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ABSTRACT

The deformation of a discontinuous rock is a sum of individual translations, rotations, and strains of the component blocks. These produce opening and slip between blocks along the discontinuities. Rather than attempting to analyze such deformations with a mathematical model representing the geometric and stress/strain properties of all points, for the back-calculation of observed displacements we have produced a novel approach to solve the inverse problem. Using a field of displacement measurements and a map of the system of discontinuities, the method determines the intrablock and interblock motions by a least squares formulation. In this brief paper we address only two-dimensional formulations. However, computer codes have also been developed for three dimensional problems.

INTRODUCTION

In many practical problems, monitoring of rock deformation provides a large amount of data on the movement of specific points at the surface or within the interior of a rock mass. Sometimes, data are also provided on differences of displacements measured directly between particular points, as for example along the line of a deformeter installed in a drill hole. To get as much as possible from such measurements, we have developed a backward modelling analysis. Forward modelling of deformation data from rock slopes and underground excavations is also possible using a variety of engineering approaches, including: comparison with closed-form solutions of appropriate boundary value problems in elasticity, or plasticity; observations with an analogous physical model; and numerical modelling with eg. finite element or finite difference analysis to solve force/displacement coupling for the boundary conditions of interest. Observations of displacement in a deforming rock mass present the engineer with a set of <u>facts</u>, whose source mechanisms are known only after interpretation. This is a problem of the inverse type - and therefore a backward modelling analysis seems more direct and applicable in practice.

Since failures and large deformations of excavations and foundations in rock usually involve relative movements along joints and faults, discontinuous deformation analysis applies directly to such problems. It also applies to tectonic analysis of crustal strain data. If the modes of deformation are known a priori then the discontinuous deformation analysis should also be able to analyze deformations to determine material properties or environmental factors, assuming one or the other is known.

In this brief introduction to the method, we will derive the underlying formulas and report on the uniqueness of the solutions obtained when a sufficient set of measurements is input. We will conclude with a few modest examples worked from synthetic data.

Notation

(x,y)	coordinates of a point
(x_0,y_0)	center of rotation
(m_1,m_2)	measured displacements of point (x,y)
(u,v)	computed displacements of point (x,y)
(u_0,v_0)	displacement of point (x_0,y_0)
(ω)	angular rotation about (x_0,y_0)
$(\varepsilon_x\ \varepsilon_y\ \gamma_{xy})$	strains
$(i,\ j,\ k)$	unit vectors parallel to the x,y,z area respectively.

FORMULATION OF DISPLACEMENTS

The sources of deformation of points in a single block are parallel translation, rotation, and strain. These will be accumulated in a single unknown vector as follows.

Parallel Translation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \tag{1}$$

Rotation ω About $(x_0,\ y_0)$

$$(u \quad v \quad w) \begin{pmatrix} i \\ j \\ k \end{pmatrix} = \begin{pmatrix} i & j & k \\ 0 & 0 & \omega \\ x-x_0 & y-y_0 & z-z_0 \end{pmatrix}$$

or

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -(y-y_0) \\ (x-x_0) \end{pmatrix} \omega \qquad (2)$$

<u>Normal Strain</u>

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x-x_0 & 0 \\ 0 & y-y_0 \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \end{pmatrix} \qquad (3)$$

<u>Shear Strain</u>

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} y-y_0 \\ x-x_0 \end{pmatrix} \gamma_{xy} \qquad (4)$$

Total Displacement (for the i^{th} block)

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -(y-y_0) & (x-x_0) & 0 & (y-y_0) \\ x-x_0 & 0 & y-y_0 & x-x_0 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ \omega \\ \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \qquad (5)$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \begin{matrix} (t) & (d) \\ (2\times6) & (6\times1) \end{matrix}$$

(the above corresponds to a perfect first order approximation of the displacement field: $u = a_1x + b_1y + d_1$, $v = a_2x + b_2y + d_2$ and with any location x_0, y_0).

LEAST SQUARES ANALYSIS

At each point of measurement, we now compute displacements as functions of the vector (d). Differentiating these functions and equating the results to zero establishes simultaneous equations to determine the terms of (d). This procedure is generalized for measurements of relative displacement between blocks.

