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DIRICHLET SPACES AND STRONG MARKOV PROCESSES

BY
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Abstract. We show that there exists a suitable strong Markov process on the underlying space of each regular Dirichlet space. Potential theoretic concepts due to A. Beurling and J. Deny are then described in terms of the associated strong Markov process. The proof is carried out by developing potential theory for Dirichlet spaces and symmetric Ray processes and by using a method of transformation of underlying spaces.

Introduction. This paper is a continuation of [10]. We will use those notions and terminologies adopted in [10].

Let $(X, m, \mathscr{F}, \mathscr{E})$ be a D-space. We define (α_0-) *capacity* of an open set $A\subset X$ by

$$(0.1)\qquad \begin{aligned} \operatorname{Cap}(A) &= \inf_{u\in\mathscr{L}_A} \mathscr{E}^{\alpha_0}(u,u) \quad \text{if } \mathscr{L}_A \neq \varnothing, \\ &= +\infty \qquad\qquad\qquad \text{otherwise,} \end{aligned}$$

where α_0 is a fixed positive number and

$$(0.2)\qquad \mathscr{L}_A = \{u\in\mathscr{F};\, u \geqq 1 \ m\text{-a.e. on } A\}.$$

The capacity of an arbitrary set $A\subset X$ is defined by

$$(0.3)\qquad \operatorname{Cap}(A) = \inf_{A\subset B,\, B \text{ open}} \operatorname{Cap}(B).$$

We show in subsection 1.1 that this definition gives us a *Choquet capacity*[2]. A set $A\subset X$ is said to be *polar* if A has zero capacity. If A is polar, then $m(A)=0$.

From subsection 1.2 to the end of this paper, we will concentrate our attention on regular D-spaces. According to Definition 2.3 of [10], a D-space is called *regular* if m is everywhere dense on X and the space $\mathscr{F}\cap C(X)$ is dense both in $\mathscr{F}$ with norm $\mathscr{E}^{\alpha_0}$ and in $C(X)$ with uniform norm, $C(X)$ being the space of all continuous functions vanishing at infinity on X. Our goal in this paper is to establish the following existence theorem of a strong Markov process.

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(1) This research was supported by the Center for Advanced Study of the University of Illinois.

(2) This fact has been proved by J. Deny [3] under a kind of regularity condition for a function space.

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THEOREM 4.1. *Let $(X, m, \mathscr{F}, \mathscr{E})$ be a regular D-space. There exist then a (possibly empty) Borel polar set $B \subset X$ and a right continuous strong Markov process $M = (\Omega, \mathscr{M}, \mathscr{M}_t, X_t, P_x)$ with state space $X \cup \partial - B$ such that the resolvent of the process M generates the given D-space $(\mathscr{F}^*, \mathscr{E})$*(3): *if we put*

$$R_\alpha f(x) = E_x\left(\int_0^{+\infty} e^{-\alpha t} f(X_t)\, dt\right), \qquad x \in X - B, \tag{0.4}$$

for $f \in L^2(X; m) \cap C(X)$ under the convention that $f(\partial) = 0$, then the function $R_\alpha f$ belongs to the space $\mathscr{F}^*$ and the equation

$$\mathscr{E}^\alpha(R_\alpha f, v) = (f, v)_X \tag{0.5}$$

holds for every $v \in \mathscr{F}$. Furthermore the state space $X \cup \partial - B$ has no branch point and M is quasi-left continuous on $[0, +\infty)$.

In §4 we will prove this theorem by constructing these objects $(\Omega, \mathscr{M}, \mathscr{M}_t, X_t, P_x)$ in a specific way and describing more detailed properties that they possess. It turns out that the process M is actually a *Hunt process*(4).

Here we give a brief account of our procedure.

§1 will provide some basic facts related to a regular D-space most of which are well known as the contents of Beurling-Deny's potential theory. We reproduce them because our definition of the regularity is slightly more general than Beurling-Deny's and further our approach to the potential theory is based on the concept of quasi-supermedian functions.

Theorem 2.1 of §2 will state that, if two regular D-spaces are equivalent in the sense of Definition 4.1 of [10], then their underlying spaces are related by a *capacity preserving quasi-homeomorphism*(5). We need the regular representation theorem [10] for the proof of Theorem 2.1.

In §3 we examine the relationship between two aspects of a strongly regular D-space—the potential theoretic one developed in §1 and the probability theoretic one corresponding to the associated Ray process. For instance, we prove in Theorem 3.12 that a set A is polar if and only if there is an m-negligible Borel set $B \supset A$ such that almost all sample paths of the Ray process starting at any point of $X - B$ will never contact with B.

The proof of Theorem 4.1 is accomplished in the following way. Let $(X, m, \mathscr{F}, \mathscr{E})$ be a regular D-space. Then by virtue of Theorem 3 of [10], there is a strongly regular D-space $(\tilde{X}, \tilde{m}, \tilde{\mathscr{F}}, \tilde{\mathscr{E}})$ which is equivalent to $(X, m, \mathscr{F}, \mathscr{E})$. Owing to Theorem 2.1, X is related to $\tilde{X}$ by a capacity preserving quasi-homeomorphism q. q will transform the associated Ray process on $\tilde{X}$ into a process on X which turns out to have the properties of Theorem 4.1.

(3) $\mathscr{F}^*$ is the quasi-continuous modification of $\mathscr{F}$ (subsection 1.2).

(4) See P. A. Meyer [16, Chapitre XVI]. The state space of the process M is not necessarily a locally compact set but a Borel subset of the compactum $X \cup \partial$.

(5) We can find an analogous reasoning in M. Nakai [17].

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Thus every regular D-space is endowed with a probabilistic structure and we can see that all theorems of §3 are generalized at once to the case of the regular D-space. Subsection 4.2 collects some of the generalizations—an identification of decomposition of the D-space and that of an associated Hunt process, an identification of quasi-continuity and q.e. fine continuity, etc. In particular our notion of polar sets turns out to be weaker in general than the usually adopted probabilistic one. They are identical, however, as we will see in subsections 3.6 and 4.2, if and only if the underlying measure m is a reference measure for the process.

Although we go no further at present, it may be asserted that sample paths governed by a Dirichlet space will run along "the roads" indicated by the 0-order Dirichlet form $\mathscr{E}$ and with "speed" indicated by the underlying measure m.

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1. **Potential theory for D-spaces.** Let $(X, m, \mathscr{F}, \mathscr{E})$ be a D-space. We do not assume any regularity condition in the first subsection. From subsection 1.2 throughout §1 we will assume that $(X, m, \mathscr{F}, \mathscr{E})$ is regular.

1.1. *Capacity.*

THEOREM 1.1. *The capacity defined by* (0.1) *and* (0.3) *for all subsets of X is a Choquet capacity, that is,*

(a) *it is increasing,*

(b) *for any increasing sequence of subsets A_n of X,*

$$\operatorname{Cap}\left(\bigcup_n A_n\right) = \sup_n \operatorname{Cap}(A_n),$$

and

(c) *for any decreasing sequence of compact subsets A_n of X,*

$$\operatorname{Cap}\left(\bigcap_n A_n\right) = \inf_n \operatorname{Cap}(A_n).$$

Furthermore it has the property that

(d) *it is nonnegative and countably subadditive.*

Our capacity is evidently nonnegative and increasing. Property (c) is also clear. In fact, for any $\varepsilon>0$, there exists an open set $E\supset\bigcap_n A_n$ such as $\operatorname{Cap}(\bigcap_n A_n) \geqq \operatorname{Cap}(E)-\varepsilon$. However, $E\supset A_n$ for some n and we have $\operatorname{Cap}(\bigcap_n A_n)\geqq \inf_n \operatorname{Cap}(A_n)$.

According to P. A. Meyer [15, III, T23], the other assertions of Theorem 1.1 follow from the next lemma.

LEMMA 1.1. *The capacity defined by* (0.1) *for all open sets of X has the following properties. Denote by $\mathscr{U}$ the class of all open sets A for which $\mathscr{L}_A \neq \varnothing$.*

(i) *It is finite, nonnegative and increasing on $\mathscr{U}$.*

(ii) *It is strongly subadditive on $\mathscr{U}$: for any $A, B\in\mathscr{U}$,*

$$\operatorname{Cap}(A\cup B)+\operatorname{Cap}(A\cap B) \leqq \operatorname{Cap}(A)+\operatorname{Cap}(B).$$

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(iii) *If* $A_n \in \mathcal{U}$ *is increasing and* $\bigcup_n A_n \in \mathcal{U}$, *then*

$$\mathrm{Cap}\left(\bigcup_n A_n\right) = \sup_n \mathrm{Cap}\,(A_n).$$

(iv) *For any open set* A *belonging to* $\mathcal{U}_\sigma$,

$$\mathrm{Cap}\,(A) = \sup_{B\subset A, B\in\mathcal{U}} \mathrm{Cap}\,(B).$$

Proof. For $A \in \mathcal{U}$, there exists a unique element $p_A \in \mathcal{L}_A$ minimizing the quadratic form $\mathcal{E}^{\alpha_0}(u, u)$ in $\mathcal{L}_A$, since $\mathcal{L}_A$ is a nonempty convex set of $\mathcal{F}$ closed with norm $\mathcal{E}^{\alpha_0}$. Evidently,

$$\mathrm{Cap}\,(A) = \mathcal{E}^{\alpha_0}(p_A, p_A). \tag{1.1}$$

Since $(0 \vee p_A) \wedge 1$, being a normal contraction of p_A, is identical with p_A, we have

$$0 \leqq p_A \leqq 1 \qquad m\text{-a.e. on } X, \tag{1.2}$$

$$p_A = 1 \qquad m\text{-a.e. on } A. \tag{1.3}$$

Further we have

$$\mathcal{E}^{\alpha_0}(p_A, v) \geqq 0 \tag{1.4}$$

for any $v \in \mathcal{F}$ which is nonnegative m-a.e. on A. This follows from

$$\mathcal{E}^{\alpha_0}(p_A+\varepsilon v, p_A+\varepsilon v) \geqq \mathcal{E}^{\alpha_0}(p_A, p_A), \qquad \varepsilon > 0.$$

It is easy to see that $p_A \in \mathcal{F}$ is characterized by two conditions (1.3) and (1.4). Keeping these in mind, let us prove Lemma 1.1.

(i) Trivial.

(ii) Since $|p_A - p_B|$ is a normal contraction of $p_A - p_B$, we have

$$\mathcal{E}^{\alpha_0}(p_A \vee p_B, p_A \vee p_B)+\mathcal{E}^{\alpha_0}(p_A \wedge p_B, p_A \wedge p_B) \leqq \mathcal{E}^{\alpha_0}(p_A, p_A)+\mathcal{E}^{\alpha_0}(p_B, p_B),$$

which implies the desired inequality.

(iii) For $n > m$,

$$\mathcal{E}^{\alpha_0}(p_{A_n} - p_{A_m}, p_{A_n} - p_{A_m}) = \mathrm{Cap}\,(A_n) - \mathrm{Cap}\,(A_m).$$

Since $\mathrm{Cap}\,(A_n)$ is bounded from above (by the capacity of $A = \bigcup_n A_n$), the preceding equality means that p_{A_n} converges to a $u_0 \in \mathcal{F}$ in norm $\mathcal{E}^{\alpha_0}$. $u_0 = 1$ m-a.e. on A because $p_{A_n} = 1$ m-a.e. on A_n. Moreover $\mathcal{E}^{\alpha_0}(u_0, v) = \lim_{n\to+\infty} \mathcal{E}^{\alpha_0}(p_{A_n}, v) \geqq 0$ for every $v \in \mathcal{F}$ which is nonnegative m-a.e. on A. Thus $u_0 = p_A$ and

$$\lim_{n\to+\infty} \mathrm{Cap}\,(A_n) = \lim_{n\to+\infty} \mathcal{E}^{\alpha_0}(p_{A_n}, p_{A_n}) = \mathcal{E}^{\alpha_0}(p_A, p_A) = \mathrm{Cap}\,(A).$$

(iv) Consider an element A of $\mathcal{U}_\sigma$ and put $c = \sup_{B\subset A, B\in\mathcal{U}} \mathrm{Cap}\,(B)$. There exists an increasing sequence of open sets $A_n \in \mathcal{U}$ such that $\bigcup_n A_n = A$. By making use of statement (iii), we easily obtain the equality $c = \lim_{n\to+\infty} \mathrm{Cap}\,(A_n) \leqq +\infty$. Now this equality combined with exactly the same argument as in the proof of (iii) leads us to

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the conclusion that c is finite if and only if $A \in \mathscr{U}$ and in this case $c = \text{Cap}(A)$. Hence, if $c = +\infty$, then $A \notin \mathscr{U}$ and $\text{Cap}(A) = +\infty$ by definition. In any case, we get the desired equality. The proof of Lemma 1.1 is complete.

Theorem 1.1 combined with Choquet's theorem implies that, for any analytic set $A \subset X$,

$$\text{Cap}(A) = \sup_{K \subset A, K \text{ compact}} \text{Cap}(K). \tag{1.5}$$

In subsection 1.5, we will give some characterizations of the capacity for compact sets in the case of the regular D-space.

A subset A of X is called *polar* if $\text{Cap}(A) = 0$. The expression "*quasi-everywhere*" or "*q.e.*" means "except for a polar set". Let E be an open set of X. A function u defined q.e. on E is called *quasi-continuous on* E if, for any $\varepsilon > 0$, there exists an open set $\omega \subset E$ such that $\text{Cap}(\omega) < \varepsilon$ and the restriction of u to $X - \omega$ is continuous there. Quasi-continuous functions on X are simply said to be quasi-continuous.

THEOREM 1.2. (i) *If A is polar, then $m(A) = 0$.*

(ii) *If u_1 and u_2 are quasi-continuous on an open set $E \subset X$ and $u_1 \geqq u_2$ m-a.e. on E, then $u_1 \geqq u_2$ q.e. on E.*

Proof. (i) This is evident in view of the inequality $\text{Cap}(A) \geqq \alpha_0 m(A)$ for the open set A, which is immediate from (0.1).

(ii) Fix an $\varepsilon > 0$. There exists then an open set $\omega \subset E$ with $\text{Cap}(\omega) < \varepsilon$ such that u_1 and u_2 are continuous on $E - \omega$. Put $\omega' = \{x \in E;$ there exists a neighborhood $U(x)$ of x such that $U(x) \subset E$ and $m(U(x) - \omega) = 0\}$. It is easy to see that ω' is an open set, $\omega \subset \omega' \subset E$ and $m(\omega' - \omega) = 0$(6). Hence $\mathscr{L}\omega' = \mathscr{L}\omega$ and, by (0.1), $\text{Cap}(\omega') = \text{Cap}(\omega) < \varepsilon$. Now let us show $A \subset \omega'$, where $A = \{x \in E; u_1(x) < u_2(x)\}$. Suppose that there is an element $x \in A \cap (E - \omega')$. Since $x \in A \cap (E - \omega)$, there exists a $U(x) \subset E$ such that $u_1 < u_2$ on $U(x) - \omega$. However, $m(U(x) - \omega) \neq 0$ because $x \in E - \omega'$. This contradicts the assumption that $u_1 \geqq u_2$ m-a.e. on E. Thus $A \subset \omega'$ and $\text{Cap}(A) < \varepsilon$, proving that A is polar.

1.2. *Quasi-continuous modification* $\mathscr{F}^*$. From now on we assume that the given D-space $(X, m, \mathscr{F}, \mathscr{E})$ is regular.

Theorem 1.2(ii) then implies that $\text{Cap}(A) > 0$ for every nonempty open set A. Moreover, if a subset $A \subset X$ has a compact closure, then $\text{Cap}(A)$ is finite. In fact, A is then included in an open set E with compact closure. $\mathscr{L}_E$ is not empty for such an E.

THEOREM 1.3. *For any $u \in \mathscr{F}$, there exists $u_n \in \mathscr{F} \cap C(X)$ and increasing closed subsets F_m such that $\mathscr{E}^{\alpha_0}(u_n - u, u_n - u) \to 0$, $\text{Cap}(\bigcap_{m=1}^{\infty} F_m^c) = 0$ and u_n converges*

(6) Since X is assumed to be separable, we can use the Lindelöf covering theorem to prove this point. Cf. Hilfssatz 5.9 in C. Constantinescu and A. Cornea, *Ideal ränder Riemannscher flächen*, Springer, 1963.

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uniformly on each F_m. The limit function u^* of u_n is quasi-continuous and equal to u m-a.e.

Proof. By means of (0.1), we have

$$\text{Cap}\{x; |v(x)| > \varepsilon\} \leqq \mathscr{E}^{\alpha_0}(v, v)/\varepsilon^2 \tag{1.6}$$

for any $\varepsilon > 0$ and $v \in \mathscr{F} \cap C(X)$. Take $u \in \mathscr{F}$ and find $u_n \in \mathscr{F} \cap C(X)$ converging to u with $\mathscr{E}^{\alpha_0}$-norm. Subtracting a suitable subsequence if necessary, we can assume that $\text{Cap}(G_n) \leqq 1/2^n$ for the open set $G_n = \{x; |u_n(x) - u_{n+1}(x)| > 1/2^n\}$. The statement of Theorem 1.3 holds for $F_m = \bigcap_{n=m}^{+\infty} G_n^c$.

If a function u is defined m-a.e. on X and if u^* is quasi-continuous and equal to u m-a.e. on X, then u^* is called a *quasi-continuous modification of u*. Denote by $\mathscr{F}^*$ the set of all quasi-continuous modifications of functions of $\mathscr{F}$. We regard two functions of $\mathscr{F}^*$ to be equivalent if they are identical q.e. on X. On account of Theorems 1.2 and 1.3, the equivalence classes of $\mathscr{F}^*$ with inner product $\mathscr{E}^\alpha$ form a real Hilbert space which is just identical with the space $(\mathscr{F}, \mathscr{E}^\alpha)$, two functions of $\mathscr{F}$ being identified if they coincide m-a.e.

The next lemma can be proved exactly in the same manner as in J. Deny and J. Lions [5, II, Lemme 4.1 and Théorème 4.1].

LEMMA 1.2. (i) *The estimate* (1.6) *holds for any* $\varepsilon > 0$ *and* $v \in \mathscr{F}^*$.

(ii) *If u_n is a Cauchy sequence in $(\mathscr{F}^*, \mathscr{E}^{\alpha_0})$, then u_n converges to a function $u \in \mathscr{F}^*$ with $\mathscr{E}^{\alpha_0}$-norm. Further there exists a subsequence n_k such that* $\lim_{n_k \to +\infty} u_{n_k}(x) = u(x)$ *q.e. on X.*

1.3. *Quasi-supermedian functions and potentials.* Let $\{G_\alpha, \alpha > 0\}$ be the L^2-resolvent associated with the D-space $(\mathscr{F}, \mathscr{E})$. Each G_α is a linear operator from $L^2(X; m)$ into $\mathscr{F}$. From now on, however, we regard G_α as a linear operator from $L^2(X; m)$ into the space $\mathscr{F}^*$, as the preceding subsection 1.2 admits us to do.

We call a function $u \in L^2(X; m)$ (α_0-) quasi-supermedian if the following two conditions are satisfied.

(1.7) u is quasi-continuous and $u \geqq 0$ q.e.

(1.8) $\beta G_{\beta+\alpha_0} u \leqq u$ q.e., $\beta > 0$.

