Discontinuities in the Electromagnetic Field

M. Mithat Idemen

IEEE Antennas and Propagation Society, *Sponsor*

The IEEE Press Series on Electromagnetic Wave Theory
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Preface

Without any hesitation we can claim that our civilization is largely based on the capacity and effectiveness of our knowledge pertaining to the natural phenomena collected under the name of *electromagnetism*. Our effective knowledge on these phenomena, with which we are familiar in fragments for nearly 2000 years, since the era of Thales, goes back only one and a quarter centuries ago. The years of 1873 and 1887 in the nineteenth century were two very important milestones in the intellectual evolution, as well as in the technological achievements of mankind. Indeed, the first one was the year in which the *electricity, magnetism*, and *optics*, which had been considered to be different natural phenomena until then, were unified under a common framework called the *electromagnetism*. To this end, Maxwell wrote a system of partial differential equations in England and claimed that those phenomena are all some particular aspects of a unique phenomenon which satisfies his equations. The equations introduced by Maxwell were not only what are reduced to the already known equations written 40 years ago by Faraday in England and 50 years ago by Ampère in France, but also claimed that the phenomenon in question consisted of a wave, which propagates with a finite velocity. What is very interesting is that physicists and engineers of those days were not familiar with such a kind of wave. To believe in the existence of the wave in question, one had to wait 14 years to witness the modest experiment carried out by Hertz in 1887 in Germany. That experiment, which revealed the so-called electromagnetic wave, showed also that energy transmission is possible through this wave. That was the indicator of a new direction for the civilization and thusly intensified the interest in the electromagnetic wave in both theoretical and experimental domains. Within the three years just after the Hertz experiment, the first radio receiver was realized by Branly in Germany in 1890, and soon after this, many crucial applications in communication, control, remote sensing, medicine, radio-astronomy, space communication, heating, and so on, began to flow. Among them we can mention, for example, the radio communication between two sides of the Atlantic Ocean (in 1901), radar (in 1940), laser (in 1957), tomography (in 1984), etc.

By considering new products of the actual technology, which permit us to use extremely short waves whose frequencies increase day by day, we can hope that electromagnetic waves will continue to be the basis of new, even unexpected, applications. These applications will require solutions of the Maxwell equations under new supplementary relations (boundary conditions, edge conditions, tip conditions, etc). Therefore, a thorough investigation of these supplementary relations is very important from both theoretical and practical points of view. The aim of this short monograph is to fulfill this job. I think (and hope) that readers who are fond of the *theory* will find several enjoyable points in the book. However, the meticulous colleagues will certainly find many points to criticize. As a scientist who imagines science as a continuously evolving struggle, I look forward with deep gratitude to all constructive comments and critiques.

What seems also important to me is that by claiming the existence of the electromagnetic wave before its experimental observation, the theory of Maxwell epitomizes an example for the "*theory before experiment.*" The electromagnetic *wave* is not the unique example presented by the Maxwell equations in this sense. Indeed, by claiming a nonuniversal time concept, they became also the main instigator for the Special Theory of Relativity established by Einstein nearly 30 years later than the Maxwell's Theory. The equations proposed by Maxwell conceal inside themselves too many secrets of the nature: reflected waves, refracted waves, creeping modes, whispering gallery modes, edge-excited waves, tip-excited waves, shadow boundaries, reflection boundaries, refraction boundaries, caustics, traveling waves, standing waves, Cherenkov waves, trapped waves, boundary conditions, edge conditions, tip conditions, space-time transformation under uniform motions, Doppler effect, aberration, and so on. They permit us to discover all these secretes as well as the interrelations among them if we interrogate with appropriate mathematical tools. I believe that the most effective tool for the investigation of the discontinuities in *natural phenomena* of any kind is the concept of *distribution*. This concept, which was introduced by Schwartz in 1950 (three quarters of a century later than the Maxwell equations) in France to the contemporary mathematics, extended the meaning of the Maxwell equations. If one assumes that the differential operations taking place in the Maxwell equations are all in the sense of distributions, then the discontinuities can be discussed pretty easily and rigorously. This monograph adopts this approach. When the surfaces which carry the discontinuities are in uniform motion, formulas of the Special Theory of Relativity help to reveal behaviors of the discontinuities by transforming the expressions pertinent to the surfaces at rest.

