# Dirty Paper Coding vs. Linear Precoding for MIMO Broadcast Channels

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***Abstract*— We study the MIMO broadcast channel and compare the achievable throughput for the optimal strategy of dirty paper coding to that achieved with sub-optimal and lower complexity linear precoding (e.g., zero-forcing and block diagonalization) transmission. Both strategies utilize all available spatial dimensions and therefore have the same multiplexing gain, but an absolute difference in terms of throughput does exist. The sum rate difference between the two strategies is analytically computed at asymptotically high SNR, and it is seen that this asymptotic statistic provides an accurate characterization at even moderate SNR levels. Weighted sum rate maximization is also considered, and a similar quantification of the throughput difference between the two strategies is computed. In the process, it is shown that allocating user powers in direct proportion to user weights asymptotically maximizes weighted sum rate.**

## I. Introduction

The multiple antenna broadcast channel (BC) has recently been the subject of tremendous interest, primarily due to the realization that such a channel can support multiple data streams, and therefore realize MIMO spatial multiplexing benefits, without requiring multiple antenna elements at the mobile devices [1]. Indeed, it is now well known that dirty paper coding (DPC) achieves the capacity region of the multiple antenna BC [2]. However, implementation of DPC requires significant additional complexity at both transmitter and receiver, and the problem of finding practical dirty paper codes that approach the capacity limit is still unsolved.

On the other hand, linear precoding is a low complexity but sub-optimal transmission technique (with complexity roughly equivalent to point-to-point MIMO) that is able to transmit the same number of data streams as a DPC-based system. Linear precoding therefore achieves the same multiplexing gain (which characterizes the slope of the capacity vs. SNR) curve) as DPC, but does incur an absolute rate/power offset relative to DPC. The contribution of this work is the quantification of this rate/power offset.

The key analytical tool used in this paper is the affine approximation to capacity at high SNR recently developed by Shamai and Verdú [3]:

$$C(P) = \mathcal{S}_\infty \left(\log_2 P - \mathcal{L}_\infty\right) + o(1), \tag{1}$$

where $\mathcal{S}_\infty$ refers to the multiplexing gain (i.e., how many additional bps/Hz for every 3 dB power gain) and $\mathcal{L}_\infty$ refers to the rate offset. Although this approximation is exact only at asymptotically high SNR, it is seen to provide very accurate results for a wide range of SNR values, e.g., on the order of 5 dB and higher. This affine approximation is evaluated for point-to-point MIMO channels for a number of different fading models in [4].

In this work, we apply the high SNR approximation to the sum rate capacity (DPC) and to the linear precoding sum rate. Both approximations have the same $\mathcal{S}_\infty$, but by characterizing the difference in the $\mathcal{L}_\infty$ terms the rate/power offset between the two strategies is determined. By averaging the per-channel realization rate offset over the iid Rayleigh fading distribution we are able to derive very simple expressions for the average rate offset as a function of only the number of transmit and receive antennas and users for systems in which the aggregate number of receive antennas is no larger than the number of transmit antennas.

Note that previous work has analyzed the *ratio* between the sum rate capacity and the linear precoding sum rate [5][6]. In this work we alternatively study the *absolute difference* between these quantities, which appears to be a more meaningful metric precisely because both strategies provide the same multiplexing gain.

In addition to sum rate, we also study weighted sum rate maximization (using DPC and linear precoding) and provide simple expressions for the rate offsets in this scenario. One of the most interesting results is that weighted sum rate (for either DPC or linear precoding) is maximized at asymptotically high SNR by *allocating power directly proportional to user weights.* This result appears to generalize the well-known property that equal power allocation (across users, fading states, and eigenmodes, for example) asymptotically maximizes sum rate in a number of different single-user and multi-user settings. To illustrate the usefulness of this asymptotically optimal power allocation policy, we apply it to a system employing queue-based scheduling (at finite SNR's) and see that it performs extremely close to the true optimal weighted sum rate maximization.

