Disappearing Solutions for the Dissipative Wave Equation

ANDREW MAJDA

Communicated by RALPH PHILLIPS

Introduction. In this work we study a special class of solutions for the dissipative wave equation in either a bounded or exterior region Ω with smooth boundary $\partial\Omega$. The systems we study have the form

$$\begin{aligned} \square\, u + c(x)u_t &= 0 \\ \frac{\partial u}{\partial n} + \gamma(x)u_t + \sigma u \,|_{\partial\Omega} &= 0, \qquad \sigma \geqq 0 \\ u(0) &= f_1 \\ u_t(0) &= f_2\,. \end{aligned} \tag{1.1}$$

The dissipation of energy in the system in (1.1) is a consequence of the requirements $c(x) \geqq 0$ and $\gamma(x) \geqq 0$. We also assume that c and γ are smooth functions and that c vanishes near infinity. If one introduces the energy norm for the Cauchy data $f = (f_1\,, f_2)$ by

$$\|f\|_E^2 = \int_\Omega |\nabla f_1|^2 + \int |f_2|^2 + \int_{\partial\Omega} \sigma\, |f_1|^2,$$

then it follows in a standard fashion that the mappings $T(t)f \equiv (u(t),\, u_t(t))$ for $t \geqq 0$ form a contraction semigroup in the energy norm with infinitesimal generator A given by

$$A = \begin{bmatrix} 0 & 1 \\ \triangle & c \end{bmatrix}$$

on a domain $\mathfrak{D}(A)$ given by the closure in the graph norm of the smooth functions which satisfy $\partial f_1/\partial n + \gamma f_2 + \sigma f_1\,|_{\partial\Omega} = 0$. We remark for future reference that the adjoint semigroup, $T^*(t)$, has generator

Indiana University Mathematics Journal, ©, Vol. 24, No. 12 (1975)

$$A^* = \begin{bmatrix} 0 & -1 \\ -\triangle & c \end{bmatrix}$$

with domain $\mathfrak{D}(A^*)$ the closure in the graph norm of the smooth functions which satisfy $\partial f_1/\partial n - \gamma f_2 + \sigma f_1 |_{\partial\Omega} = 0$.

The special class of solutions to (1.1) that we study has the property that $u(x, t) \equiv 0$ for all $t \geqq T_0$. We call such solutions $u(x, t)$ *disappearing solutions.* We remark that for the systems in (1.1) if $f = (f_1, f_2)$ is an initial state associated with a disappearing solution so that $T(t)f \equiv 0$ for $t \geqq T_0$, then $\tilde{f} = (f_1, -f_2)$ is an initial state associated with $T^*(t)$ so that $T^*(t)\tilde{f} \equiv 0$ for $t \geqq T_0$.

In the first section of this paper we give some examples which prove that disappearing solutions exist for particular regions with special choices of the parameters γ, c, and σ. One consequence of these examples is that the results of Rauch and Taylor proved in [10] for energy-conserving boundary conditions are false for general dissipative boundary conditions. Lax and Phillips in [5] have developed a scattering theory for the above dissipative systems in exterior regions. In a recent lecture (see [9]) Phillips posed the problem of determining the *controllability* of these systems. This concept is defined in Section 4. Disappearing solutions are intimately connected with this problem since it is a fact that disappearing solutions exist in an exterior region if and only if the system fails to be controllable. We formulate and prove this connection in Section 4 by using the Paley–Weiner theorem for the Radon transform and Holmgren's uniqueness theorem. When Ω is a bounded region if disappearing solutions exist, the spectral properties of the infinitesimal generator A are pathological. In this case we prove in Section 5 that when disappearing solutions exist even though A has compact resolvent ($(\beta - A)^{-K}$ is even Hilbert–Schmidt for some β and integer $K > 0$), the span of the generalized eigenfunctions for A has an infinite dimensional orthogonal complement.

Our two main technical results appear in Section 3. They both have the following theme: If $\gamma(x) \not\equiv 1 \; \forall \; x \in \partial\Omega$, then disappearing solutions cannot exist. The main ideas of the proof are a Holmgren-type argument adapted to the mixed problems considered here. Technical difficulties in this straightforward approach appear because the backward problem is illposed in the sense of energy estimates—this difficulty can be overcome under two restrictions, either $\gamma(x) < 1 \; \forall \; x \in \partial\Omega$ or that $\partial\Omega$, γ, and σ are real analytic and $\gamma(x) \not\equiv 1 \; \forall \; x \in \partial\Omega$. The condition $\gamma(x) \not\equiv 1 \; \forall \; x \in \partial\Omega$ enters crucially as a consequence of the following geometric fact: under these circumstances the direction of differentiation $\partial/\partial n + \gamma(\partial/\partial t)$ is not tangent to the backward characteristic surface emanating from $\partial\Omega$. The examples in Section 2 show that the theorems proved here are fairly sharp.

The author thanks Ralph Phillips for suggesting the proofs used in Section 4 in a private conversation.

Section 2. Examples of disappearing solutions. We begin by collecting a few simple examples which show that under various circumstances disappearing solutions to the equations in (1.1) do exist.

Consider first the case of one space variable. If $\Omega = \{x \mid x < 1\}$, the boundary conditions assume the form $\partial u/\partial x + \gamma(\partial u/\partial t) |_{x=1} = 0$ where $\gamma > 0$. Suppose additionally that the distance between supp $c(x)$ and $x = 1$ is some positive number, r_0. Then if we choose $\gamma = 1$ and $\sigma = 0$ and let $u(x, t) = f(x - t)$ where $f \in C^{\infty}_{(0)}((1 - r_0, 1))$, then $(u(0), u_t(0)) = f$ is an initial state such that $T(t) \equiv 0$ for $t \geqq r_0$. Similarly, $(u(0), -u_t(0)) = \tilde{f}$ is an initial state for the adjoint equation with $T^*(t)\tilde{f} \equiv 0$ for $t \geqq r_0$. The same example works for interior regions by placing another wall at $x = 0$ and assigning any energy conserving boundary condition there. In particular if we choose $c \equiv 0$ and the boundary condition $-(\partial u/\partial x) + \partial u/\partial t |_{x=0} = 0$ on this wall, a trivial calculation shows that the action of $T(t)$ $(T^*(t))$ for any initial data is given by D'Alemberts formula for the free-space wave equation. Thus, because of Huygen's principle, every initial state for this dissipative system vanishes in a finite amount of time. In other words,