LEMMA 1.3. *A function $u \in \mathscr{F}^*$ is quasi-supermedian if and only if $\mathscr{E}^{\alpha_0}(u, v) \geqq 0$ for every $v \in \mathscr{F}^*$ such that $v \geqq 0$ q.e.*

Proof. If $u \in \mathscr{F}^*$ is quasi-supermedian then according to Lemma 2.1 of [10],

$$\mathscr{E}^{\alpha_0}(u, v) = \lim_{\beta \to +\infty} \beta(u - \beta G_{\beta+\alpha_0} u, v)_X \geqq 0 \quad \text{for } v \in \mathscr{F}$$

such that $v \geqq 0$ m-a.e.

Conversely assume that $u \in \mathscr{F}^*$ satisfies the inequality $\mathscr{E}^{\alpha_0}(u, v) \geqq 0$ for every $v \in \mathscr{F}^*$ which is nonnegative q.e. on X. Then u is the unique element minimizing the norm $\mathscr{E}^{\alpha_0}(w, w)$ in the convex set $\mathscr{L}_u = \{w \in \mathscr{F}^*; w \geqq u \text{ q.e.}\}$. $|u|$ is a normal

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contraction of u and belongs to $\mathscr{L}_u$. Thus $u=|u|\geqq 0$ q.e. proving (1.7). Furthermore we have, for any $v\in L^2(X; m)$ such as $v\geqq 0$ m-a.e.,

$$(u-\beta G_{\beta+\alpha_0}u, v)_X = \mathscr{E}^{\beta+\alpha_0}(u, G_{\beta+\alpha_0}u)-\beta(u, G_{\beta+\alpha_0}v)_X = \mathscr{E}^{\alpha_0}(u, G_{\beta+\alpha_0}v)$$

which is nonnegative because $G_{\beta+\alpha_0}v\in\mathscr{F}^*$ and $G_{\beta+\alpha_0}v\geqq 0$ q.e. (Theorem 1.2(ii)). This proves (1.8). The proof of Lemma 1.3 is complete.

Denote by M_0^+ the set of all nonnegative Borel measures μ on X satisfying the following two conditions:

(1.9) $$\mathscr{F}\cap C(X)\subset L^1(X; \mu).$$

(1.10) There exists a function $u\in\mathscr{F}^*$ such that

$$\mathscr{E}^{\alpha_0}(u, v) = \int_X v(x)\mu(dx) \quad \text{for any } v\in\mathscr{F}\cap C(X).$$

The function u of (1.10) is uniquely determined by $\mu\in M_0^+$. It is called the (α_0-) *potential of* μ and denoted by $U\mu$.

Every $\mu\in M_0^+$ is a Radon measure on X, namely, μ is finite for any compactum.

Any $u\in\mathscr{F}^*$ defines a linear functional l_u on $\mathscr{F}\cap C(X)$ by $l_u(v)=\mathscr{E}^{\alpha_0}(u, v)$, $v\in\mathscr{F}\cap C(X)$. Meanwhile, $\mathscr{F}\cap C(X)$ is closed under lattice operations and $v\wedge 1\in\mathscr{F}\cap C(X)$ for any $v\in\mathscr{F}\cap C(X)$ [10, Lemma 4.1]. Therefore by the general theory of Daniell integral [14, Chapter 3], l_u is an integral by means of the Baire measure with respect to the class $\mathscr{F}\cap C(X)$ if and only if l_u is a positive functional and continuous under monotone limits. Since the Baire family generated by $\mathscr{F}\cap C(X)$ is the set of Borel functions, we get the following

LEMMA 1.4. *$u\in\mathscr{F}^*$ is a potential if and only if*

(1.11) $\mathscr{E}^{\alpha_0}(u, v)\geqq 0$ *for any nonnegative $v\in\mathscr{F}\cap C(X)$.*

(1.12) $\mathscr{E}^{\alpha_0}(u, v_n)\downarrow 0$ *if $v_n\in\mathscr{F}\cap C(X)$ converges monotonically to zero.*

When X is compact, condition (1.12) is superfluous. If u is a potential, u determines the associated measure $\mu\in M_0^+$ uniquely.

Now we will state the relation of quasi-supermedian functions and potentials.

THEOREM 1.4. *A function $u\in\mathscr{F}^*$ is a potential if and only if u is a quasi-supermedian function satisfying condition* (1.12).

Proof. It suffices to show that condition (1.11) implies the stronger condition of Lemma 1.3. Suppose that $u\in\mathscr{F}^*$ satisfies (1.11). Let $v\in\mathscr{F}^*$ be nonnegative q.e. and $v_n\in\mathscr{F}\cap C(X)$ be a sequence converging to v in $\mathscr{E}^{\alpha_0}$-norm. Consider $v_n^+ = v_n\vee 0\in\mathscr{F}\cap C(X)$. Then $\mathscr{E}^{\alpha_0}(v_n^+, v_n^+)\leqq\mathscr{E}^{\alpha_0}(v_n, v_n)$ is bounded in n and

$$\mathscr{E}^{\alpha_0}(v_n^+, G_{\alpha_0}f) = (v_n^+, f)_X\to(v, f)_X = \mathscr{E}^{\alpha_0}(v, G_{\alpha_0}f) \quad \text{for any } f\in L^2(X; m).$$

Since $G_{\alpha_0}(L^2)$ is dense in $\mathscr{F}$, v_n^+ converges to v weakly. In particular,

$$\mathscr{E}^{\alpha_0}(u, v) = \lim_{n\to+\infty}\mathscr{E}^{\alpha_0}(u, v_n^+)\geqq 0.$$

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1.4. *Basic properties of potentials.* Define the support $S\mu$ of $\mu \in M_0^+$ by $S\mu = \{x \in X; \mu(U_x) \neq 0$ for any neighborhood U_x of $x\}$. $S\mu$ is a closed set. Let us begin with an approximation lemma.

LEMMA 1.5. *Suppose that $\mu \in M_0^+$ has a compact support $S\mu$. Then for any open set E such that $E \supset S\mu$ and $\bar{E}$ is compact, there exist nonnegative functions $f_n \in L^2(X; m)$ which vanish m-a.e. on $X-E$ and satisfy*

(1.13) $f_n \cdot m \to \mu$ vaguely as measures,

(1.14) $G_{\alpha_0} f_n \to U\mu$ weakly in $(\mathscr{F}, \mathscr{E}^{\alpha_0})$.

Proof. By virtue of Theorem 1.4, the potential $U\mu$ is quasi-supermedian, and so

$$g_\beta = \beta(U\mu - \beta G_{\beta+\alpha_0}(U\mu)) \tag{1.15}$$

is nonnegative. Let us prove the equality

$$\lim_{\beta \to +\infty} \int_X v(x) g_\beta(x) m(dx) = \int_K v(x)\mu(dx) \tag{1.16}$$

for every continuous function v such as $|v| \leqq v_0$, where $K = S\mu$ and v_0 is an arbitrarily fixed function in $\mathscr{F} \cap C(X)$. According to Lemma 2.1 of [10] and (1.10), the equality (1.16) is true for every $v \in \mathscr{F} \cap C(X)$. Incidentally the measures $g_\beta \cdot m$ are uniformly bounded in β on any compactum. Turning to the case of general v, choose $v_k \in \mathscr{F} \cap C(X)$ such as $|v_k| \leqq v_0$ and $\|v_k - v\|_\infty \to 0$, $k \to +\infty$, and observe the following inequality:

$$\begin{aligned}
&\left| \int_X v(x) g_\beta(x) m(dx) - \int_K v(x)\mu(dx) \right| \\
&\qquad \leqq \left| \int_X v_k(x) g_\beta(x) m(dx) - \int_K v_k(x)\mu(dx) \right| + \int_K |v(x) - v_k(x)| \mu(dx) \\
&\qquad\qquad + \int_F |v(x) - v_k(x)| g_\beta(x) m(dx) + 2 \int_{X-F} v_0(x) g_\beta(x) m(dx).
\end{aligned}$$

For any $\varepsilon > 0$, take a compactum F such that $v_0 < \varepsilon$ on $X - F$, then the superior limit in β of the last term of the right-hand side is less than $2\int_K (v_0 \wedge \varepsilon)\mu(dx) \leqq 2\varepsilon \cdot \mu(K)$. Now by taking sufficiently large k, we can make the superior limit in β of the right-hand side arbitrarily small.

It is clear that (1.16) implies (1.13) with

$$f_n(x) = g_n(x)\chi_E(x), \qquad x \in X, \tag{1.17}$$

χ_E being the indicator function of the open set E. It follows from (1.13) that

$$\mathscr{E}^{\alpha_0}(G_{\alpha_0} f_n, v) = (f_n, v)_X \to \int_K v(x)\mu(dx) = \mathscr{E}^{\alpha_0}(U\mu, v) \qquad \text{for } v \in \mathscr{F} \cap C(X).$$

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Since

$$\mathscr{E}^{\alpha_0}(G_{\alpha_0}f_n, G_{\alpha_0}f_n) = (f_n, G_{\alpha_0}f_n)_X \leqq (g_n, G_{\alpha_0}g_n)_X$$
$$= (g_n, nG_{n+\alpha_0}U\mu)_X \leqq (g_n, U\mu)_X$$

is uniformly bounded by $\mathscr{E}^{\alpha_0}(U\mu, U\mu)$, we arrive at (1.14).

We will point out here that, for any $\mu \in M_0^+$ and compactum K, the measure μ_K defined by $\mu_K(\cdot)=\mu(K \cap \cdot)$ is also in M_0^+ and

$$\mathscr{E}^{\alpha_0}(U\mu_K, U\mu_K) \leqq \mathscr{E}^{\alpha_0}(U\mu, U\mu). \tag{1.18}$$

Indeed, the inequality

$$\left| \int_X v(x)\mu_K(dx) \right| \leqq \int_X |v(x)|\mu(dx) = \mathscr{E}^{\alpha_0}(U\mu, |v|)$$
$$\leqq \sqrt{(\mathscr{E}^{\alpha_0}(U\mu, U\mu))}\sqrt{(\mathscr{E}^{\alpha_0}(v, v))}, \qquad v \in \mathscr{F} \cap C(X),$$

implies the existence of the potential $U\mu_K \in \mathscr{F}$ satisfying (1.10). Further we have $\mu - \mu_K \in M_0^+$. Therefore, by means of Lemma 1.3 and Theorem 1.4,

$$\mathscr{E}^{\alpha_0}(U(\mu-\mu_K), U\mu_K) \geqq 0,$$

which means (1.18). Keeping this in mind, let us proceed to

THEOREM 1.5(7). (i) *If A is polar, then $\mu(A)=0$ for every $\mu \in M_0^+$.*
(ii) *If $\mu \in M_0^+$, then $\mathscr{F}^* \subset L^1(X; \mu)$ and*

$$\mathscr{E}^{\alpha_0}(U\mu, u) = \int_X u(x)\mu(dx), \qquad u \in \mathscr{F}^*. \tag{1.19}$$

Proof. (i) Suppose that A is polar and $\mu \in M_0^+$. For any $\varepsilon > 0$, there is an open set $E \supset A$ with Cap $(E) < \varepsilon$. Take any compactum K included in E and choose f_n satisfying conditions of the preceding lemma for μ_K and an open set $E_1 \subset E$ with compact closure $\bar{E}_1$. Then,

$$\mathscr{E}^{\alpha_0}(U\mu_K, p_E) = \lim_{n\to+\infty} \mathscr{E}^{\alpha_0}(G_{\alpha_0}f_n, p_E) = \lim_{n\to+\infty} \int_{E_1} f_n(x)m(dx) = \mu(K).$$

Hence

$$\mu(K) \leqq \sqrt{(\mathscr{E}^{\alpha_0}(U\mu_K, U\mu_K))}\sqrt{(\mathscr{E}^{\alpha_0}(p_E, p_E))} \leqq \sqrt{(\mathscr{E}^{\alpha_0}(U\mu, U\mu))}\cdot\sqrt{(\mathrm{Cap}\,(E))}$$

and

$$\mu(E) \leqq \sqrt{\varepsilon}\cdot\sqrt{(\mathscr{E}^{\alpha_0}(U\mu, U\mu))}.$$

Thus $\mu(A)=0$.

(7) This is a version of Theorem 4(iii) of [1] whose proof was recently given in [4].

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(ii) Consider $\mu \in M_0^+$ and $u \in \mathscr{F}^*$. It is clear from assertion (i) that u is μ-measurable. Let $u_n \in \mathscr{F} \cap C(X)$, $F_k \subset X$ and u^* be those of Theorem 1.3 for the present u. It suffices to show that $u^* \in L^1(X; \mu)$ and

$$\mathscr{E}^{\alpha_0}(U\mu, u) = \int_X u^*(x)\mu(dx) \tag{1.20}$$

because $u = u^*$ q.e. and the right-hand sides of (1.19) and (1.20) are identical.

Let us prove (1.20). We may assume that each F_k is compact. Consider the sequence of measures $\mu_k = \mu_{F_k}$, then

$$\mathscr{E}^{\alpha_0}(U\mu_k, v) = \int_{F_k} v(x)\mu(dx) \xrightarrow[k\to+\infty]{} \int_{\bigcup_k F_k} v(x)\mu(dx)$$
$$= \int_X v(x)\mu(dx) = \mathscr{E}^{\alpha_0}(U\mu, v), \qquad v \in \mathscr{F} \cap C(X),$$

which, combined with (1.18), implies that $U\mu_k$ converges to $U\mu$ weakly in $(\mathscr{F}, \mathscr{E}^{\alpha_0})$. On the other hand,

$$\mathscr{E}^{\alpha_0}(U\mu_k, u) = \lim_{n\to+\infty} \mathscr{E}^{\alpha_0}(U\mu_k, u_n)$$
$$= \lim_{n\to+\infty} \int_{F_k} u_n(x)\mu(dx) = \int_{F_k} u^*(x)\mu(dx).$$

Therefore,

$$\mathscr{E}^{\alpha_0}(U\mu, u) = \lim_{k\to+\infty} \mathscr{E}^{\alpha_0}(U\mu_k, u) = \int_{\bigcup_k F_k} u^*(x)\mu(dx) = \int_X u^*(x)\mu(dx).$$

THEOREM 1.6. *Let K be a compact set. Then, for $u \in \mathscr{F}^*$, the next three conditions are mutually equivalent:*

(i) *u is a potential $U\mu$ with $S\mu \subset K$.*

(ii) *$\mathscr{E}^{\alpha_0}(u, v) \geqq 0$ for any $v \in \mathscr{F} \cap C(X)$ which is nonnegative on K.*

(iii) *$\mathscr{E}^{\alpha_0}(u, v) \geqq 0$ for any $v \in \mathscr{F}^*$ which is nonnegative q.e. on K.*

Proof. Owing to Theorem 1.5, (i) implies (iii). Trivially, (iii) implies (ii). All we have to do is to derive (i) from (ii). Suppose that $u \in \mathscr{F}^*$ satisfies condition (ii). We will first prove that u has the properties (1.11) and (1.12). (1.11) is trivial. Let w be a function of $\mathscr{F} \cap C(X)$ which is no less than 1 on K. If $v_n \in \mathscr{F} \cap C(X)$ is decreasing to zero, then v_n converges uniformly on X and $a_n = \sup_{x\in K} v_n(x)$ decreases to zero. Since $v_n \leqq a_n w$ on K,

$$\mathscr{E}^{\alpha_0}(u, v_n) \leqq a_n \mathscr{E}^{\alpha_0}(u, w) \to 0, \qquad n \to +\infty.$$

Thus, u satisfies (1.12). By means of Lemma 1.4, u is a potential of a measure $\mu \in M_0^+$. In view of equation (1.10) and condition (ii), we have $S\mu \subset K$.

1.5. *Equilibrium potential and capacity for the compact set.* We will first define equilibrium potentials for open sets in the class $\mathscr{U}$ of the subsection 1.1 and study

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their properties. Let A be in $\mathscr{U}$ and p_A be the function of $\mathscr{F}$ which is characterized by (1.3) and (1.4). Denote by e_A any quasi-continuous modification of p_A. We call e_A the (α_0-) *equilibrium potential for the open set* $A \in \mathscr{U}$. According to Theorem 1.2, e_A has the following properties:

(1.21) $\operatorname{Cap}(A)=\mathscr{E}^{\alpha_0}(e_A, e_A)$.

(1.22) $e_A=1$ q.e. on A.

(1.23) $\mathscr{E}^{\alpha_0}(e_A, v)\geqq 0$ for any $v \in \mathscr{F}^*$ which is nonnegative q.e. on A.

$e_A \in \mathscr{F}^*$ is characterized by (1.22) and (1.23) and indeed, it is a unique element which minimizes the norm $\mathscr{E}^{\alpha_0}(u, u)$ in the convex set $\{u \in \mathscr{F}^*; u\geqq 1 \text{ q.e. on } A\}$ of $\mathscr{F}^*$. Obviously e_A is a quasi-supermedian function.

In the particular case when the closure $\bar{A}$ of A is compact, we can see by Theorem 1.6 and (1.23) that e_A is a potential of a measure $\nu_A \in M_0^+$ with $S_{\nu_A}\subset \bar{A}$. We call ν_A the *equilibrium distribution for the open set* A. We have

$$\operatorname{Cap}(A) = \nu_A(\bar{A}), \tag{1.24}$$

because there is a function $w \in \mathscr{F} \cap C(X)$ which is equal to 1 on $\bar{A}$ and we get $\operatorname{Cap}(A)=\mathscr{E}^{\alpha_0}(e_A, w)=\nu_A(\bar{A})$.

Now consider any compact set K of X and put $\mathscr{L}_K^*=\{u \in \mathscr{F}^*; u\geqq 1 \text{ q.e. on } K\}$. $\mathscr{L}_K^*$ is a nonempty convex set of $\mathscr{F}^*$ and closed in norm $\mathscr{E}^{\alpha_0}$ according to Lemma 1.2. Therefore there is a unique element e_K of $\mathscr{L}_K^*$ which minimizes the quadratic form $\mathscr{E}^{\alpha_0}(u, u)$ in $\mathscr{L}_K^*$. We call e_K the (α_0-) *equilibrium potential for the compactum K*. It is easy to see that e_K is characterized as an element of $\mathscr{F}^*$ which has the following two properties:

(1.25) $e_K=1$ q.e. on K.

(1.26) $\mathscr{E}^{\alpha_0}(e_K, v)\geqq 0$ for any $v \in \mathscr{F}^*$ which is nonnegative q.e. on K.

By virtue of Theorem 1.6 and (1.26), we see that e_K is a potential of a measure $\nu_K \in M_0^+$ with $S\nu_K\subset K$. We call ν_K the *equilibrium distribution for the compactum K*.

THEOREM 1.7. *Let K be compact.*

(i) *The equilibrium potential e_K is characterized as an element of $\mathscr{F}^*$ possessing properties* (1.25) *and*

$$\mathscr{E}^{\alpha_0}(e_K, v) \geqq 0 \quad \textit{for any } v \in \mathscr{F} \cap C(X) \tag{1.27}$$

which is nonnegative on K.

(ii) *The next equalities hold*:

$$\operatorname{Cap}(K) = \mathscr{E}^{\alpha_0}(e_K, e_K) = \nu_K(K). \tag{1.28}$$

$$\operatorname{Cap}(K) = \inf_{u\in\mathscr{C}_K} \mathscr{E}^{\alpha_0}(u, u), \tag{1.29}$$

where $\mathscr{C}_K=\{u \in \mathscr{F} \cap C(X); u\geqq 1 \textit{ on } K\}$.