With their elegances and powers, the above-mentioned two theories (i.e., the Theory of Electromagnetism—including the special theory of relativity—and the Theory of Distributions) excite feelings of admiration. For me, they are not a heap of spiritless mathematical symbols but rather the *self-consistent* intellectual and *fine-artistic* productions of human mind. Since the Theory of Distributions as well as that of the Special Theory of Relativity are not thoroughly included into the undergraduate curriculum, in order to offer a self-contained book, the necessary (and sufficient) material which helps to clarify the essentials of these theories, are also be included into the book.

It is a great pleasure for me to confess that the publication of this monograph could not have been realized if sincere encouragements and supports by certain colleagues had not existed. Among them I mention especially Professor Tayfun Akgül (Istanbul Technical University, Turkey), Professor Ross Stone (IEEE Antennas and Propagation Society Publishing Board member), Professor Robert Mailloux (IEEE Antennas and Propagation Society Press Liason Committee Chair) and Professor Andreas C. Cangellaris (editor of the IEEE Press Series on Electromagnetic Wave Theory). I am indebted to all of them for their invaluable interests and supports.

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Heybeliada, Istanbul
March 2011

M. Mithat Idemen

CHAPTER 1

Introduction

A man may imagine things that are false, but he can only understand things that are true.
Isaac Newton

Almost all mathematical problems connected with the electromagnetic phenomena require solutions of the Maxwell differential equations

$$\mathrm{curl}\mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} = \mathbf{J}, \qquad \mathrm{curl}\mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} \qquad (1.1\mathrm{a,b})$$

$$\mathrm{div}\mathbf{D} = \rho, \qquad \mathrm{div}\mathbf{B} = 0 \qquad (1.1\mathrm{c,d})$$

under certain supplementary restrictions stipulated on certain surfaces. In (1.1a–d) **E, D, H**, and **B** stand for the *electric field, electric displacement, magnetic field*, and *magnetic induction*, respectively, while ρ and **J** are the volume densities of the charges and currents. As to the parameter t that appears in (1.1a,b), it is, as usual, the time. The surfaces that bear the above-mentioned supplementary restrictions are either the interfaces between bodies of different constitutive parameters or surfaces that support surface charges and currents[†] or *material sheets* that model very thin layers. The supplementary restrictions in

[*]Michael White, *Isaac Newton: The Last Sorcerer*, Basic Books, New York, 1977, p. 5. See also *Sir Isaac Newton's Theological Manuscripts*, H. McLachlan (Ed.), Liverpool University Press, Liverpool, 1950, p. 17.
[†]Line and point charges are also carried by certain appropriately defined surfaces.

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question are the relations that state the *physically admissible discontinuities* that may occur on these surfaces. They consist, in general, of the so-called *boundary conditions* that give the jump discontinuities on the surfaces in questions. If the discontinuity surfaces also involve sharp edges and/or sharp tips, then some components of the field become infinitely large at some points. In this case, in addition to the boundary conditions, one also has to know the physically admissible asymptotic behaviors of the field near those points because, as was shown more than 60 years ago by Bouwkamp [1], one can construct many solutions to the Maxwell equations under the given boundary conditions. Of course some of these solutions are not acceptable from physics point of view. Depending on the nature of the singular point, the relations that state the asymptotic behaviors in question are called the *edge conditions* or the *tip conditions*.

It is worthwhile to remark here that any relation written on a surface cannot be treated as a boundary condition for the electromagnetic field. In order to be so, it must also be compatible with the Maxwell equations. The spectrum of the electromagnetic waves used in the telecommunication is enlarging every day more and more toward very short waves. Hence many types of roughness, which had been assumed to be negligible in earlier investigations in order to reduce mathematical difficulties, became today unavoidable. In studies to be made in forthcoming days, one will have to consider the effects of these types of roughness on the propagation of waves. Therefore rigorous and detailed investigations of the boundary, edge, and tip conditions for various geometrical and physical structures are of crucial importance from both pure scientific and technological applications points of view. The aim of the present monograph is to study the discontinuities (i.e., singularities) in questions in their most general framework and discuss the validity of some particular relations that are in use in current literature. Our fundamental basis will be the so-called *distributions* (or *generalized functions*). In order to clarify the crucial role of this concept in the present study, it will be useful to reexamine the derivation of the *classical* boundary conditions in *almost* all textbooks.