$$T(t) \equiv T^*(t) \equiv 0 \quad \text{for} \quad t \geqq 1.$$

We can easily build similar examples in three space dimensions using incoming and outgoing spherical waves. Choose Ω to be the interior of the unit sphere and let f be any smooth function which vanishes for all negative values and in a neighborhood of the origin. If $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ and $u(x, t) = f(r - t)/r \in C^{\infty}(\mathbf{R}^3)$ $\forall t \geqq 0$, then u satisfies the wave equation in Ω and the boundary condition, $(\partial/\partial n + \gamma(\partial/\partial t) + \sigma)u |_{|x|=1} = 0$ with the special choices $\gamma \equiv 1$ and $\sigma \equiv 1$. Thus, again we have initial states f and $\tilde{f}$ such that $T(t) = T^*(t)\tilde{f} \equiv 0$ for $t \geqq 1$. We remark that the geometry of Ω can be modified somewhat and the same type of example will work. The previous construction also yields disappearing solutions for an annular region with the outer piece of the boundary the unit sphere and the inner piece of the boundary smooth but arbitrary.

For the exterior of the unit sphere we attempt to mimic the previous example by taking incoming waves of the form $u(x, t) = f(r + t)/r$ where f vanishes for large positive values. Then u satisfies the wave equation and the boundary conditions, $(\partial/\partial n + \gamma(\partial/\partial t) + \sigma)u |_{\partial\Omega} = 0$ with the choices $\gamma = 1$ and $\sigma = -1$. Since $\sigma = -1$ the energy form conserved by the associated system in (1.1) fails to be positive definite. Nevertheless, since the energy form is positive definite on a subspace with finite codimension, it seems reasonable that the techniques of Lax and Phillips developed in [7] can be modified to handle dissipative systems of this form. Since we prove in the next section that the space of disappearing solutions if non-trivial is necessarily infinite dimensional, the above example would yield disappearing solutions for these systems.

We remark again that a common feature is apparent in all examples of disappearing solutions we have constructed. If one constructs the characteristic surface emanating backwards in time from a portion of $\partial\Omega$, then the principal part of the boundary condition geometrically represents a direction of differentiation tangent ot this surface (*i.e.* $\gamma = 1$). The theorems we prove in the next sction indicate that this is not a coincidence.

Section 3. Conditions which guarantee the non-existence of disappearing solutions. It is evident that disappearing solutions can exist only when the semigroup generated by $T(t)$ cannot be expanded into a group of operators—in other words, the process determined by the system in (1.1) is irreversible. Thus, the most straightforward circumstance where disappearing solutions cannot exist is if $T(t)$ generates a group of operators. Since $T(t)$ is a group of operators if and only if $T^*(t)$ is a group, it suffices to determine when the following Cauchy problem is well-posed in the energy norm,

$$\frac{dT^*(t)f}{dt} = -A^*T^*(t)f \qquad (3.1)$$

$$T(0)f = f.$$

We remark that the abstract Cauchy problem in (3.1) is equivalent to the concrete mixed problem

$$\begin{aligned} \square\, u - c(x)u_t &= 0 \\ \left(\frac{\partial u}{\partial n} - \gamma u_t + \sigma u\right)\Big|_{\partial\Omega} &= 0 \\ u(0) &= f_1 \\ u_t(0) &= f_2 \end{aligned} \qquad (3.2)$$

where (f_1, f_2) has finite energy and $\gamma, c \geqq 0$.

In one space dimension if $\gamma \neq 1$, the problem in (3.2) is always well-posed. (For $\gamma \neq 1$, use the multiplier $u_t - \tilde{\gamma}u_x$ where $2\gamma/(1 + \gamma^2) < \tilde{\gamma} < 1$ to derive a variant of the standard energy estimate.) For higher space dimensions if one checks the algebraic criterion for well-posedness of the mixed problem in (3.2) according to the general theory developed by Kreiss, Sakamoto, *et al.*, (see [11]), one discovers the following: For $\gamma \geqq 1$ the problem is strongly ill-posed in the sense that Hadamard-like counterexamples exist for the mixed problem; for $0 \leqq \gamma < 1$ the boundary conditions lie on the boundary of the Kreiss well-posed problems. Thus, under the conditions $0 < \gamma < 1$, well-posedness in the energy norm is possible but requires a special detailed analysis. We plan to examine these properties in a future paper but only a weaker form of well-posedness is necessary for the applications to disappearing solutions. For $0 \leqq \gamma < 1$ the problems in (3.2) satisfy the weaker hypotheses developed by R. Beals in [2] (see pp. 145–151 of [2]). Thus if Ω is bounded and additionally γ, σ, and $\partial\Omega$ are infinitely differentiable, then the problem in (3.2) has a C^∞ solution for a particular family of Cauchy data of the form $\{(0, \psi_i)\}^\infty_{i=1}$ where ψ_i is C^∞ and $\{\psi_i\}^\infty_{i=1}$ is dense in $L^2(\Omega)$. (In fact, $\psi_i \in \bigcap^\infty_{N=1} \mathfrak{D}(A^{*N})$). This weaker existence theorem suffices for the Holmgren type arguments given below. When $\gamma > 1$ and $n > 1$ the problems in (3.2) are ill-posed even in this weaker sense; nevertheless, under appropriate analyticity hypotheses, we prove that disappearing solutions cannot exist.