Proof. (i) is evident, since (1.27) is equivalent to (1.26) by virtue of Theorem 1.6. The second equality of (1.28) is immediate from (1.19) and (1.25).

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Let us prove the first equality of (1.28). For any $\varepsilon>0$, there is an open set $A\supset K$ such that $\mathrm{Cap}\,(K)+\varepsilon>\mathrm{Cap}\,(A)$. A is in $\mathscr{U}$. By (1.21), (1.22), (1.25) and (1.26), we have $\mathscr{E}^{\alpha_0}(e_A, e_K)=\mathscr{E}^{\alpha_0}(e_K, e_K)$ and $0\leqq\mathscr{E}^{\alpha_0}(e_A-e_K, e_A-e_K)=\mathrm{Cap}\,(A)-\mathscr{E}^{\alpha_0}(e_K, e_K)$. Hence we get the inequality $\mathrm{Cap}\,(K)\geqq\mathscr{E}^{\alpha_0}(e_K, e_K)$. In order to obtain the converse inequality, let us take a sequence of open sets A_n such that $\bar{A}_n$ is compact, $A_n\supset\bar{A}_{n+1}$ and $\bigcap_{n=1}^{\infty}A_n=K$. Let e_n and ν_n be the equilibrium potential and distribution for A_n respectively. Since $\mathscr{E}^{\alpha_0}(e_n-e_m, e_n-e_m)=\mathrm{Cap}\,(A_n)-\mathrm{Cap}\,(A_m)$, $n<m$, e_n converges to some $e_0\in\mathscr{F}^*$ in $\mathscr{E}^{\alpha_0}$-norm. Since $e_n=1$ q.e. on A_n, e_0 has the property (1.25). On the other hand, ν_n concentrates on $\bar{A}_n$ and $\nu_n(\bar{A}_n)=\mathrm{Cap}\,(A_n)\leqq\mathrm{Cap}\,(A_1)$ by (1.24). Therefore a subsequence of ν_n converges weakly to a measure ν_0 whose support is in K. Now the equality $\mathscr{E}^{\alpha_0}(e_n, v)=\int_{\bar{A}_n}v(x)\nu_n(dx)$ leads us to $\mathscr{E}^{\alpha_0}(e_0, v)=\int_K v(x)\nu_0(dx)$, $v\in\mathscr{F}\cap C(X)$, which enables us to conclude that e_0 has the property (1.27). Thus, by statement (i), we see that $e_0=e_K$ and $\mathscr{E}^{\alpha_0}(e_K, e_K)=\lim_{n\to+\infty}\mathscr{E}^{\alpha_0}(e_n, e_n)=\lim_{n\to+\infty}\mathrm{Cap}\,(A_n)\geqq\mathrm{Cap}\,(K)$.

Finally, we will show the equality (1.29). Put $c=\inf_{u\in\mathscr{C}_K}\mathscr{E}^{\alpha_0}(u, u)$ and take a minimizing sequence $u_n\in\mathscr{C}_K$: $\lim_{n\to+\infty}\mathscr{E}^{\alpha_0}(u_n, u_n)=c$. It is easy to see that u_n then forms a Cauchy sequence in norm $\mathscr{E}^{\alpha_0}$ and the limit function $u_0\in\mathscr{F}^*$ does not depend on the choice of the minimizing sequence u_n. Since $u_n\wedge 1\in\mathscr{C}_K$ forms a minimizing sequence as well, we have $u_0=1$ q.e. on K according to Lemma 1.2. Further the property (1.27) for u_0 can be derived from the inequality

$$\mathscr{E}^{\alpha_0}(u_n+\varepsilon v, u_n+\varepsilon v)\ \geqq\ \mathscr{E}^{\alpha_0}(u_0, u_0)$$

which holds for any $\varepsilon>0$ and $v\in\mathscr{F}\cap C(X)$ such as $v\geqq 0$ on K. Therefore, statement (i) means that $u_0=e_K$ and $c=\mathscr{E}^{\alpha_0}(u_0, u_0)=\mathrm{Cap}\,(K)$. The proof of Theorem 1.7 is complete.

2. **Transformation of underlying spaces.** Consider two regular D-spaces $(X, m, \mathscr{F}, \mathscr{E})$ and $(\tilde{X}, \tilde{m}, \tilde{\mathscr{F}}, \tilde{\mathscr{E}})$. The concepts corresponding to the latter will be denoted with tilde ~.

DEFINITION 2.1. A mapping q defined q.e. on X taking values in $\tilde{X}$ is said to be a *quasi-homeomorphism* between X and $\tilde{X}$ if, for any $\varepsilon>0$, there exist closed sets $F\subset X$, $\tilde{F}\subset\tilde{X}$ such that $\mathrm{Cap}\,(X-F)<\varepsilon$, $\mathrm{Cap}^\sim\,(\tilde{X}-\tilde{F})<\varepsilon$ and the restriction of q to F is a homeomorphism onto $\tilde{F}$. X and $\tilde{X}$ are said to be quasi-homeomorphic if there exists a quasi-homeomorphism between X and $\tilde{X}$.

It is clear that q is a quasi-homeomorphism if and only if there exist increasing sequences of closed sets $F_k\subset X$ and $\tilde{F}_k\subset\tilde{X}$ with $\lim_{k\to+\infty}\mathrm{Cap}\,(X-F_k)=0$, $\lim_{k\to+\infty}\mathrm{Cap}^\sim\,(\tilde{X}-\tilde{F}_k)=0$ such that q is one-to-one from $X_0=\bigcup_{k=1}^{\infty}F_k$ onto $\tilde{X}_0=\bigcup_{k=1}^{\infty}\tilde{F}_k$ and its restriction to each F_k is a homeomorphism onto $\tilde{F}_k$. The domain of definition of a quasi-homeomorphism q will always be considered to be such an F_σ-set X_0. q and q^{-1} are then Borel measurable transformations between X_0 and $\tilde{X}_0$. Hence the images by q and q^{-1} of analytic sets are also analytic sets(8).

(8) Cf. [15, III, T11].

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A quasi-homeomorphism q is said to be *capacity preserving* if, for any analytic set $A \subset X_0$,

$$\text{Cap}\,(A) = \text{Cap}^{\sim}\,(q(A))(^9). \tag{2.1}$$

We will write as $X \cong \tilde{X}$ if there exists a capacity preserving quasi-homeomorphism between X and $\tilde{X}$.

LEMMA 2.1. *Consider the underlying spaces X, $\hat{X}$, $\tilde{X}$ of three regular D-spaces. If $X \cong \hat{X}$ and $\hat{X} \cong \tilde{X}$, then $X \cong \tilde{X}$.*

Proof. Suppose that X and $\hat{X}$ (resp. $\hat{X}$ and $\tilde{X}$) are related by the map q_1 (resp. q_2). For any $\varepsilon > 0$, there exist closed sets $F \subset X$, $\hat{F}_1 \subset \hat{X}$, $\hat{F}_2 \subset \hat{X}$ and $\tilde{F} \subset \tilde{X}$ satisfying the following: $\text{Cap}\,(X-F) < \varepsilon$, $\text{Cap}^{\wedge}\,(\hat{X}-\hat{F}_1) < \varepsilon$, $\text{Cap}^{\wedge}\,(\hat{X}-\hat{F}_2) < \varepsilon$, $\text{Cap}^{\sim}\,(\tilde{X}-\tilde{F}) < \varepsilon$ and q_1 (resp. q_2) is homeomorphic from F (resp. $\hat{F}_2$) onto $\hat{F}_1$ (resp. $\tilde{F}$). Put $F' = q_1^{-1}(\hat{F}_1 \cap \hat{F}_2)$ and $\tilde{F}' = q_2(\hat{F}_1 \cap \hat{F}_2)$. Then, $q = q_2 \cdot q_1$ is homeomorphic from F' onto $\tilde{F}'$ and

$$\begin{aligned} \text{Cap}\,(X-F') &\leqq \text{Cap}\,(X-F) + \text{Cap}\,(q_1^{-1}(\hat{F}_1 - \hat{F}_2)) \\ &= \text{Cap}\,(X-F) + \text{Cap}^{\wedge}\,(\hat{F}_1 - \hat{F}_2) < 2\varepsilon. \end{aligned}$$

In the same way, we have $\text{Cap}^{\sim}\,(\tilde{X}-\tilde{F}') < 2\varepsilon$. Thus, X and $\tilde{X}$ are quasi-homeomorphic by the map q. Evidently q is capacity preserving.

According to Definition 4.1 of [10], two D-spaces $(X, m, \mathscr{F}, \mathscr{E})$ and $(\tilde{X}, \tilde{m}, \tilde{\mathscr{F}}, \tilde{\mathscr{E}})$ are called equivalent if there exists an algebraic isomorphism Φ from $\mathscr{F} \cap L^{\infty}(X; m)$ onto $\tilde{\mathscr{F}} \cap L^{\infty}(\tilde{X}; \tilde{m})$ which preserves three kinds of metrics—L^{∞}-norm, L^2-norm and $\mathscr{E}$-norm. Notice that we always regard the normed algebra $\mathscr{F} \cap L^{\infty}(X; m)$ (resp. $\tilde{\mathscr{F}} \cap L^{\infty}(\tilde{X}; \tilde{m})$) as the set of equivalence classes in the sense that two functions of $\mathscr{F} \cap L^{\infty}(X; m)$ (resp. $\tilde{\mathscr{F}} \cap L^{\infty}(\tilde{X}; \tilde{m})$) are identified if they coincide m-a.e. ($\tilde{m}$-a.e.). The isomorphism Φ is viewed to transform each equivalence class to an equivalence class.

The isomorphism Φ can be uniquely extended to three kinds of transformations: a unitary map Φ_1 from $(\mathscr{F}, \mathscr{E}^{\alpha})$ onto $(\tilde{\mathscr{F}}, \tilde{\mathscr{E}}^{\alpha})$, a unitary map Φ_2 from $L_0^2(X)$ onto $L_0^2(\tilde{X})$ and an isometric isomorphism Φ_3 from $L_0^{\infty}(X)$ onto $L_0^{\infty}(\tilde{X})$, where $L_0^2(X)$ (resp. $L_0^{\infty}(X)$) is the closure of $\mathscr{F} \cap L^{\infty}(X)$ in the metric space $L^2(X)$ (resp. $L^{\infty}(X)$). $L_0^2(\tilde{X})$ and $L_0^{\infty}(\tilde{X})$ are defined in the same way. Suppose that two D-spaces are regular. Then Φ_1 is regarded as a unitary map from $(\mathscr{F}^*, \mathscr{E}^{\alpha})$ onto $(\tilde{\mathscr{F}}^*, \tilde{\mathscr{E}}^{\alpha})$, two functions being identified if they coincide q.e. Moreover we have in this case $L_0^2(X) = L^2(X)$ and $L_0^{\infty}(X) \supset C(X)$ because $\mathscr{F} \cap C(X)$ is dense in the metric space $L^2(X)$ (resp. $C(X)$) (see (5.4) of [10]). We also have $L_0^2(\tilde{X}) = L^2(\tilde{X})$ and $L_0^{\infty}(\tilde{X}) \supset C(\tilde{X})$.

Now we will state the theorem of this section.

(9) This definition does not depend on the choice of set X_0.

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THEOREM 2.1. *Assume that two regular D-spaces* $(X, m, \mathscr{F}, \mathscr{E})$ *and* $(\tilde{X}, \tilde{m}, \tilde{\mathscr{F}}, \tilde{\mathscr{E}})$ *are equivalent under an isomorphism* Φ. *Then* $X \simeq \tilde{X}$ *under a capacity preserving quasi-homeomorphism q which has the following properties.*

(q.1) *q induces the extension of the given isomorphism* Φ: *put*

$$(\Phi^* u)(x) = u(q^{-1}\tilde{x}), \tag{2.2}$$

where u is a function on X and $\tilde{x}$ *is a point of* $\tilde{X}$ *for which* $u(q^{-1}\tilde{x})$ *makes sense, then* Φ^* *defines a transformation of functions which coincides on* $\mathscr{F}^*$ *with* Φ_1.

(q.2) *q is m-measure preserving*: $m(A) = m(q(A))$ *for any Borel set* $A \subset X_0$.

Before proceeding to the proof of Theorem 2.1, we need several notions related to a regular D-space $(X, m, \mathscr{F}, \mathscr{E})$. For a set $A \subset X$, we put

$$A' = \{x \in X; m(U(x) \cap A) \neq 0 \text{ for every neighbourhood } U(x) \text{ of } x\}. \tag{2.3}$$

Obviously A' is closed. We say a closed set F is *m-regular* if $F = F'$.

Consider any closed set F. Then F' is a closed set contained in F, $m(F-F')=0$ and Cap $(X-F')=$ Cap $(X-F)$. We can see this in the same manner as in the proof of Theorem 1.2(ii). Furthermore F' is necessarily m-regular because $m(U(x) \cap F') \geqq m(U(x) \cap F) - m(F-F') > 0$ for any neighborhood $U(x)$ of $x \in F'$.

Denote by X^Δ the compact space obtained from X by adjoining the point at infinity Δ. If X is already compact, we regard Δ to be isolated. For each set $A \subset X$, we put $A^\Delta = A \cup \Delta$ and consider this to be a topological subspace of X^Δ. A set $F \subset X$ is closed in X if and only if F^Δ is compact.

We further use the notion $|u|_A$ defined by $|u|_A = \sup_{x \in A} |u(x)|$ for a function u on $A \subset X$. Since m is everywhere dense, we have

$$\|u\|_\infty = |u|_X, \qquad u \in C(X). \tag{2.4}$$

Finally let $\{F_k\}$ be an increasing sequence of m-regular closed sets of X such that Cap $(X-F_k) \to 0$. Put $C(\{F_k\}) = \{u;\ u$ is defined on $X_0 = \bigcup_{k=1}^\infty F_k$, $|u|_{X_0}$ is finite, the restriction of u to each F_k is continuous there and continuously extendable to F_k^Δ by setting $u(\Delta)=0\}$. Obviously $C(X) \subset C(\{F_k\}) \subset L^\infty(X; m)$. $C(\{F_k\})$ is a Banach algebra with norm $|\ |_{X_0}$. Further

$$\|u\|_\infty = |u|_{X_0}, \qquad u \in C(\{F_k\}). \tag{2.5}$$

This is clear from m-ess-$\sup_{x \in F_k} |u(x)| = |u|_{F_k}$, $k = 1, 2, \ldots$, which is due to the definition of m-regularity of F_k.

Each element u of $C(X)$ (resp. $C(\{F_k\})$) will always be regarded as a function on X^Δ (resp. X_0^Δ) by setting $u(\Delta)=0$.

LEMMA 2.2. *Let Q be any countable subcollection of* $\mathscr{F} \cap L^\infty$. *Then there exists an increasing sequence of m-regular closed sets* F_k *with* Cap $(X-F_k) \to 0$ *such that each element of Q has a unique modification belonging to* $C(\{F_k\})$.

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Proof. For $Q=\{u_n\}$, $u_n \in \mathscr{F} \cap L^\infty$, $n=1,2,\ldots$, we denote by u_n^* a quasi-continuous modification of u_n specified in Theorem 1.3. Thanks to the countable subadditivity of the capacity, we can select an increasing sequence of closed sets $F_k \subset X$ with Cap $(X-F_k) \to 0$ such that every function u_n^* has the following property: the restriction of u_n^* to each F_k is continuous there. By virtue of the special manner of the construction of u_n^* stated in Theorem 1.3, we may further assume that u_n^* is continuously extendable from F_k to F_k^Δ by setting $u_n^*(\Delta)=0$. In order to complete the proof of Lemma 2.2, we only have to replace F_k with its m-regularization F_k'. After the replacement, we can see by (2.5) that u_n^* becomes a unique element of $C(\{F_k\})$ which coincides with u_n m-a.e.

Now we will prove Theorem 2.1 by means of the next three lemmas.

Lemma 2.3. *Under the assumption of Theorem* 2.1, *there exists an increasing sequence of m-regular closed sets* $F_k \subset X$, $k=1,2,\ldots$, *with* $\lim_{k\to+\infty}$ Cap $(X-F_k)=0$ *which satisfies the following. We put* $X_0=\bigcup_{k=1}^\infty F_k$.

(i) *There is an algebraic isomorphic and isometric transformation* ψ *from* $(C(\tilde{X}), |\ |_{\tilde{X}})$ *into* $(C(\{F_k\}), |\ |_{X_0})$. ψ *is just the restriction of the transform* Φ_3^{-1} *to* $C(\tilde{X})$.

(ii) *There is a mapping q from* X_0^Δ *into* $\tilde{X}^{\tilde{\Delta}}$ *such that* $q(\Delta)=\tilde{\Delta}$ *and the restriction of q to each* F_k^Δ *is continuous there. For each* $x \in X^\Delta$, qx *is characterized by*

$$\tilde{u}(qx) = (\psi\tilde{u})(x), \qquad \tilde{u} \in C(\tilde{X}). \tag{2.6}$$

Proof. (i) Since $\mathscr{F} \cap C(\tilde{X})$ is a dense subalgebra of $C(\tilde{X})$, we can find a countable subset $\tilde{C}_1 \subset \mathscr{F} \cap C(\tilde{X})$ such that the algebra $\mathscr{A}(\tilde{C}_1)$ generated by $\tilde{C}_1$ is dense in $C(\tilde{X})$ with maximum norm. Applying Lemma 2.2 to $\Phi^{-1}\tilde{C}_1 \subset \mathscr{F} \cap L^\infty(X;m)$, we get an increasing sequence $\{F_k\}$ of m-regular closed sets of X with Cap $(X-F_k) \to 0$ such that, for every $\tilde{u} \in \tilde{C}_1$, $\Phi^{-1}\tilde{u}$ has a unique modification belonging to $C(\{F_k\})$. Denote this modification by $\psi\tilde{u}$. The map ψ is extended to an algebraic isomorphism on $\mathscr{A}(\tilde{C}_1)$ which is consistent because of

$$|\tilde{u}|_{\tilde{X}} = |\psi\tilde{u}|_{X_0}, \qquad \tilde{u} \in \mathscr{A}(\tilde{C}_1), \tag{2.7}$$

where $X_0=\bigcup_{k=1}^\infty F_k$. The equality (2.7) follows from (2.4) and (2.5) as $|\tilde{u}|_X = \|\tilde{u}\|_\infty^{\sim} = \|\Phi_3^{-1}\tilde{u}\|_\infty = |\psi\tilde{u}|_{X_0}$. Now ψ is readily extended to a map from $C(\tilde{X})$ into $C(\{F_k\})$ satisfying conditions of the first statement of the present lemma.

(ii) For each $x \in X_0^\Delta$, $l_x(\tilde{u})=(\psi\tilde{u})(x)$, $\tilde{u} \in C(\tilde{X})$, is a character (a linear multiplicative functional) on $C(\tilde{X})$. Hence there exists a unique element $qx \in \tilde{X}^{\tilde{\Delta}}$ such as $l_x(\tilde{u})=\tilde{u}(qx)$, $\tilde{u} \in C(\tilde{X})$. Since $l_\Delta(\tilde{u})\equiv 0$, we have $q\Delta=\tilde{\Delta}$. Suppose that $x_n \in F_k^\Delta$ converges to $x \in F_k^\Delta$. Then $\tilde{u}(qx_n)=(\psi\tilde{u})(x_n)$ converges to $(\psi\tilde{u})(x)=\tilde{u}(qx)$, $\tilde{u} \in C(\tilde{X})$, which implies $qx_n \to qx$, $n \to \infty$, and hence the restriction of q to F_k^Δ is continuous there.