On those days when the Electromagnetic Theory had been established, one had a huge heap of scientific knowledge in theoretical physics, especially in fluid mechanics. With this potential, the scientists of that time (i.e., mathematicians, physicists, and engineers) had formulated and solved many mathematical problems that could have interesting and important interpretations in terms of the electromagnetic notions. First problems were those that needed to find the explicit expressions of the fields created by various sources distributed in the vacuum. They were very easy. Soon later, one had considered the problems connected with bodies having simple geometrical shapes such as infinite planes, infinitely long cylinders, whole spheres, whole ellipsoids, and so on. They were also rather easy and tractable with known techniques provided that the boundary conditions to be satisfied on the surfaces of the bodies were known beforehand. It was at this stage that the struggles to reveal the discontinuities of the electromagnetic field were started. The methods that seemed most propitious were what are based on the applications of the Gauss–Ostrogradski and Stokes theorems (see Smirnow [2, pp. 177, 197]). Although these applications are repeated in almost all textbooks, we want to recapitulate them here in order to clarify the philosophy of the method adopted in the present study. To this end, consider, for example, (1.1c) and integrate it inside a volume ϑ bounded by planar boundaries of very small areas (see Fig. 1.1). Assume that $\mathbf{D}$ as well as its partial derivatives of the first orders are all *continuous inside* ϑ *except on a regular surface* ΔS. To avoid useless complexities, assume also that the field depends only on two space coordinates. Then a *careless* application (without regarding the validity conditions) of the Gauss–Ostrogradski theorem to

$$\int_{\vartheta} \mathrm{div}\mathbf{D}\, d\vartheta = \int_{\vartheta} \rho\, d\vartheta = Q \tag{1.2a}$$

yields

$$\int_{\Delta S_1 + \Delta S_2 + \Delta S_3 + \Delta S_4} \mathbf{D} \cdot \mathbf{n}\, dS = Q. \tag{1.2b}$$

Here Q stands for the total charge existing inside ϑ. If Q consists of the surface charge distributed with density ρ_S on S, then by making

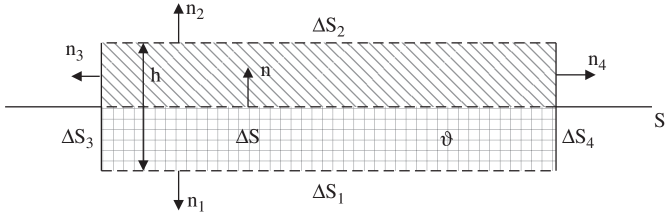

Figure 1.1. A neighborhood of a surface of discontinuity.

$h \to 0$ in (1.2b) one gets

$$\int_{\Delta S} [D_n^{(2)} - D_n^{(1)}]\, dS = \int_{\Delta S} \rho_S \, dS \tag{1.2c}$$

or

$$D_n^{(2)} - D_n^{(1)} = \rho_S . \tag{1.2d}$$

Here $D_n^{(1)}$ and $D_n^{(2)}$ show the values of the normal component of **D** which are observed when one approaches S from lower and upper sides, respectively.

Quite similarly, from (1.1a,b,d) one gets

$$B_n^{(2)} - B_n^{(1)} = 0, \tag{1.2e}$$

$$\mathbf{n} \times \mathbf{E}^{(2)} - \mathbf{n} \times \mathbf{E}^{(1)} = 0, \tag{1.2f}$$

$$\mathbf{n} \times \mathbf{H}^{(2)} - \mathbf{n} \times \mathbf{H}^{(1)} = \mathbf{J}_S . \tag{1.2g}$$

The field $\mathbf{J}_S$ appearing in (1.2g) is the density of the surface current that flows on S.