## II. System Model

We consider a $K$-user Gaussian MIMO BC in which the transmitter has $M$ antennas and each receiver has $N$ antennas with $M \geq KN$, i.e., the number of transmit antennas is no smaller than the aggregate number of received antennas. The received signal $\mathbf{y}_k$ for user $k$ is given by

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x} + \mathbf{n}_k, \qquad k = 1, \cdots, K, \tag{2}$$

where $\mathbf{H}_k(\in \mathbb{C}^{N\times M})$ is the channel gain matrix for user $k$, $\mathbf{x}$ is the transmit signal vector having a power constraint $\mathrm{tr}(E[\mathbf{x}\mathbf{x}^H]) \leq P$, and $\mathbf{n}_k$ $(k = 1,\cdots,K)$ is complex Gaussian noise with unit variance per vector component (i.e., $E[\mathbf{n}_k^H\mathbf{n}_k] = \mathbf{I}$). We assume that the transmitter has perfect knowledge of all channel matrices and each receiver has perfect knowledge of its own channel matrix. For the sake of notation, the concatenation of the channels is denoted by $\mathbf{H}^H = [\mathbf{H}_1^H\ \mathbf{H}_2^H\ \cdots\ \mathbf{H}_K^H](\in \mathbb{C}^{KN\times M})$, which can be decomposed by row vectors as $\mathbf{H}^H = [\mathbf{h}_{1,1}^H\ \mathbf{h}_{1,2}^H\ \cdots\ \mathbf{h}_{1,N}^H\ \mathbf{h}_{2,1}^H\ \mathbf{h}_{2,2}^H\ \cdots\ \mathbf{h}_{2,N}^H\ \cdots\ \mathbf{h}_{K,N}^H]$, where $\mathbf{h}_{k,n}(\in \mathbb{C}^{1\times M})$ is the $n$th row vector of $\mathbf{H}_k$.

*Notations*: Boldface letters denote matrix-vector quantities. The operation $\mathrm{tr}(\cdot)$ and $(\cdot)^H$ represents the trace and the Hermitian transpose of a matrix, respectively. The operation $|\cdot|$ and $\|\cdot\|$ denote the determinant of a matrix and the Euclidean norm of a vector, respectively.

## III. Sum Rate by Dirty Paper Coding

The DPC sum rate, which achieves the sum capacity of the MIMO BC, can be expressed from the MIMO BC-MAC duality by the following [7]:

$$\mathcal{C}_{\mathrm{DPC}}(\mathbf{H},P) = \max_{\sum_k \mathrm{tr}(\mathbf{Q}_k)\leq P} \log_2 \left| \mathbf{I} + \sum_{k=1}^{K} \mathbf{H}_k^H \mathbf{Q}_k \mathbf{H}_k \right|. \quad (3)$$

No closed-form solution to (3) is known to exist, but it has been shown that $\mathcal{C}_{\mathrm{DPC}}(\mathbf{H},P)$ converges (absolutely) to the capacity of the point-to-point MIMO channel with transfer matrix $\mathbf{H}$ whenever $M \geq KN$:

*Theorem 1 ([1]):* When $M \geq KN$,

$$\lim_{P\to\infty} \left[ \mathcal{C}_{\mathrm{DPC}}(\mathbf{H},P) - \log_2 \left| \mathbf{I} + \frac{P}{KN}\mathbf{H}^H\mathbf{H} \right| \right] = 0. \quad (4)$$

A direct corollary of this result is the fact that choosing each of the covariance matrices as $\mathbf{Q}_k = \frac{P}{KN}\mathbf{I}$ in (3) is asymptotically optimal [8].