Lemma 2.4. *In addition to the assumption of Theorem* 2.1, *we assume*

$$\Phi(\mathscr{F} \cap C(X)) \subset \mathscr{F} \cap C(\tilde{X}). \tag{2.8}$$

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Then all the conclusions of Theorem 2.1 *are valid for the map q of Lemma* 2.3.

Proof. By assumption (2.8), there exists an algebraic isomorphic and isometric transformation φ from $C(X)$ into $C(\tilde{X})$: φ is just the restriction of the transform Φ_3 to $C(X)\subset L_0^\infty(X)$. Therefore there is a continuous map γ from $\tilde{X}^{\tilde{\Delta}}$ onto X^Δ such that, for each $\tilde{x}\in\tilde{X}^{\tilde{\Delta}}$, $\gamma\tilde{x}$ is characterized by

$$u(\gamma\tilde{x}) = \varphi u(\tilde{x}), \qquad u\in C(X). \tag{2.9}$$

On the other hand, the map ψ of Lemma 2.3 is the inverse of φ in the sense that $\psi\varphi u(x)=u(x)$, $x\in X_0^\Delta$, for every $u\in C(X)$. Indeed $u\in C(X)$ and $\psi\varphi u$ $(=\Phi_3^{-1}\cdot\Phi_3 u)$ $\in C(\{F_k\})$ are in the same class of $L_0^\infty(X)$ and so they are identical on X_0^Δ by virtue of (2.5). Hence, in view of (2.6) and (2.9), the map γ is the inverse of q of Lemma 2.3:

$$\gamma\cdot qx = x, \qquad x\in X_0^\Delta. \tag{2.10}$$

In particular $q(X_0)\subset\tilde{X}$ because $\gamma(\tilde{\Delta})=\Delta$. We put

$$\tilde{F}_k = q(F_k),\quad k=1,2,\ldots, \qquad \tilde{X}_0 = \bigcup_{k=1}^{\infty}\tilde{F}_k. \tag{2.11}$$

Since the restriction of q to the compactum $F_k^\Delta\subset X^\Delta$ is a continuous map, its image $q(F_k^\Delta)=\tilde{F}_k^{\tilde{\Delta}}$ is a compact set of $\tilde{X}^{\tilde{\Delta}}$. $\tilde{F}_k$ is therefore a closed subset of $\tilde{X}$.

From now on, let us restrict the domain of the definition of q (resp. γ) to X_0 (resp. $\tilde{X}$) and study the detailed properties they possess.

First of all we know from (2.10) that q is one-to-one from X_0 onto $\tilde{X}_0$ and its restriction to each F_k is a homeomorphism onto $\tilde{F}_k$, the inverse being γ.

We will prove that q is measure preserving between X_0 and $\tilde{X}_0$. It is enough to show

$$m(q^{-1}(\tilde{K})) = \tilde{m}(\tilde{K}) \tag{2.12}$$

for any compact set $\tilde{K}$ contained in some $\tilde{F}_k$. To see (2.12), choose a sequence $\tilde{u}_n\in\tilde{\mathscr{F}}\cap C(\tilde{X})$ converging to the indicator function of $\tilde{K}$ everywhere on $\tilde{X}$ as well as in $L^2(\tilde{X};\tilde{m})$-sense. This is possible because $\tilde{\mathscr{F}}\cap C(\tilde{X})$ is a lattice and a dense subset of $C(\tilde{X})$. Then $\psi\tilde{u}_n(x)=\tilde{u}_n(qx)$ converges to the indicator function of $q^{-1}(\tilde{K})$ $\subset F_k$ for each $x\in X_0$ and hence m-a.e. on X. Since ψ on $\tilde{\mathscr{F}}\cap C(\tilde{X})$ is a modification of Φ^{-1} which preserves L^2-norm, $\{\psi\tilde{u}_n\}$ also forms a Cauchy sequence in $L^2(X;m)$ and further

$$\tilde{m}(\tilde{K}) = \lim_{n\to+\infty}(\tilde{u}_n,\tilde{u}_n)_{\tilde{X}} = \lim_{n\to+\infty}(\psi\tilde{u}_n,\psi\tilde{u}_n)_X = m(q^{-1}(\tilde{K})),$$

getting (2.12).

Exactly in the same way as above, we can prove

$$m(K) = \tilde{m}(\gamma^{-1}(K)) \tag{2.13}$$

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for any compact set $K \subset X$. Moreover, combining (2.12) and (2.13), we come to the conclusion that

$$\tilde{m}(\gamma^{-1}(F_k)-\tilde{F}_k) = 0, \qquad k = 1, 2, \ldots. \tag{2.14}$$

Indeed, fix a number k and take any compact set $\tilde{K} \subset \gamma^{-1}(F_k)-\tilde{F}_k$. Then put $K=\gamma(\tilde{K})$ and $\tilde{K}_1=q(K)$. K and $\tilde{K}_1$ are compact sets in F_k and $\tilde{F}_k$ respectively. Since $\gamma^{-1}(K) \supset \tilde{K} \cup \tilde{K}_1$, we have

$$\tilde{m}(\tilde{K}_1) = m(q^{-1}(\tilde{K}_1)) = m(K) = \tilde{m}(\gamma^{-1}(K)) \geqq \tilde{m}(\tilde{K} \cup \tilde{K}_1)$$

from which follows $\tilde{m}(\tilde{K})=0$.

Next we have to show

$$\tilde{m}(\gamma^{-1}(\Delta)) = 0. \tag{2.15}$$

Observe that $\gamma^{-1}(\Delta)=\{\tilde{x} \in \tilde{X};\ \varphi u(\tilde{x})=0 \text{ for every } u \in C(X)\}$. Notice further that, since $\mathscr{F} \cap C(X)$ is dense in $L^2(X; m)$, the space $\varphi(\mathscr{F} \cap C(X))$ $(=\Phi(\mathscr{F} \cap C(X)))$ is dense in $L^2(\tilde{X}; \tilde{m})$ $(=\Phi_2(L^2(X; m)))$. Hence for any compactum $\tilde{K} \subset \gamma^{-1}(\Delta)$ there is a sequence $u_n \in \mathscr{F} \cap C(X)$ such that φu_n converges $\tilde{m}$-a.e. on $\tilde{X}$ to the indicator function of $\tilde{K}$. But $\varphi u_n(\tilde{x})=0$, $\tilde{x} \in \tilde{K}$, $n=1, 2, \ldots$, and we have $\tilde{m}(\tilde{K})=0$ proving (2.15).

We are in a position to complete the proof of Lemma 2.4. Let us derive the inequality

$$\mathrm{Cap}^{\sim}(\tilde{K}) \leqq \mathrm{Cap}(K), \tag{2.16}$$

where $\tilde{K}$ is any compact subset of $\gamma^{-1}(X)$ and $K=\gamma(\tilde{K})$. Since γ is continuous, K is a compact set of X. Consider the sets $\mathscr{C}_K=\{u \in \mathscr{F} \cap C(X);\ u \geqq 1 \text{ on } K\}$ and $\mathscr{C}_{\tilde{K}}=\{\tilde{u} \in \tilde{\mathscr{F}} \cap C(\tilde{X});\ \tilde{u} \geqq 1 \text{ on } \tilde{K}\}$, and observe the inclusion $\varphi(\mathscr{C}_K) \subset \mathscr{C}_{\tilde{K}}$. Since φ coincides with Φ on $\mathscr{F} \cap C(X)$ and Φ preserves $\mathscr{E}^{\alpha_0}$-norm, we get from (1.29) that

$$\begin{aligned}\mathrm{Cap}(K) &= \inf_{u \in \mathscr{C}_K} \mathscr{E}^{\alpha_0}(u, u) = \inf_{u \in \mathscr{C}_K} \tilde{\mathscr{E}}^{\alpha_0}(\varphi u, \varphi u) \\ &= \inf_{\tilde{u} \in \varphi(\mathscr{C}_K)} \tilde{\mathscr{E}}^{\alpha_0}(\tilde{u}, \tilde{u}) \geqq \mathrm{Cap}^{\sim}(\tilde{K}).\end{aligned}$$

We can now show that q is capacity preserving on X_0. On account of Theorem 1.1(b) and (1.5), it suffices to prove for any compact subset $K \subset F_k$ with a fixed k,

$$\mathrm{Cap}(K) = \mathrm{Cap}^{\sim}(\tilde{K}), \tag{2.17}$$

where $\tilde{K}=q(K)$. Noting the inclusion

$$\psi(\tilde{\mathscr{C}}_{\tilde{K}}) \subset \{u \in \mathscr{F} \cap C(\{F_k\});\ u \geqq 1 \text{ on } K\} \subset \mathscr{L}_K^*,$$

we have

$$\begin{aligned}\mathrm{Cap}^{\sim}(\tilde{K}) &= \inf_{\tilde{u} \in \tilde{\mathscr{C}}_K} \tilde{\mathscr{E}}^{\alpha_0}(\tilde{u}, \tilde{u}) = \inf_{\tilde{u} \in \tilde{\mathscr{C}}_{\tilde{K}}} \mathscr{E}^{\alpha_0}(\psi\tilde{u}, \psi\tilde{u}) \\ &= \inf_{u \in \psi(\tilde{\mathscr{C}}_{\tilde{K}})} \mathscr{E}^{\alpha_0}(u, u) \geqq \mathrm{Cap}(K),\end{aligned}$$

which combined with (2.16), proves (2.17).

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For the proof that q is a capacity preserving quasi-homeomorphism and measure preserving, it only remains to show

$$\text{Cap}^{\sim}\,(\tilde{X}-\tilde{X}_0) = 0. \tag{2.18}$$

Choose any $\varepsilon>0$ and fix a number k such as Cap $(X-F_k)<\varepsilon$. We are going to show

$$\text{Cap}^{\sim}\,(\tilde{X}-\tilde{F}_k) < \varepsilon. \tag{2.19}$$

Observe that $\tilde{X}-\tilde{F}_k$ is an open set of $\tilde{X}$ consisting of three disjoint parts: $\tilde{X}-\tilde{F}_k = \gamma^{-1}(X-F_k)+(\gamma^{-1}(F_k)-\tilde{F}_k)+\gamma^{-1}(\Delta)$. By (2.14) and (2.15), $\tilde{m}$-measures of the last two terms of the right-hand side are zero. $\gamma^{-1}(X-F_k)$ is open and contained in $\tilde{X}-\tilde{F}_k$. Hence by definition (0.1) of the capacity, we have

$$\text{Cap}^{\sim}\,(\tilde{X}-\tilde{F}_k) = \text{Cap}^{\sim}\,(\gamma^{-1}(X-F_k)). \tag{2.20}$$

On the other hand (2.16) and (1.5) mean the following:

$$\begin{aligned}\text{Cap}^{\sim}\,(\gamma^{-1}(X-F_k)) &= \sup_{\tilde{K}} \text{Cap}^{\sim}\,(\tilde{K}) \leqq \sup_{K=\gamma(\tilde{K})} \text{Cap}\,(K)\\ &\leqq \text{Cap}\,(X-F_k) < \varepsilon,\end{aligned}$$

the supremum being taken for all compact set $\tilde{K}\subset\gamma^{-1}(X-F_k)$. Thus we arrive at (2.19).

It is easy to see that our q possesses the property (q.1) of Theorem 2.1: (2.9) and (2.10) mean, for $u \in \mathscr{F} \cap C(X)$,

$$\Phi^* u = \Phi_1 u \quad \text{q.e.}, \tag{2.21}$$

which can be extended to $\mathscr{F}^*$ by virtue of Lemma 1.2. We have completed the proof of Lemma 2.4.

LEMMA 2.5. *Under the assumption of Theorem* 2.1, *there exists a regular D-space* $(\hat{X}, \hat{m}, \hat{\mathscr{F}}, \hat{\mathscr{E}})$ *satisfying the following*:

(1) *Both the given regular D-spaces are equivalent to* $(\hat{X}, \hat{m}, \hat{\mathscr{F}}, \hat{\mathscr{E}})$ *by isomorphisms, say,* Φ' *and* Φ''. Φ *is equal to* $(\Phi'')^{-1}\cdot\Phi'$.

(2) $\Phi'(\mathscr{F} \cap C(X))\subset\hat{\mathscr{F}} \cap C(\hat{X})$, $\Phi''(\tilde{\mathscr{F}} \cap C(\tilde{X}))\subset\hat{\mathscr{F}} \cap C(\hat{X})$.

Proof. This lemma is an application of the regular representation theorem of [10].

First of all we will establish the inclusion

$$\Phi_3(C_0(X)) \subset L^1(\tilde{X}; \tilde{m}), \qquad \Phi_3^{-1}(C_0(\tilde{X})) \subset L^1(X; m). \tag{2.22}$$

It is enough to prove the first. For any function $u \in C_0(X)$, there is a nonnegative function $v \in \mathscr{F} \cap C(X)$ such as $v \geqq \sqrt{|u|}$ on X. Since Φ_3 is a lattice isomorph as well as an algebraic isomorph and since $\Phi_3 v \in \tilde{\mathscr{F}}\subset L^2(\tilde{X}; \tilde{m})$, we have $\Phi_3(\sqrt{|u|}) \in L^2(\tilde{X}; \tilde{m})$ and $|\Phi_3 u| = (\Phi_3(\sqrt{|u|}))^2 \in L^1(\tilde{X}; \tilde{m})$.

Now denote by L the closed subalgebra in $L_0^\infty(X)$ generated by $C(X) \cup \Phi_3^{-1}C(\tilde{X})$. Then L satisfies the condition (C) of [10, §5]. (C.1) and (C.2) are clear. By (2.22), $L^1(X; m) \cap L$ includes the algebra generated by $C_0(X) \cup \Phi_3^{-1}(C_0(\tilde{X}))$ which is

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dense in L, proving (C.3). Therefore we can take as $(\hat{X}, \hat{m}, \hat{\mathscr{F}}, \hat{\mathscr{E}})$ the regular representation of $(X, m, \mathscr{F}, \mathscr{E})$ with respect to L (Theorem 2 of [10]). The algebraic isomorphism Φ' associated with this representation is translating $\mathscr{F} \cap L$ onto $\hat{\mathscr{F}} \cap C(\hat{X})$ getting the first inclusion of (2). The second is also clear because $(\hat{X}, \hat{m}, \hat{\mathscr{F}}, \hat{\mathscr{E}})$ is the regular representation of $(\tilde{X}, \tilde{m}, \tilde{\mathscr{F}}, \tilde{\mathscr{E}})$ with respect to $\tilde{L}$ under the isomorphism $\Phi' \cdot \Phi^{-1}$, $\tilde{L}$ being the closed subalgebra of $L_0^\infty(\tilde{X}; \tilde{m})$ generated by $\Phi(C(X)) \cup C(\tilde{X})$.

Proof of Theorem 2.1. Lemmas 2.1, 2.4 and 2.5 admit us to conclude that $X \cong \tilde{X}$ under a capacity preserving quasi-homeomorphism q possessing the property (q.1). (q.2) is a consequence of (q.1) because Φ^* is L^2-norm preserving from $\mathscr{F}^*$ onto $\tilde{\mathscr{F}}^*$. The proof of Theorem 2.1 is complete.

If two D-spaces are equivalent and if one of them is regular, then it is said to be a regular representation of the other.

COROLLARY TO THEOREM 2.1. *The underlying space of a regular representation of a given D-space is unique up to a capacity preserving quasi-homeomorphism.*

3. **Potential theory for symmetric Ray processes.** Let $(X, m, \mathscr{F}, \mathscr{E})$ be a strongly regular D-space and $\{R_\alpha(x, E), \alpha > 0\}$ be its associated symmetric Ray resolvent kernel on X. For a function u on X, put

$$R_\alpha u(x) = \int_X R_\alpha(x, dy) u(y), \qquad x \in X, \tag{3.1}$$

whenever the right-hand side makes sense. The images by R_α of Borel (universally) measurable functions are also Borel (universally) measurable. By definition, $(\mathscr{F}, \mathscr{E})$ is generated by $\{R_\alpha(x, E), \alpha > 0\}$, that is, $R_\alpha(L^2(X; m) \cap C(X)) \subset \mathscr{F} \cap C(X)$ and $R_\alpha u$, $u \in L^2(X; m) \cap C(X)$, satisfies the equation

$$\mathscr{E}^\alpha(R_\alpha u, v) = (u, v)_X \tag{3.2}$$

for every $v \in \mathscr{F}$. Moreover $\mathscr{F} \cap C(X)$ includes a set C_1 attached to the Ray resolvent (Definition 2.5 of [10]).

3.1. *Supermedian and excessive functions.*

LEMMA 3.1. *For any nonnegative measurable function u of $L^2(X; m)$, the function $R_\alpha u$ defined by* (3.1) *belongs to the space $\mathscr{F}^*$ and satisfies the equation* (3.2) *for each* $\alpha > 0$.

We will prove this lemma by making use of the following proposition:

PROPOSITION. *Suppose that a set H of real-valued functions on X satisfies the next conditions.*

(H.1) *If $f_1, f_2 \in H$ and $c_1 f_1 + c_2 f_2 \geqq 0$ with some constants c_1, c_2, then $c_1 f_1 + c_2 f_2 \in H$.*

(H.2) *If $f_n \in H$ increases to $f \in L^2(X; m)$, then $f \in H$.*

(H.3) $C_0^+(X) \subset H$.

Then H contains all nonnegative Borel measurable functions of $L^2(X; m)$.

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For the proof of the Proposition, it is enough to take any open set $E \subset X$ with compact closure and consider the class S of all Borel subsets of E whose indicator functions are in H. S contains all open subsets of E. Since S is a λ-system relative to E, it contains all Borel subsets of E (Lemma 0.1 of [7]). The rest of the proof is clear.

Proof of Lemma 3.1. Let H be the set of all nonnegative Borel measurable functions u of $L^2(X; m)$ such that $R_\alpha u$ belongs to $\mathscr{F}^*$ and satisfies equation (3.2) for each $\alpha > 0$. H satisfies (H.3) because $L^2(X; m) \cap C(X) \subset H$. Suppose that $u_n \in H$ increases to $u \in L^2(X; m)$. Then

$$\begin{aligned}\mathscr{E}^\alpha(R_\alpha u_n - R_\alpha u_m, R_\alpha u_n - R_\alpha u_m) &= (u_n - u_m, R_\alpha(u_n - u_m))_X \\ &\leqq (1/\alpha)(u_n - u_m, u_n - u_m)_X \to 0, \qquad n, m \to +\infty.\end{aligned}$$

By virtue of Lemma 1.2, a subsequence of $R_\alpha u_n \in \mathscr{F}^*$ converges to a function $\mathscr{F}^*$ q.e. on X as well as in $\mathscr{E}^\alpha$-norm. However $R_\alpha u_n(x)$ converges to $R_\alpha u(x)$ for each $x \in X$. Therefore $R_\alpha u$ belongs to $\mathscr{F}^*$ and satisfies (3.2). Condition (H.2) is verified. Thus we see by the proposition that Lemma 3.1 is valid for any nonnegative Borel measurable function u of $L^2(X; m)$.