The relations in (1.2.d–g) are the *classical boundary conditions* that are satisfied on the interface between two regions filled with different materials. They are in use since the first days of the Theory to find solutions of the electromagnetic problems, which are in complete agreements with experiments. But the mathematical analysis made to reveal them is obviously *not legitimate*. Indeed, in order for the Gauss–Ostrogradski and Stokes theorems to be applicable, the fields **E, D, H**, and **B** have to be continuous inside ϑ. But, as the results themselves show, this is not the case. However, as we have already stated, the results are *correct from physics point of view*. A few writers who are familiar with this contrast warn readers by indicating that the results to be obtained by this kind of faulty applications are *assumed* to be acceptable for physics. D. S. Jones [3, p. 46] and S. A. Schelkunoff [4] epitomize this group of meticulous scientists.* It goes without saying that this assumption is nothing but an *additional postulate* to the Maxwell equations. Regarding the swift developments in the contemporary technology, which create and use very different materials and geometries, one can easily grasp that this approach cannot enable one

*Schelkunoff characterizes this kind of an approach to be “a proof which is not a proof but a swindle” (see Schelkunoff [4, Section 5]).

to overcome the difficulties completely. Hence one has to contrive a general and robust method.

One could also think that the difficulty in deriving (1.2d) resulted from the derivatives existing in (1.1c); if one started from the integral equation (Gauss' law)

$$\oint_S \mathbf{D} \cdot \mathbf{n}\, dS = Q, \tag{1.3a}$$

which is equivalent to (1.1c), one would not have the same difficulty. Indeed, in this case from (1.3a) one writes directly

$$\int_{\Delta S_1+\Delta S_2+\Delta S_3+\Delta S_4} \mathbf{D} \cdot \mathbf{n}\, dS = \int_{\Delta S} \rho_S\, dS, \tag{1.3b}$$

which, after applications of the Gauss–Ostrogradski theorem to the partial regions lying above and below the surface ΔS, yields (1.2d). But this approach too, which at first glance seems to be propitious to overcome the difficulty, has severe defects. For example, in the case when S consists of a double sheet that carries only dipoles, the right-hand side of (1.3b) becomes naught and claims $D_n^{(2)} - D_n^{(1)} = 0$, which is not correct.* Furthermore, in cases of material sheets that are represented, for example, by impedance or resistive or other more general type of boundary conditions one cannot guess the right-hand side of (1.3b).

To avoid the difficulties mentioned above, we propose to add the following assumption to the Maxwell equations [5]:

Maxwell equations are valid in the whole of the four-dimensional space in the sense of distribution.

The results of this assumption (or postulate) will be seen after Chapter 2. Here we confine ourselves to make only the following remarks that are of importance.

i. This assumption not only legitimizes the use of the Dirac delta *functions* to represent surface (or line or point) charges and currents (i.e., ρ and $\mathbf{J}$) but also claims that the field components **E,**

*Correct expression is $\lambda[D_n^{(2)} - D_{1n}] = -\text{div}\{(1/\lambda)\rho_1\mathbf{n}\}$. Here ρ_1 stands for the density of the dipole distribution while λ is a parameter depending on the curvature of S and $\mathbf{n}$ is the unit normal vector to S [see Section 3.2.1, formula (3.19c)].

D, H, and **B** are also distributions. In other words, the field components themselves can contain *singular* terms concentrated on certain surfaces.

ii. This assumption concerns not only the space coordinates and boundary conditions but also the time parameter and initial values. That means that the field components can involve also singular terms concentrated at certain isolated instants (such as flash of lightning).

iii. In 1873, when Maxwell had established his theory, as well as during the following 75 years, the concept of distribution did not exist in the scientific literature. It appeared after 1950 first in mathematics and then in physics and engineering sciences. Hence, the claim that the Maxwell equations are valid in the sense of distribution is in fact a new postulate added to the Maxwell Theory.