As a result of Theorem 1, an affine approximation for the sum rate can be found as:

$$\mathcal{C}_{\mathrm{DPC}}(\mathbf{H},P) \cong KN\log_2 P - KN\log_2 KN + \log_2 \left|\mathbf{H}^H\mathbf{H}\right|, \quad (5)$$

where $\cong$ refers to equivalence in the limit (i.e., the difference between both sides converges to zero as $P \to \infty$). Notice that the high SNR sum rate capacity only depends on the product of $K$ and $N$ and not on their specific values; this is not the case for linear precoding.

## IV. Sum Rate by Linear Precoding

In this section we compute the affine approximation to the linear precoding sum rate, and quantify the asymptotic rate/power offset relative to DPC.

When linear precoding is used, the transmit signal vector $\mathbf{x}$ is a linear function of the symbols intended for the $K$ users $\mathbf{s}_k(\in \mathbb{C}^{N\times 1})$, $k = 1,\cdots,K$:

$$\mathbf{x} = \sum_{k=1}^{K} \mathbf{V}_k \mathbf{s}_k, \quad (6)$$

where $\mathbf{V}_k(\in \mathbb{C}^{M\times N})$ is the precoding matrix for user $k$. Since each receiver has $N$ $(\leq M)$ antennas, $N$ symbols (data streams) are transmitted to each receiver.

The resulting received signal for user $k$ is given by

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{V}_k\mathbf{s}_k + \sum_{j\neq k} \mathbf{H}_k\mathbf{V}_j\mathbf{s}_j + \mathbf{n}_k, \quad (7)$$

where the second term in (7) represents the multi-user interference.

In this paper we consider two linear precoding schemes to eliminate the multi-user interference when $M \geq KN$: zero-forcing (ZF) and block diagonalization (BD). The precoding matrices $\{\mathbf{V}_j\}_{j=1}^K$ for BD are chosen such that for all $j(\neq k) \in [1,K]$,

$$\mathbf{H}_k\mathbf{V}_j = \mathbf{O}, \quad (8)$$

while those for ZF are chosen so that

$$\mathbf{h}_{k,n}\mathbf{v}_{j,l} = 0, \qquad \forall j(\neq k) \in [1,K],\ \forall n,l \in [1,N], \quad (9)$$

$$\mathbf{h}_{k,n}\mathbf{v}_{k,l} = 0, \qquad \forall l(\neq n) \in [1,N], \quad (10)$$

where $\mathbf{v}_{j,l}$ denotes the $l$th column vector of $\mathbf{V}_j$. Consequently, performing ZF in a system with $K$ users with $N(>1)$ antennas is equivalent to performing ZF in a channel with $KN$ single antenna receivers.

### *A. Zero-forcing*

Since zero-forcing eliminates multi-user and inter-antenna interference, the received signal at the $n$th antenna of user $k$ is given by

$$y_{k,n} = \mathbf{h}_{k,n}\mathbf{v}_{k,n}\mathbf{s}_k + n_{k,n}, \qquad n = 1,\cdots,N. \quad (11)$$

Thus, ZF converts the system into $KN$ parallel channels with effective channel $g_{k,n} = \mathbf{h}_{k,n}\mathbf{v}_{k,n}$. Sum rate is maximized by optimizing power allocation across these parallel channels:

$$\mathcal{C}_{\mathrm{ZF}}(\mathbf{H},P) = \max_{\sum_k\sum_n P_{k,n}\leq P} \sum_{k=1}^{K}\sum_{n=1}^{N} \log_2\left(1 + P_{k,n}|g_{k,n}|^2\right). \quad (12)$$

Since the optimum power allocation policy converges to uniform power at asymptotically high SNR [8], we have:

$$\mathcal{C}_{\mathrm{ZF}}(\mathbf{H},P) \cong KN\log_2 P - KN\log_2 KN + \log_2 \prod_{k=1}^{K}\prod_{n=1}^{N} |g_{k,n}|^2. \quad (13)$$

This approximation is identical to that for DPC in (5) except for the final constant term.