Finally let u be a nonnegative universally measurable function of $L^2(X; m)$. There exist nonnegative Borel measurable functions u_1 and u_2 such that $u_1 \leqq u \leqq u_2$ on X and $u_1 = u_2$ m-a.e. on X. We have $R_\alpha u_1 \leqq R_\alpha u \leqq R_\alpha u_2$ on X. Further, by the symmetry of R_α,

$$\begin{aligned}0 &\leqq \int_X (R_\alpha u_2 - R_\alpha u_1)(x) m(dx) = \int_X R_\alpha 1(x)(u_2(x) - u_1(x)) m(dx) \\ &\leqq \frac{1}{\alpha}\int_X (u_2(x) - u_1(x)) m(dx) = 0,\end{aligned}$$

which implies $R_\alpha u_1 = R_\alpha u_2$ m-a.e. on X. Since $R_\alpha u_1$ and $R_\alpha u_2$ are quasi-continuous, we see by Theorem 1.2(ii) that $R_\alpha u_1 = R_\alpha u = R_\alpha u_2$ q.e. on X and consequently $R_\alpha u \in \mathscr{F}^*$. The equation (3.2) for u can be derived from that for u_1. The proof of Lemma 3.1 is complete.

DEFINITION 3.1. A function u on X is said to be (α_0-) supermedian if the following two conditions are satisfied:

(3.3) u is nonnegative and universally measurable,

(3.4) $\beta R_{\beta+\alpha_0} u(x) \leqq u(x)$, $x \in X$, $\beta > 0$.

A supermedian function u is said to be (α_0-) excessive if

(3.5) $\lim_{\beta \to +\infty} \beta R_{\beta+\alpha_0} u(x) = u(x)$, $x \in X$.

If u is nonnegative and universally measurable, then $R_{\alpha_0} u$ is excessive.

If u is a nonnegative universally measurable function and $\lim_{\beta \to \infty} \beta R_{\beta+\alpha_0} u(x) = \tilde{u}(x)$ exists for every $x \in X$, then the limit function $\tilde{u}$ is said to be the *regularization of u*. Every supermedian function has its regularization which turns out to be excessive.

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THEOREM 3.1. *If a function u is nonnegative universally measurable, belongs to the space $\mathscr{F}$ and has its regularization $\tilde{u}$, then $\tilde{u}$ is a quasi-continuous modification of u. In particular any excessive function belonging to $\mathscr{F}$ is an element of $\mathscr{F}^*$.*

Proof. We see by Lemma 3.1, that $R_\alpha u \in \mathscr{F}^*$ and the operator R_α applied to u is identical with L^2-resolvent associated with $(\mathscr{F}, \mathscr{E})$. Hence by taking Lemma 2.1(iii) of [10] and Lemma 1.2 of the present paper into account, we see that a subsequence of $\beta R_{\beta+\alpha_0} u$ converges q.e. on X to a quasi-continuous modification of u. Thus we get Theorem 3.1.

REMARK 3.1. Every supermedian function belonging to the space $\mathscr{F}^*$ is quasi-supermedian in the sense of subsection 1.3. According to Theorem 3.1, every excessive function belonging to the space $\mathscr{F}$ is quasi-supermedian.

3.2. *The associated Ray process and the branch set.* Put $\bar{X} = X \cup \partial$ where ∂ is adjoined to X as the point at infinity if X is noncompact and as an isolated point if X is compact. Extend the kernel $\{R_\alpha(x, E), \alpha > 0\}$ to $\bar{X}$ in the manner of Remark 2.2(ii) of [10]. Then the extended kernel becomes a conservative Ray resolvent over the compactum $\bar{X}$ to which the original set C_1 is still attached if we extend each function u of C_1 to $\bar{X}$ by setting $u(\partial) = 0$.

Therefore the results of D. Ray [18, Theorem I, II and III] concerning resolvents on compact spaces and their improvements by H. Kunita and T. Watanabe [13, §2] can be brought over to our situation and we get the following conclusions.

The first conclusion is about the branch set. For each $x \in X$, the measure $\alpha R_\alpha(x, \cdot)$ on X converges to a unique substochastic measure $\mu(x, \cdot)$:

$$\lim_{\alpha \to +\infty} \alpha R_\alpha f(x) = \int_X \mu(x, dy) f(y) \quad \text{for any } f \in C(X).$$

A point $x \in X$ is said to be a branch point if the measure $\mu(x, \cdot)$ is not a unit distribution at x. The set X_b of all branch points of X is called the branch set. The measure $\mu(x, \cdot)$ is not supported by X_b for any $x \in X$. X_b is characterized as follows:

$$X_b = \bigcup_{g \in C_1'} \left\{ x; g(x) > \int_X g(y) \mu(x, dy) \right\}, \tag{3.6}$$

where $C_1' = \{g = f \wedge c; f \in C_1, c \text{ is any positive rational number}\}$.

The second is about the transition function. There is a unique sub-Markov transition function $P_t(x, E)$ on X such that

$$P_t f(x) = \int_X P_t(x, dy) f(y), \qquad f \in C(X), x \in X, \tag{3.7}$$

defines a right continuous function of $t > 0$ with

$$\int_0^{+\infty} e^{-\alpha t} P_t f(x)\, dt = R_\alpha f(x), \qquad \alpha > 0. \tag{3.8}$$

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The third is the existence of a right continuous strong Markov process on X with transition function P_t. This is called the *Ray process* associated with $\{R_\alpha(x, E), \alpha>0\}$. We can adopt as the Ray process the *canonical realization* $M=(W, \mathscr{B}_t^0, P_x)$ of $\{R_\alpha(x, E), \alpha>0\}$ in the following sense(10). W consists of paths $\omega=\omega(t)$, $t\in[0, +\infty)$, taking values in $\bar{X}$ such that $\omega(t)$ is right continuous in $t\in[0, +\infty)$, has the left limit at any $t\in(0, +\infty)$ and stays at ∂ after its lifetime $\zeta(\omega)$. The tth coordinate $\omega(t)$ of ω is denoted by $X_t(\omega)$. $\zeta(\omega)$ is defined by $\inf\{t\geqq 0, X_t(\omega)=\partial\}$. $\mathscr{B}_t^0$ is the σ-field of subsets of W generated by $\{X_s\in E\}$ with $0\leqq s\leqq t$ and Borel set $E\subset\bar{X}$. For each x, P_x is a unique probability measure on $\mathscr{B}^0=\bigvee_{t\geqq 0}\mathscr{B}_t^0$ which satisfies

$$
\begin{aligned}
&P_x(X_{t_1}\in E_1, X_{t_2}\in E_2, \dots, X_{t_n}\in E_n)\\
(3.9)\qquad &= \int_{E_1}\int_{E_2}\cdots\int_{E_n}\bar{P}_{t_1}(x, dy_1)\bar{P}_{t_2-t_1}(y_1, dy_2)\cdots\bar{P}_{t_n-t_{n-1}}(y_{n-1}, dy_n)
\end{aligned}
$$

for $0<t_1<t_2<\cdots<t_n$ and Borel sets $E_1, E_2, \dots, E_n$ of $\bar{X}$, where $\bar{P}_t(x, E)=P_t(x, E\cap X)+(1-P_t(x, X))\delta_{\{\partial\}}(E)$.

The Ray process $M=(W, \mathscr{B}_t^0, P_x)$ has the following properties:

(M.1) $P_x(X_0\in E)=\mu(x, E)$ for $x\in X$ and Borel $E\subset X$. Define $\mathscr{B}$ to be the completion of $\mathscr{B}^0$ with respect to the family of measures $\{P\mu(\cdot)=\int_{\bar{X}}\mu(dx)P_x(\cdot); \mu$ is a finite measure on $\bar{X}\}$ and $\mathscr{B}_t$ to be the completion of $\mathscr{B}_t^0$ in $\mathscr{B}$ with respect to the same family(11).

(M.2) Strong Markov property with respect to the augmented fields $\{\mathscr{B}_t\}$: for any stopping time T, $t>0$ and Borel $E\subset\bar{X}$, $P_x(X_{T+t}\in E\,|\,\mathscr{B}_T)=P_t(X_T, E)$, P_x-almost everywhere for each $x\in X$. Here, T is said to be a stopping time if $\{T\leqq t\}\in\mathscr{B}_t$ for any $t\geqq 0$ and $\mathscr{B}_T$ is defined as the collection of those sets $\Lambda\in\mathscr{B}$ such that $\{T\leqq t\}\cap\Lambda\in\mathscr{B}_t$ for all t.

(M.3) Quasi-left continuity in the restricted sense: if stopping times T_n increase to T, then $X_T=\lim_{n\to+\infty}X_{T_n}$ P_x-almost everywhere on the set

$$\left\{T<+\infty, \lim_{n\to+\infty}X_{T_n}\in\bar{X}-X_b\right\}\quad\text{for each } x\in X.$$

(M.4) $P_x(X_t\in\bar{X}-X_b$ for any $t\geqq 0)=1$, $x\in X$.

For a set $A\subset X$, we define the first entry time $\sigma_A(\omega)$ and the hitting time $\sigma'_A(\omega)$ by

$$
(3.10)\qquad
\begin{aligned}
\sigma_A(\omega) &= \inf\{t\geqq 0; X_t\in A\},\\
\sigma'_A(\omega) &= \inf\{t>0; X_t\in A\}.
\end{aligned}
$$

We define $\sigma_A(\omega)$ or $\sigma'_A(\omega)$ to be $\zeta(\omega)$ when the set in the braces is empty. If A is analytic, then random times σ_A, σ'_A and $\tau=\inf\{t>0; X_{t-}\in A\}$ are $\mathscr{B}_t$-stopping times. We can see this by [15, Chapter IV, T47 and 53].

(10) See [16, XIII] for the canonical realization of a Feller semigroup.

(11) See R. M. Blumenthal and R. K. Getoor [2, p. 26] for the terminology.

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THEOREM 3.2. *The branch set X_b is polar in the sense of* §1.

Proof. By Lemma 4.1 of [10] and by the inclusion $C_1 \subset \mathscr{F} \cap C(X)$, we see that $\mathscr{F} \cap C(X)$ includes the countable collection C_1' which appeared in (3.6). The members of C_1' will be numbered as $g_1, g_2, \ldots, g_k, \ldots$. Put $X_{\alpha,n}^k = \{x; g_k(x) - \alpha R_{\alpha+\alpha_0} g_k(x) > 1/n\}$, which includes the set $\{x; g_k(x) > \int_X g_k(y)\mu(x, dy) + 1/n\}$ for every $\alpha > 0$. Lemma 2.1(iii) of [10] and the estimate (1.6) lead us to

$$\mathrm{Cap}\,(X_{\alpha,n}^k) \leqq n^2 \mathscr{E}^{\alpha_0}(g_k - \alpha R_{\alpha+\alpha_0} g_k, g_k - \alpha R_{\alpha+\alpha_0} g_k) \to 0, \qquad \alpha \to +\infty.$$

For any $\varepsilon > 0$, take $\varepsilon_k > 0$ such as $\sum_{k=1}^{+\infty} \varepsilon_k < \varepsilon$. For each k and n, choose α such that the open set $Y_n^k = X_{\alpha,n}^k$ has the capacity less than $\varepsilon_k/2^{n+1}$. Now X_b is included in the open set $\bigcup_k \bigcup_n Y_n^k$ whose capacity is less than ε, as was to be proved.

3.3. *Symmetry of the process.* Here we will observe how the behaviours of the associated Ray process reflect the symmetry of our Ray resolvent. It is clear that the symmetry of $\{R_\alpha(x, E), \alpha > 0\}$ implies the symmetry of the associated transition function $\{P_t(x, E), t > 0\}$: for any $t > 0$ and nonnegative Borel measurable functions f and g on X

$$\int_X P_t f(x) \cdot g(x) m(dx) = \int_X f(x) \cdot P_t g(x) m(dx) \leqq +\infty. \tag{3.11}$$

LEMMA 3.2. *For $0 < t_1 < \cdots < t_{n-1} < t_n$ and nonnegative Borel measurable functions $f_0, f_1, \ldots, f_{n-1}, f_n$ on X,*

$$\begin{aligned} &\int_X E_x(f_0(X_0) f_1(X_{t_1}) \cdots f_{n-1}(X_{t_{n-1}}) f_n(X_{t_n})) m(dx) \\ &\quad = \int_X E_x(f_n(X_0) f_{n-1}(X_{t_n - t_{n-1}}) \cdots f_1(X_{t_n - t_1}) f_0(X_{t_n})) m(dx). \end{aligned} \tag{3.12}$$

Proof. Notice that $P_x(X_0 = x) = 1$ for m-a.e. $x \in X$ because of (**M**.1) and Theorem 3.2 of the preceding subsection and Theorem 1.2(i). We will prove this lemma by induction. Suppose that (3.12) holds for a given n. Then

$$\begin{aligned} &\int_X E_x(f_0(X_0) \cdots f_n(X_{t_n}) f_{n+1}(X_{t_{n+1}})) m(dx) \\ &\qquad = \int_X E_x(f_0(X_0) \cdots (f_n \cdot P_{t_{n+1} - t_n} f_{n+1})(X_{t_n})) m(dx) \\ &\qquad = \int_X P_{t_{n+1} - t_n} f_{n+1}(x) E_x(f_n(X_0) f_{n-1}(X_{t_n - t_{n-1}}) \cdots f_0(X_{t_n})) m(dx) \end{aligned}$$

which is equal to

$$\begin{aligned} &\int_X f_{n+1}(x) P_{t_{n+1} - t_n}(E \cdot (f_n(X_0) f_{n-1}(X_{t_n - t_{n-1}}) \cdots f_0(X_{t_n})))(x) m(dx) \\ &\qquad = \int_X E_x(f_{n+1}(X_0) f_n(X_{t_{n+1} - t_n}) \cdots f_0(X_{t_{n+1}})) m(dx) \end{aligned}$$

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by virtue of (3.11) and the Markov property. Thus (3.12) holds for $n+1$, completing the proof of Lemma 3.2.

Since ∂ is not a branch point of $\boldsymbol{M}$, (**M**.3) implies as in [2, (9.3)] that the left limits of sample paths must lie in X up to their lifetimes almost surely. In the following we assume without loss of generality that every $\omega \in W$ has the property that $X_{t-}(\omega) \in X$ for every $t < \zeta(\omega)$.

Fix a positive number $c>0$. Denote by $\mathfrak{L}$ the set of all functions $\varphi(t)$ $(0 \leqq t \leqq c)$ taking values in X. The (time reversal) transformation $\boldsymbol{q}$ of the space $\mathfrak{L}$ is defined by $\boldsymbol{q}\varphi(t)=\varphi(c-t)$, $0 \leqq t \leqq c$. For $\omega \in W$ such as $X_{c-}(\omega) \in X$, we define $\nu_r\omega$ and $\nu_l\omega \in \mathfrak{L}$ by

$$(\nu_r\omega)(t) = X_t(\omega), \quad 0 \leqq t < c, \qquad (\nu_l\omega)(t) = X_0(\omega), \quad t = 0,$$
$$\quad = X_{c-}(\omega), \quad t = c; \qquad\qquad = X_{t-}(\omega), \quad 0 < t \leqq c.$$

Finally we put for $\Gamma \subset \{X_{c-} \in X\}$,

$$\gamma\Gamma = \nu_l^{-1}\boldsymbol{q}\nu_r\Gamma \text{ [12]}. \tag{3.13}$$

Denote by $\mathscr{B}_{(0,c)}$ the restriction to $\{X_{c-} \in X\}$ of the σ-field $\bigvee_{t<c} \mathscr{B}_t^0$.

THEOREM 3.3[13]. *If* $\Gamma \subset \mathscr{B}_{(0,c)}$, *then* $\gamma\Gamma \in \mathscr{B}_{(0,c)}$ *and*

$$\int_X P_x(\gamma\Gamma)m(dx) = \int_X P_x(\Gamma)m(dx) \leqq +\infty. \tag{3.14}$$

Proof. It suffices to prove the theorem for the set

$$\Gamma = \{X_0 \in E_0, X_{t_1} \in E_1, \ldots, X_{t_{n-1}} \in E_{n-1}, X_{c-} \in E_n\}, \tag{3.15}$$

where $0 < t_1 < t_2 < \cdots < t_{n-1} < c$ and $E_0, \ldots, E_n$ are Borel subsets of X. Clearly

$$\gamma\Gamma = \{X_0 \in E_n, X_{(c-t_{n-1})-} \in E_{n-1}, \ldots, X_{(c-t_1)-} \in E_1, X_{c-} \in E_0\}.$$

By Lemma 3.2 we have for sufficiently small $\varepsilon>0$ and $\delta>0$,

$$\int_X E_x(f_0(X_0)f_1(X_{t_1+\varepsilon})\cdots f_{n-1}(X_{t_n+\varepsilon})f_n(X_{c-\delta}))m(dx)$$
$$= \int_X E_x(f_n(X_0)f_{n-1}(X_{c-t_{n-1}-\varepsilon-\delta})\cdots f_1(X_{c-t_1-\varepsilon-\delta})f_0(X_{c-\delta}))m(dx).$$

Assume that $f_0, f_1, \ldots, f_n \in C_0^+(X)$ and let ε and δ tend to zero. Then after a routine procedure we get the equality (3.14) for Γ of (3.15).

---

(12) The operator γ was introduced by E. B. Dynkin [7, IV, §4] in connection with the multidimensional Brownian motion. The present author used a similar notion in the analysis of a reflecting Brownian motion [8]. However the notion γ defined in [8, p. 206] was insufficient for the situation there and he likes to correct it here: it must be replaced by the present definition (3.13).

(13) Cf. Theorem 4.12 of [7].

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Here we give two applications of Theorem 3.3.

According to the proof of IV, T52 of P. A. Meyer [15] we observe that, for any Borel set $B \subset X$ and $t > 0$, the set $\{\sigma'_B < t\}$ is in the completion of the σ-field $\mathscr{B}_t^0$ relative to P_μ, μ being an arbitrary probability measure on X. This fact will be used in the proof of the following theorem:

THEOREM 3.4([14]). *For q.e.* $x \in X$,

$$P_x(X_{t-} \in X - X_b \text{ for every } t \in (0, \zeta)) = 1. \tag{3.16}$$

COROLLARY. *If* T_n *are increasing stopping times with limit* T, *then*

$$P_x\Big(\lim_{n \to +\infty} X_{T_n} = X_T, T < \zeta\Big) = P_x(T < \zeta) \tag{3.17}$$

for q.e. $x \in X$.

This corollary is immediate from Theorem 3.4 and property (**M**.3). Here, the exceptional points $x \in X$ do not depend on the choice of $\{T_n\}$.

Proof of Theorem 3.4. Put $\Gamma_c = \{\sigma'_{X_b} < c, X_{c-} \in X\}$ and $\Lambda_c = \{X_{t-} \in X_b$ for some $t \in (0, c), X_{c-} \in X\}$. X_b is a Borel set (actually an F_σ-set) and $P_x(\Gamma_c) = 0$, $x \in X$, according to (**M**.4). Hence there exists a set $\Gamma'_c \in B_{(0,c)}$ such that $\Gamma_c \subset \Gamma'_c$ and $\int_X P_x(\Gamma'_c) m(dx) = 0$. Since $\Lambda_c = \gamma\Gamma_c \subset \gamma\Gamma'_c$, Theorem 3.3 implies that $P_x(\Lambda_c) = 0$ for m-a.e. $x \in X$. Notice that $\Lambda = \{X_{t-} \in X_b$, for some $t \in (0, \zeta)\} = \bigcup_{c \in Q} \Lambda_c$, Q being the set of all positive rational numbers. We have therefore $P_x(\Lambda) = 0$ for m-a.e. $x \in X$. On the other hand $u(x) = P_x(\Lambda)$ is an excessive function: u is universally measurable and

$$\exp(-\alpha_0 s) P_s u(x) = \exp(-\alpha_0 s) P_x\{X_{t-} \in X_b \text{ for some } t \in (s, \zeta)\} \uparrow u(x), \quad s \downarrow 0.$$

Thus $u(x) = 0$ for q.e. $x \in X$ in view of Theorem 3.1 and Theorem 1.2.