The arrangement of the present monograph is as follows: In Chapter 2 the concepts of distribution and derivatives in the sense of distribution are explained. Grad, curl, and div operators as well as distributions concentrated on a surface are discussed in some detail. In Chapter 3 the Maxwell equations are reconsidered and discussed in the framework of these new concepts. The so-called *universal boundary conditions* are derived as a natural result. Chapter 4 is devoted to an extensive analysis of the boundary conditions on a *material sheet* at rest. The so-called impedance and resistive-type particular conditions are discussed and their validity conditions are derived. The case of moving boundaries is considered in Chapter 5. In this chapter the connection with the Special Theory of Relativity is also established whenever the motion is uniform. Chapter 6 is devoted to the edge conditions on a wedge bounded by planar walls. In this chapter, one shows also that the origin of the logarithmic-type singularities is the confluence of two algebraic singularities. In Chapter 7, one considers the tip singularities that occur at the apex of a rotationally curved material cone. For this kind of geometry, also the logarithmic singularities are derived as a result of confluence of two algebraic singularities. In Chapter 8, one considers the temporal discontinuities localized at certain times.

CHAPTER 2

Distributions and Derivatives in the Sense of Distribution

2.1 FUNCTIONS AND DISTRIBUTIONS

Even in the early days of the second half of the nineteenth century, one had observed that the mathematics, especially the concepts of functions and derivatives which had been established and maturated during the eighteenth and nineteenth centuries and permitted us to investigate various natural phenomena with deep insight, is not sufficient to grasp certain *singularities* in natural phenomena. To overcome the difficulties, one tried, from time to time, to introduce some concepts and to derive some formulas that were not based on solid mathematical basis. For example, Gustav Robert Kirchhoff,* a famous German physicist of the nineteenth century, had tried to define a force that acts on a very small area as follows [6]:

$$F = \left(\frac{\mu}{\sqrt{\pi}}\right) e^{-\mu^2 x^2} \qquad (\mu \text{ is a very large positive constant}). \quad (2.1)$$

*G. R. Kirchhoff (Königsberg 1824–Berlin 1887).

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For the same purpose Paul Dirac,* a very celebrated English physicist of the last century, had introduced his famous function† $\delta(x)$, which had the following properties [7]:

$$\delta(x) = 0 \qquad \text{when } x \neq 0 \tag{2.2a}$$

$$\delta(x) = \infty \qquad \text{when } x = 0 \tag{2.2b}$$

such that

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{2.2c}$$

By using this exotic δ-*function*, Dirac had obtained some results that were adopted by physicists with enthusiasm. But the mathematicians of those days were watching the progress with certain reserve because they had known that the requirements stipulated by (2.2a)–(2.2c) were not compatible with the classical definitions of function and integral (in the Lebesgue sense).

Although it did not exist as a function in the proper sense of function, in the first half of the last century $\delta(x)$ was extensively used with its properties given in (2.2a–c) by physicists and engineers to produce many interesting results that could all be interpreted in an acceptable manner. This achievement encouraged mathematicians of that time to establish a rigorous basis for the exotic $\delta(x)$, which, from one perspective will ensure its adoption by mathematicians without any reserve and, from the other perspective, will permit one to interpret correctly the results obtained through it. This goal was achieved by the famous French mathematician Laurent Schwartz‡ in 1950 [8]. The result was the introduction of some new entities and concepts into the mathematics of the twentieth century. These new entities are the so-called *generalized functions*, which involve also all locally integrable functions. Considering previous uses of $\delta(x)$ to represent charge distributions, localized on point sets or lines or surfaces, these new entities were also referred to as the *distribution functions* or, more simply, *distributions*. In this monograph we will

*P. A. M. Dirac (Bristol 1902–Tallahassee 1984).

†The symbol δ had been used first by Kirchhoff and then later by Dirac (see Jones [3, p. 35]).

‡L. Schwartz (Paris 1915–2002). For this brilliant achievement he was honored by the Fields Medal in 1950. This award, which has been established in 1936 and devoted only to mathematics, is a counterpart of the Nobel Prize, which does not involve mathematics.

follow the founder of the theory and use this name—that is, *distribution*. In what follows we will recapitulate some basic notions and properties of distributions, which will constitute the main basis of our investigation.

