the target $\mathcal{H}yp(\mathbf{c})$ and the environment $\mathcal{H}yp(\mathbf{b})$ are closed and the complement of the target in the environment is a repeller: This is the case because the first component of the system is $\tau'(t) = -1$ implies that all solutions $(t - s, x(s), y(s))$ starting from any (t, x, y) leave $\mathbb{R}_+ \times X \times \mathbb{R}$, and thus, $\mathcal{H}yp(\mathbf{b}) \subset \mathbb{R}_+ \times X \times \mathbb{R}$. Since we have proved that
$$\text{Capt}_{(18)}(\mathcal{H}yp(\mathbf{b}, \mathcal{H}yp(\mathbf{c}))) \ = \ \text{Capt}_{(18)}(\mathcal{H}yp(\mathbf{b}, \mathcal{H}yp(\mathbf{c}))) - \{0\} \times \{0\} \times \mathbb{R}_+$$
we infer that $\text{Capt}_{(18)}(\mathcal{H}yp(\mathbf{b}, \mathcal{H}yp(\mathbf{c})))$ is an hypograph, and thus, the hypograph of the viability hyposolution. ■

7 Contingent Frankowska Solution to the Hamilton-Jacobi Equation

We shall prove that the viability hyposolution to Hamilton-Jacobi equation (2), p.5 (see Definition 3.1, p.6) is the *contingent Frankowska solution* by characterizing them in terms of tangent cones and translating them in terms of contingent Frankowska hyposolutions.

Theorem 7.1 Contingent Frankowska Solution. *The viability hyposolution* $\mathbf{N}$ *is the **largest** upper semicontinuous solution satisfying*

$$\psi(v(t)) \;\geq\; \inf_{u\in \mathrm{Dom}(\varphi^\star)} (\varphi^\star(u) - D_\downarrow \mathbf{N}(t,x)(-1,u)) \tag{22}$$

and the initial/Dirichlet conditions and the inequality constraints. If the functions ψ*,* $\varphi^\star$ *and* v *are furthermore Lipschitz, then* $\mathbf{N}$ *is the **smallest** upper semi continuous solution satisfying*

1. *If* $\mathbf{N}(t,x) < \mathbf{b}(t,x)$*, then*

$$\psi(v(t)) \;\leq\; \inf_{u\in \mathrm{Dom}(\varphi^\star)} (D_\downarrow \mathbf{N}(t,x)(1,-u) + \varphi^\star(u)) \tag{23}$$

2. *If* $\mathbf{N}(t,x) = \mathbf{b}(t,x)$*, then*

$$\psi(v(t)) \;\leq\; \inf_{\{u|\psi(v(t))\,\leq\, D_\downarrow \mathbf{b}(t,x)(1,-u)+\varphi^\star(u)\}} (D_\downarrow \mathbf{N}(t,x)(1,-u) + \varphi^\star(u)) \tag{24}$$

Proof — Observe first that

$$(t,x,y) \;\in\; \mathcal{H}yp(\mathbf{N})\backslash\mathcal{H}yp(\mathbf{c}) \text{ if and only if } t>0,\; x\in\partial K \text{ and } y \;\leq\; \mathbf{N}(t,x)$$

Indeed, $\mathcal{H}yp(\mathbf{N})\backslash\mathcal{H}yp(\mathbf{c})$ is the set of (t,x,y) such that $\mathbf{c}(t,x) < y \leq \mathbf{N}(t,x)$. This is automatically satisfied when $t>0$ and $x\in\partial K$ whenever $y\leq \mathbf{N}(t,x)$ since in this case, $\mathbf{c}(t,x) = -\infty$. It is impossible otherwise since, by Theorem 5.1, p.13, $\mathbf{N}(t,x) = \mathbf{c}(t,x)$.
Theorem 4.6 of [7] states that since F is Marchaud by Lemma 6.3, the capture basin is the largest closed subset between the hypograph of $\mathbf{c}$ and $\mathbb{R}_+ \times X \times \mathbb{R}$ such that $\mathcal{H}yp(\mathbf{N})\backslash\mathcal{H}yp(\mathbf{c})$ is locally viable under F.

Theorems 3.2.4 and 3.3.4 of [7] state that $\mathcal{H}yp(\mathbf{N})\backslash\mathcal{H}yp(\mathbf{c})$ is locally viable under F if and only if $\forall t > 0, \forall x\in X, \forall\, y\leq \mathbf{N}(t,x),\;\; \exists u\in \mathrm{Dom}(\varphi^\star),\; \exists \pi \in [0, \alpha + \psi(v(\tau)) - \varphi^\star(u)]$, such that

$$(-1, u, -\psi(v(t)) + \varphi^\star(u) + \pi) \;\in\; T_{\mathcal{H}yp(\mathbf{N})}(t,x,y)$$

If $y = \mathbf{N}(t,x)$, then

$$T_{\mathcal{H}yp(\mathbf{N})}(t,x,\mathbf{N}(t,x)) \;=:\; \mathcal{H}yp(D_\downarrow \mathbf{N}(t,x))$$

so that we infer that there exists $u\in \mathrm{Dom}(\varphi^\star)$

$$-\psi(v(t)) + \varphi^\star(u) + \pi \;\leq\; D_\downarrow \mathbf{N}(t,x)(-1,u)$$

from which inequality (22) ensues.