Finally for an open or a closed set $A \subset X$ we define

$$P_t^0(x, E) = P_x(X_t \in E, t < \sigma_A), \qquad t \geqq 0. \tag{3.18}$$

$\{P_t^0(x, E), t \geqq 0\}$ is a sub-Markov transition probability on X. Since $P_t^0 f(x) = \int_X P_t^0(x, dy) f(y)$ is right continuous in $t \geqq 0$ for $f \in C_0(X)$, $P_t^0(x, E)$ is measurable in $(t, x) \in [0, +\infty) \times X$ for each fixed Borel set $E \subset X$. Put

$$R_\alpha^0(x, E) = \int_0^{+\infty} e^{-\alpha t} P_t^0(x, E)\, dt. \tag{3.19}$$

$\{R_\alpha^0(x, E), \alpha > 0\}$ is a sub-Markov resolvent on X. Obviously $R_\alpha^0 f(x) = \int_X R_\alpha^0(x, dy) f(y)$ satisfies

$$R_\alpha^0 f(x) = E_x\Big(\int_0^{\sigma_A} e^{-\alpha t} f(X_t)\, dt\Big), \qquad x \in X, \tag{3.20}$$

if f is Borel measurable and $R_\alpha^0 f(x)$ is well defined.

([14]) Cf. Lemma 3.7(iv) of [8].

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THEOREM 3.5[15]. *For an open or a closed set* $A \subset X$, *the transition probability and the resolvent kernel defined by* (3.18) *and* (3.19) *are m-symmetric*:

$$\int_X f(x)\cdot P_t^0 g(x)m(dx) = \int_X P_t^0 f(x)\cdot g(x)m(dx), \qquad t \geqq 0, \tag{3.21}$$

$$\int_X f(x)\cdot R_\alpha^0 g(x)m(dx) = \int_X R_\alpha^0 f(x)\cdot g(x)m(dx), \qquad \alpha > 0, \tag{3.22}$$

for $f, g \in C^+(X)$.

Proof. (3.22) is a direct consequence of (3.21). Let us show (3.21) when A is an open set. Fix a positive number c and Borel sets F, $G \subset X$. Consider the set $\Gamma = \{X_0 \in F, c \leqq \sigma_A, X_{c-} \in G\}$. Since A is open, $\Gamma \in \mathscr{B}_{(0,c)}$ and

$$\gamma\Gamma = \{X_0 \in G, c \leqq \sigma_A, X_{c-} \in F\}.$$

Noting that $P_x(X_0 = x) = 1$ m-a.e., we get by Theorem 3.3,

$$\int_F P_x(X_{c-} \in G, c \leqq \sigma_A)m(dx) = \int_G P_x(X_{c-} \in F, c \leqq \sigma_A)m(dx)$$

from which follows the equality

$$\int_X f(x)E_x(g(X_{c-}), c \leqq \sigma_A)m(dx) = \int_X E_x(f(X_{c-}), c \leqq \sigma_A)g(x)m(dx)$$

for $f, g \in C_0(X)$. By putting $c = t + 1/n$ and letting n tend to infinity in this equality, we obtain (3.21).

Next suppose that A is closed and choose a sequence of open sets A_n such as $A_n \supset \bar{A}_{n+1} \supset A$ and $A_n \downarrow A$. By virtue of the quasi-left continuity (Corollary to Theorem 3.4)

$$P_x(\sigma_{A_n} \uparrow \sigma_A) = 1, \qquad \text{q.e. } x \in X. \tag{3.23}$$

Hence (3.21) for the closed set A follows from those for open sets A_n.

3.4. *Probabilistic decomposition of* $(\mathscr{F}^*, \mathscr{E}^\alpha)$. Let A be an open or a closed set of X. We put

$$\begin{aligned} H_\alpha(x, E) &= E_x(e^{-\alpha\sigma_A}; X\sigma_A \in E), \\ H'_\alpha(x, E) &= E_x(e^{-\alpha\sigma'_A}; X\sigma'_A \in E). \end{aligned} \tag{3.24}$$

In this subsection we will consider the kernels H_α and H'_α on X as well as the localized resolvent R_α^0 defined by (3.19) and reveal the roles they play in the strongly regular D-space $(\mathscr{F}, \mathscr{E})$.

Any Borel measurable function on X is extended to $\bar{X} = X \cup \partial$ by defining its value at ∂ to be zero. It holds under this convention that $H_\alpha f(x) = E_x(e^{-\alpha\sigma_A} f(X_{\sigma_A}))$ and $R_\alpha f(x) = E_x(\int_0^\zeta e^{-\alpha t} f(X_t)\,dt)$ for all $x \in \bar{X}$ when f is Borel measurable and

[15] Cf. Lemma 14.1 of [7].

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$H_\alpha f(x)=\int_X H_\alpha(x, dy)f(y)$ and $R_\alpha f(x)$ are well defined for $x \in X$. $H'_\alpha f$ can be expressed in a similar way. If a Borel measurable function f is excessive, then $H_{\alpha_0} f$ is supermedian and $H'_{\alpha_0} f$ is excessive. When A is open, $\sigma_A=\sigma'_A$ and $H_\alpha=H'_\alpha$.

LEMMA 3.3. *If f is a bounded Borel measurable function on X, then for each $x \in X$*

$$R_\alpha f(x) = R^0_\alpha f(x)+H_\alpha R_\alpha f(x), \qquad \alpha > 0, \tag{3.25}$$

$$H_\alpha f(x)-H_\beta f(x)+(\alpha-\beta)R^0_\alpha H_\beta f(x) = 0, \qquad \beta > 0. \tag{3.26}$$

Proof. We can use the strong Markov property (**M**.2) to obtain the formula (3.25). (3.26) is a result of the Markov property.

By virtue of Theorem 3.5, the kernel $\{R^0_\alpha(x, E), \alpha>0\}$ is an m-symmetric sub-Markov resolvent kernel on X. Let $(\mathcal{F}^{(0)}, \mathcal{E}^{(0)})$ be the Dirichlet space generated by $\{R^0_\alpha(x, E), \alpha>0\}$: $R^0_\alpha(L^2(X; m) \cap C(X))\subset\mathcal{F}^{(0)}$ and the function $R^0_\alpha f, f\in L^2(X; m) \cap C(X)$, satisfies

$$\mathcal{E}^{(0),\alpha}(R^0_\alpha f, v) = (f, v)_X \quad \text{for every } v \in \mathcal{F}^{(0)}. \tag{3.27}$$

THEOREM 3.6. *$\mathcal{F}^{(0)}\subset\mathcal{F}$ and $\mathcal{E}^{(0)}$ is the restriction of $\mathcal{E}$ to $\mathcal{F}^{(0)}$:*

$$\mathcal{E}^{(0)}(u, v) = \mathcal{E}(u, v), \qquad u, v \in \mathcal{F}^{(0)}. \tag{3.28}$$

Proof. Denote by $\mathfrak{B}$ the set of all bounded Borel measurable functions on X. Since $R^0_{\alpha_0}(L^2 \cap C)$ is dense in $\mathcal{F}^{(0)}$ with metric $\mathcal{E}^{(0),\alpha_0}$, it suffices to prove the following:

$$\text{For every } f\in \mathfrak{B} \cap L^2,\ R^0_{\alpha_0} f\in\mathcal{F} \text{ and } \mathcal{E}^{\alpha_0}(R^0_{\alpha_0} f, R^0_{\alpha_0} f) = (f, R^0_{\alpha_0} f)_X. \tag{3.29}$$

We can observe by Lemma 1 of [9] that $u \in L^2(X; m)$ is an element of $\mathcal{F}$ if and only if $\lim_{\beta\to+\infty} \beta(u-\beta R_{\beta+\alpha_0}u, u)_X$ is finite and in this case the limit is equal to $\mathcal{E}^{\alpha_0}(u, u)$. Thus the relation (3.29) is equivalent to

$$\lim_{\beta\to+\infty} \beta(u-\beta R_{\beta+\alpha_0}u, u)_X = (f, u)_X \tag{3.30}$$

where $u=R^0_{\alpha_0} f, f\in \mathfrak{B} \cap L^2$.

Let us show (3.30) for an open set A. By (3.25) and the resolvent equation for R^0_α,

$$\beta(u-\beta R_{\beta+\alpha_0}u, u)_X = \beta(R^0_{\beta+\alpha_0} f, u)_X-\beta^2(H_{\beta+\alpha_0}R_{\beta+\alpha_0}u, u)_X.$$

Since R^0_α is symmetric and its L^2-norm is no greater than $1/\alpha$,

$$\beta(R^0_{\beta+\alpha_0} f, u)_X = (f, R^0_{\alpha_0} f-R^0_{\beta+\alpha_0} f)_X \to (f, u)_X, \qquad \beta\to+\infty.$$

We have to prove

$$\beta^2(H_{\beta+\alpha_0}R_{\beta+\alpha_0}u, u)_X \to 0, \qquad \beta\to+\infty. \tag{3.31}$$

We may assume without loss of generality that f is nonnegative. By the symmetry of $R^0_{\alpha_0}$ and the formula (3.26), the left-hand side of (3.31) is no greater than

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$\beta(H_{\alpha_0}R_{\beta+\alpha_0}u, f)_X$. Notice that u is the difference of two excessive functions: $u = R_{\alpha_0}f - H_{\alpha_0}(R_{\alpha_0}f)$. Hence $\lim_{\beta\to+\infty}\beta(H_{\alpha_0}R_{\beta+\alpha_0}u, f)_X = (H_{\alpha_0}u, f)_X$. However, since $\sigma_A(\theta_{\sigma_A}\omega) = 0$ for $\omega \in \{\sigma_A < \zeta\}$, we have $H_{\alpha_0}\cdot H_{\alpha_0}(R_{\alpha_0}f)(x) = H_{\alpha_0}(R_{\alpha_0}f)(x)$, $x \in X$, which means $H_{\alpha_0}u(x) = 0$, $x \in X$, yielding (3.31). Here θ denotes the usual translation operator on W (cf. [16]).

Thus (3.30) and hence (3.29) are established when A is open. (3.29) for a closed set A is now to be proved. Find open sets A_n such that $A_n \supset \bar{A}_{n+1}$ and $A_n \downarrow A$. Denote by ${}^nR_\alpha^0$ and ${}^nH_\alpha$ kernels corresponding to A_n. Put $u_n = {}^nR_{\alpha_0}^0 f$ for a non-negative $f \in \mathfrak{B} \cap L^2$. Then owing to (3.20) and (3.23), u_n increases to $R_{\alpha_0}^0 f$ q.e. as $n \to +\infty$. Now observe the following equality: for $m \leqq n$,

$$\beta(u_n - \beta R_{\beta+\alpha_0}u_n, u_m)_X$$
$$= \beta({}^mR_{\beta+\alpha_0}^0 f, u_m)_X + \beta({}^mH_{\beta+\alpha_0}\,{}^nR_{\beta+\alpha_0}^0 f, u_m)_X - \beta^2({}^nH_{\beta+\alpha_0}R_{\beta+\alpha_0}u_n, u_m)_X.$$

Here we used the identity ${}^nR_\beta^0 f = {}^mR_\beta^0 f + {}^mH_\beta\,{}^nR_\beta^0 f$. Let β tend to infinity. Then the first term of the right-hand side of the equality tends to $(f, u_m)_X$ as was proved earlier. The second term is, in view of the symmetry of ${}^mR_{\alpha_0}^0$ and (3.26), no greater than $({}^mH_{\alpha_0}\,{}^nR_{\beta+\alpha_0}^0 f, f)_X$, which decreases to zero. The absolute value of the last term is no greater than $\beta^2({}^nH_{\beta+\alpha_0}R_{\beta+\alpha_0}u_n, u_n)_X$, which also tends to zero by (3.31). What we have proved is $\mathscr{E}^{\alpha_0}(u_n, u_m) = (f, u_m)_X$, $m \leqq n$, which in turn tells us that u_n converges to $R_{\alpha_0}^0 f$ in $\mathscr{E}^{\alpha_0}$-norm, arriving at (3.29) for the closed set A. The proof of Theorem 3.6 is complete.

On account of Theorem 3.6, $\mathscr{F}^{(0)}$ is a closed subspace of $\mathscr{F}$ with metric $\mathscr{E}^{\alpha_0}$. Let us denote by $\mathscr{H}_{\alpha_0}$ the orthogonal complement of $\mathscr{F}^{(0)}$ in the Hilbert space $(\mathscr{F}, \mathscr{E}^{\alpha_0})$.

LEMMA 3.4[16]. *If u is either an element of $\mathscr{F} \cap C(X)$ or of the form $R_{\alpha_0}h$, $h \in \mathfrak{B} \cap L^2$, then $H_{\alpha_0}u$ is quasi-continuous and*

$$H_{\alpha_0}u = P_{\mathscr{H}_{\alpha_0}}u, \tag{3.32}$$

$P_{\mathscr{H}_{\alpha_0}}$ being the projection on the space $\mathscr{H}_{\alpha_0}$.

Proof. Let us first show this for $u = R_{\alpha_0}h$, $h \in \mathfrak{B} \cap L^2$. By (3.25), (3.29) and Lemma 3.1, we have

$$\mathscr{E}^{\alpha_0}(H_{\alpha_0}R_{\alpha_0}h, R_{\alpha_0}^0 g) = \mathscr{E}^{\alpha_0}(R_{\alpha_0}h - R_{\alpha_0}^0 h, R_{\alpha_0}^0 g)$$
$$= (h, R_{\alpha_0}^0 g)_X - (h, R_{\alpha_0}^0 g)_X = 0$$

for every $g \in \mathfrak{B} \cap L^2$. This means that the formula (3.25) represents the direct decomposition of $u = R_{\alpha_0}h$ into the sum of elements of $\mathscr{F}^{(0)}$ and $\mathscr{H}_{\alpha_0}$, getting (3.32). The quasi-continuity of $H_{\alpha_0}u$ is clear if A is open, because it is then the difference

[16] We have further $H_{\alpha_0}u = H'_{\alpha_0}u$, q.e. for $u \in \mathscr{F} \cap C(X)$. Since $H'_{\alpha_0}R_{\alpha_0}u$ is the regularization of the quasi-continuous function $H_{\alpha_0}R_{\alpha_0}u$, these two are equal q.e. From this we can get the desired equality.

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of excessive functions $H_{\alpha_0}(R_{\alpha_0}h^+)$ and $H_{\alpha_0}(R_{\alpha_0}h^-)$ of $\mathscr{F}$ to which we can apply Theorem 3.1. Coming to the case when A is closed, consider open sets A_n and corresponding kernels ${}^nR^0_{\alpha_0}$ and ${}^nH_{\alpha_0}$ just as in the second part of the proof of Theorem 3.6. Then the quasi-continuous functions ${}^nH_{\alpha_0}u=u-{}^nR^0_{\alpha_0}h$ converge to $u-R^0_{\alpha_0}h=H_{\alpha_0}u$ q.e. on X as well as in $\mathscr{E}^{\alpha_0}$-norm. Hence the latter must be quasi-continuous (Lemma 1.2).

Next take any $u\in\mathscr{F}\cap C(X)$. Since, for each $\beta>0$, $R_\beta u$ is equal to $R_{\alpha_0}h$ with some $h\in\mathfrak{B}\cap L^2$, we have $H_{\alpha_0}(\beta R_\beta u)=P_{\mathscr{H}_{\alpha_0}}(\beta R_\beta u)$.

By Lemma 2.1 of [10], $\beta R_\beta u\to u$ and hence

$$P_{\mathscr{H}_{\alpha_0}}(\beta R_\beta u)\to P_{\mathscr{H}_{\alpha_0}}u, \qquad \beta\to+\infty,$$

with respect to $\mathscr{E}^{\alpha_0}$-norm. On the other hand,

$$\begin{aligned}\lim_{\beta\to+\infty} H_{\alpha_0}(\beta R_\beta u)(x) &= \lim_{\beta\to+\infty} E_x(\exp(-\alpha_0\sigma_A)\beta G_\beta u(X_{\sigma_A});\ X_{\sigma_A}\notin X_b)\\ &= E_x(\exp(-\alpha_0\sigma_A)u(X_{\sigma_A})) = H_{\alpha_0}u(x), \qquad x\in X,\end{aligned}$$

by virtue of property (**M**.4). Thus we get (3.32). $H_{\alpha_0}u$ is quasi-continuous because it is the limit of quasi-continuous functions $H_{\alpha_0}(\beta R_\beta u)$ in $\mathscr{E}^{\alpha_0}$-norm as well as in the pointwise sense.

LEMMA 3.5. *Suppose that A is compact. Any quasi-supermedian function belonging to the space $\mathscr{H}_{\alpha_0}$ is a potential of a measure whose support is concentrated on A.*

Proof. Assume that u is quasi-supermedian and $u\in\mathscr{H}_{\alpha_0}$. Then $u\in\mathscr{F}^*$ and we have by Lemma 1.3 that $\mathscr{E}^{\alpha_0}(u,v)\geqq 0$ for all $v\in\mathscr{F}^*$ such as $v\geqq 0$ q.e. Let v be any function of $\mathscr{F}\cap C(X)$ which is nonnegative on A. By Lemma 3.4, $\mathscr{E}^{\alpha_0}(u,v)$ $=\mathscr{E}^{\alpha_0}(u,H_{\alpha_0}v)$ which is nonnegative because $H_{\alpha_0}v(x)\geqq 0$, $x\in X$. According to Theorem 1.6, we arrive at Lemma 3.5.

The next two are the main theorems of this subsection.

THEOREM 3.7([17]). *Put*

$$\mathscr{F}^*_{X-A}=\{u\in\mathscr{F}^*;\ u=0\ q.e.\ on\ A\}. \tag{3.33}$$

Then $\mathscr{F}^*_{X-A}=(\mathscr{F}^{(0)})^*$, where $(\mathscr{F}^{(0)})^*$ denotes the set of all quasi-continuous modifications of functions in the space $\mathscr{F}^{(0)}$.

Proof. On account of Theorem 3.6 and Lemma 1.2, $(\mathscr{F}^{(0)})^*$ and $\mathscr{F}^*_{X-A}$ are closed subspaces of the Hilbert space $(\mathscr{F}^*,\mathscr{E}^{\alpha_0})$. If $f\in L^2\cap C$, then $R^0_{\alpha_0}f(x)=0$ on $A-X_b$ and $R^0_{\alpha_0}f=R^0_{\alpha_0}f-H_{\alpha_0}(R_{\alpha_0}f)$ is quasi-continuous in view of Lemma 3.4. Hence $\mathscr{F}^*_{X-A}$ contains $R^0_{\alpha_0}(L^2\cap C)$ which is dense in $(\mathscr{F}^{(0)})^*$. Thus, $\mathscr{F}^*_{X-A}\supset(\mathscr{F}^{(0)})^*$.