Suppose a disappearing solution exists; in other words that there exists an f with finite energy such that $T(t)f \equiv 0$ for $t \geqq t_0$. Then it is trivial to verify that the set defined by

$$H(t_0) \equiv \{g \mid T(t)g = 0 \text{ for } t \geqq t_0\}$$

is a closed linear subspace under the energy norm. ($\tilde{H}(t_0) \equiv \{g \mid T^*(t)g = 0$ for $t \geqq t_0\}$ can be defined similarly and has analogous properties.) Furthermore, $H(t_0)$ is invariant under the action of $T(\tau)$ for all $\tau \geqq 0$. We remark for future reference that if $H(t_0)$ is non-trivial, then $H(t_0)$ is necessarily infinite dimensional. For suppose $H(t_0)$ is finite dimensional. As a finite dimensional invariant subspace of $T(t)$, $H(t_0)$ necessarily contains a non-trivial eigenvector v_0 for $T(t)$ by the theory of semigroups acting on finite dimensional spaces. Thus, there exists a λ_0 with $T(t)v_0 = e^{\lambda_0 t}v_0$, and if we choose $t > t_0$ we conclude $v_0 = 0$, a contradiction. Let $\phi \geqq 0$ belong to $C^\infty{}_0(\frac{1}{2}, 1)$ with $\int^\infty{}_{-\infty} \phi = 1$, and let $\phi_\epsilon(t) = \epsilon^{-1}\phi(t/\epsilon)$. Given $g \in H(t_0)$, let $g_\epsilon(t) \equiv \int^\infty{}_{-\infty} \phi_\epsilon(\tau - t)T(\tau)g\, d\tau$. As constructed $g_\epsilon(0)$ belongs to $\bigcap^\infty{}_{n=1} \mathfrak{D}(A^n)$ and a standard approximate identity argument together with the strong continuity of $T(t)$ implies that $\|g_\epsilon(0) - g\|_E \to 0$ as $\epsilon \to 0$. Furthermore, because $H(t_0)$ is a closed invariant subspace under $T(t)$, $g_\epsilon(0)$ belongs to $H(t_0)$. Thus, to establish the statement that $H(t_0) = \{0\}$, it suffices to prove that

$$\bigcap_{n=1}^{\infty} \mathfrak{D}(A^n) \cap H(t_0) = \{0\}.$$

The fundamental lemma in our argument is the following:

Proposition 3.1. *Assume* i) Ω *is bounded*

ii) $\partial\Omega$, γ, *and* σ *are real analytic*

iii) $\gamma(x) \neq 1 \ \forall x \in \partial\Omega$

iv) supp $c(x) \cap \partial\Omega = \varnothing$.

Then one can find a fixed number $\delta_0 > 0$ *and a set of real analytic functions* $\{\phi_\alpha\}_{\alpha \in A}$ *dense in* $L^2(\Omega)$ *such that the boundary-value problem*

$$\Box\, v - c(x)v_t = 0$$

$$v(t_1) = 0 \tag{3.3}$$

$$v_t(t_1) = \phi_\alpha$$

$$\frac{\partial v}{\partial n} - \gamma v_t + \sigma v \,\Big|_{\partial\Omega\times[t_1, t_1+\delta_0]} = 0$$

has a C^2 *solution* v *defined in the region* $[t_1, t_1 + \delta_0] \times \bar{\Omega}$. *Furthermore, as a consequence of this construction,* v *satisfies the compatibility conditions* $v|_{\partial\Omega \times \{t_1\}} = 0$ *and* $\partial v/\partial n + \sigma v|_{\partial\Omega\times\{t_1\}} = 0$.

We will now use Proposition 3.1 and the above remarks to prove the following theorems. We will prove Proposition 3.1 afterwards.

Theorem 3.1. *Assume that* $\partial\Omega$, γ, *and* σ *are* C^∞ *and that* $\gamma(x)$ *satisfies* $\gamma(x) < 1$, $\forall x \in \partial\Omega$. *Then no disappearing solutions to the system in* (1.1) *can exist.*

Theorem 3.2 *Assume hypotheses* ii)–iv) *of* Proposition 3.1. *Then no disappearing solutions to the system in* (1.1) *can exist.*

Remark. We omit the proof of Theorem 3.1 since it follows from the previously mentioned results of R. Beals in the same way that Theorem 3.2 follows from Proposition 3.1. We have included Theorem 3.1 to indicate to the reader that analyticity is only a technical assumption in Theorem 3.2.

Proof of Theorem 3.2. Let f be an initial state so that $(u, u_t) = T(t)f$ is a disappearing solution, *i.e.* $f \in H(t_0)$ for some $t_0 > 0$. We must prove that f is zero. Because the mixed problem defined in (1.1) has finite propagation speed, we can assume that there exists a fixed numbr $r_0 > 0$ so that $\forall t \geqq 0$, supp $(T(t)f) \subseteq \Omega \cap \{x \mid |x| \leqq r_0\}$ and that $T(t)f$ satisfies Dirichlet conditions on the analytic surface, $\{x \mid |x| = r_0\}$. For notational convenience we will denote $\Omega \cap \{x \mid |x| \leqq r_0\}$ by Ω but assume Ω is bounded. Furthermore, by the above remarks it suffices to assume $T(t)f \in \bigcap_{N=1}^{\infty} \mathfrak{D}(A^N)$, $t \geqq 0$. Under these circumstances $T(t)f$ belongs to $C^2([0, t_0], \mathfrak{D}(A))$ where $\mathfrak{D}(A)$ is viewed as a Hilbert space under the graph norm of A. ($C^m([0, t], H)$ denotes the space of m-times continuously differentiable functions with values in the Hilbert space H.) Standard elliptic estimates guarantee that the inclusion $i : \mathfrak{D}(A) \to H^2(\Omega) \oplus H^1(\Omega)$ is continuous. Therefore, $u(t)$ belongs to $C([0, t_0]), H^2(\Omega))$. Choose $\epsilon \leqq \delta_0$ and let v be a C^2 solution from Proposition 3.1 with initial value ϕ_α assigned on $t_1 = t_0 - \epsilon$. Since the trace mapping of the Cauchy data is continuous from $H^2(\Omega)$ into $H^{1/2}(\partial\Omega) \oplus H^{3/2}(\partial\Omega)$ and vector-valued integration commutes with bounded linear maps, Green's formula is valid. Therefore,

$$(3.4) \quad \int_{t_0-\epsilon}^{t_0} \int_\Omega \triangle u\bar{v}\, dx\, dt - \int_{t_0-\epsilon}^{t_0} \int_\Omega \overline{u \triangle v}\, dx\, dt = \int_{t_0-\epsilon}^{t_0} \int_{\partial\Omega} \left(\frac{\partial u}{\partial n}\bar{v} - \frac{\partial \bar{v}}{\partial n}\right) ds\, dt$$
$$= -\int_{t_0-\epsilon}^{\epsilon} \int_{\partial\Omega} (\gamma u_t\bar{v} + \gamma u\bar{v}_t)\, ds\, dt$$
$$= -\int_{t_0-\epsilon}^{t_0} \frac{\partial}{\partial t}\left(\int_{\partial\Omega} \gamma u\bar{v}\right)$$
$$= -\int_{\partial\Omega} \gamma u\bar{v}\, \Big|_{t_0-\epsilon}^{t_0} = 0.$$

The last term vanishes because $u(t_0) \equiv 0$, while by construction $v(t_0 - \epsilon)|_{\partial\Omega} = 0$.