Conversely, since $D_\downarrow \mathbf{N}(t,x)(-1,\cdot)$ is upper semicontinuous and the support of $\varphi^\star$ is compact, inequality (22) implies the existence of $u\in \mathrm{Dom}(\varphi^\star)$ such that

$$(-1, u, -\psi(v(t)) + \varphi^\star(u) + \pi) \;\in\; T_{\mathcal{H}yp(\mathbf{N})}(t,x,\mathbf{N}(t,x))$$

When $y < \mathbf{N}(t,x)$, then Lemma 9.10 implies that

$$(-1, u, -\psi(v(t)) + \varphi^\star(u) + \pi) \;\in\; T_{\mathcal{H}yp(\mathbf{N})}(t,x,y)$$

because $(-1,u)$ belongs to the domain of $D_{\downarrow}\mathbf{N}(t,x)$.

By Theorems 4.7 and 4.10 of [7], the capture basin is the smallest closed subset between the hypographs of $\mathbf{c}$ and $\mathbf{c}$ such that $\mathcal{H}yp(\mathbf{N})$ is backward invariant with respect to $\mathcal{H}yp(\mathbf{b})$. Since F_∞ is Lipschitz by Lemma 6.3 whenever the functions ψ, $\varphi^\star$ and v are Lipschitz, the invariance theorem (Theorem 5.3.4 in [5]) states that $\mathcal{H}yp(\mathbf{N})$ is backward invariant with respect to $\mathcal{H}yp(\mathbf{b})$ under F_∞ if and only if

$$\forall\ (t,x,y)\in\mathcal{H}yp(\mathbf{N}),\ \ F_\infty(t,x,y)\cap T_{\mathcal{H}yp(\mathbf{b})}(t,x,y)\ \subset\ T_{\mathcal{H}yp(\mathbf{N})}(t,x,y)$$

Since the function $\mathbf{b}$ is assumed to be continuous,

$$\mathrm{Int}(\mathcal{H}yp(\mathbf{N}))\ =\ \{(t,x,y)\ \text{ such that } y\ <\ \mathbf{b}(t,x)\}$$

Therefore, we have to investigate two cases:

1. either for all $(t,x,y)\in\mathcal{H}yp(\mathbf{N})\cap\mathrm{Int}(\mathcal{H}yp(\mathbf{b}))$.

   Then

   $\forall t\geq 0,\forall x\in X,\forall\ y\leq\mathbf{N}(t,x),\ \ \forall\ u\in\mathrm{Dom}(\varphi^\star),\ \forall\ \pi\in[0,\alpha+\psi(v(\tau))-\varphi^\star(u)],$

   $$(1,-u,\psi(v(t))-\varphi^\star(u)-\pi)\ \in\ T_{\mathcal{H}yp(\mathbf{N})}(t,x,y)$$

   If $y=\mathbf{N}(t,x)$, then so that we infer that for all $u\in\mathrm{Dom}(\varphi^\star)$

   $$\psi(v(t))-\varphi^\star(u)\ \leq\ D_{\downarrow}\mathbf{N}(t,x)(1,-u)$$

   from which we derive inequality (23). Conversely, since for all $u\in\mathrm{Dom}(\varphi^\star)$, $(1,-u)$ belongs to the domain of $D_{\downarrow}\mathbf{N}(t,x)$, we derive that

   $$(1,-u,\psi(v(t))-\varphi^\star(u)-\pi)\ \in\ T_{\mathcal{H}yp(\mathbf{N})}(t,x,y)$$

   holds true.

2. or for all $(t,x,y)\in\mathcal{H}yp(\mathbf{N})\cap\partial(\mathcal{H}yp(\mathbf{b}))$, and in this case, $y=\mathbf{N}(t,x)=\mathbf{b}(t,x)$. Then,

   $\forall t\geq 0,\forall x\in X,\ \ \forall\ u\in\mathrm{Dom}(\varphi^\star),\forall\ \pi\geq 0$ such that

   $$(1,-u,\psi(v(t))-\varphi^\star(u)-\pi)\ \in\ T_{\mathcal{H}yp(\mathbf{b})}(t,x,y)$$

   we have

   $$(1,-u,\psi(v(t))-\varphi^\star(u)-\pi)\ \in\ T_{\mathcal{H}yp(\mathbf{N})}(t,x,y)$$

   This means that whenever

   $$\psi(v(t))\ \leq\ D_{\downarrow}\mathbf{b}(t,x)(1,-u)+\varphi^\star(u)$$

   then

   $$\psi(v(t))\ \leq\ D_{\downarrow}\mathbf{N}(t,x)(1,-u)+\varphi^\star(u)$$

   which is (24), p.19. ■

Theorem 4.5, p.11 states that the viability hyposolution is the valuation function 16, p.11 of the underlying optimal control problem (8), p.6.
The *associated regulation map* R for regulating the optimal evolutions is thus defined by

$$\forall\, t > 0,\; x \in X,\;\; R(t,x) \;:=\; \{u \mid 0 \;\le\; D_{\downarrow}\mathbf{N}(t,x)(-1,u) - \varphi^{\star}(u) + \psi(v(t))\}$$

One can prove that the optimal solutions of the control problem are governed by the control system

$$\begin{cases} \tau'(s) \;=\; -1 \\ x'(s) \;=\; u(s) \;\in\; R(\tau(s), x(s)) \\ y'(s) \;=\; \varphi^{\star}(u(s)) - \psi(v(\tau(s))) \end{cases}$$

This motivates a further study of the regulation map. If the solution $\mathbf{N}$ is differentiable, the regulation map can be written in the form

$$R(t,x) \;:=\; \left\{u \mid \;0 \;\le\; -\frac{\partial \mathbf{N}(t,x)}{\partial t} + \frac{\partial \mathbf{N}(t,x)}{\partial x} u - \varphi^{\star}(u) + \psi(v(t))\right\}$$

The elements u maximizing the right-hand side are the elements belonging to $-\partial_{+}\psi\left(\frac{\partial \mathbf{N}(t,x)}{\partial x}\right)$, Consequently,

$$-\partial_{+}\psi\left(\frac{\partial \mathbf{N}(t,x)}{\partial x}\right) \;\subset\; R(t,x)$$

Actually, approximations of the regulation map and thus, optimal evolutions, as well as the solution to the Hamilton-Jacobi-Bellman equation are provided by the Capture Basin Algorithm.

