

Discontinuous Galerkin finite element methods for dynamic linear solid viscoelasticity problems

Béatrice Rivière^{*}, Simon Shaw[†], JR Whiteman[‡]

November 8, 2005[‡]

Abstract

We consider the usual linear elastodynamics equations augmented with evolution equations for viscoelastic internal stresses. Both semi- and fully discrete approximations are defined, based on a discontinuous Galerkin finite element method, and error estimates are given.

1 Introduction

This is the second in a series of papers, [12, 11], extending spatially discontinuous Galerkin methods to viscoelasticity problems.

We consider a model for the dynamic response of linear viscoelastic solids. This comprises the usual equations of elastodynamics, but augmented with evolution equations for the viscoelastic internal stresses. The spatial discretisation is effected by a discontinuous Galerkin finite element method (DG FEM), which can be taken as either a symmetric or non-symmetric scheme, and the time discretisation is a standard finite difference method of Crank-Nicolson type.

For the analogous quasistatic problem considered in [12] we represented the viscoelasticity through a hereditary integral. Here we have chosen

^{*}Computational Mathematics Research Group, Department of Mathematics, 301 Thackeray, University of Pittsburgh, Pittsburgh, PA 15260. (riviere@math.pitt.edu).

[†]BICOM (Brunel Institute of Computational Mathematics), Brunel University, Uxbridge UB8 3PH, England. ([simon.shaw](mailto:simon.shaw@brunel.ac.uk), [john.whiteman](mailto:john.whiteman@brunel.ac.uk)). Shaw and Whiteman would like to acknowledge the support of the Engineering and Physical Sciences Research Council, GR/R10844/01, and also the US Army Research Office, DAAD19-00-1-0421.

[‡]Revision of the version originally dated 17 October, 2004.

the alternative representation through internal variables. The reasons for this are, firstly, to show that the error estimates can be extended to this case and, secondly, because some practitioners prefer to work with internal variables rather than history integrals (see e.g. [6, 5]).

For background to viscoelasticity and the assumptions we make we refer back to [12], and for more general background to the application of DG methods we refer to [9, 8, 3, 13, 10] and, in particular, to the elastic problem studied in [10].

This article is arranged as follows. We finish this section with some notation and then in Section 2 describe the model problem and the spatial discretisation. Estimates for the semidiscrete scheme are then given in Section 3 and for the fully discrete scheme in Section 4. We then conclude with Section 5.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an open bounded domain with polygonal/polyhedral boundary and let $I = (0, T)$ be a finite time interval. The following notation is standard. For $\omega \subseteq \bar{\Omega}$,

$$(\mathbf{v}, \mathbf{w})_\omega := \int_\omega \mathbf{v} \cdot \mathbf{w} \, d\omega,$$

but we drop the subscript when $\omega = \Omega$. We use $\|\cdot\|_{p,\omega}$ to denote the $\mathbf{H}^p(\omega) := (H^p(\omega))^d$ norm and again abbreviate, $\|\cdot\|_m = \|\cdot\|_{m,\Omega}$, when $\omega = \Omega$. Since we are dealing with time dependent functions we take the usual approach of treating these as maps from time into a Banach space and set,

$$\|v\|_{L_p(0,t;X)} := \left(\int_0^t \|v(t)\|_X^p \, dt \right)^{1/p},$$

for $t \leq T$, $1 \leq p < \infty$ and with the obvious modification for $p = \infty$. When $t = T$ we abbreviate: $\|\cdot\|_{L_2(L_2)} := \|\cdot\|_{L_2(0,T;L_2(\Omega))}$ and so on.

We need also to deal with scalar- and tensor-valued functions and, to ease notation, we make no distinction with the inner products and norms in these cases.

2 Model problem and spatial discretisation

The basic equations are,

$$\rho \mathbf{u}_{tt}(\mathbf{x}, t) - \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}; \mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t) \quad \text{in } \Omega \times I, \quad (1)$$

$$\mathbf{u}(\mathbf{x}, 0) = \bar{\mathbf{u}}(\mathbf{x}) \quad \text{in } \bar{\Omega}, \quad (2)$$

$$\mathbf{u}_t(\mathbf{x}, 0) = \bar{\mathbf{z}}(\mathbf{x}) \quad \text{in } \bar{\Omega}, \quad (3)$$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{0}, \quad \text{on } \Gamma_D \times \bar{I}, \quad (4)$$

$$\boldsymbol{\sigma}(\mathbf{u}; \mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{g}(\mathbf{x}, t), \quad \text{on } \Gamma_N \times \bar{I}. \quad (5)$$

In these $\Gamma_D \cup \Gamma_N = \partial\Omega$ and $\Gamma_D \cap \Gamma_N = \emptyset$, and we assume that Γ_D is a closed set with positive surface measure. We do not explicitly display the \mathbf{x} dependence in most of what follows. Also, to ease notation, we denote partial time differentiation with either a subscript, as above, or a dot. Thus $\dot{\mathbf{u}} = \mathbf{u}_t$, $\ddot{\mathbf{u}} = \mathbf{u}_{tt}$, and so on. We also assume that the boundary and initial data are compatible at $t = 0$.

The symmetric second-order stress tensor satisfies the constitutive relation,

$$\boldsymbol{\sigma}(\mathbf{u}(t)) = \mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}(t)) - \sum_{i=1}^{N_\varphi} \gamma_i {}^* \boldsymbol{\sigma}_i(t),$$

where: $\varepsilon_{ij}(\mathbf{u}) := (u_{i,j} + u_{j,i})/2$; for $i = 1, \dots, N_\varphi$,

$${}^* \boldsymbol{\sigma}_i(t) = \int_0^t \gamma_i e^{-(t-s)/\tau_i} \mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}(s)) ds : \quad (6)$$

and, the fourth order Hooke's tensor, \mathbf{D} , satisfies the symmetries,

$$D_{ijkl} = D_{jikl} = D_{ijlk} = D_{klij},$$

and is positive definite over symmetric second order tensors. Also,

$$\gamma_i = \left(\frac{\varphi_i}{\tau_i} \right)^{1/2},$$

where the φ_i and τ_i are positive constants, and we impose the normalisation,

$$\sum_{i=0}^{N_\varphi} \varphi_i = 1$$

with, additionally (see (52) later), $\varphi_0 > 0$ (note that we have a φ_0 but not a τ_0). Then it follows that,

$$\sum_{i=1}^{N_\varphi} \gamma_i^2 \tau_i = 1 - \varphi_0 > 0. \quad (7)$$

From (6), we see that each of the internal stress tensors, ${}^* \boldsymbol{\sigma}_i$, satisfies an initial value problem,

$${}^* \dot{\boldsymbol{\sigma}}_i(t) + \frac{1}{\tau_i} {}^* \boldsymbol{\sigma}_i(t) = \gamma_i \mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \quad (8)$$

$${}^* \boldsymbol{\sigma}_i(0) = \mathbf{0}, \quad (9)$$

and on the other hand, if we eliminate the viscous stresses, our basic equation becomes a second-order hyperbolic partial differential equation with a

fading memory Volterra integral. For the well-posedness of these types of equations we refer to [2], and for numerical analysis we cite, for example, [14, 7, 15, 4, 1]. All of these deal with the Volterra form of the problem whereas, here, we include the viscoelasticity through the evolution equations for the internal variables, (8). We are not aware of literature containing error estimates for this approach.

From our definitions we obtain the following regularity estimates.

Lemma 2.1 *For each $i = 1, \dots, N_\varphi$ we have,*

$$\left\| \frac{\partial^{n*} \boldsymbol{\sigma}_i}{\partial t^n} \right\|_{L_2(0,t;L_2(\Omega))} \leq C \sum_{j=0}^{n-1} \left\| \frac{\partial^j \mathbf{u}}{\partial t^j} \right\|_{L_2(0,t;\mathbf{H}^1(\Omega))},$$

for $n = 1, 2, \dots$

Proof. Taking norms in (6) and using Hölder's inequality for convolutions gives, $\|\boldsymbol{\sigma}_i\|_{L_2(0,t;L_2(\Omega))} \leq C \|\mathbf{u}\|_{L_2(0,t;\mathbf{H}^1(\Omega))}$. Now use successive differentiation on (8) and recursively apply the estimates obtained. \square

We also have the following.

Lemma 2.2 *For each $i = 1, \dots, N_\varphi$ we have,*

$$\|\boldsymbol{\sigma}_i(t)\|_r \leq C \|\mathbf{u}(t)\|_{r+1}$$

for $r \geq 0$.

The first step towards spatial discretisation is to establish some more notation. Let $\mathcal{E}_h = \{E_1, E_2, \dots, E_{N_h}\}$ be a nondegenerate quasiuniform subdivision of Ω , where E_j is a triangle if $d = 2$, or a tetrahedron if $d = 3$. The nondegeneracy requirement is that there exists $\rho > 0$ such that if $h_j = \text{diam}(E_j)$, then E_j contains a ball of radius ρh_j in its interior. Let $h = \max\{h_j : 1 \leq j \leq N_h\}$, the quasiuniformity requirement is that there exists $\tau > 0$ such that $h/h_j \leq \tau$ for all $j \in \{1, \dots, N_h\}$. We denote the set of interior edges (faces for $d = 3$) of \mathcal{E}_h by Γ_h . With each edge (or face) e , we associate a unit normal vector \mathbf{n}_e . For a boundary edge (or face), \mathbf{n}_e is taken to be the unit outward vector normal to $\partial\Omega$.

We now define the average and the jump operators. For each of the interior edges suppose the neighbouring elements of e are E_e^1 and E_e^2 so that $e = \partial E_e^1 \cap \partial E_e^2$, and for a boundary edge suppose that E_e is the neighbouring element. We define the averaging operator $\{\cdot\}$ by,

$$\{\mathbf{w}\} := \begin{cases} \frac{1}{2}(\mathbf{w}|_{E_e^1})|_e + \frac{1}{2}(\mathbf{w}|_{E_e^2})|_e & \text{if } e \subset \Omega, \\ (\mathbf{w}|_{E_e})|_e & \text{if } e \subset \partial\Omega. \end{cases}$$

and the jump operator $[\cdot]$ by,

$$[\mathbf{w}] := \begin{cases} (\mathbf{w}|_{E_e^1})|_e - (\mathbf{w}|_{E_e^2})|_e & \text{if } e \subset \Omega, \\ (\mathbf{w}|_{E_e})|_e & \text{if } e \subset \partial\Omega. \end{cases}$$

The distinction between $[\cdot]$ and $-\cdot$ can be made because each edge e has a unit normal associated with it. The “direction” in which the jump takes place is unimportant.

These operators are well defined if $\mathbf{w}|_{E_e^i} \in (H^{\frac{1}{2}+\epsilon}(E_e^i))^d$ for $i = 1, 2$ and $\epsilon > 0$. Below, we use $|e|$ to denote the $(d-1)$ -dimensional surface measure of the edge/face e . We also frequently use the estimate, $|e| \leq Ch^{d-1}$ which arises as a consequence of our assumptions.

Define the broken spaces for any integer $r \geq 0$,

$$\mathcal{D}_r(\mathcal{E}_h) = \{v \in L_2(\Omega) : v|_E \in \mathcal{P}_r(E) \quad \forall E \in \mathcal{E}_h\}, \quad (10)$$

$$\mathbf{D}_r(\mathcal{E}_h) = \mathcal{D}_r(\mathcal{E}_h)^d, \quad (11)$$

$$\mathbf{L}_r(\mathcal{E}_h) = \mathcal{D}_r(\mathcal{E}_h)^{d \times d}. \quad (12)$$

For these finite element spaces we have the following interpolation-error estimates. If $\mathbf{v} \in \mathbf{H}^n(\mathcal{E}_h) \cap C(\bar{\Omega})^d$ and $\mu = \min\{r+1, n\}$ then there is an interpolant $\hat{\mathbf{v}} \in \mathbf{D}_r(\mathcal{E}_h) \cap C(\bar{\Omega})^d$ such that for each $E \in \mathcal{E}_h$,

$$\|\mathbf{v} - \hat{\mathbf{v}}\|_{m,E} \leq Ch_E^{\mu-m} \|\mathbf{v}\|_{n,E} \quad \text{for } n \geq m \geq 0, \quad (13)$$

$$\|\mathbf{v} - \hat{\mathbf{v}}\|_{m,\gamma} \leq Ch_E^{\mu-m-1/2} \|\mathbf{v}\|_{n,E} \quad \text{for } m = 0, 1 \text{ and } n \geq m, \quad (14)$$

where $\gamma \subseteq \partial E$.

For positive constants, δ and β , define the bilinear forms,

$$J_0^{\delta,\beta}(\mathbf{w}, \mathbf{v}) = \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{\delta}{|e|^\beta} \int_e [\mathbf{w}] \cdot [\mathbf{v}], \quad (15)$$

$$\begin{aligned} A(\mathbf{w}, \mathbf{v}) &= \sum_E \int_E \mathbf{D}\boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\varepsilon}(\mathbf{v}) - \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\boldsymbol{\varepsilon}(\mathbf{w})\mathbf{n}_e\} \cdot [\mathbf{v}] \\ &\quad + \kappa \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\boldsymbol{\varepsilon}(\mathbf{v})\mathbf{n}_e\} \cdot [\mathbf{w}] + J_0^{\delta,\beta}(\mathbf{w}, \mathbf{v}). \end{aligned} \quad (16)$$

Here κ is a switch: we set $\kappa = 1$ to obtain the non-symmetric DG scheme, and $\kappa = -1$ to obtain the symmetric scheme.