Since $u(t)$ belongs to the space $C^2([0, t_0], \mathfrak{D}(A))$, integration by parts is also valid in the t-direction so that by (3.4) we obtain

$$\begin{aligned} 0 &= \int_{\Omega \times [t_0-\epsilon, t_0]} \Box u\bar{v} - u \Box \bar{v} \, dx \, dt \\ &= \int_\Omega u_t \bar{v} - u\bar{v}_t \, dx \,\Big|_{t_0-\epsilon}^{t_0} \\ &= \int_\Omega u_t \,\Big|_{t_0-\epsilon} \bar{\phi}_\alpha \, dx. \end{aligned}$$

Once again the integrals at $t = t_0$ vanish because $T(t_0) \equiv 0$. Since the collection $\{\phi_\alpha\}$ is dense in $L^2(\Omega)$, the above equation implies that $u_t \,|_{t_0-\epsilon} = 0$ for any ϵ with $0 \leqq \epsilon \leqq \delta_0$. Since u_t belongs to $C([0, t_0], H^1(\Omega))$, $u = -\int^{t_0}{}_{t_0} u_\tau \, d\tau$ vanishes for $t \geqq t_0 - \delta_0$. Thus, $T(t)f$ actually vanishes in the shorter time, $t_0 - \delta_0$. Repeating this argument K times where $K\delta_0 \geqq t_0$, we conclude that $T(0)f = f \equiv 0$ as desired.

Proof of Proposition 3.1. We will prove Proposition 3.1 through a series of lemmas.

Lemma 3.1. *Given $\Omega \subseteq \mathbf{R}^n$, a bounded domain with real analytic boundary $\partial\Omega$, there exists a set of functions $\{\phi_\alpha\}$ with the following properties:*

i) *The collection $\{\phi_\alpha\}$ is dense in $L^2(\Omega)$*

ii) $\phi_\alpha \,|_{\partial\Omega} = 0$ *and* $(\partial/\partial n + \sigma)\phi_\alpha \,|_{\partial\Omega} = 0$

iii) *There exists a fixed open set $\mathcal{V}$ with $\bar{\Omega} \subseteq \mathcal{V}$ and with each ϕ_α real analytic on $\mathcal{V}$.*

Remark. The condition in ii) may seem *ad hoc* at the moment, but in fact they are the natural compatibility conditions on the initial data necessary to guarantee a C^2 solution for the boundary value problem in (3.3)

Proof. Choose λ_0 so that Im $\lambda_0 \neq 0$ and consider the following Dirichlet problem for the biharmonic operator in Ω:

$$(3.5) \qquad \begin{aligned} (\triangle\triangle - \lambda_0)\phi_\alpha &= p_\alpha \\ \phi_\alpha \,|_{\partial\Omega} &= 0 \\ \left(\frac{\partial}{\partial n} + \sigma\right)\phi_\alpha \,|_{\partial\Omega} &= 0. \end{aligned}$$

The biharmonic operator with boundary conditions (3.5) is a self-adjoint operator on $L^2(\Omega)$ with dense domain consisting of the $H^4(\Omega)$ functions which satisfy the boundary conditions. Thus, when Im $\lambda_0 \neq 0$ the resolvent $R(\lambda_0)$ exists and is a continuous map from $L^2(\Omega)$ to $L^2(\Omega)$ satisfying the estimate

$$(3.6) \qquad ||R(\lambda_0)f||_{L^2} \leqq C \, ||f||_{L^2} .$$

Choose $\{p_\alpha\}$ to be the restrictions to Ω of polynomials in $\mathbf{R}^n$—then $\{p_\alpha\}$ is dense in $L^2(\Omega)$. We claim that the corresponding $\{\phi_\alpha\}$ from (3.5) satisfy all the conditions of the lemma. Given $\psi \in C^\infty{}_0(\Omega)$, there exists polynomials, p_j, such that $\|p_j - (\Delta\Delta - \lambda_0)\psi\|_{L^2(\Omega)} \to 0$ as $j \to \infty$. Since $R(\lambda_0)p_j = \phi_j$ and $R(\lambda_0)\,(\Delta\Delta - \lambda_0)\psi = \psi$, the estimate in (3.6) implies $\|\phi_j - \psi\|_{L^2(\Omega)} \to 0$. Since $C^\infty{}_0(\Omega)$ is dense in $L^2(\Omega)$, this proves condition i). Property ii) is immediate from the construction of the ϕ_α. Since the system in (3.5), the $\{p_\alpha\}$, and $\partial\Omega$ are real analytic, it follows from a classical theorem of Morrey and Nirenberg (see [8]) that ϕ_α is real analytic in Ω and $\partial^j\phi_\alpha/\partial n^j\,|_{\partial\Omega}$ is a real analytic function on $\partial\Omega$ for any α and $j = 0, 1, 2, 3$. Consider the following initial value problem

$$(3.7)\qquad (\Delta\Delta - \lambda_0)\tilde{\psi}_\alpha = p_\alpha$$
$$\frac{\partial^j}{\partial n^j}\tilde{\psi}_\alpha\,|_{\partial\Omega} = \frac{\partial^j}{\partial n^j}\phi_\alpha\,|_{\partial\Omega}, \qquad j = 0, 1, 2, 3.$$

Since $\partial\Omega$ is real analytic and non-characteristic for $\Delta\Delta - \lambda_0$, we conclude from the Cauchy–Kowalesky theorem that there exists a neighborhood $\mathcal{V}(\alpha) \supseteq \partial\Omega$ where $\tilde{\psi}_\alpha$ is defined. Actually, $\mathcal{V}(\alpha)$ can be chosen independent of α since the size of $\mathcal{V}(\alpha)$ depends only on the region of convergence of both p_α and $\partial^j\psi_\alpha/\partial n^j\,|_{\partial\Omega}$ and these quantities are independent of α—call this fixed region $\mathcal{V}$. By the uniqueness of solutions to the equations in (3.7), $\tilde{\psi}_\alpha|_\Omega \equiv \phi_\alpha$ so that $\tilde{\psi}_\alpha$ extends ϕ_α to the fixed larger region $\mathcal{V} \supseteq \bar{\Omega}$. This verifies condition iii) of the lemma.