We define the rate loss as the asymptotic (in SNR) difference between the sum rate capacity and the zero forcing sum rate:

$$\beta_{\text{DPC-ZF}}(\mathbf{H}) \triangleq \lim_{P\to\infty} \left[\mathcal{C}_{\mathrm{DPC}}(\mathbf{H},P) - \mathcal{C}_{\mathrm{ZF}}(\mathbf{H},P)\right]. \quad (14)$$

From (5) and (13), the rate loss incurred by ZF is:

$$\beta_{\text{DPC-ZF}}(\mathbf{H}) = \log_2 \frac{|\mathbf{H}^H\mathbf{H}|}{\prod_{k=1}^{K}\prod_{n=1}^{N}|g_{k,n}|^2}. \quad (15)$$

Due to the affine behavior of sum rate at high SNR, this rate offset (i.e., the vertical offset between capacity vs. SNR curves) can be immediately translated into a power offset (i.e., a horizontal offset): $\frac{3}{K}\beta_{\text{DPC-ZF}}(\mathbf{H})$ dB.

While the above metric is the rate loss per realization, we are more interested in the average rate offset across the fading distribution:

$$\bar{\beta}_{\text{DPC-ZF}} = E_{\mathbf{H}}\left[\beta_{\text{DPC-ZF}}(\mathbf{H})\right], \tag{16}$$

which allows a comparison of average (over the fading distribution) throughput. Under iid Rayleigh fading, the matrix $\mathbf{H}^H\mathbf{H}$ clearly is Wishart with $2M$ degrees of freedom. Furthermore, it can be shown that the square of the norm of the effective channel gains $|g_{k,n}|^2$ are identically $\chi^2_{2(M-K+1)}$. Utilizing the expression for the expected log-determinant of Wishart matrices and of chi-squared variables in terms of Euler's digamma function [9], we can compute the expected offset in closed form:

*Theorem 2:* The expected loss in Rayleigh fading due to zero-forcing is given by

$$\boxed{\bar{\beta}_{\text{DPC-ZF}} = \log_2 e \sum_{j=1}^{K-1} \frac{j}{M-j} \quad \text{(bps/Hz).}} \tag{17}$$

*Proof:* See [8]. ■

Furthermore, when $M = K$, the loss is approximately found as

$$\bar{\beta}_{\text{DPC-ZF}}(M) \approx M \log_2 M \quad \text{(bps/Hz).} \tag{18}$$

### B. Block Diagonalization

Since the precoding matrix for BD is chosen to be $\mathbf{H}_k\mathbf{V}_j = \mathbf{O}$ for $k \neq j$, the received signal for user $k$ is given by

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{V}_k\mathbf{s}_k + \mathbf{n}_k. \tag{19}$$

Thus, BD converts the system into $K$ parallel MIMO channels with effective channel matrices $\mathbf{G}_k = \mathbf{H}_k\mathbf{V}_k$, $k = 1, \cdots, K$. The BD sum rate is given by [10][11]

$$\mathcal{C}_{\text{BD}}(\mathbf{H}, P) = \max_{\mathbf{Q}_k: \sum_{k=1}^{K} \text{tr}\{\mathbf{Q}_k\} \leq P} \sum_{k=1}^{K} \log_2 \left|\mathbf{I} + \mathbf{G}_k\mathbf{Q}_k\mathbf{G}_k^H\right|, \tag{20}$$

and the optimal rate is achieved asymptotically by uniform power allocation at high SNR since the channel can be decomposed into parallel channels. Hence, the sum rate is asymptotically given by

$$\mathcal{C}_{\text{BD}}(\mathbf{H}, P) \cong KN \log_2 P - KN \log_2 KN + \log_2 \prod_{k=1}^{K} |\mathbf{G}_k^H\mathbf{G}_k|. \tag{21}$$