Defining $\mathbf{z}(t) := \mathbf{u}_t(t)$, we first note that if $\mathbf{z}(t), \mathbf{u}(t) \in C(\bar{\Omega})^d$ for each

t , then we have,

$$\begin{aligned}
 & (\rho \dot{\mathbf{z}}(t), \mathbf{v}) + A(\mathbf{u}(t), \mathbf{v}) + J_0^{\delta, \beta}(\mathbf{z}(t), \mathbf{v}) \\
 & + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \gamma_i \{ {}^* \boldsymbol{\sigma}_i(t) \mathbf{n}_e \} \cdot [\mathbf{v}] - \sum_{i=1}^{N_\varphi} \sum_E \int_E \gamma_i {}^* \boldsymbol{\sigma}_i(t) : \boldsymbol{\varepsilon}(\mathbf{v}) \\
 & = L(t; \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h), \tag{17}
 \end{aligned}$$

where

$$L(t; \mathbf{v}) := (\mathbf{f}(t), \mathbf{v}) + (\mathbf{g}(t), \mathbf{v})_{\Gamma_N},$$

and, for each $i = 1, \dots, N_\varphi$,

$$\begin{aligned}
 & \sum_E ({}^* \dot{\boldsymbol{\sigma}}(t) + \frac{1}{\tau_i} {}^* \boldsymbol{\sigma}(t), \mathbf{w}_i)_E = \sum_E \gamma_i (\mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \mathbf{w}_i)_E \\
 & - \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{ \mathbf{D}\mathbf{w}_i \mathbf{n}_e \} \cdot [\mathbf{u}(t)] \quad \forall \mathbf{w}_i \in \mathbf{L}_{r-1}(\mathcal{E}_h), \tag{18}
 \end{aligned}$$

and,

$$(\rho \mathbf{z}(t), \mathbf{v})_E = (\rho \dot{\mathbf{u}}(t), \mathbf{v})_E \quad \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h). \tag{19}$$

Equations (17) and (18) arise from elementwise integration by parts, see [12], and ‘adding zero’.

We will also use the following norm and semi-norm,

$$\begin{aligned}
 \|\mathbf{v}\|_{\mathcal{A}} & := \left(|\mathbf{v}|_{\mathcal{E}}^2 + J_0^{\delta, \beta}(\mathbf{v}, \mathbf{v}) \right)^{\frac{1}{2}}, \\
 |\mathbf{v}|_{\mathcal{E}} & := \left(\sum_{E \in \mathcal{E}_h} \int_E \mathbf{D}\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v}) \right)^{\frac{1}{2}}.
 \end{aligned}$$

3 Semidiscrete error estimates

For $r > 0$ the semidiscrete DG scheme is as follows: find $\mathbf{u}^h : I \rightarrow \mathbf{D}_r(\mathcal{E}_h)$ and ${}^* \boldsymbol{\sigma}_i^h : I \rightarrow \mathbf{L}_{r-1}(\mathcal{E}_h)$ such that,

$$\begin{aligned}
 & (\rho \mathbf{u}_{tt}^h(t), \mathbf{v}) + A(\mathbf{u}^h(t), \mathbf{v}) + J_0^{\delta, \beta}(\mathbf{u}_t^h(t), \mathbf{v}) \\
 & + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{ {}^* \boldsymbol{\sigma}_i^h(t) \mathbf{n}_e \} \cdot [\mathbf{v}] - \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \int_E {}^* \boldsymbol{\sigma}_i^h(t) : \boldsymbol{\varepsilon}(\mathbf{v}) \\
 & = L(t; \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h), \tag{20}
 \end{aligned}$$

along with, for $i = 1, \dots, N_\varphi$,

$$\begin{aligned} & (*\dot{\boldsymbol{\sigma}}_i^h(t) + \frac{1}{\tau_i} * \boldsymbol{\sigma}_i^h(t), \mathbf{w}) = \sum_E \gamma_i (\mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}^h(t)), \mathbf{w})_E \\ & - \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\mathbf{D}\mathbf{w}\mathbf{n}_e\} \cdot [\mathbf{u}^h(t)] \quad \forall \mathbf{w} \in \mathbf{L}_{r-1}(\mathcal{E}_h). \end{aligned} \quad (21)$$

The initial conditions are given by $*\boldsymbol{\sigma}_i(0) = \mathbf{0}$ and,

$$A(\mathbf{u}^h(0), \mathbf{v}) = A(\bar{\mathbf{u}}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h), \quad (22)$$

$$(\rho \mathbf{u}_t^h(0), \mathbf{v}) = (\rho \bar{\mathbf{z}}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h). \quad (23)$$

We first prove an optimal error estimate for the natural ‘energy’ norm, and then follow it with an error estimate in the for the $L_2(\Omega)$ norm.

Theorem 3.1 (semidiscrete ‘energy’ error estimate) *If $h \leq \hat{h}$, $\beta \geq (d-1)^{-1}$, $\bar{\mathbf{u}} \in \mathbf{H}^{r+1}(\Omega)$, $\bar{\mathbf{z}} \in \mathbf{H}^r(\Omega)$ and $\mathbf{u} \in H^2(\mathbf{H}^{r+1}) \cap C^1(C(\bar{\Omega})^d)$ then, for $\kappa = \pm 1$ and $\hat{h}^{(d-1)\beta-1}/\delta$ small enough, there exists a constant, C , independent of h , such that,*

$$\begin{aligned} & \|\rho^{1/2}(\mathbf{u}_t(\tau) - \mathbf{u}_t^h(\tau))\|_0 + \|\mathbf{u}(\tau) - \mathbf{u}^h(\tau)\|_{\mathcal{A}} \\ & + \left(\varphi_0 \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2}(*\boldsymbol{\sigma}_i(\tau) - *\boldsymbol{\sigma}_i^h(\tau))\|_0^2 \right)^{1/2} \leq Ch^r, \end{aligned}$$

for all $\tau \in I$.

Proof: Set,

$$\begin{aligned} \boldsymbol{\chi} &= \mathbf{u}^h - \tilde{\mathbf{u}}, & \boldsymbol{\eta}_i &= *\boldsymbol{\sigma}_i^h - *\tilde{\boldsymbol{\sigma}}_i, \\ \boldsymbol{\xi} &= \mathbf{u} - \tilde{\mathbf{u}}, & \boldsymbol{\theta}_i &= *\boldsymbol{\sigma}_i - *\tilde{\boldsymbol{\sigma}}_i, \end{aligned}$$

then $\mathbf{u}^h - \mathbf{u} = \boldsymbol{\chi} - \boldsymbol{\xi}$ and $*\boldsymbol{\sigma}_i^h - *\boldsymbol{\sigma}_i = \boldsymbol{\eta}_i - \boldsymbol{\theta}_i$.

We will define later what $\tilde{\mathbf{u}} \in \mathbf{D}_r(\mathcal{E}_h)$ and $*\tilde{\boldsymbol{\sigma}}_i \in \mathbf{L}_{r-1}(\mathcal{E}_h)$ are. The error equations follow by subtracting (17), (18) from (20), (21). They are,

$$\begin{aligned} & (\rho \boldsymbol{\chi}_{tt}, \mathbf{v}) + A(\boldsymbol{\chi}, \mathbf{v}) + J_0^{\delta, \beta}(\boldsymbol{\chi}_t, \mathbf{v}) \\ & + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\boldsymbol{\eta}_i \mathbf{n}_e\} [\mathbf{v}] - \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \int_E \boldsymbol{\eta}_i : \boldsymbol{\varepsilon}(\mathbf{v}) \\ & = (\rho \boldsymbol{\xi}_{tt}, \mathbf{v}) + A(\boldsymbol{\xi}, \mathbf{v}) + J_0^{\delta, \beta}(\boldsymbol{\xi}_t, \mathbf{v}) + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\boldsymbol{\theta}_i \mathbf{n}_e\} [\mathbf{v}] \\ & - \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \int_E \boldsymbol{\theta}_i : \boldsymbol{\varepsilon}(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h). \end{aligned} \quad (24)$$

And, for each i ,

$$\begin{aligned}
 & \sum_E (\dot{\boldsymbol{\eta}}_i + \frac{1}{\tau_i} \boldsymbol{\eta}_i, \mathbf{w}_i)_E - \sum_E \gamma_i (\mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi}), \mathbf{w}_i)_E + \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\mathbf{D}\mathbf{w}_i \mathbf{n}_e\} \cdot [\boldsymbol{\chi}] \\
 &= \sum_E (\dot{\boldsymbol{\theta}}_i + \frac{1}{\tau_i} \boldsymbol{\theta}_i, \mathbf{w}_i)_E - \sum_E \gamma_i (\mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}), \mathbf{w}_i)_E \\
 &+ \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\mathbf{D}\mathbf{w}_i \mathbf{n}_e\} \cdot [\boldsymbol{\xi}] \quad \forall \mathbf{w}_i \in \mathbf{L}_{r-1}(\mathcal{E}_h). \quad (25)
 \end{aligned}$$

Choose $\mathbf{v} = \boldsymbol{\chi}_t$ in (24) and $\mathbf{w}_i = \mathbf{D}^{-1} \dot{\boldsymbol{\eta}}_i$ in (25) and add the resulting equations to get,

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|\rho^{1/2} \boldsymbol{\chi}_t\|_0^2 + \frac{1}{2} \frac{d}{dt} \|\boldsymbol{\chi}\|_{\mathcal{A}}^2 + J_0^{\delta, \beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t) \\
 &+ \sum_{i=1}^{N_\varphi} \|\mathbf{D}^{-1/2} \dot{\boldsymbol{\eta}}_i\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{1}{2\tau_i} \frac{d}{dt} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i\|_0^2 \\
 &= - \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\boldsymbol{\eta}_i \mathbf{n}_e\} \cdot [\boldsymbol{\chi}_t] + \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \int_E \boldsymbol{\eta}_i : \boldsymbol{\varepsilon}(\boldsymbol{\chi}_t) \\
 &+ \sum_{i=1}^{N_\varphi} \sum_E (\dot{\boldsymbol{\theta}}_i + \frac{1}{\tau_i} \boldsymbol{\theta}_i, \mathbf{D}^{-1} \dot{\boldsymbol{\eta}}_i)_E - \sum_{i=1}^{N_\varphi} \sum_E \gamma_i (\boldsymbol{\varepsilon}(\boldsymbol{\xi}), \dot{\boldsymbol{\eta}}_i)_E \\
 &+ \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\dot{\boldsymbol{\eta}}_i \mathbf{n}_e\} \cdot [\boldsymbol{\xi}] + (\rho \boldsymbol{\xi}_{tt}, \boldsymbol{\chi}_t) + A(\boldsymbol{\xi}, \boldsymbol{\chi}_t) + J_0^{\delta, \beta}(\boldsymbol{\xi}_t, \boldsymbol{\chi}_t) \\
 &+ \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\boldsymbol{\theta}_i \mathbf{n}_e\} [\boldsymbol{\chi}_t] - \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \int_E \boldsymbol{\theta}_i : \boldsymbol{\varepsilon}(\boldsymbol{\chi}_t) \\
 &+ \sum_{i=1}^{N_\varphi} \sum_E \gamma_i (\boldsymbol{\varepsilon}(\boldsymbol{\chi}), \dot{\boldsymbol{\eta}}_i)_E - \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\dot{\boldsymbol{\eta}}_i \mathbf{n}_e\} \cdot [\boldsymbol{\chi}] \\
 &+ \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi}) \mathbf{n}_e\} \cdot [\boldsymbol{\chi}_t] - \kappa \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi}_t) \mathbf{n}_e\} \cdot [\boldsymbol{\chi}].
 \end{aligned}$$

Now, if we choose $\tilde{\mathbf{u}}$ to be the elliptic projection:

$$A(\boldsymbol{\xi}, \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h),$$

and if we choose ${}^* \tilde{\boldsymbol{\sigma}}_i$ to be the L_2 projection into $\mathbf{L}_{r-1}(\mathcal{E}_h)$ we have,

$$\begin{aligned} A(\boldsymbol{\xi}, \boldsymbol{\chi}_t) &= 0, \\ \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \int_E \boldsymbol{\theta}_i : \boldsymbol{\varepsilon}(\boldsymbol{\chi}_t) &= 0, \\ \sum_{i=1}^{N_\varphi} \sum_E (\dot{\boldsymbol{\theta}}_i + \frac{1}{\tau_i} \boldsymbol{\theta}_i, \mathbf{D}^{-1} \dot{\boldsymbol{\eta}}_i)_E &= 0. \end{aligned}$$

Thus, we are left with

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\rho^{1/2} \boldsymbol{\chi}_t\|_0^2 + \frac{1}{2} \frac{d}{dt} \|\boldsymbol{\chi}\|_{\mathcal{A}}^2 + J_0^{\delta, \beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t) \\ & + \sum_{i=1}^{N_\varphi} \|\mathbf{D}^{-1/2} \dot{\boldsymbol{\eta}}_i\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{1}{2\tau_i} \frac{d}{dt} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i\|_0^2 \\ & = - \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\boldsymbol{\eta}_i \mathbf{n}_e\} \cdot [\boldsymbol{\chi}_t] + \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \int_E \boldsymbol{\eta}_i : \boldsymbol{\varepsilon}(\boldsymbol{\chi}_t) \\ & - \sum_{i=1}^{N_\varphi} \sum_E \gamma_i (\boldsymbol{\varepsilon}(\boldsymbol{\xi}), \dot{\boldsymbol{\eta}}_i)_E + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\dot{\boldsymbol{\eta}}_i \mathbf{n}_e\} \cdot [\boldsymbol{\xi}] \\ & + (\rho \boldsymbol{\xi}_{tt}, \boldsymbol{\chi}_t) + J_0^{\delta, \beta}(\boldsymbol{\xi}_t, \boldsymbol{\chi}_t) + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\boldsymbol{\theta}_i \mathbf{n}_e\} [\boldsymbol{\chi}_t] \\ & + \sum_{i=1}^{N_\varphi} \sum_E \gamma_i (\boldsymbol{\varepsilon}(\boldsymbol{\chi}), \dot{\boldsymbol{\eta}}_i)_E - \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\dot{\boldsymbol{\eta}}_i \mathbf{n}_e\} \cdot [\boldsymbol{\chi}] \\ & + \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D} \boldsymbol{\varepsilon}(\boldsymbol{\chi}) \mathbf{n}_e\} \cdot [\boldsymbol{\chi}_t] - \kappa \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D} \boldsymbol{\varepsilon}(\boldsymbol{\chi}_t) \mathbf{n}_e\} \cdot [\boldsymbol{\chi}]. \end{aligned}$$