8 Barron-Jensen/Frankowska Solution to the Hamilton-Jacobi Equation

Instead of characterizing capture basins in terms of tangent cones and translating them in terms of contingent Frankowska hyposolutions, we translate them in the equivalent formulation of Frankowska-Barron/Jensen solutions, a weaker concept of viscosity solutions requiring only the upper semicontinuity of the solution instead of its continuity.

Theorem 8.1 Barron-Jensen/Frankowska Solution. *The viability hyposolution* $\mathbf{N}$ *is the* ***largest*** *upper semicontinuous solution between* $\mathbf{c}$ *and* $\mathbf{b}$ *satisfying*

$$\left\{\begin{array}{lll} (i) & \forall t>0,\ \forall x\in \mathrm{Int}(K),\ \forall (p_t,p_x)\in\partial_+\mathbf{N}(t,x), & p_t+\psi(p_x)\ \leq\ \psi(v(t)) \\ (ii) & \forall t>0,\ \forall x\in \mathrm{Int}(K),\ \forall (p_t,p_x)\in (\mathrm{Dom}(D_\downarrow \mathbf{N}(t,x)))^-, & p_t-\sigma(\mathrm{Dom}(\varphi^\star),p_x)\ \leq\ 0 \end{array}\right. \tag{25}$$

If the functions ψ*,* $\varphi^\star$ *and* v *are furthermore Lipschitz, then* $\mathbf{N}$ *is the* ***smallest*** *upper semi continuous solution between* $\mathbf{c}$ *and* $\mathbf{b}$ *satisfying*

1. *If* $\mathbf{N}(t,x)<\mathbf{b}(t,x)$*, then*

$$\left\{\begin{array}{ll} (i) & \forall t\geq 0,\ \forall x\in K \text{ such that } \mathbf{N}(t,x)<\mathbf{b}(t,x),\ \forall (p_t,p_x)\in\partial_+\mathbf{N}(t,x), \\ & p_t+\psi(p_x)\ \geq\ \psi(v(t)) \\ (ii) & \forall t\geq 0,\ \forall x\in K \text{ such that } \mathbf{N}(t,x)<\mathbf{b}(t,x),\ \forall (p_t,p_x)\in (\mathrm{Dom}(D_\downarrow \mathbf{N}(t,x)))^-, \\ & p_t-\sigma(\mathrm{Dom}(\varphi^\star),p_x)\ \geq\ 0 \end{array}\right. \tag{26}$$

2. *If* $\mathbf{N}(t,x)=\mathbf{b}(t,x)$*, then*

$$\left\{\begin{array}{ll} & \forall\ (p_t,p_x)\in\partial_+\mathbf{N}(t,x),\ \ \exists\ (q_t,q_x)\in\partial_+\mathbf{b}(t,x) \text{ and } 0<\mu<1 \textit{ such that} \\ \textit{either} & p_t-q_t-\sigma(\mathrm{Dom}(\varphi^\star),p_x-q_x)\ \geq\ 0 \\ \textit{or} & \dfrac{p_t-\mu q_t}{1-\mu}+\psi\left(\dfrac{p_x-\mu q_x}{1-\mu}\right)\ \geq\ \psi(v(t)) \end{array}\right. \tag{27}$$

and thus, the ***unique*** *upper semicontinuous solution satisfying all these properties.*

Observe that under the Lipschitz assumptions, the hypo viability solution satisfies

$$\left\{\begin{array}{ll} & \forall t>0,\ \forall x\in \mathrm{Int}(K)\ \text{ such that } \mathbf{N}(t,x)<\mathbf{b}(t,x), \\ (i) & \forall (p_t,p_x)\in\partial_+\mathbf{N}(t,x),\ \ p_t+\psi(p_x)\ =\ \psi(v(t)) \\ (ii) & \forall (p_t,p_x)\in (\mathrm{Dom}(D_\downarrow \mathbf{N}(t,x)))^-,\ \ p_t-\sigma(\mathrm{Dom}(\varphi^\star),p_x)\ =\ 0 \end{array}\right. \tag{28}$$

Proof — Proposition 6.4, p.18 states that the hypograph of the viability hyposolution satisfies:

$$\mathcal{H}yp(\mathbf{N})\ =\ \mathrm{Capt}_{(18)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c}))\ =\ \mathrm{Capt}_{(19)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c}))$$

Theorem 9.4, p.26 states that $\mathrm{Capt}_{(18)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c}))$ is the *largest* subset $\mathcal{D}$ between $\mathcal{C}$ and $\mathcal{K}$ such that $\mathcal{D}\backslash\mathcal{C}$ is locally invariant.
Taking $\mathcal{D}:=\mathcal{H}yp(\mathbf{N})$, Theorems 3.2.4 and 3.3.4 of [5] state that $\mathcal{H}yp(\mathbf{N})\backslash\mathcal{H}yp(\mathbf{c})$ is locally viable under F if and only if $\forall t>0,\forall x\in\mathrm{Int}(K),\forall\ y\leq\mathbf{N}(t,x),\ \ \exists u\in\mathrm{Dom}(\varphi^\star),\ \exists\pi\in[0,\alpha+\psi(v(t))-\varphi^\star(u)]$, such that $\forall(-p_t,-p_x,\lambda)\in N_{\mathcal{H}yp(\mathbf{N})}(t,x,y)$,

$$\left\{\begin{array}{l} \langle(-p_t,-p_x,\lambda),(-1,u,-\psi(v(t))+\varphi^\star(u)+\pi)\rangle \\ =\ p_t-\langle p_x,u\rangle+\lambda(-\psi(v(t))+\varphi^\star(u)+\pi)\ \leq\ \ 0 \end{array}\right. \tag{29}$$