Let us define the loss from the DPC sum rate as

$$\beta_{\text{DPC-BD}}(\mathbf{H}) \triangleq \lim_{P\to\infty} \left[\mathcal{C}_{\text{DPC}}(\mathbf{H}, P) - \mathcal{C}_{\text{BD}}(\mathbf{H}, P)\right], \tag{22}$$

and denote the expected loss as $\bar{\beta}_{\text{DPC-BD}} \triangleq E_{\mathbf{H}}[\beta_{\text{DPC-BD}}(\mathbf{H})]$. Similar to the analysis for ZF, we can calculate the loss terms for a fixed channel and also average over Rayleigh fading. In order to compute the average rate loss, we use the fact that each of the effective channels is equivalent to an $N \times (M - (K-1)N)$ iid Rayleigh MIMO channel [10].

*Theorem 3:* The expected loss in Rayleigh fading due to block diagonalization is given by

$$\boxed{\bar{\beta}_{\text{DPC-BD}} = (\log_2 e)\left(\sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\sum_{i=kN+1}^{(K-1)N} \frac{1}{M-n-i}\right) \quad \text{(bps/Hz).}} \tag{23}$$

*Proof:* See [12]. ■

Eq. (23) simplifies to (17) when $N = 1$; i.e., zero-forcing is a special case of block diagonalization. If $M$ is kept fixed but $N$ is increased and $K$ is decreased such that $KN$ remains constant, i.e., the number of antennas per receiver increases but the aggregate number of antennas is kept constant, then the rate offset decreases. To understand this, consider that the sum rate capacity (DPC) is the same as the capacity of a $KN \times M$ MIMO channel (and thus only depends on the product $KN$). However, the BD sum rate is $K$ times the capacity of an $N \times (M - KN + 1)$ MIMO channel, while the ZF sum rate is $KN$ times the capacity of an $(M - KN + 1) \times 1$ MISO channel.

The above expression is somewhat unwieldy, but it is possible to gain more intuition by considering the offset between BD ($K$ receivers with $N$ antennas each) and ZF (equivalent to $KN$ receivers with 1 antenna each).

*Theorem 4:* If $M = \alpha KN$ with $N > 1$ and $\alpha > 1$, the difference of the expected losses by BD and ZF is given by

$$\begin{aligned}\bar{\beta}_{\text{BD-ZF}} &\triangleq \bar{\beta}_{\text{DPC-ZF}} - \bar{\beta}_{\text{DPC-BD}} \\ &= (\log_2 e)K \sum_{i=1}^{N-1} \frac{(N-i)}{(\alpha-1)KN + i} \quad \text{(bps/Hz).}\end{aligned}$$

*Proof:* See [12]. ■

A direct corollary of this is an expression for the expected power offset when $M = KN$:

$$\boxed{\bar{\Delta}_{\text{BD-ZF}} = \frac{3(\log_2 e)}{N} \sum_{j=1}^{N-1} \frac{N-j}{j} \quad \text{(dB).}} \tag{24}$$

The most important feature of this expression is that it depends only on the number of receive antennas. For example, consider two system configurations: (i) $\frac{M}{2}$ users each have two receive antennas, and (ii) $M$ users each have one receive antenna. Equation (24) indicates that the power advantage of using BD in the $N = 2$ system is $\bar{\Delta}_{\text{BD-ZF}} = 2.1640$ (dB) relative to performing ZF. Since this offset is independent of $M$, it is the same for $M = 4$ and $K = 4, N = 1$ vs. $K = 2, N = 2$ systems as well as for $M = 6$ and $K = 6, N = 1$ vs. $K = 3, N = 2$ systems. To illustrate the utility of the asymptotic rate offsets, sum rates are plotted in Fig. 1 for systems with $M = 6$ and $K = 6, N = 1$, $K = 3, N = 2$, and $K = 2, N = 3$. Notice that the asymptotic offsets provide insight at even moderate SNR levels (e.g., 10 dB).