The Squared Deviation of Displacements at Data Points

Let (m_1,m_2) be the displacements measured at point $a = (x,y)$, which lies within block i. The displacements are computed for this point using (6). The squared deviation of point a is Φ_a, defined as

$$\Phi_a = (m_1-u)^2 + (m_2-v)^2. \tag{7}$$

Inserting (6) in (7) gives

$$\Phi_a = (m_1 - \sum_{\ell=1}^{6} t^i_{1\ell}\ (x,y)\ d^i_\ell)^2 + (m_2 - \sum_{\ell=1}^{6} t^i_{2\ell}\ (x,y)\ d^i_\ell)^2 \tag{8}$$

The Squared Deviation of Extensions Between Data Points

Suppose point P_1 (x_1,y_1) belongs to block i and P_2 (x_2,y_2) belongs to block j. Let m be the extension measured between these points. The unit vector from P_1 to P_2 is $\hat{g} = (g_x,g_y)$ determined by

$$\hat{g} = \frac{x_2 - x_1}{R}, \frac{y_2 - y_1}{R} \tag{9}$$

where $R = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$.

At P_1, the projection of the measured displacement along $\hat{g}$ is p^i given by

$$p^i = (g_x,g_y)\ (t)\ (d) \tag{10}$$

or

$$\underset{}{p^i} = \underset{(1x6)}{(c^i)}\ \underset{(6x1)}{(d^i)} \tag{11}$$

where $c^i_\ell = g_x\ t^i_{1\ell} + g_y\ t^i_{2\ell}$.

Similarly, at P_2 the projection of the measured displacement along $\hat{g}$ is p^j given by

$$p^j = \underset{(1x6)}{(c^j)}\ \underset{(6x1)}{(d^j)} \tag{12}$$

where

$$c^j = g_x\, t^j_{1\ell} + g_y\, t^j_{2\ell}\ . \qquad (13)$$

The squared deviation of extension between P_1 and P_2 is defined as

$$\chi_b = [m - (p^j - p^i)]^2. \qquad (14)$$

Inserting (11) and (12) in (14) gives

$$\chi_b = (m + \sum_{\ell=1}^{6} c^i_\ell\, d^i_\ell - \sum_{\ell=1}^{6} c^j_\ell\, d^j_\ell)^2 \qquad (15)$$

<u>The Squared Deviation at Block Corners</u>

Suppose c(x,y) is an end point of a boundary segment (B) between blocks i and j. Let (x_1,y_1) and (x_2,y_2) be any two points along B. The unit normal to B is $\hat{n}$ given by

$$\hat{n} = \left(\frac{y_1 - y_2}{R}, \frac{x_2 - x_1}{R}\right) \qquad (16)$$

where $R = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$.

The subscripts are ordered to assure that $\hat{n}$ points into block i (and $-\hat{n}$ points into block j).

The displacement of point c in block i, projected along $\hat{n}$ is e^i given by

$$e^i = (n_x\ n_y) \begin{pmatrix} u \\ v \end{pmatrix}^i \qquad (17)$$

inserting (6),

$$e^i = \underset{(1\text{x}6)}{(b^i)}\ \underset{(6\text{x}1)}{(d^i)} \qquad (18)$$

where

$$b^i_\ell = n_x\, t^i_{1\ell} + n_y\, t^i_{2\ell}\ .$$

Similarly, the displacement of block j at point c, projected along $(-\hat{n})$ is

$$e^j = \underset{(1\times 6)}{(b^j)} \quad \underset{(6\times 1)}{(d^j)} \tag{19}$$

$$b^j_\ell = -n_x\, t^j_{1\ell} - n_y\, t^j_{2\ell}. \tag{20}$$

The boundary segment B at c will open if $e^i + e^j > 0$ and will tend to close if $e^i + e^j < 0$.

For the least squares analysis, we define an object function ψ_c analogous to ψ_a and χ_b, to express the relative normal displacement across the joint.

$$\psi_c = (e^i + e^j)^2 \cdot P \tag{21}$$

where P is $\begin{cases} 0 \text{ if } e^i + e^j \geq 0 \\ \text{a large number if } e^i + e^j < 0 \end{cases}$

P will be termed "the punishment factor". Introducing (19) and (20) in the above

$$\psi_c = P\left(\sum_{\ell=1}^{6} b^i_\ell\ (x,y)\ d^i_\ell + \sum_{\ell=1}^{6} b^j_\ell\ (x,y)\ d^j_\ell\right)^2 \tag{22}$$

<u>General Equations for n Blocks</u>

We seek to minimize $\Sigma_a\, \phi_a + \Sigma_b\, \chi_b + \Sigma_c\, \psi_c$ for all points a, b, c.