We now integrate the equation over time from 0 to τ , and certain terms simplify by integration by parts:

$$\begin{aligned} & \int_0^\tau \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \int_E \boldsymbol{\eta}_i : \boldsymbol{\varepsilon}(\boldsymbol{\chi}_t) + \int_0^\tau \sum_{i=1}^{N_\varphi} \sum_E \gamma_i (\boldsymbol{\varepsilon}(\boldsymbol{\chi}), \dot{\boldsymbol{\eta}}_i)_E \\ & = \sum_{i=1}^{N_\varphi} \sum_E \gamma_i (\boldsymbol{\varepsilon}(\boldsymbol{\chi})(\tau), \boldsymbol{\eta}_i(\tau))_E - \sum_{i=1}^{N_\varphi} \sum_E \gamma_i (\boldsymbol{\varepsilon}(\boldsymbol{\chi})(0), \boldsymbol{\eta}_i(0))_E, \end{aligned}$$

and

$$\begin{aligned}
 -\kappa \int_0^\tau \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi}_t) \mathbf{n}_e\} \cdot [\boldsymbol{\chi}] &= \kappa \int_0^\tau \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi}) \mathbf{n}_e\} \cdot [\boldsymbol{\chi}_t] \\
 &\quad - \kappa \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})(\tau) \mathbf{n}_e\} \cdot [\boldsymbol{\chi}(\tau)] \\
 &\quad + \kappa \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})(0) \mathbf{n}_e\} \cdot [\boldsymbol{\chi}(0)].
 \end{aligned}$$

Thus, the error equation becomes

$$\begin{aligned}
 &\frac{1}{2} \|\rho^{1/2} \boldsymbol{\chi}_t(\tau)\|_0^2 + \frac{1}{2} \|\boldsymbol{\chi}(\tau)\|_A^2 + \int_0^\tau J_0^{\delta, \beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t) \\
 &\quad + \int_0^\tau \sum_{i=1}^{N_\varphi} \|\mathbf{D}^{-1/2} \dot{\boldsymbol{\eta}}_i\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{1}{2\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i(\tau)\|_0^2 \\
 &\quad = \frac{1}{2} \|\rho^{1/2} \boldsymbol{\chi}_t(0)\|_0^2 + \frac{1}{2} \|\boldsymbol{\chi}(0)\|_A^2 \\
 &\quad + \sum_{i=1}^{N_\varphi} \frac{1}{2\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i(0)\|_0^2 - \int_0^\tau \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\boldsymbol{\eta}_i \mathbf{n}_e\} [\boldsymbol{\chi}_t] \\
 &\quad + \sum_{i=1}^{N_\varphi} \sum_E \gamma_i (\boldsymbol{\varepsilon}(\boldsymbol{\chi})(\tau), \boldsymbol{\eta}_i(\tau))_E - \sum_{i=1}^{N_\varphi} \sum_E \gamma_i (\boldsymbol{\varepsilon}(\boldsymbol{\chi})(0), \boldsymbol{\eta}_i(0))_E \\
 &\quad + (1 + \kappa) \int_0^\tau \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi}) \mathbf{n}_e\} \cdot [\boldsymbol{\chi}_t] \\
 &\quad - \kappa \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})(\tau) \mathbf{n}_e\} \cdot [\boldsymbol{\chi}(\tau)] \\
 &\quad + \kappa \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})(0) \mathbf{n}_e\} \cdot [\boldsymbol{\chi}(0)] \\
 &\quad + \int_0^\tau (\rho \boldsymbol{\xi}_{tt}, \boldsymbol{\chi}_t) + \int_0^\tau J_0^{\delta, \beta}(\boldsymbol{\xi}_t, \boldsymbol{\chi}_t) \\
 &\quad + \int_0^\tau \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\boldsymbol{\theta}_i \mathbf{n}_e\} [\boldsymbol{\chi}_t] - \int_0^\tau \sum_{i=1}^{N_\varphi} \sum_E \gamma_i (\boldsymbol{\varepsilon}(\boldsymbol{\xi}), \dot{\boldsymbol{\eta}}_i)_E \\
 &\quad + \int_0^\tau \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\dot{\boldsymbol{\eta}}_i \mathbf{n}_e\} \cdot [\boldsymbol{\xi}] - \int_0^\tau \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\boldsymbol{\eta}_i \mathbf{n}_e\} \cdot [\boldsymbol{\chi}] \\
 &\quad = T_1 + \dots + T_{15}.
 \end{aligned}$$

We now bound each of the terms T_i , beginning with T_4 ,

$$\begin{aligned}
 |T_4| &\leq \int_0^\tau \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \|\{\boldsymbol{\eta}_i \mathbf{n}_e\}\|_{0,e} \frac{|e|^{\beta/2}}{\delta^{1/2}} \|\boldsymbol{\chi}_t\|_{0,e} \frac{\delta^{1/2}}{|e|^{\beta/2}}, \\
 &\leq C \int_0^\tau \sum_{i=1}^{N_\varphi} \gamma_i J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t)^{1/2} h^{(d-1)\beta/2-1/2} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i\|_0, \\
 &\leq \frac{C}{\epsilon} h^{(d-1)\beta-1} \int_0^\tau \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i\|_0^2 + \frac{\epsilon}{2} \int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t),
 \end{aligned}$$

where we used (7). For T_5 we have,

$$\begin{aligned}
 |T_5| &\leq \sum_{i=1}^{N_\varphi} \gamma_i \|\mathbf{D}^{1/2} \boldsymbol{\varepsilon}(\boldsymbol{\chi})(\tau)\|_0 \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i(\tau)\|_0, \\
 &\leq \sum_{i=1}^{N_\varphi} \frac{\bar{\epsilon} \gamma_i^2}{2\varphi_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i(\tau)\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{\varphi_i}{2\bar{\epsilon}} \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2, \\
 &\leq \sum_{i=1}^{N_\varphi} \frac{\bar{\epsilon}}{2\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i(\tau)\|_0^2 + \frac{1-\varphi_0}{2\bar{\epsilon}} \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2.
 \end{aligned}$$

If $\kappa = -1$, then the term T_7 vanishes. Otherwise if $\kappa = 1$, we have,

$$\begin{aligned}
 |T_7| &\leq 2 \int_0^\tau \sum_{e \in \Gamma_h \cup \Gamma_D} \|\{\mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})\mathbf{n}_e\}\|_{0,e} \frac{|e|^{\beta/2}}{\delta^{1/2}} \|\boldsymbol{\chi}_t\|_{0,e} \frac{\delta^{1/2}}{|e|^{\beta/2}}, \\
 &\leq Ch^{(d-1)\beta/2-1/2} \int_0^\tau \|\boldsymbol{\chi}\|_{\mathcal{A}} J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t)^{1/2}, \\
 &\leq \frac{C}{2\epsilon} h^{(d-1)\beta-1} \int_0^\tau \|\boldsymbol{\chi}\|_{\mathcal{A}}^2 dt + \frac{\epsilon}{2} \int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t) dt.
 \end{aligned}$$

For T_8 ,

$$\begin{aligned}
 |T_8| &\leq \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \|\{\mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})(\tau)\mathbf{n}_e\}\|_{0,e} \|\boldsymbol{\chi}(\tau)\|_{0,e} \left(\frac{\delta|e|^\beta}{\delta|e|^\beta} \right)^{1/2}, \\
 &\leq Ch^{(d-1)\beta/2-1/2} \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}} J_0^{1,\beta}(\boldsymbol{\chi}(\tau), \boldsymbol{\chi}(\tau))^{1/2}, \\
 &\leq \frac{\hat{\epsilon}}{2} \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 + \frac{C}{2\hat{\epsilon}} h^{(d-1)\beta-1} J_0^{1,\beta}(\boldsymbol{\chi}(\tau), \boldsymbol{\chi}(\tau)).
 \end{aligned}$$

Because we are using the elliptic projection on the initial value of \mathbf{u} we

have that $\boldsymbol{\chi}(0) = \mathbf{0}$, and $T_9 = 0$. For T_{10} ,

$$\begin{aligned} |T_{10}| &\leq \int_0^\tau \|\rho^{1/2} \boldsymbol{\xi}_{tt}\|_0 \|\rho^{1/2} \boldsymbol{\chi}_t\|_0, \\ &\leq \frac{1}{2} \int_0^\tau \|\rho^{1/2} \boldsymbol{\chi}_t\|_0^2 + Ch^{2r} \int_0^\tau \|\mathbf{u}_{tt}\|_{r+1}^2, \end{aligned}$$

where we used the estimate,

$$\|\rho^{1/2} \boldsymbol{\xi}_{tt}\|_0^2 \leq C(\rho) h^{2r} \|\mathbf{u}_{tt}\|_{r+1}^2.$$

For T_{11} we have,

$$\begin{aligned} |T_{11}| &\leq \int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\xi}_t, \boldsymbol{\xi}_t)^{1/2} J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t)^{1/2}, \\ &\leq \frac{\epsilon}{2} \int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t) + \frac{C}{2\epsilon} h^{2r} \int_0^\tau \|\mathbf{u}_t\|_{r+1}^2, \end{aligned}$$

where we used,

$$|J_0^{\delta,\beta}(\boldsymbol{\xi}_t, \boldsymbol{\xi}_t)| \leq Ch^{2r} \|\mathbf{u}_t\|_{r+1}^2.$$

For T_{12} ,

$$\begin{aligned} |T_{12}| &\leq C \int_0^\tau \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i h^{(d-1)\beta/2-1/2} \|\boldsymbol{\theta}_i\|_0 J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t)^{1/2}, \\ &\leq \frac{\epsilon}{2} \int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t) + C \int_0^\tau \sum_{i=1}^{N_\varphi} \gamma_i \frac{h^{(d-1)\beta-1+2r}}{2\epsilon} \|\mathbf{u}\|_{r+1}^2, \end{aligned}$$

where we used,

$$\|\boldsymbol{\theta}_i\|_0 \leq Ch^r \|\boldsymbol{\sigma}_i\|_r,$$

and Lemma 2.2. For T_{13} ,

$$\begin{aligned} |T_{13}| &\leq \int_0^\tau \sum_{i=1}^{N_\varphi} \gamma_i \|\mathbf{D}^{1/2} \boldsymbol{\varepsilon}(\boldsymbol{\xi})\|_0 \|\mathbf{D}^{-1/2} \dot{\boldsymbol{\eta}}_i\|_0, \\ &\leq \frac{C}{2\epsilon^*} \int_0^\tau h^{2r} \|\mathbf{u}\|_{r+1}^2 + \int_0^\tau \sum_{i=1}^{N_\varphi} \frac{\epsilon^*}{2} \|\mathbf{D}^{-1/2} \dot{\boldsymbol{\eta}}_i\|_0^2. \end{aligned}$$

For T_{14} we use,

$$|J_0^{\delta,\beta}(\boldsymbol{\xi}, \boldsymbol{\xi})| \leq \|\boldsymbol{\xi}\|_{\mathcal{A}}^2,$$

and obtain,

$$\begin{aligned}
 |T_{14}| &\leq \int_0^\tau \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \|\{\dot{\boldsymbol{\eta}}_i \mathbf{n}_e\}\|_{0,e} \|[\boldsymbol{\xi}]\|_{0,e}, \\
 &\leq \int_0^\tau \sum_{i=1}^{N_\varphi} \frac{\epsilon^*}{2} \|\mathbf{D}^{-1/2} \dot{\boldsymbol{\eta}}_i\|_0^2 + \int_0^\tau \frac{C}{2\epsilon^*} h^{2r} |\mathbf{u}|_{r+1}^2.
 \end{aligned}$$

Finally, for T_{15} we get,

$$\begin{aligned}
 |T_{15}| &\leq \int_0^\tau \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \|\{\dot{\boldsymbol{\eta}}_i \mathbf{n}_e\}\|_{0,e} \|[\boldsymbol{\chi}]\|_{0,e}, \\
 &\leq \int_0^\tau \sum_{i=1}^{N_\varphi} \frac{\epsilon^*}{2} \|\mathbf{D}^{-1/2} \dot{\boldsymbol{\eta}}_i\|_0^2 + \int_0^\tau \frac{C}{2\epsilon^*} J_0^{1,\beta}(\boldsymbol{\chi}, \boldsymbol{\chi}).
 \end{aligned}$$

Combining all bounds above and using the initial condition $\boldsymbol{\chi}(0) = \mathbf{0}$, we are left with,

$$\begin{aligned}
 &\frac{1}{2} \|\rho^{1/2} \boldsymbol{\chi}_t(\tau)\|_0^2 + \left(\frac{1}{2} - \frac{1 - \varphi_0}{2\bar{\epsilon}}\right) \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 \\
 &\quad - \frac{C}{2\hat{\epsilon}} h^{(d-1)\beta-1} J_0^{1,\beta}(\boldsymbol{\chi}(\tau), \boldsymbol{\chi}(\tau)) \\
 &\quad - \frac{\hat{\epsilon}}{2} \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 + (1 - 2\epsilon) \int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t) \\
 &+ \left(1 - \frac{3\epsilon^*}{2}\right) \int_0^\tau \sum_{i=1}^{N_\varphi} \|\mathbf{D}^{-1/2} \dot{\boldsymbol{\eta}}_i\|_0^2 + (1 - \bar{\epsilon}) \sum_{i=1}^{N_\varphi} \frac{1}{2\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i(\tau)\|_0^2 \\
 &\leq \frac{1}{2} \|\rho^{1/2} \boldsymbol{\chi}_t(0)\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{1}{2\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i(0)\|_0^2 \\
 &\quad + \frac{C}{\epsilon} h^{(d-1)\beta-1} \int_0^\tau (1 - \varphi_0) \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i\|_0^2 \\
 &\quad + \frac{C}{2\epsilon} h^{(d-1)\beta-1} \int_0^\tau \|\boldsymbol{\chi}\|_{\mathcal{A}}^2 dt + \frac{1}{2} \int_0^\tau \|\rho^{1/2} \boldsymbol{\chi}_t\|_0^2 \\
 &+ \int_0^\tau \frac{C}{2\epsilon^*} J_0^{1,\beta}(\boldsymbol{\chi}, \boldsymbol{\chi}) + Ch^{2r} \int_0^\tau \|\mathbf{u}_{tt}\|_{r+1}^2 + \frac{C}{2\epsilon} \int_0^\tau h^{2r} \|\mathbf{u}_t\|_{r+1}^2 \\
 &\quad + C \int_0^\tau \sum_{i=1}^{N_\varphi} \gamma_i \frac{h^{(d-1)\beta-1+2r}}{2\epsilon} \|\mathbf{u}\|_{r+1}^2 + \frac{C}{\epsilon^*} \int_0^\tau h^{2r} \|\mathbf{u}\|_{r+1}^2.
 \end{aligned}$$