By Lemma 9.10, if $y=\mathbf{N}(t,x)$, $(-p_t,-p_x,\lambda)\in N_{\mathcal{H}yp(\mathbf{N})}(t,x,y)$ means that either $\lambda>0$, and that, taking $\lambda=1$, $(p_t,p_x)\in\partial_+\mathbf{N}(t,x)$ or that $\lambda=0$, and that $(p_t,p_x)\in(\mathrm{Dom}(D_\downarrow\mathbf{N}(t,x)))^-$. If $y<\mathbf{N}(t,x)$, $(-p_t,-p_x,\lambda)\in N_{\mathcal{H}yp(\mathbf{N})}(t,x,y)$ means also that $\lambda=0$, and that $(p_t,p_x)\in(\mathrm{Dom}(D_\downarrow\mathbf{N}(t,x)))^-$.

Consequently, condition (29) can be written in the following form:

- Case when $y = \mathbf{N}(t,x)$ and $\lambda = 1$:

$$\begin{cases} \forall t > 0, \forall x \in \mathrm{Int}(K),\ \forall\ (p_t, p_x) \in \partial_+ \mathbf{N}(t,x),\ \text{then} \\ p_t - \psi(v(t)) + \inf_{u \in \mathrm{Dom}(\varphi^\star)}[\varphi^\star(u) - \langle p_x, u\rangle] \\ =\ p_t - \psi(v(t)) + \psi(p_x) \le 0 \end{cases}$$

- Case when $y \le \mathbf{N}(t,x)$ and $\lambda = 0$:

$$\begin{cases} \forall t > 0, \forall x \in \mathrm{Int}(K),\ \forall\ (p_t, p_x) \in (\mathrm{Dom}(D_\downarrow \mathbf{N}(t,x)))^-,\ \text{then} \\ p_t - \sup_{u \in \mathrm{Dom}(\varphi^\star)} \langle p_x, u\rangle\ =\ p_t - \sigma(\mathrm{Dom}(\varphi^\star), p_x) \le 0 \end{cases}$$

(Recall that this condition disappears whenever the viability hyposolution $\mathbf{N}$ is hypo-differentiable, and, in particular, when the hyposolution is Lipschitz).

Proof of inequalities (28) and (27): Theorem 9.4, p.26 states that $\mathrm{Capt}_{(19)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c}))$ is the *smallest* subset $\mathcal{D}$ between $\mathcal{C}$ and $\mathcal{K}$ such that $\mathcal{D}$ is backward invariant with respect to $\mathcal{K}$. Theorem 9.7, p.27 and Lemma 9.8, p.27 states that $\mathcal{D} := \mathcal{H}yp(\mathbf{N})$ is backward invariant with respect to $\mathcal{H}yp(\mathbf{b})$ under (19) if and only if

1. either for all $(t,x,y) \in \mathcal{H}yp(\mathbf{N}) \cap \mathrm{Int}(\mathcal{H}yp(\mathbf{N}))$,

$$\forall\ (-p_t - p_x, \lambda) \in N_{\mathcal{H}yp(\mathbf{N})}(t,x,y),\ \ \sigma(F_\infty(x), (p_t, p_x, -\lambda))\ \le\ 0$$

   Since the function $\mathbf{b}$ is assumed to be continuous,

$$\mathrm{Int}(\mathcal{H}yp(\mathbf{N}))\ =\ \{(t,x,y)\ \text{ such that } y\ <\ \mathbf{b}(t,x)\}$$

   the first case means that $y \le \mathbf{N}(t,x) < \mathbf{b}(t,x)$ and the above condition that

$$\begin{cases} \forall\ (-p_t - p_x, \lambda) \in N_{\mathcal{H}yp(\mathbf{N})}(t,x,y), \\ =\ -p_t + \langle p_x, u\rangle + \lambda(\psi(v(t)) - \varphi^\star(u) - \pi)\ \le\ \ 0 \end{cases} \tag{30}$$

   This implies that $\lambda \ge 0$.

   Consequently, condition (30) can be written in the following form:

   - Case when $y = \mathbf{N}(t,x) < \mathbf{b}(t,x)$ and $\lambda = 1$:

$$\begin{cases} \forall t > 0, \forall x \in X,\ \forall\ (p_t, p_x) \in \partial_+ \mathbf{N}(t,x),\ \text{then} \\ -p_t + \psi(v(t)) + \sup_{u \in \mathrm{Dom}(\varphi^\star)}[\langle p_x, u\rangle - \varphi^\star(u)] \\ =\ -p_t + \psi(v(t)) - \psi(p_x) \le 0 \end{cases}$$

   - Case when $y \le \mathbf{N}(t,x)$ and $\lambda = 0$:

$$\begin{cases} \forall t > 0, \forall x \in X,\ \forall\ (p_t, p_x) \in (\mathrm{Dom}(D_\downarrow \mathbf{N}(t,x)))^-,\ \text{then} \\ -p_t + \sup_{u \in \mathrm{Dom}(\varphi^\star)} \langle p_x, u\rangle\ =\ -p_t + \sigma(\mathrm{Dom}(\varphi^\star), p_x) \le 0 \end{cases}$$

   - Case when $y = \mathbf{N}(t,x) = \mathbf{b}(t,x)$.