Differentiating each of these functions in turn and equating to zero generates a system of 6 n equations in 6 n unknowns as follows:

$$(K)\ (D) = (F) \tag{23}$$

where

each term k_{ij} of (K) is a 6 x 6 submatrix

and each term D_i of (D) and F_i of (F) is a 6 x 1 submatrix.

These matrices are constructed as follows: (1) Each measured displacement point a in block i adds 6x6 submatrix k_{ii}^{m} to submatrix k_{ii} and 6x1 submatrix F_i^m to submatrix F_i. (2) Each measured extension E_b between one point in block i and another in block j adds 4 6x6 matrices K_{ii}^{e}, K_{ij}^{e}, K_{ji}^{e}, and K_{jj}^{e} to submatrices K_{ii}, K_{ij}, K_{ji}, and K_{ji} respectively and two 6x1 matrices F_i^e and F_j^e to submatrices F_i and F_j respectively. (3) A point c on boundary B_b between block i and j adds four 6x6 matrices K_{ii}, K_{ij}, K_{ji} and K_{jj} to submatrices K_{ii}, K_{ij}, K_{ji} and K_{jj} respectively. It does not contribute to (F).

Formulas for the terms of (23) are given in Table 1. Derivations are omitted owing to space limitations (but will be presented in a forthcoming journal)*.

With prior knowledge of the location of the boundaries of all blocks, inputting the measured displacements and extensions at a sufficient number of points allows solution of (23) to determine (D). This then describes the translations, rotations, and strains of each block. At this point, if desired, deformability properties can be input for the blocks and joints to establish the force and stress distribution throughout the block system. To determine what consistitutes a "sufficient number of points" we discuss mathematical properties of (K).

*Numerical and Analytical Methods in Geomechanics (1985).

Table 1

Elements of Row r and, for K submatrices, Column s

$$K^m_{ii}: \quad t^i_{1r}\, t^i_{1s} + t^i_{2r}\, t^i_{2s}$$

$$F^m_i: \quad m_1\, t^i_{1r} + m_2\, t^i_{2r}$$

$$K^e_{ii}: \quad c^i_r\, c^i_s$$

$$K^e_{ij}: \quad -c^i_r\, c^j_s$$

$$K^e_{ji}: \quad -c^j_r\, c^i_s$$

$$K^e_{jj}: \quad c^j_r\, c^j_s$$

$$F_i: \quad -m\, c^i_r$$

$$F_j: \quad m\, c^j_r$$

$$K^b_{ii}: \quad P\, b^i_r\, b^i_s$$

$$K^b_{ij}: \quad P\, b^i_r\, b^j_s$$

$$K^b_{ji}: \quad P\, b^j_r\, b^i_s$$

$$K^b_{jj}: \quad P\, b^j_r\, b^j_s \,.$$

Properties of the Matrix (K)

It can be proved that (1) Matrix (K) is symmetric, and (2) If each block contains at least three measured points not in the same line, then (K) is positive definite. In three dimensions, the analogous matrix (K) is positive definite if there are at least four non co-planar points in each block. Proofs of these theorems are presented in a forthcoming journal article.*

Because of these properties a set of 3 displacement measurements per block, at points not along a line, will give a unique solution to a plane problem. The same is true when there are four displacement measurements per block in a three dimensional problem; the 4 points must not lie along a plane. When there are more than these minimum number of points, a solution will be obtained corresponding to the best estimate according to "least squares". When there are fewer measurements points per block, a solution may or may not be obtainable, depending on the particulars.

APPLICATIONS

Figures 1 and 2 show a simple illustration of this method. In principle, the number of blocks, n, could be large. For now we deal with a simple case of four blocks. In Figure 1, we see the displacement vectors radiating from 3 points of measurement in each block. Figure 2 shows the output deformed shape, drawn to an exaggerated scale. Error vectors measuring the difference between computed and measured displacements are also drawn at each point.

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*See note page 7.