Choosing $\epsilon = 1/4$, $\epsilon^* = 1/3$ and $\bar{\epsilon} = 1 - (\varphi_0/2)$ we then obtain,

$$\begin{aligned}
 & \frac{1}{2} \|\rho^{1/2} \boldsymbol{\chi}_t(\tau)\|_0^2 + \frac{\varphi_0}{2(2-\varphi_0)} \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 - \frac{C}{2\hat{\epsilon}} h^{(d-1)\beta-1} J_0^{1,\beta}(\boldsymbol{\chi}(\tau), \boldsymbol{\chi}(\tau)) \\
 & \quad - \frac{\hat{\epsilon}}{2} \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 + \frac{1}{2} \int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t) + \frac{1}{2} \int_0^\tau \sum_{i=1}^{N_\varphi} \|\mathbf{D}^{-1/2} \dot{\boldsymbol{\eta}}_i\|_0^2 \\
 & + \frac{\varphi_0}{2} \sum_{i=1}^{N_\varphi} \frac{1}{2\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i(\tau)\|_0^2 \leq \frac{1}{2} \|\rho^{1/2} \boldsymbol{\chi}_t(0)\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{1}{2\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i(0)\|_0^2 \\
 & \quad + Ch^{(d-1)\beta/2-1/2} \int_0^\tau (1-\varphi_0) \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_i\|_0^2 \\
 & + Ch^{(d-1)\beta-1} \int_0^\tau \|\boldsymbol{\chi}\|_{\mathcal{A}}^2 + \frac{1}{2} \int_0^\tau \|\rho^{1/2} \boldsymbol{\chi}_t\|_0^2 + C \int_0^\tau J_0^{1,\beta}(\boldsymbol{\chi}, \boldsymbol{\chi}) \\
 & \quad + Ch^{2r} \left(\|\mathbf{u}\|_{L_2(H^{r+1})}^2 + \|\mathbf{u}_t\|_{L_2(H^{r+1})}^2 + \|\mathbf{u}_{tt}\|_{L_2(H^{r+1})}^2 \right) \\
 & \quad + Ch^{2r} \int_0^\tau \sum_{i=1}^{N_\varphi} \gamma_i h^{(d-1)\beta-1} \|\mathbf{u}\|_{r+1}^2.
 \end{aligned}$$

Now we choose $\hat{\epsilon} = \varphi_0/(4 - 2\varphi_0)$ and find a constant, C^* , independent of h , such that for small enough $\hat{h}^{(d-1)\beta-1}/\delta$,

$$\begin{aligned}
 & \frac{\varphi_0}{2(2-\varphi_0)} \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 - \frac{\hat{\epsilon}}{2} \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 - \frac{Ch^{(d-1)\beta-1}}{2\hat{\epsilon}} J_0^{1,\beta}(\boldsymbol{\chi}(\tau), \boldsymbol{\chi}(\tau)) \\
 & = \frac{\varphi_0}{4(2-\varphi_0)} \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 - \frac{Ch^{(d-1)\beta-1}(4-2\varphi_0)}{2\delta\varphi_0} J_0^{\delta,\beta}(\boldsymbol{\chi}(\tau), \boldsymbol{\chi}(\tau)), \\
 & \geq \left(\frac{\varphi_0}{4(2-\varphi_0)} - \frac{C\hat{h}^{(d-1)\beta-1}(4-2\varphi_0)}{2\delta\varphi_0} \right) \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2, \\
 & = \frac{C^*}{2} \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2.
 \end{aligned}$$

Multiplying the error equation by 2 and using this lower bound then yields,

$$\begin{aligned}
 & \|\rho^{1/2}\boldsymbol{\chi}_t(\tau)\|_0^2 + C^*\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 + \int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t) \\
 & + \int_0^\tau \sum_{i=1}^{N_\varphi} \|\mathbf{D}^{-1/2}\dot{\boldsymbol{\eta}}_i\|_0^2 + \varphi_0 \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2}\boldsymbol{\eta}_i(\tau)\|_0^2 \\
 & \leq \|\rho^{1/2}\boldsymbol{\chi}_t(0)\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2}\boldsymbol{\eta}_i(0)\|_0^2 \\
 & + Ch^{(d-1)\beta/2-1/2} \int_0^\tau (1-\varphi_0) \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2}\boldsymbol{\eta}_i\|_0^2 \\
 & + Ch^{(d-1)\beta-1} \int_0^\tau \|\boldsymbol{\chi}\|_{\mathcal{A}}^2 + \int_0^\tau \|\rho^{1/2}\boldsymbol{\chi}_t\|_0^2 + C \int_0^\tau J_0^{1,\beta}(\boldsymbol{\chi}, \boldsymbol{\chi}) \\
 & + Ch^{2r} \left(\|\mathbf{u}\|_{L_2(H^{r+1})}^2 + \|\mathbf{u}_t\|_{L_2(H^{r+1})}^2 + \|\mathbf{u}_{tt}\|_{L_2(H^{r+1})}^2 \right) \\
 & + Ch^{2r} \int_0^\tau \sum_{i=1}^{N_\varphi} \gamma_i h^{(d-1)\beta-1} \|\mathbf{u}\|_{r+1}^2.
 \end{aligned}$$

Now we apply Gronwall's lemma, the initial condition $\boldsymbol{\eta}_i(0) = \mathbf{0}$, and the fact that $(d-1)\beta-1 \geq 0$ and arrive at,

$$\begin{aligned}
 & \|\rho^{1/2}\boldsymbol{\chi}_t(\tau)\|_0^2 + C^*\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 + \int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t) + \int_0^\tau \sum_{i=1}^{N_\varphi} \|\mathbf{D}^{-1/2}\dot{\boldsymbol{\eta}}_i\|_0^2 \\
 & + \varphi_0 \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2}\boldsymbol{\eta}_i(\tau)\|_0^2 \leq C\|\rho^{1/2}\boldsymbol{\chi}_t(0)\|_0^2 + Ch^{2r}.
 \end{aligned}$$

Now, subtracting like terms from both sides of (23) we have,

$$(\rho\boldsymbol{\chi}_t(0), \mathbf{v}) = (\rho(\mathbf{u}_t^h(0) - \tilde{\mathbf{u}}_t(0)), \mathbf{v}) = (\rho(\mathbf{u}_t(0) - \tilde{\mathbf{u}}_t(0)), \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h).$$

Taking $\mathbf{v} = \boldsymbol{\chi}_t(0)$ and using (13) we then get,

$$\|\rho^{1/2}\boldsymbol{\chi}_t(0)\|_0 \leq \|\rho^{1/2}(\mathbf{u}_t(0) - \tilde{\mathbf{u}}_t(0))\|_0 \leq Ch^r \|\mathbf{u}_t(0)\|_r.$$

The theorem is obtained by using approximation results for $\boldsymbol{\xi}$ and the $\boldsymbol{\theta}_i$, and using the triangle inequality. \square

The next result is concerned with the error in $\mathbf{L}_2(\Omega)$. It applies only to the symmetric formulation where $\kappa = -1$, and relies on error representation

through the following dual problem:

$$\rho \Phi_{tt} - \nabla \cdot D\varepsilon(\Phi) = \mathbf{g} - \sum_{i=1}^{N_\varphi} \gamma_i \nabla \cdot \Psi_i^* \quad \text{in } \Omega \times I, \quad (26)$$

$$\Phi = \mathbf{0} \quad \text{on } \Gamma_D, \quad (27)$$

$$D\varepsilon(\Phi)\mathbf{n} - \sum_{i=1}^{N_\varphi} \gamma_i \Psi_i^* \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_N, \quad (28)$$

$$-\dot{\Psi}_i^* + \frac{1}{\tau_i} \Psi_i^* = \gamma_i D\varepsilon(\Phi) \quad \text{in } \Omega \times I, \quad (29)$$

$$\Psi_i^*(T) = \mathbf{0}, \quad \Phi(T) = \mathbf{0}, \quad \Phi_t(T) = \mathbf{0} \quad \text{in } \Omega, \quad (30)$$

for $i = 1, \dots, N_\varphi$.

Theorem 3.2 (semidiscrete $L_2(\Omega)$ error estimate) *Let Theorem 3.1 hold and assume that in the dual problem we have $\Phi(t) \in C(\bar{\Omega})^d$ for each $t \in \bar{I}$ and also that,*

$$\|\Phi(t)\|_2 + \|\Phi_t(t)\|_1 \leq C\|\mathbf{g}(t)\|_0 \quad \text{and} \quad \|\Psi_i^*(t)\|_1 \leq C\|\mathbf{g}(t)\|_0,$$

for each $i = 1, \dots, N_\varphi$. Assume further that the dual stress is in equilibrium across each element edge,

$$\left[\left(D\varepsilon(\Phi(t)) - \sum_{i=1}^{N_\varphi} \gamma_i \Psi_i^* \right) \mathbf{n}_e \right] = \mathbf{0} \quad \text{for each } e \in \Gamma_h.$$

Also assume that $\bar{\mathbf{u}} \in \mathbf{D}_r(\mathcal{E}_h) \cap C(\bar{\Omega})^d$. Then, for the symmetric formulation (where $\kappa = -1$),

$$\|\mathbf{u} - \mathbf{u}^h\|_{L_2(L_2)} \leq Ch^{r+1}$$

Proof: Apply the dual problem with $\mathbf{g} = \boldsymbol{\zeta} = \mathbf{u} - \mathbf{u}^h$ and use the fact that $[D\varepsilon(\Phi)\mathbf{n} - \sum_{i=1}^{N_\varphi} \gamma_i \Psi_i^* \mathbf{n}] = \mathbf{0}$ to get,

$$\begin{aligned} \|\boldsymbol{\zeta}\|_0^2 &= (\boldsymbol{\zeta}, \rho \Phi_{tt} - \nabla \cdot (D\varepsilon(\Phi) - \sum_{i=1}^{N_\varphi} \gamma_i \Psi_i^*)), \\ &= (\boldsymbol{\zeta}, \rho \Phi_{tt}) + \sum_E ((D\varepsilon(\Phi) - \sum_{i=1}^{N_\varphi} \gamma_i \Psi_i^*), \varepsilon(\boldsymbol{\zeta}))_E \\ &\quad - \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{(D\varepsilon(\Phi) - \sum_{i=1}^{N_\varphi} \gamma_i \Psi_i^*) \mathbf{n}_e\} \cdot [\boldsymbol{\zeta}]. \end{aligned}$$

Thus, using the fact that $[\Phi] = \mathbf{0}$ and $\Phi = 0$ on Γ_D :

$$\begin{aligned} \|\zeta\|_0^2 &= (\zeta, \rho \Phi_{tt}) + \sum_E (\mathbf{D}\varepsilon(\Phi), \varepsilon(\zeta))_E - \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\Psi_i^*, \varepsilon(\zeta))_E \\ &\quad - \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{\mathbf{D}\varepsilon(\Phi) \mathbf{n}_e\} \cdot [\zeta] + \kappa^* \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\varepsilon(\zeta) \mathbf{n}_e\} \cdot [\Phi] \\ &\quad + J_0^{\delta, \beta}(\zeta, \Phi) + J_0^{\delta, \beta}(\zeta_t, \Phi) + \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{\Psi_i^* \mathbf{n}_e\} \cdot [\zeta]. \end{aligned}$$

Using the boundary conditions (27) and (28) and integrating over time, we obtain,

$$\begin{aligned} \int_0^T \|\zeta\|_0^2 &= \int_0^T (\zeta, \rho \Phi_{tt}) + \int_0^T \sum_E (\mathbf{D}\varepsilon(\Phi), \varepsilon(\zeta))_E \\ &\quad - \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\Psi_i^*, \varepsilon(\zeta))_E - \int_0^T \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\varepsilon(\Phi) \mathbf{n}_e\} \cdot [\zeta] \\ &\quad + \kappa^* \int_0^T \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\varepsilon(\zeta) \mathbf{n}_e\} \cdot [\Phi] \\ &\quad + \int_0^T J_0^{\delta, \beta}(\zeta, \Phi) + \int_0^T J_0^{\delta, \beta}(\zeta_t, \Phi) + \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\Psi_i^* \mathbf{n}_e\} \cdot [\zeta]. \end{aligned}$$

Let us consider the symmetric case ($\kappa = \kappa^* = -1$) then, using the definition of the bilinear form A ,

$$\begin{aligned} \int_0^T \|\zeta\|_0^2 &= \int_0^T (\zeta, \rho \Phi_{tt}) + \int_0^T A(\zeta, \Phi) + \int_0^T J_0^{\delta, \beta}(\zeta_t, \Phi) \\ &\quad - \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\Psi_i^*, \varepsilon(\zeta))_E + \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\Psi_i^* \mathbf{n}_e\} \cdot [\zeta]. \end{aligned}$$

Integrating by parts twice and using the conditions $\Phi(T) = \Phi_t(T) = 0$ results in,

$$\begin{aligned} \int_0^T (\zeta, \rho \Phi_{tt}) &= (\zeta(T), \rho \Phi(T)) - (\zeta(0), \rho \Phi_t(0)) - \int_0^T (\zeta_t, \rho \Phi_t) \\ &= -(\zeta(0), \rho \Phi_t(0)) + (\zeta_t(0), \rho \Phi(0)) + \int_0^T (\rho \zeta_{tt}, \Phi), \end{aligned}$$

and so,

$$\begin{aligned}
 \int_0^T \|\zeta\|_0^2 &= \int_0^T ((\zeta_{tt}, \rho\Phi) + A(\zeta, \Phi) + J_0^{\delta, \beta}(\zeta_t, \Phi)) \\
 &- \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\Psi_i^*, \varepsilon(\zeta))_E + \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\Psi_i^* \mathbf{n}_e\} \cdot [\zeta] \\
 &- (\zeta(0), \rho\Phi_t(0)) + (\zeta_t(0), \rho\Phi(0)). \tag{31}
 \end{aligned}$$