## V. Generalization to Weighted Sum Rates

In this section we generalize the sum rate to weighted sum rate maximization for single antenna receivers ($N = 1$). We first show that allocating power in proportion to user weights is asymptotically optimal, and then use this result to compute the associated rate offsets. Finally, we show the utility of our

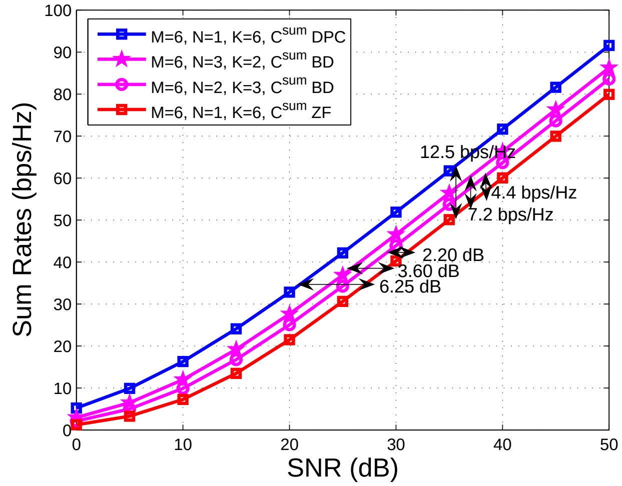


Fig. 1. Simulated rate losses and power offsets between the sum rate by DPC and the sum rate by linear precoding.

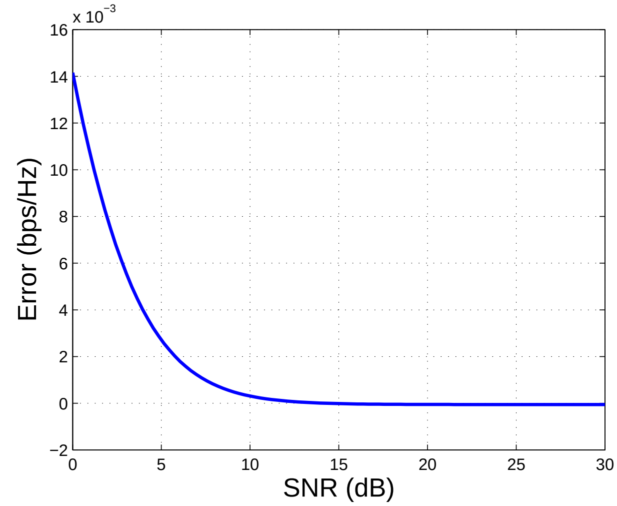


Fig. 2. Weighted sum rate difference between the exact solution (25) and the asymptotic solution (30) when $\mu_1 = 0.6$ and $\mu_2 = 0.4$ for $\mathbf{h}_1 = [0.8576 + 1.3809i\ \ 0.3070 - 0.7095i\ \ 0.2146 - 0.5851i\ \ -0.1260 + 0.6932i]$, and $\mathbf{h}_2 = [-1.2592 - 0.5144i\ \ 0.5696 - 0.2819i\ \ -0.8380 - 0.1668i\ \ 0.0919 + 0.4464i]$.

simple power allocation policy via application to queue-based scheduling.

### A. Asymptotic Optimal Power Allocation

Without loss of generality, we assume user weights are in descending order: $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_K \geq 0$ with $\sum_{k=1}^{K} \mu_k = 1$. The maximum weighted sum rate problem (DPC), which is defined as the maximum of $\sum_{k=1}^{K} \mu_k R_k$ over the capacity region, can be written as:

$$\mathcal{C}_{\text{DPC}}(\mu, \mathbf{H}, P) = \max_{\sum_{k=1}^{K} P_k \leq P} \sum_{k=1}^{K} \mu_k \log_2 \left(1 + P_k \mathbf{h}_k (\mathbf{A}^{(k-1)})^{-1} \mathbf{h}_k^H \right), \tag{25}$$

where $\mathbf{A}^{(k-1)} = \mathbf{I} + \sum_{j=1}^{k-1} P_j \mathbf{h}_j^H \mathbf{h}_j$. Since $N = 1$, the channel is a row vector and is written as $\mathbf{h}_k$.