CONCLUSION

This is a first presentation of a novel analysis for back-calculation of deformations in discontinuous rock. The user need not create a complete model of the rock mass by loading constitutive data into elements. The program simply finds a deformed shape consistent with observed deformations. The list of deformation functions (D) can be expanded to include additional contributers to displacement, for example from temperature variations. After a solution is obtained, the known mode of the deformation could be analyzed by finite element analysis to provide a more complete picture of the flow of stress throughout the blocks. We believe that discontinuous deformation analysis will have many meaningful applications in rock mechanics and in other fields.

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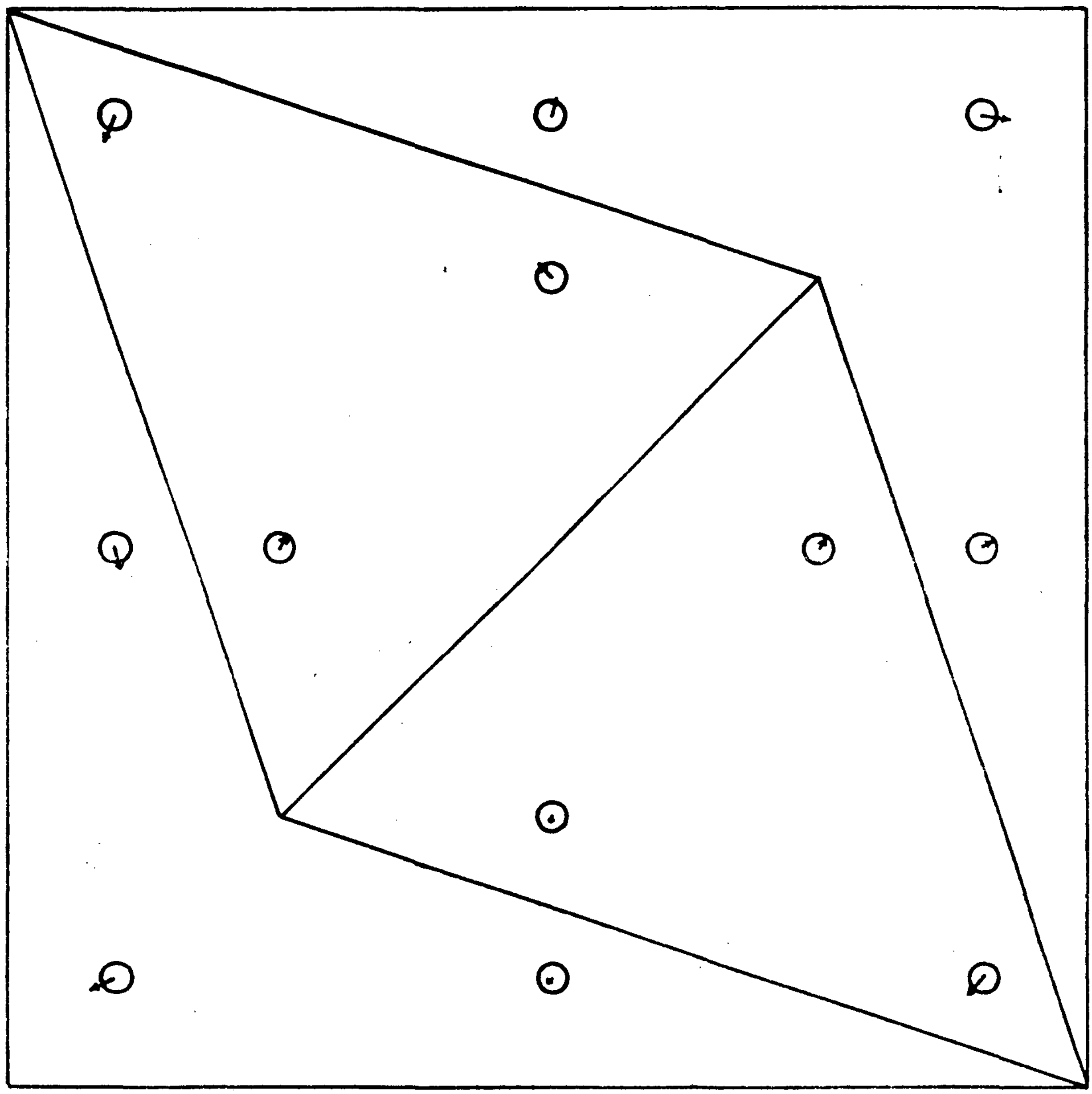

Figure 1. Input displacements and block system.