But the error equation (by the Galerkin orthogonality resulting from (17) and (20)), with $\Lambda_i = {}^* \sigma_i - {}^* \sigma_i^h$ is,

$$\begin{aligned}
 &(\rho\zeta_{tt}, \Phi_h) + A(\zeta, \Phi_h) + J_0^{\delta, \beta}(\zeta_t, \Phi_h) - \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\Lambda_i, \varepsilon(\Phi_h))_E \\
 &+ \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\Lambda_i \mathbf{n}_e\} \cdot [\Phi_h] = 0, \quad \forall \Phi_h \in \mathbf{D}_r(\mathcal{E}_h).
 \end{aligned}$$

Integrate this from 0 to T and subtract the result from (31) and we get, for all $\Phi_h \in \mathbf{D}_r(\mathcal{E}_h)$

$$\begin{aligned}
 \|\zeta\|_{L_2(L_2)}^2 &= \int_0^T ((\rho\zeta_{tt}, \Phi - \Phi_h) + A(\zeta, \Phi - \Phi_h) + J_0^{\delta, \beta}(\zeta_t, \Phi - \Phi_h)) \\
 &- \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\Psi_i^*, \varepsilon(\zeta))_E + \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\Psi_i^* \mathbf{n}_e\} \cdot [\zeta] \\
 &- (\zeta(0), \rho\Phi_t(0)) + (\zeta_t(0), \rho\Phi(0)) \\
 &+ \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\Lambda_i, \varepsilon(\Phi_h))_E - \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\Lambda_i \mathbf{n}_e\} \cdot [\Phi_h]. \tag{32}
 \end{aligned}$$

Now multiply (29) by a test function $\mathbf{w}_i \in \mathbf{L}_{r-1}(\mathcal{E}_h)$ and integrate by parts,

$$\begin{aligned}
 &-(\Psi_i^*(T), \mathbf{w}_i(T)) + (\Psi_i^*(0), \mathbf{w}_i(0)) + \int_0^T (\Psi_i^*, \dot{\mathbf{w}}_i + \frac{1}{\tau_i} \mathbf{w}_i) \\
 &= \int_0^T \gamma_i (\mathbf{D}\varepsilon(\Phi), \mathbf{w}_i).
 \end{aligned}$$

Take $\mathbf{w}_i = \mathbf{D}^{-1} \boldsymbol{\eta}_i$ where $\boldsymbol{\eta}_i := {}^* \sigma_i^h - {}^* \tilde{\sigma}_i$, and ${}^* \tilde{\sigma}_i$ is the $L_2(\Omega)$ projection

of ${}^* \boldsymbol{\sigma}_i$ onto $\mathbf{L}_{r-1}(\mathcal{E}_h)$, sum over i , and use the condition $\boldsymbol{\Psi}_i^*(T) = \mathbf{0}$,

$$\begin{aligned} & \sum_{i=1}^{N_\varphi} (\boldsymbol{\Psi}_i^*(0), \mathbf{D}^{-1} \boldsymbol{\eta}_i(0)) + \int_0^T \sum_{i=1}^{N_\varphi} (\mathbf{D}^{-1} \boldsymbol{\Psi}_i^*, \dot{\boldsymbol{\eta}}_i + \frac{1}{\tau_i} \boldsymbol{\eta}_i) \\ & - \sum_{i=1}^{N_\varphi} \int_0^T \gamma_i(\boldsymbol{\varepsilon}(\boldsymbol{\Phi}), \boldsymbol{\eta}_i) = 0. \end{aligned} \quad (33)$$

From (18) and (21) the error $\boldsymbol{\Lambda}_i$ satisfies:

$$\begin{aligned} & (\dot{\boldsymbol{\Lambda}}_i + \frac{1}{\tau_i} \boldsymbol{\Lambda}_i, \mathbf{w}_i) - \gamma_i(\mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\zeta}), \mathbf{w}_i) \\ & + \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{ \mathbf{D}\mathbf{w}_i \mathbf{n}_e \} \cdot [\boldsymbol{\zeta}] = 0 \quad \forall \mathbf{w}_i \in \mathbf{L}_{r-1}(\mathcal{E}_h). \end{aligned}$$

Choose $\mathbf{w}_i = \mathbf{D}^{-1} \boldsymbol{\Psi}_i^h$, where $\boldsymbol{\Psi}_i^h$ is the L^2 projection of $\boldsymbol{\Psi}_i^*$, sum over i and integrate from 0 to T :

$$\begin{aligned} & \sum_{i=1}^{N_\varphi} \int_0^T (\dot{\boldsymbol{\Lambda}}_i + \frac{1}{\tau_i} \boldsymbol{\Lambda}_i, \mathbf{D}^{-1} \boldsymbol{\Psi}_i^h) - \sum_{i=1}^{N_\varphi} \int_0^T \gamma_i(\boldsymbol{\varepsilon}(\boldsymbol{\zeta}), \boldsymbol{\Psi}_i^h) \\ & + \sum_{i=1}^{N_\varphi} \int_0^T \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{ \boldsymbol{\Psi}_i^h \mathbf{n}_e \} \cdot [\boldsymbol{\zeta}] = 0. \end{aligned} \quad (34)$$

Combining (32), (34) and (33), setting $\boldsymbol{\theta}_i := {}^* \boldsymbol{\sigma}_i - {}^* \tilde{\boldsymbol{\sigma}}_i$, and noting that

$\Lambda_i = \boldsymbol{\theta}_i - \boldsymbol{\eta}_i$, we obtain,

$$\begin{aligned}
\|\zeta\|_{L_2(L_2)}^2 &= \int_0^T ((\rho\zeta_{tt}, \boldsymbol{\Phi} - \boldsymbol{\Phi}_h) + A(\zeta, \boldsymbol{\Phi} - \boldsymbol{\Phi}_h) + \mathcal{J}_0^{\delta,\beta}(\zeta_t, \boldsymbol{\Phi} - \boldsymbol{\Phi}_h)) \\
&\quad - \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E ((\boldsymbol{\Psi}_i^* - \boldsymbol{\Psi}_i^h), \boldsymbol{\varepsilon}(\zeta))_E \\
&\quad + \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{(\boldsymbol{\Psi}_i^* - \boldsymbol{\Psi}_i^h) \mathbf{n}_e\} \cdot [\zeta] \\
&\quad - (\zeta(0), \rho\boldsymbol{\Phi}_t(0)) + (\zeta_t(0), \rho\boldsymbol{\Phi}(0)) - \sum_{i=1}^{N_\varphi} (\boldsymbol{\Psi}_i^*(0), \mathbf{D}^{-1}\boldsymbol{\eta}_i(0)) \\
&+ \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\boldsymbol{\eta}_i, \boldsymbol{\varepsilon}(\boldsymbol{\Phi} - \boldsymbol{\Phi}_h))_E - \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\Lambda_i \mathbf{n}_e\} \cdot [\boldsymbol{\Phi}_h] \\
&- \int_0^T \sum_{i=1}^{N_\varphi} (\mathbf{D}^{-1}\boldsymbol{\Psi}_i^h, \dot{\boldsymbol{\theta}}_i + \frac{1}{\tau_i}\boldsymbol{\theta}_i) + \int_0^T \sum_{i=1}^{N_\varphi} (\mathbf{D}^{-1}(\boldsymbol{\Psi}_i^h - \boldsymbol{\Psi}_i^*), \dot{\boldsymbol{\eta}}_i + \frac{1}{\tau_i}\boldsymbol{\eta}_i) \\
&\quad + \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\boldsymbol{\theta}_i, \boldsymbol{\varepsilon}(\boldsymbol{\Phi}_h))_E = R_1 + \dots + R_{13}.
\end{aligned}$$

We now bound each term R_i . First, we integrate by parts in R_1 , combine it with R_7 and use the fact that $(\boldsymbol{\Phi} - \boldsymbol{\Phi}_h)(T) = \mathbf{0}$. Thus,

$$\begin{aligned}
R_1 + R_7 &= (\rho\zeta_t(T), (\boldsymbol{\Phi} - \boldsymbol{\Phi}_h)(T)) - (\rho\zeta_t(0), (\boldsymbol{\Phi} - \boldsymbol{\Phi}_h)(0)) \\
&\quad + (\rho\zeta_t(0), \boldsymbol{\Phi}(0)) - \int_0^T (\rho\zeta_t, (\boldsymbol{\Phi} - \boldsymbol{\Phi}_h)_t) \\
&= (\rho\zeta_t(0), \boldsymbol{\Phi}_h(0)) - \int_0^T (\rho\zeta_t, (\boldsymbol{\Phi} - \boldsymbol{\Phi}_h)_t) \\
&= - \int_0^T (\rho\zeta_t, (\boldsymbol{\Phi} - \boldsymbol{\Phi}_h)_t),
\end{aligned}$$

because $(\rho\zeta_t(0), \mathbf{v}) = 0, \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h)$ and so, if $\boldsymbol{\Phi}_h$ is a continuous interpolant of $\boldsymbol{\Phi}$, we have

$$\begin{aligned}
|R_1| &\leq \|\rho^{1/2}\zeta_t\|_{L_2(L_2)} \|\rho^{1/2}(\boldsymbol{\Phi} - \boldsymbol{\Phi}_h)_t\|_{L_2(L_2)}, \\
&\leq Ch \|\boldsymbol{\Phi}_t\|_{L_2(H^1)} \|\rho^{1/2}\zeta_t\|_{L_2(L_2)}, \\
&\leq Ch \|\zeta\|_{L_2(L_2)} \|\rho^{1/2}\zeta_t\|_{L_2(L_2)}.
\end{aligned}$$

We first expand R_2 ,

$$\begin{aligned} R_2 &= \int_0^T \sum_E \int_E \mathbf{D}\varepsilon(\zeta) : \varepsilon(\Phi - \Phi_h) - \int_0^T \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\varepsilon(\zeta)\mathbf{n}_e\} \cdot [\Phi - \Phi_h] \\ &\quad + \kappa \int_0^T \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\varepsilon(\Phi - \Phi_h)\mathbf{n}_e\} \cdot [\zeta] + \int_0^T J_0^{\delta, \beta}(\zeta, \Phi - \Phi_h), \\ &= R_{21} + \dots + R_{24}, \end{aligned}$$

and note that R_{22} and R_{24} are zero because Φ and Φ_h are continuous and vanish on Γ_D .

$$\begin{aligned} |R_{21} + R_{23}| &\leq C \int_0^T \|\zeta\|_{\mathcal{A}} \|\Phi - \Phi_h\|_1, \\ &\leq C \|\zeta\|_{L_2(\mathcal{A})} h \|\Phi\|_{L_2(H^2)}, \\ &\leq Ch \|\zeta\|_{L_2(\mathcal{A})} \|\zeta\|_{L_2(L_2)}. \end{aligned}$$

The terms R_3 and R_{10} vanish. For R_4 we have, since Ψ_i^h approximates Ψ_i^* optimally

$$\begin{aligned} |R_4| &\leq C \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \|\Psi_i^* - \Psi_i^h\|_0 \|\zeta\|_{\mathcal{A}}, \\ &\leq Ch \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \|\Psi_i^*\|_1 \|\zeta\|_{\mathcal{A}}, \\ &\leq C(\gamma_i) h \|\zeta\|_{L_2(L_2)} \|\zeta\|_{L_2(\mathcal{A})}. \end{aligned}$$

Similarly,

$$\begin{aligned} |R_5| &\leq C \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i (\|\Psi_i^* - \Psi_i^h\|_0 + h \|\Psi_i^* - \Psi_i^h\|_1) \|\zeta\|_{\mathcal{A}}, \\ &\leq Ch \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \|\Psi_i^*\|_1 \|\zeta\|_{\mathcal{A}}, \\ &\leq C(\gamma_i) h \|\zeta\|_{L_2(L_2)} \|\zeta\|_{L_2(\mathcal{A})}. \end{aligned}$$

Note that if $\bar{\mathbf{u}} \in \mathbf{D}_r(\mathcal{E}_h)$, then $\zeta(0) = \mathbf{0}$ and the term R_6 disappears. Also,

because $\boldsymbol{\eta}_i(0) = \mathbf{0}$, the term R_8 is zero. For R_9 ,

$$\begin{aligned} |R_9| &\leq \int_0^T \sum_{i=1}^{N_\varphi} \|\boldsymbol{\eta}_i\|_0 \|\boldsymbol{\varepsilon}(\boldsymbol{\Phi} - \boldsymbol{\Phi}_h)\|_0, \\ &\leq Ch \int_0^T \sum_{i=1}^{N_\varphi} \|\boldsymbol{\eta}_i\|_0 \|\boldsymbol{\Phi}\|_2, \\ &\leq Ch \sum_{i=1}^{N_\varphi} \|\boldsymbol{\eta}_i\|_{L_2(L_2)} \|\boldsymbol{\zeta}\|_{L_2(L_2)}. \end{aligned}$$

The terms R_{11} and R_{13} vanish because of the property of L_2 projection, and, because $\boldsymbol{\Psi}_i^h$ is the L^2 projection of $\boldsymbol{\Psi}_i^*$, the term R_{12} also disappears.