First consider the case where $c \equiv 0$. Because the system in (3.3) is translation invariant in time, we assume $t_1 = 0$. Consider the following initial value problem,

$$\Box\, u_\alpha = 0$$
$$u_\alpha\,|_{t=0} = 0$$
$$(u_\alpha)_t\,|_{t=0} = \psi_\alpha\,.$$

From Lemma 3.1 the region of analyticity of ψ_α is $\mathcal{V} \supseteq \bar{\Omega}$ independent of α so that as a consequence of the construction of the Cauchy–Kowaleski theorem, there exists δ_0 independent of α so that u_α is analytic in $\bar{\Omega} \times [0, \delta_0]$. We will impose a finite number of additional restrictions on δ_0. Express the solution to the equations in (3.3) as $v_\alpha = u_\alpha + w_\alpha$ where w_α solves

$$\Box\, w_\alpha = 0$$
$$(3.8)\qquad \left(\frac{\partial}{\partial n} - \gamma\frac{\partial}{\partial t} + \sigma\right)w_\alpha\,|_{\partial\Omega} = f_\alpha \equiv \left(\frac{\partial}{\partial n} - \gamma\frac{\partial}{\partial t} + \sigma\right)u_\alpha$$
$$w_\alpha\,|_{t=0} = 0$$
$$(w_\alpha)_t\,|_{t=0} = 0.$$

Consider the characteristic surface, C_0, emanating forward in time from $\partial\Omega \times \{0\}$ and choose δ_0 so that this surface develops no caustic singularities in the time interval $[0, \delta_0]$. Then a natural way to construct w_α is to define w_α in the "wedge"

R between C_0 and $\partial\Omega \times [0, \delta_0]$ as the solution to

$$\begin{gathered} \square\, w_\alpha = 0 \\ \left(\frac{\partial}{\partial n} - \gamma \frac{\partial}{\partial t} + \sigma\right) w_\alpha \,|_{\partial\Omega} = f_\alpha \\ w_\alpha \,|_{C_0} = 0 \end{gathered} \tag{3.9}$$

and to define w_α in the region $\Omega \times [0, \delta_0]\backslash R$ by $w_\alpha \equiv 0$. If one tries to patch together these two solutions to construct a smooth C^2 solution, then the normal derivatives to the surface C_0 of w_α up to the second order must necessarily vanish. The surface C_0 is spanned by the bicharacteristic rays emenating from $\partial\Omega \times \{0\}$ and the jump in any normal derivative to C_0 satisfies a homogeneous ordinary differential equation along the bicharacteristic curves (see [3] pp. 573–574). Thus, to construct a C^2 solution w_α, it is sufficient to construct w_α so that the initial jump in the first and second normal derivatives of w_α at $\partial\Omega \times \{0\}$ is zero. Since $\partial/\partial t$ is a transverse differentiation to C_0, these conditions on w_α will be satisfied as long as f_α from (3.8) satisfies

$$f_\alpha \,|_{\partial\Omega\times\{0\}} = 0 \quad \text{and} \quad \frac{\partial}{\partial t} f_\alpha \,|_{\partial\Omega\times\{0\}} = 0. \tag{3.10}$$

Since f_α is determined from ϕ_α by the Cauchy–Kowaleski computation, one easily verifies that f_α satisfies the conditions in (3.10) precisely when ϕ_α satisfies the conditions in ii) of Lemma 3.1. Thus, to complete the construction of Proposition 3.1 in the case when $c = 0$, what remains is to prove that the solutions w_α to the characteristic initial value problem in (3.9) are defined on a fixed region R independent of f_α as long as $f_\alpha \,|_{\partial\Omega}$ is analytic on the fixed region $\partial\Omega \times [0, \delta_0]$. Under these conditions on $f_\alpha \,|_{\partial\Omega}$, G. F. Duff in [4] has solved this classical problem.

Lemma 3.2. (*Duff*) *Suppose $f_\alpha \,|_{\partial\Omega}$ is analytic on $\partial\Omega \times [0, \delta_0]$ independent of α. Then under the hypotheses of Proposition* 3.1 *there exists a fixed region R so that the boundary-value problem in* (3.9) *has a solution w_α analytic in R.*

For the sake of completeness we sketch the proof. After a change of dependent and independent variables so that $\partial\Omega \times [0, t_0]$ corresponds to $s_1 = 0$ and so that the surface $s_2 = c$ are translates c units in time of the initial characteristic surface C_0, the problem in (3.9) has the form

$$\begin{gathered} \left(\frac{\partial^2}{\partial s_1\, \partial s_2} + L\left(s_1, s_2, y, \frac{\partial}{\partial y}\right)\right)\tilde{w}_\alpha = 0 \\ \tilde{w}_\alpha \,|_{s_2=0} = 0 \\ \left(\frac{\partial}{\partial s_2} + L\left(s_1, s_2, y, \frac{\partial}{\partial s_1}, \frac{\partial}{\partial y}\right)\right)\tilde{w}_\alpha \,|_{s_1=0} = \tilde{f}_\alpha, \end{gathered}$$

where $\tilde{f}_\alpha$ is analytic in a region independent of α and all indicated differential operators have analytic coefficients. (The condition $\gamma \neq 1$ allows one to solve for $\partial/\partial s_2$ in the boundary condition above.) Assume $\tilde{w}_\alpha$ has the form $\tilde{w}_\alpha = \sum_{n=1}^{\infty} a_n(s_1, y)s_2^n$; then

$$n \frac{\partial a_n}{\partial s_1} = \sum_{j=1}^{n-1} B_j a_j$$

and

$$a_n |_{s_1=0} = \sum_{j=1}^{n-1} C_j a_j + f_{\alpha,n} .$$

Thus, a_n is determined recursively from a_j for $j < n$ by integration of the above equation, and a variant of the method of majorants establishes the actual convergence of the power series. This completes the construction for $c \equiv 0$.