2. or for all $(t,x,y) \in \mathcal{H}yp(\mathbf{N}) \cap \partial(\mathcal{H}yp(\mathbf{N}))$, and in this case, $y = \mathbf{N}(t,x) = \mathbf{b}(t,x)$ and

$$\forall\, (-p_t - p_x, \lambda) \in N_{\mathcal{H}yp(\mathbf{N})}(t,x,y),\;\; \exists\, (-q_t - q_x, \mu) \in N_{\mathcal{H}yp(\mathbf{b})}(t,x,y) \quad \text{such that} \sigma(F_\infty(x), (p_t - q_t, p_x - q_x, \mu - \lambda)) \;\leq\; 0$$

where $\lambda \geq 0$ and $\mu > 0$ since we have assumed that $\mathbf{b}$ is Lipschitz, and thus hypodifferentiable. This can be translated in the following form

$$-p_t + q_t + \sup_u (\langle p_x - q_x, u\rangle + (\mu - \lambda)(\varphi^\star(u)) + sup_{\pi \geq 0}(\mu - \lambda)\pi - \psi(v(t))) \;\leq\; 0$$

This implies that $\lambda \geq \mu > 0$.

- case when $\lambda - \mu = 0$. It happens when both $(p_t, p_x) \in \partial_+ \mathbf{N}(t,x)$ and $(q_t, q_x) \in \partial_+ \mathbf{b}(t,x)$. In this case, the above inequality boils down to

$$-p_t + q_t + \sigma(\mathrm{Dom}(\varphi^\star), p_x - q_x) \;\leq\; 0$$

- case when $\lambda - \mu > 0$. The condition states that for every $\lambda > 0$ and $(p_t, p_x) \in \partial_+ \mathbf{N}(t,x)$, there exist $0 < \mu < \lambda$ and $(q_t, q_x) \in \partial_+ \mathbf{b}(t,x)$ such that

$$-\frac{\lambda p_t - \mu q_t}{\lambda - \mu} + \sup_u \left(\left\langle \frac{\lambda p_x - \mu q_x}{\lambda - \mu}, u \right\rangle - \varphi^\star(u) \right) + \psi(v(t)) \;\leq\; 0$$

which can be written

$$-\frac{\lambda p_t - \mu q_t}{\lambda - \mu} - \psi \left(\frac{\lambda p_x - \mu q_x}{\lambda - \mu} \right) + \psi(v(t)) \;\leq\; 0$$

This completes the proof. ■

9 Appendix

9.1 Some Prerequisites from Viability Theory

Here, $X := \mathbb{R}^n$ and $Y := \mathbb{R}^m$ denote finite dimensional vector spaces. Let $f : X \times Y \mapsto X$ be a single-valued map describing the dynamics of a control system and $U : X \rightsquigarrow Y$ the set-valued map describing the state-dependent constraints on the controls.

First, any solution to a control system with state-dependent constraints on the controls

$$\begin{cases} (i) & x'(t) \;=\; f(x(t), u(t)) \\ (ii) & u(t) \;\in\; U(x(t)) \end{cases}$$

can be regarded as a solution to the differential inclusion $x'(t) \in F(x(t))$ where the right hand side is defined by $F(x) := f(x, U(x)) := \{f(x,u)\}_{u \in U(x)}$.

We denote by $\mathcal{S}(x) \subset \mathcal{C}(0,\infty; X)$ the set of *absolutely continuous functions* $t \mapsto x(t) \in X$ satisfying

$$\text{for almost all } t \geq 0,\;\; x'(t) \;\in\; F(x(t))$$

starting at time 0 at x: $x(0) = x$. The set-valued map $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ is called the *solution map* associated with F.

Therefore, from now on, as long as we do not need to implicate explicitly the controls in our study, we shall replace control problems by differential inclusions.
We shall say that K *is locally viable under* F if from every $x \in K$ starts a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ viable in K on the nonempty interval $[0, T_x[$ in the sense

$$\forall\, t \in [0, T_x[, \;\; x(t) \in K$$

and that K is *viable* if we can take $T_x = +\infty$. It is *locally backward invariant under* F if for every $t_0 \in]0, +\infty[$, $x \in K$, for all solutions $x(\cdot)$ to the differential inclusion $x' \in F(x)$ arriving at x at time t_0, there exists $s \in [0, t_0[$ such that $x(\cdot)$ is viable in K on the interval $[s, t_0]$, and *backward invariant* if we can take $s = 0$.
We denote by

$$\text{Graph}(F) := \{(x, y) \in X \in Y \mid y \in F(x)\}$$

the *graph* of a set-valued map $F : X \rightsquigarrow Y$ and $\text{Dom}(F) := \{x \in X | F(x) \neq \emptyset\}$ its *domain.*
Most of the results of viability theory are true whenever we assume that the dynamics is Marchaud:

Definition 9.1 Marchaud Map. *We shall say that F is a* Marchaud map *if*

$$\begin{cases} (i) & \text{the graph of } F \text{ is closed} \\ (ii) & \text{the values } F(x) \text{ of } F \text{ are convex} \\ (iii) & \text{the growth of } F \text{ is linear: } \exists\, c > 0 \mid \forall\, x \in X, \|F(x)\| := \sup_{v \in F(x)} \|v\| \;\leq\; c(\|x\| + 1) \end{cases}$$

We shall say that F is λ-Lipschitz if

$$\forall\, x, y \in X, \;\; F(x) \;\subset\; F(y) + \lambda \|x - y\| B$$

where B is the unit ball.

This covers the case of *Marchaud control systems* where $(x, u) \mapsto f(x, u)$ is continuous, affine with respect to the controls u and with linear growth and when U is Marchaud.