Thus, combining the terms above and using Theorem 3.1, we conclude that

$$\|\boldsymbol{\zeta}\|_{L_2(L_2)} \leq Ch^{r+1},$$

and this completes the proof. \square

4 Fully discrete estimates

Let us define $k = T/N$ for some positive integer N and set $t_j = jk$. Setting,

$$L_j(\mathbf{v}) := \frac{1}{2} \left(L(t_j; \mathbf{v}) + L(t_{j-1}; \mathbf{v}) \right),$$

our fully discrete approximation of the problem described by (17), (18) and (19) is as follows: for each $j = 1, \dots, N$, find $\{\mathbf{z}_j^h, \mathbf{u}_j^h, \dots, \boldsymbol{\sigma}_{ij}^h, \dots\} \in \mathbf{D}_r(\mathcal{E}_h) \times \mathbf{D}_r(\mathcal{E}_h) \times \mathbf{L}_{r-1}(\mathcal{E}_h)^{N_\varphi}$ such that,

$$\begin{aligned} &\left(\rho \frac{\mathbf{z}_j^h - \mathbf{z}_{j-1}^h}{k}, \mathbf{v} \right) + A \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2}, \mathbf{v} \right) + J_0^{\delta, \beta} \left(\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \mathbf{v} \right) \\ &\quad + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \gamma_i \left\{ \frac{\boldsymbol{\sigma}_{ij}^h + \boldsymbol{\sigma}_{i, j-1}^h}{2} \mathbf{n}_e \right\} \cdot [\mathbf{v}] \\ &- \sum_{i=1}^{N_\varphi} \sum_E \int_E \gamma_i \frac{\boldsymbol{\sigma}_{ij}^h + \boldsymbol{\sigma}_{i, j-1}^h}{2} : \boldsymbol{\varepsilon}(\mathbf{v}) = L_j(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h), \end{aligned} \quad (35)$$

with, for each $i = 1, \dots, N_\varphi$,

$$\begin{aligned} & \sum_E \left(\frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} + \frac{1}{\tau_i} \frac{*\sigma_{ij}^h + *\sigma_{i,j-1}^h}{2}, \mathbf{w}_i \right)_E \\ &= \sum_E \gamma_i \left(\mathbf{D}\varepsilon \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right), \mathbf{w}_i \right)_E \\ & - \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{ \mathbf{D}\mathbf{w}_i \mathbf{n}_e \} \cdot \left[\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right] \quad \forall \mathbf{w}_i \in \mathbf{L}_{r-1}(\mathcal{E}_h), \end{aligned} \quad (36)$$

and,

$$\left(\rho \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \mathbf{v} \right)_E = \left(\rho \frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k}, \mathbf{v} \right)_E \quad \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h). \quad (37)$$

It follows from this last equation that,

$$\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2} = \frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k}. \quad (38)$$

For the initial data we set $*\sigma_{i0}^h = \mathbf{0}$, for $i = 1, \dots, N_\varphi$, and,

$$A(\mathbf{u}_0^h, \mathbf{v}) = A(\bar{\mathbf{u}}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h), \quad (39)$$

$$(\rho \mathbf{z}_0^h, \mathbf{v}) = (\rho \bar{\mathbf{z}}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h). \quad (40)$$

The first result is a basic stability estimate. Beforehand, we need some lemmas, and before these we note that if $\bar{\mathbf{u}} \in C(\bar{\Omega})^d$ then $J_0^{\delta, \beta}(\bar{\mathbf{u}}, \bar{\mathbf{u}}) = 0$ and, clearly,

$$\|\bar{\mathbf{u}}\|_{\mathcal{A}} \leq C \|\bar{\mathbf{u}}\|_1. \quad (41)$$

Lemma 4.1 *For the elliptic projection, $\mathbf{u}_0^h \in \mathbf{D}_r(\mathcal{E}_h)$, of $\bar{\mathbf{u}}$ defined by (39) and for $\beta \geq (d-1)^{-1}$, $h \leq \hat{h}$ and $\kappa = \pm 1$, we have, if $\bar{\mathbf{u}} \in \mathbf{H}^2(\Omega)$ (hence $u \in C(\bar{\Omega})^d$), that,*

$$\|\mathbf{u}_0^h\|_{\mathcal{A}} \leq C_{\mathcal{A}} \|\bar{\mathbf{u}}\|_2.$$

When $\kappa = -1$ this estimate holds only for δ large enough.

Proof. Since $[\bar{\mathbf{u}}] = \mathbf{0}$ we see that,

$$\begin{aligned} \|\mathbf{u}_0^h\|_{\mathcal{A}}^2 &= \sum_E \int_E \mathbf{D}\varepsilon(\bar{\mathbf{u}}) : \varepsilon(\mathbf{u}_0^h) + J_0^{\delta, \beta}(\bar{\mathbf{u}}, \mathbf{u}_0^h) \\ & - \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{ \mathbf{D}\varepsilon(\bar{\mathbf{u}}) \mathbf{n}_e \} \cdot [\mathbf{u}_0^h] + \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{ \mathbf{D}\varepsilon(\mathbf{u}_0^h) \mathbf{n}_e \} \cdot [\mathbf{u}_0^h] \\ & - \kappa \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{ \mathbf{D}\varepsilon(\mathbf{u}_0^h) \mathbf{n}_e \} \cdot [\mathbf{u}_0^h]. \end{aligned}$$

Now,

$$\left| \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\boldsymbol{\varepsilon}(\bar{\mathbf{u}})\mathbf{n}_e\} \cdot [\mathbf{u}_0^h] \right| \leq \frac{1}{2\epsilon} \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{|e|^\beta}{\delta} \int_e |\{\mathbf{D}\boldsymbol{\varepsilon}(\bar{\mathbf{u}})\mathbf{n}_e\}|^2 + \frac{\epsilon}{2} \|\mathbf{u}_0^h\|_{\mathcal{A}}^2.$$

Also, since $|e|^\beta \leq Ch^{(d-1)\beta}$, we have by a trace inequality that,

$$\frac{1}{2\epsilon} \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{|e|^\beta}{\delta} \int_e |\{\mathbf{D}\boldsymbol{\varepsilon}(\bar{\mathbf{u}})\mathbf{n}_e\}|^2 \leq \frac{Ch^{(d-1)\beta-1}}{2\delta\epsilon} \|\bar{\mathbf{u}}\|_2^2,$$

and so, in the case $\kappa = 1$, we arrive at,

$$\|\mathbf{u}_0^h\|_{\mathcal{A}}^2 \leq \frac{1}{2\epsilon} \|\bar{\mathbf{u}}\|_{\mathcal{A}}^2 + \epsilon \|\mathbf{u}_0^h\|_{\mathcal{A}}^2 + \frac{Ch^{(d-1)\beta-1}}{\delta\epsilon} \|\bar{\mathbf{u}}\|_2^2.$$

Choosing $\epsilon = 1/2$ and using (41) then yields the required result for any $\delta > 0$.

When $\kappa = -1$ we have one extra term to deal with:

$$\begin{aligned} & \left| 2 \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}_0^h)\mathbf{n}_e\} \cdot [\mathbf{u}_0^h] \right| \\ & \leq 2 \left(\sum_{e \in \Gamma_h \cup \Gamma_D} |e|^\beta \int_e |\{\mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}_0^h)\mathbf{n}_e\}|^2 \right)^{1/2} \left(\sum_{e \in \Gamma_h \cup \Gamma_D} \frac{1}{|e|^\beta} \int_e |[\mathbf{u}_0^h]|^2 \right)^{1/2}, \\ & \leq \epsilon |\mathbf{u}_0^h|_{\mathcal{E}}^2 + \frac{Ch^{(d-1)\beta-1}}{\delta\epsilon} J_0^{\delta,\beta}(\mathbf{u}_0^h, \mathbf{u}_0^h). \end{aligned}$$

Here we used an ‘inverse-trace’ inequality to get,

$$\begin{aligned} & \sum_{e \in \Gamma_h \cup \Gamma_D} |e|^\beta \int_e |\{\mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}_0^h)\mathbf{n}_e\}|^2 \\ & \leq Ch^{(d-1)\beta-1} \sum_{E \in \mathcal{E}_h} \left(\|\mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}_0^h)\|_{0,E}^2 + h_E^2 \|\mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}_0^h)\|_{1,E}^2 \right), \\ & \leq Ch^{(d-1)\beta-1} |\mathbf{u}_0^h|_{\mathcal{E}}^2, \end{aligned}$$

since $h_E^2 \|\mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}_0^h)\|_{1,E}^2 \leq C \|\mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}_0^h)\|_{0,E}^2$.

Putting these results together gives,

$$\begin{aligned} \|\mathbf{u}_0^h\|_{\mathcal{A}}^2 & \leq \frac{1}{2\epsilon} \|\bar{\mathbf{u}}\|_{\mathcal{A}}^2 + \epsilon \|\mathbf{u}_0^h\|_{\mathcal{A}}^2 + \frac{Ch^{(d-1)\beta-1}}{\delta\epsilon} \|\bar{\mathbf{u}}\|_{2,\Omega}^2 \\ & \quad + \epsilon |\mathbf{u}_0^h|_{\mathcal{E}}^2 + \frac{Ch^{(d-1)\beta-1}}{\delta\epsilon} J_0^{\delta,\beta}(\mathbf{u}_0^h, \mathbf{u}_0^h), \end{aligned}$$

and using (41), choosing $\epsilon = 1/4$ and δ such that,

$$\delta \geq 4C\hat{h}^{(d-1)\beta-1} \implies \frac{Ch^{(d-1)\beta-1}}{\delta\epsilon} \leq \epsilon,$$

then completes the proof. \square

Remark 4.2 *If we appeal to the error estimates given for the elliptic (elasticity) problem in [12] then, for $\kappa = 1$, we have,*

$$\|\bar{\mathbf{u}} - \mathbf{u}_0^h\|_{\mathcal{A}} \leq C\hat{h}\|\bar{\mathbf{u}}\|_2.$$

The proof of Lemma 4.1 is then very short:

$$\|\mathbf{u}_0^h\|_{\mathcal{A}} \leq \|\bar{\mathbf{u}} - \mathbf{u}_0^h\|_{\mathcal{A}} + \|\bar{\mathbf{u}}\|_{\mathcal{A}} \leq (C\hat{h} + C)\|\bar{\mathbf{u}}\|_2,$$

where we used (41). For $\kappa = -1$ we can make a similar argument but the error estimate requires that δ be ‘large enough’.

Lemma 4.3 *For every $\epsilon'_L, \epsilon_L > 0$,*

$$\begin{aligned} 2k \left| \sum_{j=1}^m L_j \left(\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) \right| &\leq C_{\mathcal{A}}^2 \|\bar{\mathbf{u}}\|_2^2 + C^2 h^{-1} \|\mathbf{g}(0)\|_{0,\Gamma_N}^2 \\ &+ \frac{C^2 h^{-1}}{\epsilon'_L} \|\mathbf{g}(t_m)\|_{0,\Gamma_N}^2 + \frac{C^2 h^{-1} k}{\epsilon_L} \sum_{j=1}^m \left\| \frac{\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})}{k} \right\|_{0,\Gamma_N}^2 \\ &+ \frac{k}{\rho\epsilon_L} \sum_{j=1}^m \left\| \frac{\mathbf{f}(t_j) + \mathbf{f}(t_{j-1})}{2} \right\|_0^2 + \epsilon_L k \sum_{j=0}^{m-1} \left(\|\rho^{\frac{1}{2}} \mathbf{z}_j^h\|_0^2 + \|\mathbf{u}_j^h\|_{\mathcal{A}}^2 \right) \\ &+ \epsilon_L k \|\rho^{\frac{1}{2}} \mathbf{z}_m^h\|_0^2 + (\epsilon'_L + \epsilon_L k) \|\mathbf{u}_m^h\|_{\mathcal{A}}^2, \end{aligned}$$

for $m = 1, 2, \dots, N$, and where $C_{\mathcal{A}}$ is from Lemma 4.1.

Proof. Using (38) we have,

$$\begin{aligned} 2k \sum_{j=1}^m L_j \left(\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) &= \frac{k}{2} \sum_{j=1}^m (\mathbf{f}(t_j) + \mathbf{f}(t_{j-1}), \mathbf{z}_j^h + \mathbf{z}_{j-1}^h) \\ &+ \sum_{j=1}^m (\mathbf{g}(t_j) + \mathbf{g}(t_{j-1}), \mathbf{u}_j^h - \mathbf{u}_{j-1}^h)_{\Gamma_N}, \\ &= \frac{k}{2} \sum_{j=1}^m (\mathbf{f}(t_j) + \mathbf{f}(t_{j-1}), \mathbf{z}_j^h + \mathbf{z}_{j-1}^h) \\ &\quad - \sum_{j=1}^m (\mathbf{g}(t_j) - \mathbf{g}(t_{j-1}), \mathbf{u}_j^h + \mathbf{u}_{j-1}^h)_{\Gamma_N} \\ &\quad + 2(\mathbf{g}(t_m), \mathbf{u}_m^h)_{\Gamma_N} - 2(\mathbf{g}(0), \mathbf{u}_0^h)_{\Gamma_N}. \end{aligned}$$

Now, for $\mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h)$ we note that by using inverse estimates and interpolation,

$$\begin{aligned}
 |(\mathbf{g}, \mathbf{v})_{\Gamma_N}| &\leq \|\mathbf{g}\|_{0,\Gamma_N} \left(\sum_{e \in \Gamma_N} \|\mathbf{v}\|_{0,e}^2 \right)^{1/2}, \\
 &\leq C \|\mathbf{g}\|_{0,\Gamma_N} \left(\sum_{\bar{E} \cap \Gamma_N \neq \emptyset} h^{-1} \|\mathbf{v}\|_{1/2,E}^2 \right)^{1/2}, \\
 &\leq Ch^{-1/2} \|\mathbf{g}\|_{0,\Gamma_N} \left(\sum_{\bar{E} \cap \Gamma_N \neq \emptyset} \|\mathbf{v}\|_{0,E} \|\mathbf{v}\|_{1,E} \right)^{1/2}, \\
 &\leq Ch^{-1/2} \|\mathbf{g}\|_{0,\Gamma_N} \|\mathbf{v}\|_{\mathcal{A}}, \tag{42}
 \end{aligned}$$

by equivalence of norms and where $|\cdot|_{1,E}$ denotes the $H^1(E)^d$ seminorm. Hence,

$$\begin{aligned}
 &2k \left| \sum_{j=1}^m L_j \left(\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) \right| \\
 &\leq k \sum_{j=1}^m \left\| \frac{\mathbf{f}(t_j) + \mathbf{f}(t_{j-1})}{2\rho^{1/2}} \right\|_0 \left(\|\rho^{1/2} \mathbf{z}_j^h\|_0 + \|\rho^{1/2} \mathbf{z}_{j-1}^h\|_0 \right) \\
 &\quad + Ch^{-1/2} k \sum_{j=1}^m \left\| \frac{\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})}{k} \right\|_{0,\Gamma_N} \left(\|\mathbf{u}_j^h\|_{\mathcal{A}} + \|\mathbf{u}_{j-1}^h\|_{\mathcal{A}} \right) \\
 &\quad + 2Ch^{-1/2} \|\mathbf{g}(t_m)\|_{0,\Gamma_N} \|\mathbf{u}_m^h\|_{\mathcal{A}} + 2Ch^{-1/2} \|\mathbf{g}(0)\|_{0,\Gamma_N} \|\mathbf{u}_0^h\|_{\mathcal{A}}.
 \end{aligned}$$

By several applications of Young's inequality and Lemma 4.1 we therefore have,

$$\begin{aligned}
 &2k \left| \sum_{j=1}^m L_j \left(\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) \right| \\
 &\leq \frac{k}{\rho \epsilon_L} \sum_{j=1}^m \left\| \frac{\mathbf{f}(t_j) + \mathbf{f}(t_{j-1})}{2} \right\|_0^2 + \epsilon_L k \sum_{j=0}^m \|\rho^{1/2} \mathbf{z}_j^h\|_0^2 + C_{\mathcal{A}}^2 \|\bar{\mathbf{u}}\|_2^2 \\
 &\quad + \frac{C^2 h^{-1} k}{\epsilon_L} \sum_{j=1}^m \left\| \frac{\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})}{k} \right\|_{0,\Gamma_N}^2 + \epsilon_L k \sum_{j=0}^m \|\mathbf{u}_j^h\|_{\mathcal{A}}^2 \\
 &\quad + \frac{C^2 h^{-1}}{\epsilon'_L} \|\mathbf{g}(t_m)\|_{0,\Gamma_N}^2 + \epsilon'_L \|\mathbf{u}_m^h\|_{\mathcal{A}}^2 + C^2 h^{-1} \|\mathbf{g}(0)\|_{0,\Gamma_N}^2.
 \end{aligned}$$

This completes the proof. \square

With these preliminary results established we can now give the stability estimate. Note that the factor ‘ h^{-1} ’ is not observed in practical computations; its presence seems to be due only to a weakness in the proof.