When $c \not\equiv 0$, let $\tilde{v}_\alpha$ solve the inhomogeneous free space problem

$$\Box \tilde{v}_\alpha - c(x)(\tilde{v}_\alpha)_t = c(x)(v_\alpha)_t$$
$$\tilde{v}_\alpha |_{t=0} = 0$$
$$(\tilde{v}_\alpha)_t |_{t=0} = 0.$$

Since supp $c(x) \cap \partial\Omega = \emptyset$ and the above equation has finite propagation speed, there exists $\delta \leqq \delta_0$ so that supp $\tilde{v}_\alpha \cap \partial\Omega = \emptyset$ — $\tilde{v}_\alpha + v_\alpha$ is the solution required in Proposition 3.1 for this case.

Section 4. An application to a problem in dissipative scattering theory. Suppose Ω is an exterior region in $\mathbf{R}^n$. In this section assume the number of space dimensions n is odd and let D^+_ρ (D^-_ρ) denote the outgoing (incoming) subspace of the Lax–Phillips scattering theory (see [6]). Then the space H_E of initial states decomposes into an orthogonal direct sum

$$H_E = D^-_\rho \oplus K_\rho \oplus D^+_\rho$$

and the action of the semigroup $T(t)$ associated with the system in (1.1) is as follows:

$$\begin{aligned} &T(t)D^-_\rho \subseteq H_E \\ (4.1) \qquad &T(t)K_\rho \subseteq K_\rho \oplus D^+_\rho \\ &T(t)D^+_\rho \subseteq D^+_\rho . \end{aligned}$$

Here K_ρ corresponds to the "black box" states associated with the perturbation effects of c, $\partial\Omega$, and γ. Recently, R. S. Phillips (see [9]) posed the following problems: It it possible for a "black box" state to remain undetected by the scattering process?; determine the circumstances under which every initial state in the black box has a component which escapes to infinity. In view of the relations in (4.1), an undetected state is an $f \in K_\rho$ such that $T(t)f$ satisfies

$$(4.2) \qquad T(t)f \perp D^+_\rho \qquad t \geqq 0$$

If no initial states with the property in (4.2) exist, the system will be called *controllable.* It is evident that if a system is controllable, then every initial state in the black box has a component which escapes to infinity.

We have the following partial answer to the above problem.

Theorem 4.1. *Suppose* Ω, γ, σ, *and* $c(x)$ *satisfy the hypotheses of Theorem* 3.1 *or Theorem* 3.2 *and that* $\Omega \backslash \operatorname{supp} c(x)$ *has only one component. Then the dissipative system defined in* (1.1) *is controllable.*

This result is an immediate consequence of Theorem 3.1 or 3.2 and the following two lemmas.

Lemma 4.1. *Given any system of the form in* (1.1) *and an* $f \in K_\rho$ *satisfying* $T(t)f \perp D^+_\rho$, $\forall t \geqq 0$, *then in fact* $T(t)f$ *satisfies the condition*

$$\operatorname{supp} T(t)f \subseteq \Omega \cap \{x \mid |x| < \rho\}, \qquad \forall t \geqq 0$$

Lemma 4.2. *Suppose additionally to the basic hypotheses in* (1.1) *that* $\operatorname{supp} c(x) \cap \partial\Omega = \varnothing$ *and that* $\Omega\backslash\operatorname{supp} c(x)$ *has only one component; if* $\operatorname{supp} T(t)f \subseteq \{x \mid |x| < \rho\} \cap \Omega$ $\forall t \geqq 0$, *then* $T(t)f$ *is a disappearing solution.*

Proof of Lemma 4.1. Since we can mollify in time and preserve the conditions $f \in K_\rho$, $T(t)f \perp D^+_\rho$ $\forall t \geqq 0$, we will assume that $T(t)f$ belongs to $C^\infty([0, \infty), H_E)$. Consider $\psi(x)T(t)f$ where $\psi \in C^\infty{}_0(\Omega)$ and $\psi \equiv 1$ for $|x| \geqq \rho$. It suffices to prove that $\operatorname{supp} \psi T(t)f \subseteq \Omega \cap \{x \mid |x| < \rho\}$. Since $(1 - \psi)T(t)f$ has compact support in $B_\rho = \{x \mid |x| < \rho\}$, $(1 - \psi)T(t)f \in K_\rho$ $\forall t \geqq 0$. By assumption $T(t)f$ satisfies $T(t)f \perp D^+_\rho$ $\forall t \geqq 0$ so that we conclude from the invariance properties of $T(t)$ in (4.1) that

$$\psi T(t)f \perp D^+_\rho \quad \text{and} \quad \psi T(t)f \mid D^-_\rho \ \forall t \geqq 0. \tag{4.3}$$

Denote $\psi T(t)f$ by $\{v, v_t\}$, then v satisfies

$$v_{tt} - \triangle v = g \tag{4.4}$$

where g is square integrable with support in B_ρ. (In fact, $g = -w\triangle\rho - 2\nabla w \nabla \rho$ where w is the first component of $T(t)f$.) Furthermore, to prove that $\operatorname{supp} \psi T(t)$ is contained in $\Omega \cap B_\rho$, it is sufficient to prove that $\operatorname{supp} \{v_t\} \subseteq \Omega \cap B_\rho$, $\forall t \geqq 0$. For under these circumstances, the first component $w(t)$ of $T(t)f$ is then independent of t for $t \geqq 0$ and $|x| > \rho$. Since Lax and Phillips in [5] have proved that the local energy decays for an initial state, *i.e.*

$$\|T(t)f\|_{\Omega\cap B_r} \to 0 \quad \text{as} \quad t \to \infty \quad \text{for any} \quad r > 0,$$

this would imply that $w(t)$ is zero in any annulus $\rho < |x| < r$ and therefore that w is identically zero for $|x| > \rho$. To prove that v_t has compact support, we use the Paley–Weiner theorem for the Radon transform. Recall that the Radon transform of the function pair $\{v, v_t\}$ with finite energy is the function in $L^2(\mathbf{R} \times S^{n-1})$ given by

$$\mathcal{R}(\{v, v_t\}) = \partial_s{}^{(n-1)/2}(\partial_s \tilde{v} - \tilde{v}_t)$$