We recall the following version of the important Theorem 3.5.2 of **Viability Theory**, [5]:

Theorem 9.2 The Stability Theorem . *Assume that $F : X \rightsquigarrow X$ is Marchaud. Then the solution map $\mathcal{S}$ is upper semicompact with nonempty values: This means that whenever $x_n \in X$ converge to x in X and $x_n(\cdot) \in \mathcal{S}(x_n)$ is a solution to the differential inclusion $x' \in F(x)$ starting at x_n, there exists a subsequence (again denoted by) $x_n(\cdot)$ converging to a solution $x(\cdot) \in \mathcal{S}(x)$ uniformly on compact intervals.*

We shall also need some other prerequisites from Viability Theory:

Definition 9.3 Capture Basin of a Target. *Let $C \subset K \subset X$ be two subsets, C being regarded as a target, K as a constrained set. The subset* $\text{Capt}(K, C)$ *of initial states $x_0 \in K$ such that C is reached in finite time before possibly leaving K by at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is called the* viable-capture basin *of C in K. A subset K is a* repeller *under F if all solutions starting from K leave K in finite time. A subset D is* locally backward invariant relatively to K *if all backward solutions starting from D viable in K are actually viable in K.*

We recall the following result of [9]:

Theorem 9.4 Fixed-Point Characterization of Capture Basins. *The viable-capture basin* $\mathrm{Capt}(K,C)$ *of a target* C *viable in* K *is*

1. *the* largest *subset* D *satisfying* $C \subset D \subset K$ *and* $D \subset \mathrm{Capt}(D,C)$,
2. *the* smallest *subset* D *satisfying* $C \subset D \subset K$ *and* $\mathrm{Capt}(K,D) \subset D$,
3. *the* unique *subset* D *satisfying* $C \subset D \subset K$ *and*

$$D = \mathrm{Capt}(K,D) = \mathrm{Capt}(D,C)$$

The subset $K\backslash C$ denotes the intersection of K and the complement of C, i.e., is the set of elements of K which do not belong to C. We can derive the following characterization of capture basin (see [7]):

Theorem 9.5 Viability Characterization of Capture Basins. *Let us assume that* F *is Marchaud and that the subsets* $C \subset K$ *and* K *are closed. If* $K\backslash C$ *is a repeller (this is the case when* K *itself is a repeller), then the viable-capture basin* $\mathrm{Capt}(K,C)$ *of the target* C *under* $\mathcal{S}$ *is the* **unique** *closed subset satisfying* $C \subset D \subset K$ *and*

$$\left\{ \begin{array}{ll} (i) & D\backslash C \text{ is } \textit{locally viable} \text{ under } \mathcal{S} \\ (ii) & D \textit{ is locally backward invariant relatively to } K \end{array} \right. \tag{31}$$

The *contingent cone* $T_L(x)$ to $L \subset X$ at $x \in L$ is the set of directions $v \in X$ such that there exist sequences $h_n > 0$ converging to 0 and v_n converging to v satisfying $x + h_n v_n \in K$ for every n (see for instance [12] or [76] for more details). The *(regular) normal cone* is the polar cone $N_L(x) := (T_L(x))^-$ of the contingent cone. We introduce the following Frankowska property that we need for deriving the system of Hamilton-Jacobi-Bellman equations of which the detector is a solution:

Definition 9.6 Frankowska Property. *Let us consider a set-valued map* $F : X \rightsquigarrow X$ *and two subsets* $C \subset K$ *and* K*. We shall say that a subset* D *between* C *and* K *satisfies the* Frankowska property *with respect to* F *if*

$$\left\{ \begin{array}{lll} (i) & \forall\, x \in D\backslash C, & F(x) \cap T_D(x) \neq \emptyset \\ (ii) & \forall\, x \in D \cap \mathrm{Int}(K), & -F(x) \subset T_D(x) \\ (iii) & \forall\, x \in D \cap \partial K, & -F(x) \cap T_K(x) \subset T_D(x) \end{array} \right. \tag{32}$$

Actually, conditions (32) (ii) and (iii), p.26 boil down to the same condition

$$\forall\, x \in D, \ \ -F(x) \cap T_K(x) \subset T_D(x)$$

When K is assumed further to be backward locally invariant, the above conditions (32) boil down to

$$\left\{ \begin{array}{lll} (i) & \forall\, x \in D\backslash C, & F(x) \cap T_D(x) \neq \emptyset \\ (ii) & \forall\, x \in D, & -F(x) \subset T_D(x) \end{array} \right. \tag{33}$$

Theorem 9.5 and the Viability[7] and Invariance Theorems imply

[7]See for instance Theorems 3.2.4, 3.3.2 and 3.5.2 of [5].

Theorem 9.7 Tangential Characterization of Capture Basins. *Let us assume that F is Marchaud, that K is closed and that a closed subset C satisfies* $\text{Viab}_F(K\backslash C) = \emptyset$. *Then the viable-capture basin* $\text{Capt}_F^K(C)$ *is*

1. *the* **largest** *closed subset D satisfying $C \subset D \subset K$ and*

$$\forall\, x \in D\backslash C, \;\; F(x) \cap T_D(x) \;\neq\; \emptyset \tag{34}$$

2. *if F is Lipschitz, the* **unique** *closed subset D satisfying the Frankowska property (32).*

We provide the dual characterization of the capture basin in terms of normal cones due to Hélène Frankowska:

Lemma 9.8 Normal Characterization of Capture Basins. *Let us assume that for*

$$\forall\, x \in K, \;\; 0 \in \text{Int}(F(x) + T_K(x))$$

Then the Frankowska property *(32), p.26 is equivalent to the* dual Frankowska property

$$\left\{ \begin{array}{lll} (i) & \forall\, x \in D\backslash C, \;\; \forall\, x \in N_D(x), \;\; \sigma(F(x), -p) & \geq 0 \\ (ii) & \forall\, x \in D \cap \text{Int}(K), \forall\, x \in N_D(x), \;\; \sigma(F(x), -p) & \leq 0 \\ (iii) & \forall\, x \in D \cap \partial K, \;\; \forall\, p \in N_D(x), \;\; \inf_{q \in N_K(x)} \sigma(F(x), q-p) & \leq \; 0 \end{array} \right. \tag{35}$$