The following lemma shows that if we limit ourselves to linear power allocation policies, then the objective function in (25) can be decoupled at high SNR:

*Lemma 1:* If $M \geq K$, then for any $\alpha_k (= P_k/P) > 0$, $k = 1, \cdots, K$ with $\sum_{k=1}^{K} \alpha_k = 1$,

$$\lim_{P \to \infty} \left[ \sum_{k=1}^{K} \mu_k \log_2 \left(1 + \alpha_k P \mathbf{h}_k (\mathbf{A}^{(k-1)})^{-1} \mathbf{h}_k^H \right) - \sum_{k=1}^{K} \mu_k \log_2 \left(1 + \alpha_k P \|\mathbf{f}_k\|^2 \right) \right] = 0, \tag{26}$$

where $\mathbf{f}_k$ is the projection of $\mathbf{h}_k$ onto the nullspace of $\{\mathbf{h}_1, \cdots, \mathbf{h}_{k-1}\}$.

*Proof:* See [12]. ■

That is, instead of solving (25) directly, the following optimization will yield an asymptotically identical solution (albeit with a restriction on allowable power policies):

$$\max_{P_k: \sum_{k=1}^{K} P_k \leq P} \sum_{k=1}^{K} \mu_k \log_2 \left(1 + P_k \|\mathbf{f}_k\|^2 \right). \tag{27}$$

The KKT conditions yield the solution to (27):

$$P_k^* = \mu_k P + \mu_k \left( \sum_i \frac{1}{\|\mathbf{f}_i\|^2} \right) - \frac{1}{\|\mathbf{f}_k\|^2}, \tag{28}$$

for $k = 1, \cdots, K$. Therefore, at high SNR we have

$$P_k^* = \mu_k P + O(1), \qquad k = 1, \cdots, K. \tag{29}$$

Since the $O(1)$ power term leads to a vanishing rate, allocating power according to:

$$\boxed{P_k = \mu_k P, \qquad k = 1, \cdots, K} \tag{30}$$

maximizes (27) at asymptotically high SNR.

In Fig. 2 the difference between the true weighted sum rate (25) and the weighted sum rate achieved using $P_k = \mu_k P$ is shown for a specific channel realization when $\mu_1 = 0.6$ and $\mu_2 = 0.4$ for $M = 4$ and $K = 2$, and the gap is seen to be negligible throughout the entire SNR range.

Meanwhile, the weighted sum rate by ZF is given by

$$\mathcal{C}_{\text{ZF}}(\mu, \mathbf{H}, P) = \max_{P_k: \sum_{k=1}^{K} P_k \leq P} \sum_{k=1}^{K} \mu_k \log_2 \left(1 + P_k \|\mathbf{g}_k\|^2 \right), \tag{31}$$

where $\mathbf{g}_k$ is the projection of $\mathbf{h}_k$ onto the null space of $\{\mathbf{h}_1, \cdots, \mathbf{h}_{k-1}, \mathbf{h}_{k+1}, \cdots, \mathbf{h}_K\}$. Notice that the optimization (31) is the same as the optimization (27) except that $\mathbf{f}_k$ is replaced by $\mathbf{g}_k$ which does not contribute to the asymptotic solution. This only affects the $O(1)$ term in (28)and thus the power allocation policy in (30) is also the asymptotic solution to (31).