([17]) We can assert even more: for any nonnegative universally measurable function $f\in L^2(X;m)$, $R^0_\alpha f$ belongs to the space $\mathscr{F}^*_{X-A}$ and the equation $\mathscr{E}^\alpha(R^0_\alpha f,v)=(f,v)_X$ holds for every $v\in\mathscr{F}^*_{X-A}$. In view of the proof of Theorem 3.7, this is true for $f\in L^2\cap C(X)$. Now the general case can be obtained exactly in the same manner as in the proof of Lemma 3.1.

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Let us prove the converse inclusion. Denote by $\mathscr{H}^*_{\alpha_0}$ the space of all quasi-continuous modifications of functions of the $\mathscr{H}_{\alpha_0}$. It suffices to show that $\mathscr{H}^*_{\alpha_0}$—the orthogonal complement of $(\mathscr{F}^{(0)})^*$ in $(\mathscr{F}^*, \mathscr{E}^{\alpha_0})$—is orthogonal to $\mathscr{F}^*_{X-A}$. Since $H_{\alpha_0}R_{\alpha_0}(L^2 \cap C)$ is in $\mathscr{H}^*_{\alpha_0}$ by Lemma 3.4 and dense there, it is enough to prove

$$\mathscr{E}^{\alpha_0}(H_{\alpha_0}R_{\alpha_0}f, v) = 0, \qquad f \in L^2 \cap C^+, v \in \mathscr{F}^*_{X-A}. \tag{3.34}$$

Assume first that A is compact. Since $H_{\alpha_0}R_{\alpha_0}f$ is supermedian, it is quasi-supermedian (Remark 3.1) and is a potential of a measure $\mu \in M_0^+$ with $S\mu \subset A$ by virtue of Lemma 3.5. Owing to Theorem 1.5, the left-hand side of (3.34) is equal to $\int_A v(x)\mu(dx) = 0$.

In the case when A is open or closed, we can find compact sets A_n such that $A_n \uparrow A$. Denote by ${}^nH_{\alpha_0}$ the kernel corresponding to A_n. It is easy to see that ${}^nH_{\alpha_0}R_{\alpha_0}f$ then converges to $H_{\alpha_0}R_{\alpha_0}f$ increasingly and in $\mathscr{E}^{\alpha_0}$-norm. If $v \in \mathscr{F}^*_{X-A}$, then $v \in \mathscr{F}^*_{X-A_n}$ and the left-hand side of (3.34) is equal to $\lim_n \mathscr{E}^{\alpha_0}({}^nH_{\alpha_0}R_{\alpha_0}f, v) = 0$. The proof of Theorem 3.7 is complete.

Owing to this theorem, we get an important conclusion about a local property of the space $\mathscr{H}^*_{\alpha_0}$: Any function in $\mathscr{H}^*_{\alpha_0}$ is determined by its restriction to the set A. In fact the space $(\mathscr{F}^*, \mathscr{E}^{\alpha_0})$ is expressed as a direct sum $\mathscr{F}^* = \mathscr{F}^*_{X-A} \oplus \mathscr{H}^*_{\alpha_0}$ and so any function of $\mathscr{H}^*_{\alpha_0}$ which vanishes q.e. on A should vanish q.e. on X. Keeping this in mind let us prove the next theorem.

THEOREM 3.8. *Suppose that A is a compact set or an open set belonging to the class $\mathscr{U}$. Then we have for q.e. $x \in X$,*

$$e_A(x) = E_x(\exp(-\alpha_0\sigma_A); \sigma_A < \zeta) = E_x(\exp(-\alpha_0\sigma'_A); \sigma'_A < \zeta). \tag{3.35}$$

Here e_A denotes the equilibrium potential of A defined in subsection 1.5.

Proof. Suppose that A is compact. By (1.25) and (1.26), e_A is an element of $\mathscr{H}^*_{\alpha_0}$ which is equal to 1 q.e. on A. The function $u(x) = E_x(\exp(-\alpha_0\sigma_A); \sigma_A < \zeta)$ has the same property. Indeed $u(x) = 1$, $x \in A - X_b$, and hence q.e. on A by Theorem 3.2. u can be expressed as $H_{\alpha_0}f(x)$ with any function $f \in \mathscr{F} \cap C(X)$ such as $f(x) = 1$ for $x \in A$. Hence $u \in \mathscr{H}^*_{\alpha_0}$ by Lemma 3.4. Thus we get the first equality of (3.35). The third term of (3.35) is the regularization of the second. Hence they are equal q.e. on X in view of Theorem 3.1.

When A is an open set of the class $\mathscr{U}$, (3.35) is also obtained by approximating A with a sequence of compact sets increasing to A and by noting (1.22) and (1.23).

3.5. *Regularity of quasi-continuous transformations along sample paths.* Let us begin with a lemma which states a probabilistic feature of polar sets.

LEMMA 3.6. *Let A be a Borel polar set and $\{A_n\}$ be a decreasing sequence of open sets such that $A_n \supset A$ and* $\lim_{n\to+\infty} \mathrm{Cap}(A_n) = 0$. *Then the equalities*

$$P_x(X_t \text{ or } X_{t-} \in A \text{ for some } t \geqq 0) = 0, \tag{3.36}$$

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(3.37) $$P_x\left(\sigma_{A_n} = \zeta \textit{ for some } n \textit{ or } \lim_{n\to+\infty} \sigma_{A_n} = +\infty\right) = 1$$

hold for q.e. $x \in X$.

Proof. (3.36) is a consequence of (3.37). In order to show (3.37), let us put $u_n(x) = E_x(\exp(-\alpha_0\sigma_{A_n}); \sigma_{A_n} < \zeta)$. By Theorem 3.8 and (1.21) we have $\mathscr{E}^{\alpha_0}(u_n, u_n) = \mathrm{Cap}(A_n)$ which decreases to zero by the assumption. Since u_n is quasi-continuous, Lemma 1.2 implies that $\lim_{n\to+\infty} u_n(x) = 0$ q.e. on X. We arrive at (3.37) on account of the identity

$$\lim_{n\to+\infty} u_n(x) = E_x\left(\exp\left(-\alpha_0 \lim_n \sigma_{A_n}\right); \bigcap_{n=1}^{+\infty} \{\sigma_{A_n} < \zeta\}\right).$$

Since the branch set X_b is polar (Theorem 3.2), we can apply the above lemma to X_b to get

(3.38) $$P_x(X_{t-} \in X_b \text{ for some } t \geqq 0) = 0,$$

for q.e. $x \in X$. Notice that (3.38) is stronger than (3.16). We will further strengthen the assertions of Lemma 3.6 as follows:

THEOREM 3.9. *Under the same assumption as in Lemma* 3.6, *there exists a Borel polar set B including A such that the equalities* (3.37) *and*

(3.39) $$P_x(X_t \textit{ or } X_{t-} \in B \textit{ for some } t \geqq 0) = 0$$

are simultaneously valid for every $x \in X - B$.

Proof. By virtue of Lemma 3.6, we see that (3.36) and (3.37) are valid for every $x \in X$ except on a polar set N_1. By replacing N_1 with a G_δ-polar set including it if necessary, we may assume that N_1 is a Borel set. Apply again Lemma 3.6 to the Borel polar set $A \cup N_1$. We get

(3.40) $$P_x(X_t \text{ or } X_{t-} \in A \cup N_1 \text{ for some } t \geqq 0) = 0$$

for every $x \in X$ except on a Borel polar set N_2. Repeating the same argument, we have a sequence $\{N_k\}$ of Borel polar sets such that for each k the equality

(3.41) $$P_x(X_t \text{ or } X_{t-} \in A \cup N_1 \cup \cdots \cup N_k \text{ for some } t \geqq 0) = 0$$

holds for every $x \in X - N_{k+1}$. Put $B = A \cup (\bigcup_{k=1}^{+\infty} N_k)$. B is polar by Theorem 1.1. If $x \in X - B$, then (3.41) is valid for every k. Letting k tend to infinity, we get (3.39).

Turning to the main task of this subsection, let us consider a quasi-continuous function q on X taking values in some nice topological space. We fix a decreasing sequence $\{A_n\}$ of open subsets of X such that q is continuous on each $X - A_n$ and

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$\lim_{n\to+\infty}$ Cap $(A_n)=0$. By virtue of Theorem 3.9, there is a Borel polar set B such that $B\supset\bigcap_{n=1}^{+\infty} A_n$ and equalities (3.37) and (3.39) hold for every $x\in\bar{X}-B$: if we put

$$W_{11} = \{\omega\in W;\ X_t(\omega) \text{ and } X_{t-}(\omega)\in\bar{X}-B \text{ for all } t\geqq 0\},$$
$$\text{(3.42)}\qquad W_{12} = \left\{\omega\in W;\ \sigma_{A_n}(\omega) = \zeta(\omega) \text{ for some } n \text{ or } \lim_{n\to+\infty}\sigma_{A_n}(\omega) = +\infty\right\},$$

then

$$\text{(3.43)}\qquad P_x(W_1) = 1, \qquad x\in\bar{X}-B,$$

where W_1 denotes the set $W_{11}\cap W_{12}$.

Now let us put

$$\text{(3.44)}\qquad \mathscr{B}^1 = \mathscr{B}\cdot W_1, \quad \mathscr{B}_t^1 = \mathscr{B}_t\cdot W_1, \qquad t\geqq 0,$$

and denote the restrictions of measures P_x, $x\in\bar{X}-B$, to $\mathscr{B}^1$ by P_x again. We also maintain the notion X_t to express its restriction to W_1. It is then clear that the process $M_1=\{W_1, \mathscr{B}^1, \mathscr{B}_t^1, X_t, P_x\}$ is a right continuous Markov process with state space $\bar{X}-B$.

THEOREM 3.10. (i) *The process M_1 is a strong Markov process with state space $\bar{X}-B$. The resolvent kernel of M_1 is the restriction to $X-B$ of the Ray resolvent kernel $\{R_\alpha(x, E), \alpha>0\}$ of the original process M. Further*

$$\text{(3.45)}\qquad R_\alpha(x, B) = 0, \qquad x\in X-B.$$

(ii) *The σ-field $\mathscr{B}^1$ (resp. $\mathscr{B}_t^1$) is the completion of $\mathscr{B}^0\cdot W_1$ (resp. $\mathscr{B}_t^0\cdot W_1$ in $\mathscr{B}^1$) with respect to the family of measures*

$$\left\{P_\mu(\cdot) = \int_{\bar{X}-B}\mu(dx)P_x(\cdot);\ \mu \text{ is a finite measure on } \bar{X}-B\right\}.$$

(iii) *Assume an additional condition that*

$$\text{(3.46)}\qquad X_b\subset\bigcap_{n=1}^{+\infty} A_n.$$

Then the process M_1 is quasi-left continuous on $[0, +\infty)$: *if $\{\mathscr{B}_t^1\}$-stopping times T_n increase to T, then $\lim_{n\to+\infty} X_{T_n}=X_T$ P_x-a.e. on $\{T<+\infty\}$ for every $x\in\bar{X}-B$.*

(iv) *Assume an additional condition that*

(3.47) *q can be extended to $\bar{X}-\bigcap_{n=1}^{+\infty} A_n$ in such a way that the restriction of q to $\bar{X}-A_n$ is continuous there for every n.*

Then, for each $\omega\in W$, $q(X_t(\omega))$ and $q(X_{t-}(\omega))$ are well defined and $Y_t(\omega)=q(X_t(\omega))$ is right continuous in $t\geqq 0$. $Y_t(\omega)$ has the left limit at every $t>0$ with

$$\text{(3.48)}\qquad Y_{t-}(\omega) = q(X_{t-}(\omega)).$$

(v) *Assume that both the conditions* (3.46) *and* (3.47) *are valid. If $\{\mathscr{B}_t^1\}$-stopping times T_n increase to T, then $\lim_{n\to+\infty} Y_{T_n}=Y_T$ P_x-a.e. on $\{T<+\infty\}$ for every $x\in\bar{X}-B$.*

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Proof. (i) The latter assertion together with (3.45) is evident. $\boldsymbol{M}_1$ is now a Markov process on $\bar{X}-B$ with right continuous sample paths and right continuous σ-fields $\{\mathscr{B}_t^1\}$. Obviously the right continuity of $R_\alpha f$, $f \in C(X)$, along the sample paths is preserved under the transfer from $\boldsymbol{M}$ to $\boldsymbol{M}_1$. Thus, $\boldsymbol{M}_1$ is a strong Markov process on $\bar{X}-B$.

(ii) Take a set $\Lambda \in \mathscr{B}^1$ and consider its property with respect to the original process $\boldsymbol{M}$. Property (**M**.1) implies that

$$P_x(\Lambda) \leqq P_x(W_{11}\cdot\{X_0 = x\}) = 0 \quad \text{if } x \in B - X_b$$

and

$$P_x(\Lambda) = \int_{\bar{X}-X_b} \mu(x, dy)P_y(\Lambda) = \int_{\bar{X}-B} \mu(x, dy)P_y(\Lambda) \quad \text{if } x \in X_b.$$

Therefore, for any finite measure μ on $\bar{X}$, $P_\mu(\Lambda)=P_{\mu_1}(\Lambda)$ with a finite measure μ_1 supported by the set $\bar{X}-B$. This means statement (ii).

(iii) By the hypothesis (3.46),

$$P_x(X_{t-} \in X_b \text{ for some } t \geqq 0) = 0, \qquad x \in \bar{X}-B.$$

Combining this with statement (i), we can prove the quasi-left continuity of $\boldsymbol{M}_1$ on $[0, +\infty)$ exactly in the same way as in [13, §2] (see [16, XIV, T15] for more information).

(iv) Fix an $\omega \in W_1$. If $\zeta(\omega) < +\infty$, then $\sigma_{A_n}(\omega)=\zeta(\omega)$ for some n and hence $X_t(\omega)$ and $X_{t-}(\omega)$ belong to the closed set $\bar{X}-A_n$ for all $t \geqq 0$. Hence we get the desired properties of Y_t by the hypothesis (3.47). If $\zeta(\omega)= +\infty$, then for any $t \geqq 0$ there exists an A_n such that $\sigma_{A_n}(\omega) > t$. Hence we get the desired conclusion in this case also.

(v) By the preceding two statements (iii) and (iv), we have $\lim_{n\to+\infty} Y_{T_n} = q(\lim_{n\to+\infty} X_{T_n})=q(X_T)= Y_T$ P_x-a.e. on $\{T< +\infty\}$ for every $x \in \bar{X}-B$. The proof of Theorem 3.10 is complete.

Remark 3.2. Here we give some remarks on the hypotheses (3.46) and (3.47) in Theorem 3.10. We can assume (3.46) without loss of generality because the branch set X_b is polar. Assertions (iv) and (v) are still valid up to the lifetime ζ without assuming (3.47). Condition (3.47) is satisfied by two important cases in which we have interest. Theorem 1.3 implies that each numerical function $u \in \mathscr{F}$ has a modification $\tilde{u}$ which is not only quasi-continuous but also satisfies (3.47) by setting $\tilde{u}(\partial)=0$. In case that q is a quasi-homeomorphism from X to the underlying space X' of some regular D-space, q satisfies (3.47) if we put $q(\partial)=\partial'$. We can see this immediately from the definition of quasi-homeomorphism.

Theorem 3.10 will be the key to prove Theorem 4.1. Here we state another application of Theorem 3.10. Consider a function u defined q.e. on X. Let us agree to say u to be *Borel* (resp. *universally*) *measurable* if there is a Borel (resp. universally) measurable function $\tilde{u}$ on X such as $u=\tilde{u}$ q.e. We call *u finely continuous q.e.* if

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there exists a nearly Borel polar set B satisfying the following: $B \supset X_b$, $X-B$ is a fine open set and u is finely continuous at each point $x \in X-B$, fine topology being defined in terms of M([18]). For instance take a quasi-continuous function u (not necessarily real valued). Then u is clearly Borel measurable in the above sense. Furthermore from the first and second remarks in Remark 3.2, we can see that u is finely continuous q.e. Thus we get the first part of the following theorem.

THEOREM 3.11. (i) *Every quasi-continuous function on X is finely continuous q.e. and Borel measurable.*

(ii) *Conversely if a function u of $\mathscr{F}$ is finely continuous q.e. and Borel (or more generally, universally) measurable, then u is quasi-continuous.*

Suppose that a function $u \in \mathscr{F}$ is finely continuous q.e. and universally measurable. Denote by $\tilde{u}$ a quasi-continuous modification of u. Then by the first part of Theorem 3.11, the m-negligible function $v=u-\tilde{u}$ is finely continuous q.e. Therefore the second part of Theorem 3.11 follows from the next lemma which is a counterpart of Theorem 1.2(ii).

LEMMA 3.7. *If a function v is finely continuous q.e. and universally measurable and if $v=0$ m-a.e., then $v=0$ q.e.*

Proof. By making use of Theorem 3.9, we see that there is a Borel polar set $B \supset X_b$ such that $X-B$ is finely open, v is finely continuous at each point of $X-B$ and v is universally measurable on $X-B$. The set $C=\{x \in X-B;\, v(x) \neq 0\}$ is then a fine open and universally measurable set which is consequently contained in the set $D=\{x \in X-B;\, R_{\alpha_0}(x, C)>0\}$. Since C is m-negligible and R_{α_0} is symmetric, D is m-negligible. Hence D becomes polar by virtue of Theorem 1.2(ii) and Lemma 3.1. v now vanishes except on the polar set $B \cup C$.

3.6. *Polar sets and absolute continuity conditions.* The first half of the preceding subsection gives us probabilistic interpretations of polar sets. Here we will complete them.

We say a Borel set $Y \subset X$ is ***M*-*invariant*** if the equality

$$P_x(X_t \text{ and } X_{t-} \in Y \cup \partial \text{ for all } t \geqq 0) = 1$$

holds for every $x \in Y$.

THEOREM 3.12. *The following statements are equivalent to each other:*

(i) *A Borel set A is polar.*

(ii) *For m-almost all $x \in X$,*

$$P_x(\sigma'_A < \zeta) = 0. \tag{3.49}$$

(iii) *There exists an m-negligible Borel set B including A such that $X-B$ is* ***M*-*invariant.***

([18]) See [2, II] or [16, XV].

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Proof. (i) implies (iii) according to Theorem 3.9. Statement (iii) means (ii). Suppose that statement (ii) is valid. Then $E_x(\exp(-\alpha_0\sigma'_K); \sigma'_K<\zeta)=0$ m-a.e., for any compact set $K\subset A$. We have $\text{Cap}(K)=0$ by Theorem 3.8 and (1.21). Owing to (1.5) we arrive at the statement (i).

It should be noticed that we cannot generally strengthen the above statement (ii) by replacing "m-almost all x" with "all x". The simplest example illustrative of this point is the case when $\mathcal{F}=L^2(X)$ and $\mathcal{E}\equiv 0$. In this case each m-negligible set is polar but every point of X is trap with respect to the corresponding Ray process. The next theorem will concern the conditions to eliminate such irregular situations.

Suppose that a Borel set A is of potential zero: $R_\alpha(x, A)=0$ for all $x\in A$ and $\alpha>0$. Then $m(A)=0$. Indeed symmetry of the kernel implies $\alpha\int_A R_\alpha 1(x)m(dx) =\alpha\int_X R_\alpha(x, A)m(dx)=0$. Letting α tend to infinity, we get $m(A)=m(A-X_b)=0$.