where $\tilde{f}(s, \omega) = (2\pi)^{(n-1)/2} \int_{x\cdot\omega=s} f(x)\, ds$. Furthermore, if $m(s, \omega) = \mathcal{R}(\{v, v_t\})$ and $\ell(s, \omega) = \mathcal{R}(\{0, g\})$, then from the equation in (4.4) we get

$$(4.5) \qquad (\partial_t + \partial_s)m(s, \omega, t) = \ell(s, \Omega, t).$$

Also recall that one can recover v_t by the inversion formula, $\partial_t v = \int_{S^{n-1}} \partial^{n-1/2}{}_s m \cdot (x\cdot\omega, \omega)\, d\omega$. In order to deduce that v_t has compact support, by the Paley–Weiner theorem (see [6] pp. 113–114) it is necessary and sufficient that $m(s, \omega)$ belong to $L^2(\mathbf{R} \times S^{n-1})$ and satisfy

i) $m(s, \omega, t) = 0$ for $|s| > \rho$

(4.6) ii) $\langle Y_m(\omega)s^a, m(s, \omega)\rangle = 0$ where a is a non-negative integer and Y_m is any spherical harmonic of degree $m > a + (1 - n)/2$.

(Here $\langle\ ,\ \rangle$ denotes the inner product on $L^2(\mathbf{R} \times S^{n-1})$.) By the easy part of this theorem, $\ell(s, \omega)$ has both of these properties. Furthermore, the conditions in (4.3) and the unitarity of the Radon transform guarantee that $m(s, \omega, t) \in L^2(\mathbf{R} \times S^{n-1})$ and that $m(s, \omega, t) = 0$ for $|s| > \rho$. We verify that the argument for the case $a = 0$ is handled in the same way as the general case. To avoid repetition we will only give the proof for the general case. Suppose Y_m is any spherical harmonic with degree $m > a + 1 + (1 - n)/2$. Taking inner products with the equation in (4.5) and integrating by parts, we obtain

$$(4.7) \qquad \begin{aligned} \langle s^{a+1}Y_m(\omega), \partial_t m\rangle &= -\langle s^{a+1}Y_m(\omega), \partial_s m\rangle + \langle s^{a+1}Y_m(\omega), \ell\rangle \\ &= (a + 1)\langle s^a Y_m(\omega), m\rangle + \langle s^{a+1}Y_m(\omega), \ell\rangle. \end{aligned}$$

Since ℓ is the Radon transform of a function with compact support, $\langle s^{a+1}Y_m(\omega), \ell\rangle = 0$ and by the induction hypotheses $\langle s^a Y_m(\omega), m\rangle = 0$; thus we conclude from (4.7) that

$$(4.8) \qquad \langle s^{a+1}Y_m(\omega), \partial_t m\rangle \equiv 0.$$

In [5] Lax and Phillips have proved the strong property that $||P^+T(t)f|| \to 0$ as $t \to \infty$ where P^+ is the orthogonal projection on the complement of $D^+{}_\rho$. If f satisfies the condition in (4.2), then $P^+T(t)f = T(t)f$. Thus, by applying the Poincare inequality to $\psi T(t)f$ and the unitarity of the Radon transform, we obtain

$$||m(s, \omega, t)||_{L^2(\mathbf{R}\times S^{n-1})} = ||\{v, v_t\}|| \leqq C\, ||T(t)f||$$

and we conclude that $||m(s, \omega, t)||_{L^2(\mathbf{R}\times S^{n-1})} \to 0$ as $t \to \infty$. Since $t \to m(s, \omega, t)$ is a smooth map from $\mathbf{R}^+$ to $L^2(\mathbf{R} \times S^{n-1})$, after integrating (4.8) from t_0 to t and using the Schwarz inequality we obtain

$$(4.9) \qquad \begin{aligned} |\langle s^{a+1}Y_m(\omega), m(\cdot, t_0)\rangle| &= |\langle s^{a+1}Y_m(\omega), m(\cdot, t)\rangle| \\ &\leqq \left(\frac{\rho^{a+2}}{a + 2}\right)^{1/2} ||m(\cdot, t)||_{L^2(\mathbf{R}\times S^{n-1})}. \end{aligned}$$

Letting t tend to ∞ in (4.9), the decay of $M(\cdot, t)$ implies that

$$\langle s^{a+1} Y_m(\ ), m(\cdot, t_0)\rangle = 0 \quad \text{for any} \quad t_0 \geqq 0$$

and $m > a + 1 + (1 - n)/2$ as required in condition ii) of (4.6).

Proof of Lemma 4.2. First consider the domain Γ formed by $\Omega \backslash \bar{\mathcal{V}}$ where $\mathcal{V}$ is an open set with smooth boundary enclosing supp $c(x)$ and satisfying $\bar{\mathcal{V}} \cap \partial\Omega = \varnothing$. Since Γ has only one component and $\partial\Gamma$ is smooth, it is easy to see that any point $x \in \Gamma$ can be joined to a fixed x_0 with $|x_0| = \rho$ by a polygonal path lying entirely in Γ with total length less than some L_0 where L_0 is a fixed number independent of $x \in \Gamma$. Since the number of space dimensions is odd, by Holmgren's theorem if u satisfies the wave equation and vanishes in $|x| \leqq R_0$ and $|t| \leqq T$, then in fact u vanishes in the larger region, $|x| \leqq R + T - |t|$. Recall that by assumption supp $u(x, t)$ is contained in $\Omega \cap \{x \mid |x| \leqq \rho\}$, $\forall t \geqq 0$. That $u(x, t)$ vanishes on Γ for $t > L_0 + 1$ follows by applying Holmgren's theorem in small steps along the polygonal path connecting x to x_0 . Since $\bar{\mathcal{V}} \cap \partial\Omega = \varnothing$ and for $t > L_0 + 1$ u can be nonzero only on this set, $w(x, t) = u(x, t + L_0 + 1)$ is a solution of the following interior mixed problem

$$\Box\, w + c(x)w_t = 0 \qquad (4.10)$$

$$w\,|_{\partial\Omega} = w\,|_{|x|=\rho} = 0.$$

But w omits a fixed open set near $\partial\Omega$ $\forall t \geqq 0$ for the system in (4.10) with conservative boundary conditions, so that by a theorem of Rauch and Taylor in [10], w must be zero so that u vanishes identically for $t \geqq L_0 + 1$.