Theorem 4.4 (discrete stability) *Assume that $\beta \geq (d-1)^{-1}$ along with $k \leq \hat{k}$ and $h \leq \hat{h}$. Then, for δ large enough, \hat{k} and \hat{h} small enough, and $m = 1, 2, \dots, N$,*

$$\begin{aligned}
 & \|\rho^{\frac{1}{2}} \mathbf{z}_m^h\|_0^2 + \|\mathbf{u}_m^h\|_{\mathcal{A}}^2 + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-\frac{1}{2}} * \boldsymbol{\sigma}_{im}^h\|_0^2 \\
 & + k \sum_{j=1}^m J_0^{\delta, \beta} \left(\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2} \right) \\
 & + k \sum_{i=1}^{N_\varphi} \sum_{j=1}^m \left\| \mathbf{D}^{-\frac{1}{2}} \left(\frac{* \boldsymbol{\sigma}_{ij}^h - * \boldsymbol{\sigma}_{i,j-1}^h}{k} \right) \right\|_0^2 \\
 & \leq C \|\rho^{\frac{1}{2}} \bar{\mathbf{z}}\|_0^2 + C \|\bar{\mathbf{u}}\|_2^2 + Ch^{-1} \|\mathbf{g}(0)\|_{0, \Gamma_N}^2 + Ch^{-1} \|\mathbf{g}(t_m)\|_{0, \Gamma_N}^2 \\
 & + Ch^{-1} k \sum_{j=1}^m \left\| \frac{\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})}{k} \right\|_{0, \Gamma_N}^2 + Ck \sum_{j=1}^m \left\| \frac{\mathbf{f}(t_j) + \mathbf{f}(t_{j-1})}{2} \right\|_0^2,
 \end{aligned}$$

where C represents a generic positive constant.

Proof: We choose

$$\mathbf{v} = (\mathbf{u}_j^h - \mathbf{u}_{j-1}^h)/k \in \mathbf{D}_r(\mathcal{E}_h)$$

in (35),

$$\mathbf{w}_i = \mathbf{D}^{-1} (* \boldsymbol{\sigma}_{ij}^h - * \boldsymbol{\sigma}_{i,j-1}^h)/k \in \mathbf{L}_{r-1}(\mathcal{E}_h)$$

in (36),

$$\mathbf{v} = (\mathbf{z}_j^h - \mathbf{z}_{j-1}^h)/k \in \mathbf{D}_r(\mathcal{E}_h)$$

in (37), and we recall (38). Equation (37) then becomes,

$$\left(\rho \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h - \mathbf{z}_{j-1}^h}{k} \right) = \left(\rho \frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k}, \frac{\mathbf{z}_j^h - \mathbf{z}_{j-1}^h}{k} \right)$$

and with this (35) becomes,

$$\begin{aligned}
 & \left(\rho \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h - \mathbf{z}_{j-1}^h}{k} \right) + A \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2}, \frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) \\
 & \quad + J_0^{\delta, \beta} \left(\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2} \right) \\
 & \quad + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \gamma_i \left\{ \frac{*\sigma_{ij}^h + *\sigma_{i,j-1}^h}{2} \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right] \\
 & - \sum_{i=1}^{N_\varphi} \sum_E \int_E \gamma_i \frac{*\sigma_{ij}^h + *\sigma_{i,j-1}^h}{2} : \boldsymbol{\varepsilon} \left(\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) = L_j \left(\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right).
 \end{aligned}$$

Also, (36) becomes,

$$\begin{aligned}
 & \sum_{i=1}^{N_\varphi} \sum_E \left(\frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k}, \mathbf{D}^{-1} \frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} \right)_E \\
 & + \sum_{i=1}^{N_\varphi} \sum_E \frac{1}{\tau_i} \left(\frac{*\sigma_{ij}^h + *\sigma_{i,j-1}^h}{2}, \mathbf{D}^{-1} \frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} \right)_E \\
 & = \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \left(\boldsymbol{\varepsilon} \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right), \frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} \right)_E \\
 & - \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \left\{ \frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right].
 \end{aligned}$$

Adding the third of these to the second and noting that,

$$\begin{aligned}
 & \left(\rho \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h - \mathbf{z}_{j-1}^h}{k} \right) + A \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2}, \frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) \\
 & = \frac{1}{2k} \|\rho^{1/2} \mathbf{z}_j^h\|_0^2 - \frac{1}{2k} \|\rho^{1/2} \mathbf{z}_{j-1}^h\|_0^2 + \frac{1}{2k} \|\mathbf{u}_j^h\|_{\mathcal{A}}^2 - \frac{1}{2k} \|\mathbf{u}_{j-1}^h\|_{\mathcal{A}}^2 \\
 & \quad - \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \left\{ \mathbf{D} \boldsymbol{\varepsilon} \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right) \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right] \\
 & \quad + \kappa \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \left\{ \mathbf{D} \boldsymbol{\varepsilon} \left(\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right],
 \end{aligned}$$

along with,

$$\begin{aligned} & \sum_{i=1}^{N_\varphi} \sum_E \frac{1}{\tau_i} \left(\frac{*\sigma_{ij}^h + *\sigma_{i,j-1}^h}{2}, \mathbf{D}^{-1} \frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} \right) \\ &= \frac{1}{2k} \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} *\sigma_{ij}^h\|_0^2 - \frac{1}{2k} \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} *\sigma_{i,j-1}^h\|_0^2, \end{aligned}$$

we obtain,

$$\begin{aligned} & \frac{1}{2k} \|\rho^{1/2} \mathbf{z}_j^h\|_0^2 + \frac{1}{2k} \|\mathbf{u}_j^h\|_{\mathcal{A}}^2 + J_0^{\delta,\beta} \left(\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2} \right) \\ &+ \sum_{i=1}^{N_\varphi} \left\| \mathbf{D}^{-1/2} \left(\frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} \right) \right\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{1}{2k\tau_i} \|\mathbf{D}^{-1/2} *\sigma_{ij}^h\|_0^2 \\ &= \frac{1}{2k} \|\rho^{1/2} \mathbf{z}_{j-1}^h\|_0^2 + \frac{1}{2k} \|\mathbf{u}_{j-1}^h\|_{\mathcal{A}}^2 + \sum_{i=1}^{N_\varphi} \frac{1}{2k\tau_i} \|\mathbf{D}^{-1/2} *\sigma_{i,j-1}^h\|_0^2 \\ &\quad + L_j \left(\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) \\ &\quad + \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \left(\boldsymbol{\varepsilon} \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right), \frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} \right)_E \\ &\quad - \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \left\{ \frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right] \\ &\quad - \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \gamma_i \left\{ \frac{*\sigma_{ij}^h + *\sigma_{i,j-1}^h}{2} \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right] \\ &\quad + \sum_{i=1}^{N_\varphi} \sum_E \int_E \gamma_i \frac{*\sigma_{ij}^h + *\sigma_{i,j-1}^h}{2} : \boldsymbol{\varepsilon} \left(\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) \\ &\quad + \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \left\{ \mathbf{D}\boldsymbol{\varepsilon} \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right) \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right] \\ &\quad - \kappa \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \left\{ \mathbf{D}\boldsymbol{\varepsilon} \left(\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right]. \end{aligned}$$

Multiplying by $2k$, summing over $j = 1, \dots, m$ and recalling that each ${}^* \boldsymbol{\sigma}_{i0}^h = \mathbf{0}$ then gives,

$$\begin{aligned} & \|\rho^{1/2} \mathbf{z}_m^h\|_0^2 + \|\mathbf{u}_m^h\|_{\mathcal{A}}^2 + 2k \sum_{j=1}^m J_0^{\delta, \beta} \left(\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2} \right) \\ & + 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \left\| \mathbf{D}^{-1/2} \left(\frac{{}^* \boldsymbol{\sigma}_{ij}^h - {}^* \boldsymbol{\sigma}_{i,j-1}^h}{k} \right) \right\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} {}^* \boldsymbol{\sigma}_{im}^h\|_0^2 \\ & = \|\rho^{1/2} \mathbf{z}_0^h\|_0^2 + \|\mathbf{u}_0^h\|_{\mathcal{A}}^2 + 2k \sum_{j=1}^m L_j \left(\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) \\ & \quad + T_1 + T_2 + T_3, \end{aligned}$$

where the T_i are given by,

$$\begin{aligned} T_1 &= -2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \left(\left\{ \frac{{}^* \boldsymbol{\sigma}_{ij}^h - {}^* \boldsymbol{\sigma}_{i,j-1}^h}{k} \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right] \right. \\ & \quad \left. + \left\{ \frac{{}^* \boldsymbol{\sigma}_{ij}^h + {}^* \boldsymbol{\sigma}_{i,j-1}^h}{2} \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right] \right), \\ T_2 &= 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \int_E \left(\boldsymbol{\varepsilon} \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right) : \frac{{}^* \boldsymbol{\sigma}_{ij}^h - {}^* \boldsymbol{\sigma}_{i,j-1}^h}{k} \right. \\ & \quad \left. + \boldsymbol{\varepsilon} \left(\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) : \frac{{}^* \boldsymbol{\sigma}_{ij}^h + {}^* \boldsymbol{\sigma}_{i,j-1}^h}{2} \right), \\ T_3 &= 2k \sum_{j=1}^m \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \left(\left\{ \mathbf{D} \boldsymbol{\varepsilon} \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right) \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right] \right. \\ & \quad \left. - \kappa \left\{ \mathbf{D} \boldsymbol{\varepsilon} \left(\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right] \right). \end{aligned}$$

We take each of these T_i in turn. Firstly, using the summation identity,

$$\sum_{j=1}^m \left((a_j - a_{j-1})(b_j + b_{j-1}) + (a_j + a_{j-1})(b_j - b_{j-1}) \right) = 2a_m b_m - 2a_0 b_0,$$

we get for T_2 that,

$$\begin{aligned} T_2 &= \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \int_E \left(\boldsymbol{\varepsilon}(\mathbf{u}_j^h + \mathbf{u}_{j-1}^h) : (*\boldsymbol{\sigma}_{ij}^h - *\boldsymbol{\sigma}_{i,j-1}^h) \right. \\ &\quad \left. + \boldsymbol{\varepsilon}(\mathbf{u}_j^h - \mathbf{u}_{j-1}^h) : (*\boldsymbol{\sigma}_{ij}^h + *\boldsymbol{\sigma}_{i,j-1}^h) \right), \\ &= 2 \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \int_E \boldsymbol{\varepsilon}(\mathbf{u}_m^h) : *\boldsymbol{\sigma}_{im}^h, \end{aligned}$$

since $*\boldsymbol{\sigma}_{i0}^h = \mathbf{0}$. Hence,

$$\begin{aligned} |T_2| &\leq 2 \sum_{i=1}^{N_\varphi} \gamma_i \|\mathbf{D}^{\frac{1}{2}} \boldsymbol{\varepsilon}(\mathbf{u}_m^h)\|_0 \|\mathbf{D}^{-\frac{1}{2}} *\boldsymbol{\sigma}_{im}^h\|_0, \\ &\leq \sum_{i=1}^{N_\varphi} \left(\frac{\bar{\varepsilon} \gamma_i^2}{\varphi_i} \|\mathbf{D}^{-\frac{1}{2}} *\boldsymbol{\sigma}_{im}^h\|_0^2 + \frac{\varphi_i}{\bar{\varepsilon}} \|\mathbf{D}^{\frac{1}{2}} \boldsymbol{\varepsilon}(\mathbf{u}_m^h)\|_0^2 \right). \end{aligned}$$

But $\gamma_i^2/\varphi_i = 1/\tau_i$, so,

$$|T_2| \leq \sum_{i=1}^{N_\varphi} \frac{\bar{\varepsilon}}{\tau_i} \|\mathbf{D}^{-\frac{1}{2}} *\boldsymbol{\sigma}_{im}^h\|_0^2 + \frac{1 - \varphi_0}{\bar{\varepsilon}} |\mathbf{u}_m^h|_{\mathcal{E}}^2,$$

because $\sum_{i=1}^{N_\varphi} \varphi_i = 1 - \varphi_0$.