THEOREM 3.13. *The following conditions are mutually equivalent*:

(i) *A Borel set A is polar if and only if* (3.49) *is satisfied for all $x\in X$.*

(ii) *m is a reference measure of $\mathbf{M}$: a set A is of potential zero if and only if $m(A)=0$.*

(iii) *$R_\alpha(x, \cdot)$ is absolutely continuous with respect to m for each $x\in X$ and $\alpha>0$.*

Proof. It suffices to show the equivalence of (i) and (iii). Suppose that condition (iii) is satisfied and consider a Borel polar set A. By Theorem 3.12, we have $u(x) =P_x(\sigma'_A<\zeta)=0$, m-a.e., and consequently $u(x)=\lim_{\beta\to+\infty}\beta R_\beta u(x)=0$ for every $x\in X$. Thus condition (i) is valid. Conversely assume that (i) is met. Let A be an m-negligible Borel set. Symmetry of the kernel and Lemma 3.1 then imply that $R_\alpha(x, A)=0$ holds for every $x\in X$ except on a polar set, which is of potential zero under condition (i). Thus we have $R_\alpha(x, A)=\lim_{\beta\to+\infty}\beta R_{\beta+\alpha}R_\alpha(x, A)=0$ for every $x\in X$ arriving at condition (iii).

REMARK 3.3. Suppose that condition (iii) of Theorem 3.13 is satisfied. Theorem 3.13 then tells us that we can adopt the set $(\bigcap_{n=1}^\infty A_n)\cup X_b$ as the set B in Theorems 3.9 and 3.10.

4. Regular Dirichlet spaces and strong Markov processes.

Let $(X, m, \mathcal{F}, \mathcal{E})$ be a regular D-space. We adjoin a point ∂ to X as the point at infinity if X is noncompact and as an isolated point if X is compact.

4.1. *Construction of a strong Markov process—proof of Theorem* 4.1. This subsection is devoted to the proof of Theorem 4.1 mentioned in the beginning of the present paper.

(I) By Theorem 3 of [10], there exists a strongly regular D-space $(\tilde{X}, \tilde{m}, \tilde{\mathcal{F}}, \tilde{\mathcal{E}})$ which is equivalent to $(X, m, \mathcal{F}, \mathcal{E})$. Every notion related to this strongly regular D-space will be written with tilde $\tilde{}$. We already have several related notions specified in §3—the Ray resolvent $\{\tilde{R}_\alpha(\tilde{x}, \tilde{E}), \alpha>0\}$, the branch set $\tilde{X}_b$, the transition function $\{\tilde{P}_t(\tilde{x}, \tilde{E}), t>0\}$ and the Ray process $(\tilde{W}, \tilde{\mathcal{B}}_t, \tilde{P}_{\tilde{x}})$ on the extended space $\tilde{X}\cup\tilde{\delta}$.

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By Theorem 2.1, there exists a capacity preserving quasi-homeomorphism q from X to $\tilde{X}$: there are decreasing sequences of open sets $A_k \subset X$ and $\tilde{A}_k \subset \tilde{X}$ such that

$$\lim_{k\to+\infty} \mathrm{Cap}\,(A_k) = 0, \qquad \lim_{k\to+\infty} \mathrm{Cap}^{\sim}\,(\tilde{A}_k) = 0$$

and the restriction of q to $X - A_k$ is homeomorphic onto $\tilde{X} - \tilde{A}_k$ for each k. The equality (2.1) holds for every analytic set $A \subset X - \bigcap_{k=1}^{+\infty} A_k$. If we extend q by setting $q(\partial) = \tilde{\partial}$, then according to Remark 3.2,

(4.1) the restriction of q to $X \cup \partial - A_k$ is a homeomorphism onto $\tilde{X} \cup \tilde{\partial} - \tilde{A}_k$.

Moreover we can assume without loss of generality that

$$\tilde{X}_b \subset \bigcap_{k=1}^{+\infty} \tilde{A}_k, \tag{4.2}$$

because we can replace $\tilde{A}_k$ (resp. A_k) with

$$\tilde{A}_k \cup \tilde{D}_k = \tilde{A}_k \cup \left(\tilde{D}_k - \bigcap_{n=1}^{+\infty} \tilde{A}_n\right) \quad \left(\text{resp. } A_k \cup q^{-1}\left(\tilde{D}_k - \bigcap_{n=1}^{+\infty} \tilde{A}_n\right)\right)$$

if necessary. Here $\{\tilde{D}_k\}$ is a decreasing sequence of open sets of $\tilde{X}$ such that $\tilde{D}_k \supset \tilde{X}_b$ and $\lim_{k\to+\infty} \mathrm{Cap}^{\sim}\,(\tilde{D}_k) = 0$.

By Theorem 3.9, there exists a Borel polar set $\tilde{B} \supset \bigcap_{k=1}^{+\infty} \tilde{A}_k$ which satisfies the following: if we put

$$\begin{aligned} \tilde{W}_{11} &= \{\tilde{\omega} \in \tilde{W};\ \tilde{X}_t(\tilde{\omega}) \text{ and } \tilde{X}_{t-}(\tilde{\omega}) \in \tilde{X} \cup \tilde{\partial} - \tilde{B} \text{ for all } t \geqq 0\}, \\ \tilde{W}_{12} &= \left\{\tilde{\omega} \in \tilde{W};\ \tilde{\sigma}_{\tilde{A}_n}(\tilde{\omega}) = \tilde{\xi}(\tilde{\omega}) \text{ for some } n \text{ or } \lim_{n\to+\infty} \tilde{\sigma}_{\tilde{A}_n}(\tilde{\omega}) = +\infty\right\}, \end{aligned} \tag{4.3}$$

and $\tilde{W}_1 = \tilde{W}_{11} \cap \tilde{W}_{12}$, then

$$\tilde{P}_{\tilde{x}}(\tilde{W}_1) = 1, \qquad \tilde{x} \in \tilde{X} \cup \tilde{\partial} - \tilde{B}. \tag{4.4}$$

According to Theorem 3.10, we have a right continuous strong Markov process $\tilde{M}_1 = (\tilde{W}_1, \tilde{\mathscr{B}}_t^1, \tilde{X}_t, \tilde{P}_x)$ with state space $\tilde{X} \cup \tilde{\partial} - \tilde{B}$ which is quasi-left continuous on $[0, +\infty)$.

(II) *Definition of* $M = (\Omega, \mathscr{M}, \mathscr{M}_t, X_t, P_x)$. Let us define a set $B \subset X$ by

$$X - B = q^{-1}(\tilde{X} - \tilde{B}). \tag{4.5}$$

Since $B = (\bigcap_{k=1}^{\infty} A_k) \cup q^{-1}(\tilde{B} - \bigcap_{k=1}^{\infty} \tilde{A}_k)$, B is a Borel polar set including the set $\bigcap_{k=1}^{\infty} A_k$. We put

$$\Omega = \tilde{W}_1, \quad \mathscr{M} = \tilde{\mathscr{B}}^1, \quad \mathscr{M}_t = \tilde{\mathscr{B}}_t^1. \tag{4.6}$$

The element of Ω (resp. $\mathscr{M}$) is denoted by ω (resp. Λ) instead of $\tilde{\omega}$ (resp. $\tilde{\Lambda}$). Define X_t and P_x by

$$\begin{aligned} X_t(\omega) &= q^{-1}(\tilde{X}_t(\omega)), \qquad && \omega \in \Omega,\ t \geqq 0, \\ P_x(\Lambda) &= \tilde{P}_{qx}(\Lambda), && x \in X \cup \partial - B,\ \Lambda \in \mathscr{M}. \end{aligned} \tag{4.7}$$

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X_t takes values in $X \cup \partial - B$. The field $\mathscr{M}_t^0 = \tilde{\mathscr{B}}_t^0 \cdot \tilde{W}_1$ is generated by functions X_s, $s \leqq t$, because it is generated by $\tilde{X}_s$, $s \leqq t$, and both q and q^{-1} are one-to-one Borel measurable between $X \cup \partial - B$ and $\tilde{X} \cup \tilde{\partial} - \tilde{B}$. The field $\mathscr{M}^0 = \tilde{\mathscr{B}}^0 \cdot \tilde{W}_1$ is of course generated by X_s, $s \geqq 0$.

By (4.4), P_x is a probability measure for each $x \in X \cup \partial - B$. $P_x(\Lambda)$ is for each $\Lambda \in \mathscr{M}^0$ a Borel measurable function of $x \in X \cup \partial - B$ because it is the composition of two Borel functions q and $\tilde{P}\cdot(\Lambda)$.

The field $\mathscr{M}$ is the completion of $\mathscr{M}^0$ with respect to the family of measures $\{P_\mu(\cdot) = \int_{X \cup \partial - B} \mu(dx) P_x(\cdot);\ \mu$ is a finite measure on $X \cup \partial - B\}$. This follows from Theorem 3.10(ii) and from the following observation: there is a one-to-one correspondence between finite measures on $X \cup \partial - B$ and those on $\tilde{X} \cup \tilde{\partial} - \tilde{B}$ by the relation $\mu(E) = \tilde{\mu}(q(E))$ and in this case we have $P_\mu(\Lambda) = \tilde{P}_{\tilde{\mu}}(\Lambda)$, $\Lambda \in \mathscr{M}^0$. The field $\mathscr{M}_t$ is the completion of $\mathscr{M}_t^0$ in $\mathscr{M}$ with respect to the same family of measures.

(III) *M is a Hunt process on $X \cup \partial - B$.* M has namely the following properties (**M**.a)$\sim$(**M**.e).

(**M**.a) The sample path X_t is right continuous for $t \geqq 0$ and has the left limit in $X \cup \partial - B$ for $t > 0$, P_x-a.e. ($x \in X \cup \partial - B$). Further $X_t = \partial$ for $t \geqq \zeta$ P_x-a.e., where $\zeta(\omega) = \inf\{t \geqq 0;\ X_t(\omega) = \partial\}$.

(**M**.b) $P_x(X_0 = x) = 1$, $x \in X \cup \partial - B$.

(**M**.c) $\mathscr{M}_t = \mathscr{M}_{t+}$, $t > 0$. $\mathscr{M}_t$ is the completion in the sense of the preceding paragraph of the σ-field generated by $\{X_s,\ t \geqq s \geqq 0\}$.

(**M**.d) Strong Markov property.

(**M**.e) Quasi-left continuity.

In the present case, the statement (**M**.a) is valid for all $\omega \in \Omega$ in view of Theorem 3.10(iv) and (4.1). By (4.2) we see $\tilde{X}_b \subset \tilde{B}$ and $P_x(X_0 = x) = \tilde{P}_{qx}$ $(\tilde{X}_0 = qx) = 1$, $x \in X \cup \partial - B$ yielding (**M**.b). The second property of (**M**.c) is evident by the observation of the preceding paragraph. The first is due to the right continuity of $\tilde{\mathscr{B}}_t$. (**M**.d) follows from Theorem 3.10(i). To see this, consider an $\mathscr{M}_t$-stopping time T and a set $\Lambda \in \mathscr{M}_T$. Then for any $t > 0$ and any Borel set $E \subset X \cup \partial - B$,

$$\begin{aligned} P_x(X_{T+t} \in E, \Lambda) &= \tilde{P}_{qx}(\tilde{X}_{T+t} \in qE, \Lambda) = \tilde{E}_{qx}(\tilde{P}_{\tilde{X}_T}(\tilde{X}_t \in qE), \Lambda) \\ &= \tilde{E}_{qx}(P_{q^{-1}(\tilde{X}_T)}(X_t \in E), \Lambda) = E_x(P_{X_T}(X_t \in E), \Lambda). \end{aligned}$$

(**M**.e) is due to Theorem 3.10(iv): if $\{T_n\}$ is an increasing sequence of $\mathscr{M}_t$-stopping times and if $T = \lim_{n \to +\infty} T_n$, then

$$\begin{aligned} P_x\left(\lim_{n \to +\infty} X_{T_n} = X_T, T < \infty\right) &= \tilde{P}_{qx}\left(\lim_{n \to +\infty} X_{T_n} = X_T, T < \infty\right) \\ &= \tilde{P}_{qx}(T < \infty) = P_x(T < \infty). \end{aligned}$$

(IV) *The resolvent of the process M generates* ($\mathscr{F}^*$, $\mathscr{E}$). If we define the resolvent kernel $\{R_\alpha(x, E), \alpha > 0\}$ on $x - B$ by

$$R_\alpha(x, E) = E_x\left(\int_0^{+\infty} e^{-\alpha t} \chi_E(X_t)\, dt\right), \tag{4.8}$$

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then for any nonnegative universally measurable function f on $X-B$ which belongs to the space $L^2(X; m)$,

$$R_\alpha f \in \mathscr{F}^* \tag{4.9}$$

and $R_\alpha f$ satisfies the equation (0.5).

In order to prove this statement, put $\tilde{f}=\Phi^* f$ with Φ^* defined by (2.2): $\Phi^* f(\tilde{x}) = f(q^{-1}\tilde{x})$, $\tilde{x} \in \tilde{X}-\tilde{B}$. $\tilde{f}$ is then a nonnegative universally measurable function on $\tilde{X}-\tilde{B}$. If we extend $\tilde{f}$ to a function on $\tilde{X}$ by setting $\tilde{f}(\tilde{x})=0$, $\tilde{x} \in \tilde{B}$, then $\tilde{f}$ becomes universally measurable on $\tilde{X}$. Further by virtue of (q.2) of Theorem 2.1, $\tilde{f} \in L^2(\tilde{X}; \tilde{m})$. Hence we can see by Lemma 3.1 that $\tilde{R}_\alpha \tilde{f}$ belongs to $\tilde{\mathscr{F}}^*$ and satisfies the equation $\tilde{\mathscr{E}}^\alpha(\tilde{R}_\alpha \tilde{f}, \tilde{v}) = (\tilde{f}, \tilde{v})_{\tilde{X}}$ for all $\tilde{v} \in \tilde{\mathscr{F}}$. On the other hand we have $\tilde{R}_\alpha \tilde{f}(\tilde{x}) = \Phi^*(R_\alpha f)(\tilde{x})$, $\tilde{x} \in \tilde{X}-\tilde{B}$, because

$$\begin{aligned} R_\alpha f(x) &= E_x\left(\int_0^{+\infty} e^{-\alpha t} f(X_t)\, dt\right) \\ &= \tilde{E}_{qx}\left(\int_0^{+\infty} e^{-\alpha t} \tilde{f}(\tilde{X}_t)\, dt\right) = (\tilde{R}_\alpha \tilde{f})(qx), \qquad x \in X-B. \end{aligned}$$

Now (q.1) of Theorem 2.1 leads us to (4.9) and the equality $\mathscr{E}^\alpha(R_\alpha f, v) = \tilde{\mathscr{E}}^\alpha(\tilde{R}_\alpha \tilde{f}, \Phi^* v) = (\tilde{f}, \Phi^* v)_{\tilde{X}}$ which is equal to $(f, v)_X$ for $v \in \mathscr{F}^*$ according to the property (q.2).

4.2. *Generalizations of theorems of* §3. All results of §3 are still valid when the strongly regular D-space and the associated Ray process of §3 are replaced with the regular D-space $(X, m, \mathscr{F}, \mathscr{E})$ and its associated Markov process due to Theorem 4.1 respectively.

We consider a Borel polar set $B \subset X$ and a Markov process

$$M = (\Omega, \mathscr{M}, \mathscr{M}_t, X_t, P_x)$$

on $X \cup \partial - B$ which enjoys the properties (III) and (IV) of subsection 4.1([19]).

Observe that in the course of arguments of §3 the speciality of the Ray process that its resolvent leaves the space $C(X)$ invariant has been essentially used nowhere except in the proof of Lemma 3.1. Besides we now have the counterpart of Lemma 3.1, namely, property (IV) of 4.1. Thus all the arguments of §3 are immediately applicable to the present context to establish following generalizations.

THEOREM 4.2. *If a function u is nonnegative universally measurable on $X-B$, belongs to the space $\mathscr{F}$ and has its regularization $\tilde{u}$ on $X-B$, then $\tilde{u}$ is a quasi-continuous modification of u. In particular any excessive function on $X-B$ belonging to $\mathscr{F}$ is an element of $\mathscr{F}^*$.*

This corresponds to Theorem 3.1. The next is a generalization of Lemma 3.4, Theorems 3.7 and 3.8.

([19]) It is sufficient to assume property (IV) only for $f \in L^2 \cap C$.

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THEOREM 4.3. *Let A be an open or a closed subset of X. Put*

$$\mathscr{F}^*_{X-A} = \{u \in \mathscr{F}^*; u = 0 \text{ q.e. on } A\}.$$

(i) *The D-space* $(\mathscr{F}^*_{X-A}, \mathscr{E})$ *is generated by the resolvent kernel* $R_\alpha(x, E) = E_x(\int_0^{\sigma_A} e^{-\alpha t}\chi_E(X_t)\, dt)$, $\alpha > 0$, *on* $X-B$: *for each nonnegative universally measurable function f on $X-B$ belonging to* $L^2(X; m)$, *the function* $R^0_\alpha f(x) = \int_{X-B} R^0_\alpha(x, dy)f(y)$, $x \in X-B$, *belongs to the space* $\mathscr{F}^*_{X-A}$ *and the equation* $\mathscr{E}^\alpha(R^0_\alpha f, v) = (f, v)_X$ *holds for every* $v \in \mathscr{F}^*_{X-A}$.

(ii) *Denote by* $\mathscr{H}^*_\alpha$ *the orthogonal complement of* $\mathscr{F}^*_{X-A}$ *in the Hilbert space* $(\mathscr{F}^*, \mathscr{E}^\alpha)$ *and define the kernel* $H_\alpha(x, E)$ *on* $X-B$ *by*

$$H_\alpha(x, E) = E_x(\exp(-\alpha\sigma_A); X_{\sigma_A} \in E).$$

Then the relation $P_{\mathscr{H}^*_\alpha} u = H_\alpha u$ *holds for every* $u \in \mathscr{F} \cap C(X)$ *where* $P_{\mathscr{H}^*_\alpha}$ *stands for the projection on* $\mathscr{H}^*_\alpha$.

(iii) *If A is an open set of the class* $\mathscr{U}$ *or a compact set, then the equality* (3.35) *holds q.e. on X.*

The first assertion of the above theorem generalizes Lemma 3.7(ii) of [8]. Finally we give generalized versions of Theorem 3.11, 12 and 13.

THEOREM 4.4. *A function* $u \in \mathscr{F}$ *is quasi-continuous if and only if u is finely continuous q.e. on X and universally measurable.*

This essentially generalizes a theorem of J. Deny and J. Lions [5, Chapitre II, Théorème 3.2] concerning BLD functions and Cartan's fine topology.

THEOREM 4.5. *The following statements are equivalent to each other*:

(i) *A Borel set A is polar.*

(ii) (3.49) *holds for m-almost all* $x \in X$.

(iii) *There exists an m-negligible Borel set* $C \supset A \cup B$ *such that* $X-C$ *is* $\boldsymbol{M}$-*invariant.*

THEOREM 4.6. *The following conditions are mutually equivalent*:

(i) *A Borel set A is polar if and only if* (3.49) *is satisfied for every* $x \in X-B$.

(ii) *m is a reference measure of the process* $\boldsymbol{M}$.

(iii) $R_\alpha(x, \cdot)$ *is absolutely continuous with respect to m for each* $x \in X-B$ *and* $\alpha > 0$.

(iv) *m is strictly positive on every nonempty finely open set.*

The last condition of Theorem 4.6 follows from condition (iii). The converse is also true because of symmetry of R_α.

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