Proof — Whenever $0 \in \text{Int}(F(x) + T_K(x))$, Proposition 3.9, p.50 of [6] implies that the support function of $-F(x) \cap T_K(x)$ is the inf-convolution of the support functions of $-F(x)$ and $T_K(x)$:

$$\sigma(-F(x) \cap T_K(x), p) \;=\; \inf_{q \in N_K(x)} \sigma(F(x), q-p)$$

Consequently, inclusion $-F(x) \cap T_K(x) \subset T_D(x)$ is equivalent to

$$\forall\, p, \;\; \inf_{q \in N_K(x)} \sigma(F(x), q-p) \;\leq\; \sigma(T_D(x), p)$$

which can be written

$$\forall\, p \in N_D(x), \;\; \inf_{q \in N_K(x)} \sigma(F(x), q-p) \;\leq\; 0$$

This concludes the proof. ■

9.2 Some Prerequisites of Convex Analysis

We gather in this section notations and some results on convex analysis for the convenience of the reader non familiar with this topic. Since the authors of most of books on convex analysis have chosen to study convex functions rather than concave ones, we have chosen to associate with the concave function ψ the Fenchel transform $\varphi^\star$ of $\varphi := -\psi$ rather than the "concave Fenchel" transform $\psi^\boxtimes$ defined by the concave function

$$\psi^\boxtimes(u) \;:=\; \inf_{p \in \text{Dom}(\psi)} [\langle p, u \rangle - \psi(p)] = -\varphi^\star(-u)$$

The basic theorem of convex analysis states that $\psi = \psi^{\boxtimes\boxtimes}$ if and only if ψ is concave, upper semicontinuous, and non trivial (i.e. $\text{Dom}(\psi) \;:=\; \{p \mid \varphi(p) > -\infty\} \neq 0$).

The epigraph $\mathcal{E}p(\varphi)$ of an extended function φ is the set of pairs $(x,\lambda) \in X \times \mathbb{R}$ such that $\varphi(x) \le \lambda$ and the hypograph $\mathcal{H}yp(\psi)$ of a function ψ is the set of pairs $(p,\mu) \in X \times \mathbb{R}$ such that $\mu \le \psi(p)$. Note that the hypograph of ψ is related to the epigraph of φ by the relation

$$(p,\lambda) \ \in \ \mathcal{H}yp(\psi) \text{ if and only if } (p,-\lambda) \ \in \ \mathcal{E}p(\varphi)$$

An extended function is lower semicontinuous if and only if its epigraph is closed and upper semicontinuous if and only if its hypograph is closed.

Definition 9.9 Hypoderivatives and Superdifferentials. *The hypoderivative $D_{\downarrow}\psi(p)$ and the epiderivative $D_{\uparrow}\varphi(p)$ are related to the tangent cones of the hypograph of ψ and epigraph of φ by the relations*

$$\mathcal{H}yp(D_{\downarrow}\psi(p)) \ := \ T_{\mathcal{H}yp(\psi)}(p,\psi(p)) \text{ and } \mathcal{E}p(D_{\uparrow}\varphi(p)) \ := \ T_{\mathcal{E}p(\varphi)}(p,\varphi(p))$$

The superdifferential $\partial_{+}\psi(p)$ of the concave function ψ at p is defined by

$$u \ \in \ \partial_{+}\psi(p) \text{ if } \forall\, v \in X, \ \ \langle u,v\rangle \ \ge \ D_{\downarrow}\psi(p)(v)$$

and the subdifferential $\partial_{-}\varphi(p)$ is defined by

$$u \ \in \ \partial_{-}\varphi(p) \text{ if } \forall\, v \in X, \ \ \langle u,v\rangle \ \le \ D_{\uparrow}\varphi(p)(v)$$

We infer that

$$\forall\, v \in X, \ \ D_{\downarrow}\psi(p)(v) = -D_{\uparrow}\varphi(p)(v)$$

and that

$$u \ \in \ \partial_{+}\psi(p) \text{ if and only if } u \ \in \ -\partial_{-}\varphi(p)$$

The *polar cone* P^{-} of a given set P is defined by:

$$P^{-} = \{p \in X^{\star} \mid \forall x \in P, \ \langle p,x\rangle \le 0\}$$

where $X^{\star}$ is the dual space of X and the normal cone $N_K(x) := T_K(x)^{-}$ to K at $x \in K$ we use in this paper is the polar cone to the contingent cone to K at $x \in K$. The superdifferential $\partial_{+}\psi(p)$ and the subdifferential $\partial_{-}\varphi(p)$ are related to the normal cones of the hypograph of ψ and epigraph of φ by the relations

$$u \ \in \ \partial_{+}\psi(p) \text{ if and only if } (-u,1) \ \in \ N_{\mathcal{H}yp(\psi)}(p,\psi(p))$$

and

$$u \ \in \ \partial_{-}\varphi(p) \text{ if and only if } (u,-1) \ \in \ N_{\mathcal{E}p(\varphi)}(p,\varphi(p))$$

Recall the Legendre inversion formula:

$$u \ \in \ -\partial_{+}\psi(p) \text{ if and only if } p \in \partial_{-}\varphi^{\star}(u)$$

and the (decreasing) monotonicity property of superdifferential maps $p \rightsquigarrow \partial_{+}\psi(p)$ of a concave function:

$$\forall\, u_i \in \partial_{+}\psi(p_i), \ i = 1,2, \ \langle u_1 - u_2, p_1 - p_2\rangle \ \le \ 0$$

The subdifferential $\partial_{-}\sigma(K,p)$ of the support function is defined by the support zone $\{u \in K$ such that $\sigma(K,p) = \langle p,u\rangle\}$ of p in K. See [6] or [76] for more details.