### B. Rate Loss

Using the asymptotically optimal power allocation policy of (30), the weighted sum rates of DPC and ZF can be expressed

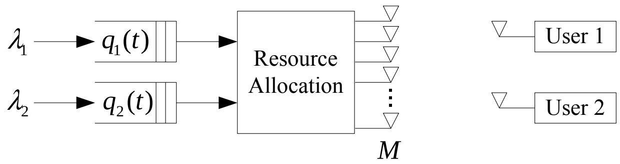


Fig. 3. MIMO BC with queues

as

$$\mathcal{C}_{\text{DPC}}(\mu, \mathbf{H}, P) \cong \sum_{k=1}^{K} \mu_k \log\left(1 + \mu_k P \|\mathbf{f}_k\|^2\right), \quad (32)$$

$$\mathcal{C}_{\text{ZF}}(\mu, \mathbf{H}, P) \cong \sum_{k=1}^{K} \mu_k \log\left(1 + \mu_k P \|\mathbf{g}_k\|^2\right). \quad (33)$$

Thus, the rate offset per realization is given by

$$\beta_{\text{DPC-ZF}}(\mu, \mathbf{H}, P) \cong \sum_{k=1}^{K} \mu_k \log \frac{\|\mathbf{f}_k\|^2}{\|\mathbf{g}_k\|^2}. \quad (34)$$

In Rayleigh fading, the distributions of $\|\mathbf{f}_k\|^2$ and $\|\mathbf{g}_k\|^2$ are $\chi^2_{2(M-k+1)}$ and $\chi^2_{2(M-K+1)}$, respectively. Therefore, the expected rate loss is given by

$$\bar{\beta}_{\text{DPC-ZF}}(\mu, M, K) \cong (\log_2 e) \sum_{k=1}^{K} \mu_k \left( \sum_{j=M-K+1}^{M-k} \frac{1}{j} \right) . \quad (35)$$

### *C. Application to Queue-based Scheduling*

Queue-based scheduling, introduced by the seminal work of Tassiulas and Ephremides [13], is one application in which it is necessary to repeatedly maximize the weighted sum rate for different user weights. Fig. 3 illustrates a queue-based scheduling system for two users. Data for the users arrive at rates $\lambda_1$ and $\lambda_2$, which are generally assumed to be unknown. During each time slot, the transmitter chooses the rate vector that maximizes the weighted sum rate over the instantaneous rate region with weights equal to the current queue sizes. If the queue lengths are denoted as $q_1(t)$ and $q_2(t)$, then the transmitter solves the following optimization during each time slot:

$$\max_{\mathbf{R} \in \mathcal{C}(\mathbf{H}, P)} q_1(t) R_1 + q_2(t) R_2, \quad (36)$$

and such a policy stabilizes any rate vector in the ergodic capacity region.

Although the weighted sum rate maximization problem for DPC stated in equation (25) is convex, it still requires considerable complexity and could be difficult to perform on a slot-by-slot basis. An alternative is to use the approximate power allocation policy from (30) is used during each time slot:

$$P_k = \frac{q_k(t)}{q_1(t) + q_2(t)} P, \quad (37)$$

and where the ordering of the queues determines the dual MAC decoding order (larger queue decoded last).

Although we do not yet have any analytical results on the performance of the asymptotically optimal power policy, numerical results indicate that such a policy performs nearly as well as actually maximizing weighted sum rate.

## VI. Conclusion

We have investigated the difference between the throughputs achieved by dirty paper coding (DPC) relative to those achieved with linear precoding strategies by utilizing the affine approximation to high SNR and computing the exact throughput/power offsets at asymptotically high SNR for MIMO broadcast channels in which the number of transmit antennas is no smaller than the total number of receive antennas. Simple expressions in terms of the number of transmit and receive antennas are provided for the average rate/power offset in a spatially white Rayleigh fading environment. When the aggregate number of receive antennas is equal or slightly less than than the number of transmit antennas, linear precoding incurs a rather significant penalty relative to DPC, but this penalty is much smaller when the number of transmit antennas is large relative to the number of receive antennas.

Furthermore, we generalized our analysis to weighted sum rate and quantified the asymptotic rate/power offsets for this scenario as well. One of the most interesting aspects of this extension is the finding that allocating power directly proportional to user weights is asymptotically optimal at high SNR. This finding appears to apply quite generally to power allocation over parallel channels, and may prove to be useful in other settings.

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