Section 5. On the completeness of generalized eigenfunctions for the interior problem. When Ω is an exterior region, in the previous section we have explained how the existence of disappearing solutions is pathological from the view-point of dissipative scattering theory. Similarly, when Ω is a bounded domain, the existence of disappearing solutions results in pathological spectral properties for the interior problem.

Recall that if A is a closed operator with compact resolvent acting on a Hilbert space H, the *generalized eigenfunctions* of A are the solutions of $(A - \lambda)^N u = 0$ for some λ and integer $N > 0$. The span of A, denoted Sp(A), is the linear subspace obtained by taking the closure of finite linear combinations of generalized eigenvectors. It is natural to expect the completeness of the generalized eigenfunctions, *i.e.* that Sp(A) $= H$. (This is the case for many concrete elliptic boundary-value problems acting on $L^2(\Omega)$—see [1].) In contrast to this situation, when disappearing solutions exist for the adjoint semigroup, *i.e.* $\tilde{H}(t_0) \neq \{0\}$ for some $t_0 > 0$, we have the following result.

Theorem 5.1. *Suppose A is a closed maximal dissipative operator with compact resolvent. Suppose $\tilde{H}(t_0) \neq \{0\}$ for some $t_0 > 0$. Then the generalized eigenfunctions for A fail to be complete. More precisely, $\tilde{H}(t_0) \subseteq \mathrm{Sp}(A)^\perp$ and $\mathrm{Sp}(A)^\perp$ is infinite dimensional.*

Proof. Suppose $f \in \tilde{H}(t_0)$ with $f \not\equiv 0$; then $T^*(t)f \equiv 0$ for $t \geqq t_0$. From the Laplace transform formula for the generator of a contraction semigroup for $\operatorname{Re} \eta > 0$.

$$(\eta - A^*)^{-1}f = \int_0^\infty e^{-\eta t}T^*(t)f\,dt = \int_0^{t_0+1} e^{-\eta t}T^*(t)f\,dt. \tag{5.1}$$

The extreme right-hand side of (5.1) is defined for all η as an entire function; therefore $(\eta - A^*)^{-1}$ is an entire vector-valued function. We claim that f is orthogonal to $\operatorname{Sp}(A)$. Let $\sum^n_{i=1} c_i e_i$ be a linear combination of generalized eigenvectors for A and let Γ be a finite closed curve enclosing the eigenvalues corresponding to the e_i. Consider the projection operator,

$$P = \frac{1}{2\pi i}\int_\Gamma (\lambda - A)^{-1}\,d\lambda.$$

By construction, $P(\sum^n_{i=1} c_i e_i) = \sum^n_{i=1} c_i e_i$. Furthermore, P^* has the form

$$P^* = \frac{1}{2\pi i}\int_{\bar\Gamma} (\eta - A^*)^{-1}\,d\eta \tag{5.2}$$

where $\bar\Gamma$ is the closed curve of conjugate points to the curve Γ. Because $(\eta - A^*)^{-1}f$ is an entire function, by Cauchy's theorem

$$\int_{\bar\Gamma} (\eta - A^*)^{-1}f\,d\eta = 0.$$

Thus we obtain

$$\left(\sum_{i=1}^n c_i e_i\,,\, f\right) = \left(P\left(\sum_{i=1}^n c_i e_i\,,\, f\right)\right) = \left(\sum_{i=1}^n c_i e_i\,,\, P^*f\right) = 0$$

so that $\tilde{H}(t_0) \subseteq \operatorname{Sp}(A)^\perp$. By our remarks in Section 3 if $\tilde{H}(t_0) \neq \{0\}$, then this space is necessarily infinite dimensional. This completes the proof.

We have the following immediate corollary:

Corollary 5.1. *Consider the concrete mixed problems on bounded regions associated with the examples in Section 2. The infinitesimal generator of these systems has compact resolvent, but the span of the generalized eigenfunctions has infinite codimension.*

Remark. The system in (1.1) have compact resolvent as a consequence of standard elliptic estimates. Also we remind the reader that for these concrete systems, $T(t)$ has disappearing solutions if and only if $T^*(t)$ does, so that the hypotheses of Theorem 5.1 are satisfied.

The author has recently proved that the generalized eigenfunctions are complete when Ω is a bounded region provided that γ satisfies $\gamma(x) < 1\ \forall x \in \partial\Omega$. These results will appear in a forthcoming paper.

REFERENCES

1. S. AGMON, *Lectures on elliptic boundary value problems*, Van Nostrand, New York, 1965.
2. R. BEALS, *Hyperbolic equations and systems with multiple characteristics*, Arch. Rat. Mech. **48** (1972), 124–152.
3. R. COURANT & D. HILBERT, *Methods of mathematical physics*, Vol. II, Wiley, New York, 1962.
4. G. F. DUFF, *A mixed problem for normal hyperbolic linear partial differential equations of the second order*, Can. J. Math. **9** (1957), 141–160.
5. P. D. LAX & R. S. PHILLIPS, *Scattering theory for dissipative hyperbolic systems*, J. Fun. Anal. **14** #2, (1973), 172–235.
6. P. D. LAX & R. S. PHILLIPS, *Scattering theory*, Academic Press, New York, 1967.
7. P. D. LAX & R. S. PHILLIPS, *The acoustic equation with an indefinite energy form and the Schrodinger equation*, J. Fun. Anal. **1** (1967), 37–83.
8. C. MORREY & L. NIRENBERG, *On the analyticity of the solutions of linear elliptic systems of partial differential equations*, Comm. Pure Appl. Math. **10** (1957), 271–290.
9. R. S. PHILLIPS, *Proceedings of N.A.T.O. conference on scattering*, Summer 1973, Denver, Colorado (to appear).
10. J. RAUCH & M. TAYLOR, *Penetrations into the shadow regions and unique continuation properties in hyperbolic mixed problems*, Indiana Univ. Math. J. **22** (1972), 277–285.
11. H. O. KREISS, *Initial boundary value problems for hyperbolic systems*, Comm. Pure Appl. Math. **13** (1970), 277–298.

This work was partially supported by National Science Foundation grant GP-37069X1.

Courant Institute of Mathematical Sciences,
New York University

Date communicated: MAY 28, 1974