We shall need the following result on tangent and normal cones to hypographs:

Lemma 9.10 Normal Cones to Hypographs.
A. *If $\psi : X \mapsto \mathbf{R}_+ \cup \{-\infty\}$ is an extended function and if $D_\downarrow\psi(p)(dp)$ is finite, then, for every $w < \psi(p)$ and every $\mu \in \mathbf{R}$, the pair (dp,μ) belongs to the contingent cone $T_{\mathcal{H}yp(\psi)}(p,w)$ to the hypograph of ψ at (p,w).*

B. *Consequently, a pair (u,λ) belongs to the normal cone $N_{\mathcal{H}yp(\psi)}(p,w)$ to the hypograph of ψ at (p,w) if and only*

1. *if $w = \psi(p)$, $\lambda > 0$ and $u \in -\lambda\partial_+\psi(p)$,*
2. *if $w \leq \psi(p)$, $\lambda = 0$ and $u \in (\mathrm{Dom}(D_\downarrow\psi(p)))^-$.*

C. *In particular, if the domain of $D_\downarrow\psi(p)$ is dense in X, then (u,λ) belongs to the normal cone $N_{\mathcal{H}yp(\psi)}(p,w)$ to the epigraph of ψ at (p,w) if and only if $\lambda > 0$ and $u \in -\lambda\partial_+\psi(p)$. This is the case whenever ψ is Lipschitz around p.*

Proof
A. Let (dp,λ) belong to $T_{\mathcal{H}yp(\psi)}(p,\psi(p))$. Then we know that there exist sequences $h_n > 0$ converging to 0, dp_n converging to dp and λ_n converging to λ such that $(p + h_n dp_n, \psi(p) + h_n\lambda_n)$ belongs to $\mathcal{H}yp(\psi)$. Therefore, for $w < \psi(p)$ and $\mu \in \mathbf{R}$ and h_n small enough,

$$(p + h_n dp_n, w + h_n\mu) \;=\; (p + h_n dp_n, \psi(p) + h_n\lambda_n) + (0, w - \psi(p) + h_n(\mu - \lambda_n)) \;\in\; \mathcal{H}yp(\psi)$$

belongs to the hypograph of ψ because $w - \psi(p) + h_n(\mu - \lambda_n) \leq 0$ for h_n small enough. Therefore, since $dp_n \to p$ and $\mu_n := \mu \to \mu$, we infer that $(dp,\mu) \in T_{\mathcal{H}yp(\psi)}(p,w)$.

B. Let us consider now a pair (u,λ) belonging to the normal cone $N_{\mathcal{H}yp(\psi)}(p,w) := (T_{\mathcal{H}yp(\psi)}(p,w))^-$ to the epigraph of ψ at (p,w): Therefore,

$$\forall\,(dp,\mu) \in T_{\mathcal{H}yp(\psi)}(p,w),\;\; \langle (dp,\mu),(u,\lambda)\rangle \;= \langle u, dp\rangle + \lambda\mu \;\leq\; 0$$

Examine first the case when $w = \psi(p)$, for which $(dp,\mu) \in T_{\mathcal{H}yp(\psi)}(p,\psi(p))$ if and only if $dp \in \mathrm{Dom}(D_\downarrow\psi(p))$ and $\mu \leq D_\downarrow\psi(p)(dp)$. If $\lambda < 0$, we obtain a contradiction because, when $\mu \to -\infty$, $\langle u, dp\rangle + \lambda\mu \to +\infty$. Hence

- either $\lambda > 0$, and thus, dividing by λ and taking $\mu := D_\downarrow\psi(p)(dp)$, we obtain

  $$\forall\, dp \in \mathrm{Dom}(D_\downarrow\psi(p)),\;\; \left\langle \frac{u}{\lambda}, dp \right\rangle + D_\downarrow\psi(p)(dp) \;\leq\; 0$$

  which means that $-\dfrac{u}{\lambda} \in \partial_+\psi(p)$
- or $\lambda = 0$ and we obtain

  $$\forall\, dp \in \mathrm{Dom}(D_\downarrow\psi(p)),\;\; \langle u, dp\rangle \;\leq\; 0$$

  which means that $u \in (\mathrm{Dom}(D_\downarrow\psi(p)))^-$ by definition of the polar cone.

When $w < \psi(p)$, inequalities

$$\forall\,(dp,\mu) \in T_{\mathcal{H}yp(\psi)}(p,w),\;\; \langle (dp,\mu),(u,\lambda)\rangle \;= \langle u, dp\rangle + \lambda\mu \;\leq\; 0$$

imply that $\lambda = 0$ because by property **A**: Otherwise, $\lambda\mu$ converges to $+\infty$ when $\mu \to +\infty$ when $\lambda > 0$ and when $\mu \to -\infty$ when $\lambda < 0$ since μ is allowed to range over $\mathbb{R}$. Therefore $u \in (\mathrm{Dom}(D_\downarrow\psi(p)))^-$ because whenever $dp \in \mathrm{Dom}(D_\downarrow\psi(p))$ and $\mu \in \mathbb{R}$, then $(dp, \mu) \in T_{\mathcal{H}yp(\psi)}(p, w)$.

C. If the domain $D_\downarrow\psi(p)$ is dense in X, then the polar cone $(\mathrm{Dom}(D_\downarrow\psi(p)))^-$ is $\{0\}$. Therefore, if $\lambda = 0$, then by **B.** , $u = 0$. Aside from this trivial case, the only possibility is thus $\lambda > 0$ and $u \in -\lambda\partial_+\psi(p)$. ■

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