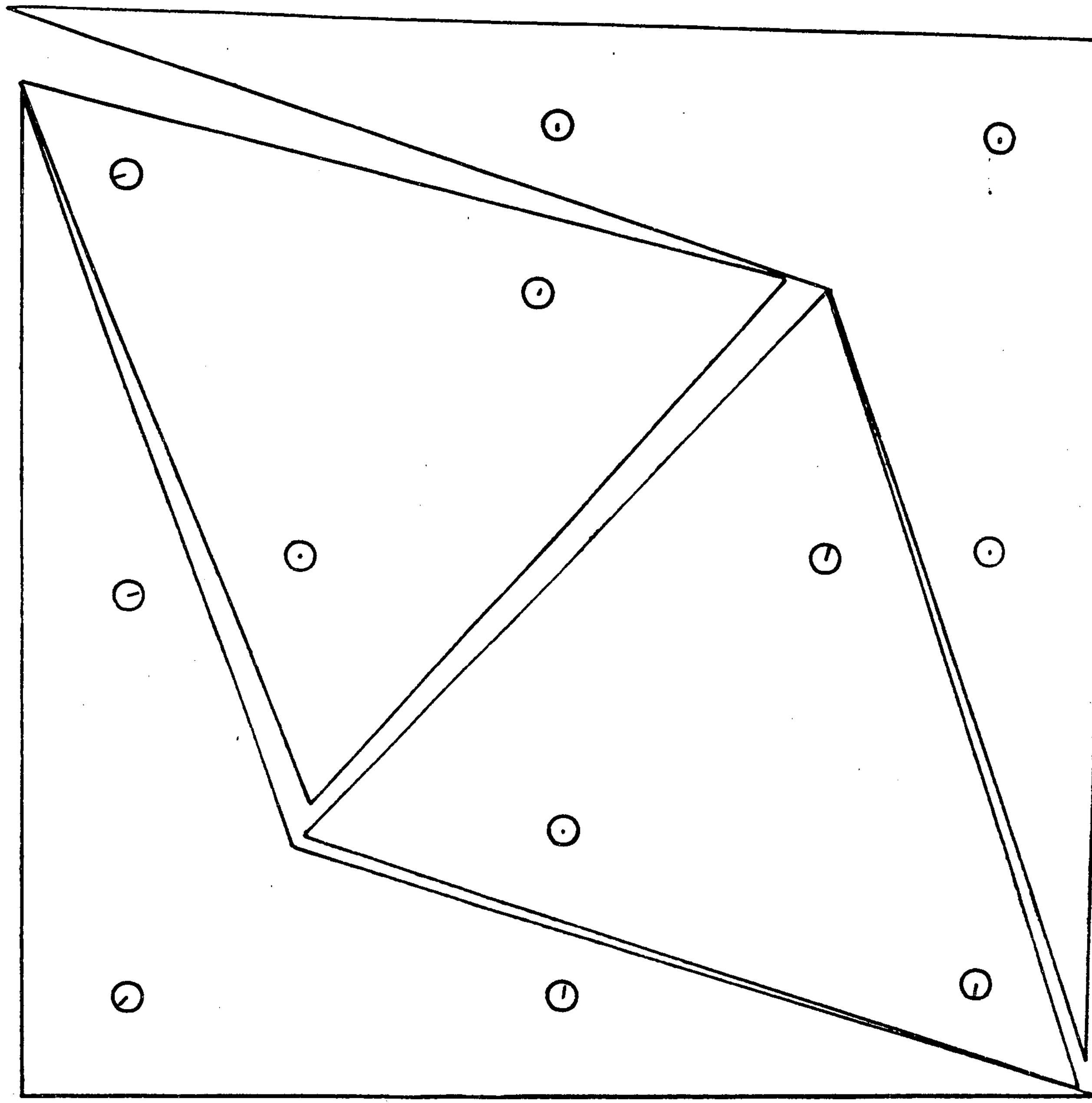

Figure 2. Output deformed system, embracing block displacement, displacement, strain and rotation.

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