2.2 TEST FUNCTIONS. THE SPACE C_0^∞

Let D be the set of real-valued functions that depend on the real variable $x \in (-\infty, \infty)$ such that beyond certain *finite* intervals they are naught and have continuous derivatives of all orders for all $x \in (-\infty, \infty)$. As we will see later on, these functions will play a crucial role in defining the distributions as well as their equalities, sums, multiplications, sequences, series, limits, derivatives, and so on. Hence the functions belonging to D are called the *test functions*. In what follows we want to clarify first the set D and its ability in the goal mentioned here.

To begin with, let us show that the set D is not empty. Indeed, the classical example given by Schwartz is as follows:

$$\varphi(x,a) = \begin{cases} \exp(-a^2/(a^2 - x^2)), & x \in [-a, a] \\ 0, & |x| \geq a. \end{cases} \tag{2.3}$$

The finite *closed* interval $[-a, a]$, outside of which $\varphi(x,a) \equiv 0$, is referred to as the *support** of $\varphi(x,a)$ and denoted by supp $\{\varphi(x,a)\}$ (see Fig. 2.1). A function having this property is called a function of *bounded support*. The set of test functions with support inside the interval (α, β) is also denoted by $C_0^\infty(\alpha, \beta)$, while $C_0(\alpha, \beta)$ shows the set of continuous functions with support inside (α, β). Note that $D = C_0^\infty(-\infty, \infty)$. We will denote D by C_0^∞ for short.

Starting from $\varphi(x,a)$ given in (2.3), we can also define many other test functions. Consider, for example, an arbitrary locally integrable† function $f(x) \in L_0^1(\alpha, \beta)$, which is identically naught beyond a certain

*Generally, the support of a function $f(x, y, \ldots)$ is the closure of the set of points $(x, y, \ldots)$ for which $f(x, y, \ldots) \neq 0$.

†If the integral of $|f(x)|$ on every finite interval exists, then $f(x)$ is said to be locally integrable.

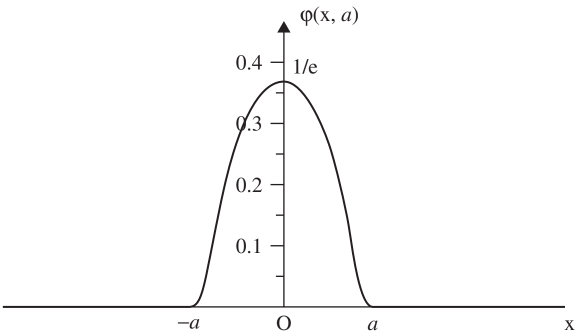

Figure 2.1. The test function $\varphi(x,a)$ and its support.

finite interval $(\alpha,\ \beta)$ and write

$$\varphi(x) = f(x) * \varphi(x,a) \tag{2.4a}$$

$$= \int_{\alpha}^{\beta} f(\xi)\varphi(x-\xi,a)\,d\xi \tag{2.4b}$$

$$= \int_{-a}^{a} f(x-\xi)\varphi(\xi,a)\,d\xi. \tag{2.4c}$$

In (2.4a), * stands for the *convolution*. By considering the support of $f(x)$, we can easily convince ourselves that $f(x-\xi) \equiv 0$ when $x > (\beta + a)$ or $x < (\alpha - a)$ if $\xi \in (-a, a)$. Indeed, from $f(x) \equiv 0$ when $x > \beta$ one gets $f(x-\xi) \equiv 0$ when $x - \xi > \beta$ or $x > \beta + \xi$. Hence $f(x-\xi) \equiv 0$ for all $x > \beta + a$. Similarly, $f(x-\xi) \equiv 0$ for all $x < \alpha - a$. Thus from (2.4c) one concludes

$$\text{supp}\{\varphi(x)\} \subseteq (\alpha - a, \beta + a),$$

which claims that $\varphi(x)$ defined by (2.4a) is a function of bounded support (see Fig. 2.2). On the other hand, the expression in (2.4b) can be differentiated as many times as desired because the range of integration is finite and the integrand has continuous derivatives of all orders. This shows that the function $\varphi(x)$ is exactly a test function. By replacing $f(x)$ in (2.4a) by different functions, one gets different test functions. Two simple examples are shown in Fig. 2.2.