Now, using (38) and $|e|^\beta \leq Ch^{(d-1)\beta}$ we have for T_1 that,

$$\begin{aligned}
 |T_1| &\leq 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \left| \left\{ \frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right] \right| \\
 &\quad + 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \left| \left(\frac{|e|^\beta}{\delta} \right)^{\frac{1}{2}} \left\{ \frac{*\sigma_{ij}^h + *\sigma_{i,j-1}^h}{2} \cdot \mathbf{n}_e \right\} \right. \\
 &\quad \quad \quad \left. \cdot \left(\frac{\delta}{|e|^\beta} \right)^{\frac{1}{2}} \left[\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right] \right|, \\
 &\leq 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \gamma_i C(\mathbf{D}) h^{(d-1)\beta/2-1/2} \left\| \mathbf{D}^{-\frac{1}{2}} \left(\frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} \right) \right\|_0 \\
 &\quad \times J_0^{1,\beta} \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2}, \frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right)^{\frac{1}{2}} \\
 &\quad + 2C(\mathbf{D}, \delta) h^{(d-1)\beta/2-1/2} k \\
 &\quad \times \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \gamma_i J_0^{\delta,\beta} \left(\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2} \right)^{\frac{1}{2}} \\
 &\quad \times \left(\|\mathbf{D}^{-\frac{1}{2}} *\sigma_{ij}^h\|_0 + \|\mathbf{D}^{-\frac{1}{2}} *\sigma_{i,j-1}^h\|_0 \right), \\
 &\leq 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \frac{\hat{\epsilon}_i}{2} \left\| \mathbf{D}^{-\frac{1}{2}} \left(\frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} \right) \right\|_0^2 \\
 &\quad + 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \frac{Ch^{(d-1)\beta-1} \gamma_i^2}{2\hat{\epsilon}_i} J_0^{1,\beta} \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2}, \frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right) \\
 &\quad + 2Ck \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \left(\frac{\tilde{\epsilon}_i h^{(d-1)\beta-1}}{2} J_0^{\delta,\beta} \left(\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2} \right) \right. \\
 &\quad \quad \quad \left. + \frac{\gamma_i^2}{4\tilde{\epsilon}_i} \left(\|\mathbf{D}^{-\frac{1}{2}} *\sigma_{ij}^h\|_0^2 + \|\mathbf{D}^{-\frac{1}{2}} *\sigma_{i,j-1}^h\|_0^2 \right) \right).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 |T_1| &\leq 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \frac{\hat{\epsilon}_i}{2} \left\| \mathbf{D}^{-\frac{1}{2}} \left(\frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} \right) \right\|_0^2 \\
 &\quad + Ck \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \frac{\gamma_i^2 h^{(d-1)\beta-1}}{\hat{\epsilon}_i} J_0^{1,\beta} \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2}, \frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right) \\
 &\quad + 2h^{(d-1)\beta-1} k \check{\epsilon} (1 - \varphi_0) \sum_{j=1}^m J_0^{\delta,\beta} \left(\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2} \right) \\
 &\quad + \frac{2Ck}{\check{\epsilon}} \sum_{j=0}^m \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-\frac{1}{2}} *\sigma_{ij}^h\|_0^2.
 \end{aligned}$$

Observing that,

$$J_0^{\delta,\beta} \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2}, \frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right) \leq \frac{1}{2} J_0^{\delta,\beta}(\mathbf{u}_j^h, \mathbf{u}_j^h) + \frac{1}{2} J_0^{\delta,\beta}(\mathbf{u}_{j-1}^h, \mathbf{u}_{j-1}^h),$$

then yields,

$$\begin{aligned}
 |T_1| &\leq 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \frac{\hat{\epsilon}_i}{2} \left\| \mathbf{D}^{-\frac{1}{2}} \left(\frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} \right) \right\|_0^2 \\
 &\quad + Ck \sum_{j=0}^m \sum_{i=1}^{N_\varphi} \frac{\gamma_i^2 h^{(d-1)\beta-1}}{\hat{\epsilon}_i} J_0^{1,\beta}(\mathbf{u}_j^h, \mathbf{u}_j^h) \\
 &\quad + 2h^{(d-1)\beta-1} k \check{\epsilon} (1 - \varphi_0) \sum_{j=1}^m J_0^{\delta,\beta} \left(\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2} \right) \\
 &\quad + \frac{2Ck}{\check{\epsilon}} \sum_{j=0}^m \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-\frac{1}{2}} *\sigma_{ij}^h\|_0^2,
 \end{aligned}$$

where $\tilde{\epsilon}_i = 2\varphi_i \check{\epsilon} / C$.

Now, for T_3 , we note first that,

$$\begin{aligned}
 &-\kappa \int_e \left\{ \mathbf{D}\boldsymbol{\varepsilon} \left(\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right) \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right] \\
 &= -\frac{\kappa}{k} \int_e \{ \mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}_j^h) \cdot \mathbf{n}_e \} \cdot [\mathbf{u}_j^h] + \frac{\kappa}{k} \int_e \{ \mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u}_{j-1}^h) \cdot \mathbf{n}_e \} \cdot [\mathbf{u}_{j-1}^h] \\
 &\quad + \kappa \int_e \left\{ \mathbf{D}\boldsymbol{\varepsilon} \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right) \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right].
 \end{aligned}$$

Hence,

$$T_3 = 2k \sum_{j=1}^m \sum_{e \in \Gamma_h \cup \Gamma_D} \left(\int_e (1 + \kappa) \left\{ \mathbf{D}\varepsilon \left(\frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right) \cdot \mathbf{n}_e \right\} \cdot \left[\frac{\mathbf{u}_j^h - \mathbf{u}_{j-1}^h}{k} \right] \right. \\ \left. - \frac{\kappa}{k} \int_e \{ \mathbf{D}\varepsilon(\mathbf{u}_j^h) \cdot \mathbf{n}_e \} \cdot [\mathbf{u}_j^h] + \frac{\kappa}{k} \int_e \{ \mathbf{D}\varepsilon(\mathbf{u}_{j-1}^h) \cdot \mathbf{n}_e \} \cdot [\mathbf{u}_{j-1}^h] \right).$$

Therefore, since $|\kappa| \leq 1$ and $|1 + \kappa| \leq 2$,

$$\begin{aligned} |T_3| &\leq 2k \sum_{j=1}^m 2Ch^{(d-1)\beta/2-1/2} \left\| \frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right\|_{\mathcal{A}} \\ &\quad \times J_0^{\delta, \beta} \left(\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2} \right)^{\frac{1}{2}} \\ &\quad + 2 \sum_{e \in \Gamma_h \cup \Gamma_D} \left| \int_e \{ \mathbf{D}\varepsilon(\mathbf{u}_m^h) \cdot \mathbf{n}_e \} \cdot [\mathbf{u}_m^h] \right| \\ &\quad + 2 \sum_{e \in \Gamma_h \cup \Gamma_D} \left| \int_e \{ \mathbf{D}\varepsilon(\mathbf{u}_0^h) \cdot \mathbf{n}_e \} \cdot [\mathbf{u}_0^h] \right|, \\ &\leq 2k \left(\sum_{j=1}^m \frac{2C^2}{\epsilon'} \left\| \frac{\mathbf{u}_j^h + \mathbf{u}_{j-1}^h}{2} \right\|_{\mathcal{A}}^2 \right. \\ &\quad \left. + \sum_{j=1}^m \frac{\epsilon' h^{(d-1)\beta-1}}{2} J_0^{\delta, \beta} \left(\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2} \right) \right) \\ &\quad + 2Ch^{(d-1)\beta/2-1/2} \|\mathbf{u}_m^h\|_{\mathcal{A}} J_0^{1, \beta}(\mathbf{u}_m^h, \mathbf{u}_m^h)^{\frac{1}{2}} \\ &\quad + 2Ch^{(d-1)\beta/2-1/2} \|\mathbf{u}_0^h\|_{\mathcal{A}} J_0^{\delta, \beta}(\mathbf{u}_0^h, \mathbf{u}_0^h)^{\frac{1}{2}}, \\ &\leq \frac{\epsilon''}{2} \|\mathbf{u}_m^h\|_{\mathcal{A}}^2 + \frac{2Ck}{\epsilon'} \sum_{j=0}^m \|\mathbf{u}_j^h\|_{\mathcal{A}}^2 \\ &\quad + k\epsilon' h^{(d-1)\beta-1} \sum_{j=1}^m J_0^{\delta, \beta} \left(\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2} \right) \\ &\quad + \frac{2C^2 h^{(d-1)\beta-1}}{\epsilon''} J_0^{1, \beta}(\mathbf{u}_m^h, \mathbf{u}_m^h) + Ch^{(d-1)\beta/2-1/2} \|\mathbf{u}_0^h\|_{\mathcal{A}}^2. \end{aligned}$$

With these bounds on T_1 , T_2 and T_3 we can return to our earlier inequality, use Lemma 4.3, and obtain,

$$\begin{aligned}
 & (1 - \epsilon_L k) \|\rho^{\frac{1}{2}} \mathbf{z}_m^h\|_0^2 \\
 & + 2k \left(1 - \left(\frac{\epsilon'}{2} + (1 - \varphi_0) \tilde{\epsilon} \right) \hat{h}^{(d-1)\beta-1} \right) \sum_{j=1}^m J_0^{\delta, \beta} \left(\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2} \right) \\
 & \quad + \left(1 - \epsilon'_L - \epsilon_L k - \frac{2Ck}{\epsilon'} - \frac{\epsilon''}{2} \right) \|\mathbf{u}_m^h\|_{\mathcal{A}}^2 \\
 & - \left(\frac{1 - \varphi_0}{\bar{\epsilon}} |\mathbf{u}_m^h|_{\tilde{\mathcal{E}}}^2 + \hat{h}^{(d-1)\beta-1} \left(\frac{2C^2}{\epsilon''} + Ck \sum_{i=1}^{N_\varphi} \frac{\gamma_i^2}{\hat{\epsilon}_i} \right) J_0^{1, \beta}(\mathbf{u}_m^h, \mathbf{u}_m^h) \right) \\
 & \quad + 2k \sum_{i=1}^{N_\varphi} \left(1 - \frac{\hat{\epsilon}_i}{2} \right) \sum_{j=1}^m \left\| \mathbf{D}^{-\frac{1}{2}} \left(\frac{*\sigma_{ij}^h - *\sigma_{i,j-1}^h}{k} \right) \right\|_0^2 \\
 & \quad + \left(1 - \bar{\epsilon} - \frac{2Ck}{\tilde{\epsilon}} \right) \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-\frac{1}{2}} *\sigma_{im}^h\|_0^2 \\
 & \leq \|\rho^{\frac{1}{2}} \bar{\mathbf{z}}\|_0^2 + C_{\mathcal{A}}^2 (2 + C \hat{h}^{(d-1)\beta/2-1/2}) \|\bar{\mathbf{u}}\|_2^2 \\
 & \quad + C^2 h^{-1} \|\mathbf{g}(0)\|_{0, \Gamma_N}^2 + \frac{C^2 h^{-1}}{\epsilon'_L} \|\mathbf{g}(t_m)\|_{0, \Gamma_N}^2 \\
 & + \frac{C^2 h^{-1} k}{\epsilon_L} \sum_{j=1}^m \left\| \frac{\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})}{k} \right\|_{0, \Gamma_N}^2 + \frac{k}{\rho \epsilon_L} \sum_{j=1}^m \left\| \frac{\mathbf{f}(t_j) + \mathbf{f}(t_{j-1})}{2} \right\|_0^2 \\
 & + k \sum_{j=0}^{m-1} \left(\epsilon_L \|\mathbf{z}_j^h\|_0^2 + \left(\epsilon_L + \frac{2C}{\epsilon'} + \sum_{i=1}^{N_\varphi} \frac{C^2 \gamma_i^2 \hat{h}^{(d-1)\beta-1}}{\delta \hat{\epsilon}_i} \right) \|\mathbf{u}_j^h\|_{\mathcal{A}}^2 \right. \\
 & \quad \left. + \frac{2C}{\tilde{\epsilon}} \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-\frac{1}{2}} *\sigma_{ij}^h\|_0^2 \right).
 \end{aligned}$$

We now choose,

$$\begin{aligned}
 \epsilon_L &= \frac{1}{\varphi_0}, & \epsilon'_L &= \frac{\varphi_0}{8 - 4\varphi_0}, & \bar{\epsilon} &= 1 - \frac{\varphi_0}{2}, & \hat{\epsilon}_i &= 1, \\
 \epsilon'' &= \frac{\varphi_0/4}{1 - \varphi_0/2}, & \epsilon' &= 2\varphi_0, & \tilde{\epsilon} &= \frac{1}{2},
 \end{aligned}$$

and insist that,

$$\delta \geq \frac{4C^2 (2 - \varphi_0)^2 \hat{h}^{(d-1)\beta-1}}{(2 - 2\varphi_0)\varphi_0}.$$

Then, observing that,

$$\begin{aligned} & - \left(\frac{1 - \varphi_0}{\bar{\epsilon}} |\mathbf{u}_m^h|_{\mathcal{E}}^2 + \frac{2C^2 \hat{h}^{(d-1)\beta-1}}{\epsilon''} J_0^{1,\beta}(\mathbf{u}_m^h, \mathbf{u}_m^h) \right) \\ &= - \left(\frac{2 - 2\varphi_0}{2 - \varphi_0} |\mathbf{u}_m^h|_{\mathcal{E}}^2 + \frac{4C^2(2 - \varphi_0) \hat{h}^{(d-1)\beta-1}}{\delta\varphi_0} J_0^{\delta,\beta}(\mathbf{u}_m^h, \mathbf{u}_m^h) \right) \\ & \geq \frac{2\varphi_0 - 2}{2 - \varphi_0} \|\mathbf{u}_m^h\|_{\mathcal{A}}^2, \end{aligned}$$

our stability estimate can now be re-cast as,

$$\begin{aligned} & \left(1 - \frac{\hat{k}}{\varphi_0} \right) \|\rho^{\frac{1}{2}} \mathbf{z}_m^h\|_0^2 + \left(\frac{\varphi_0}{4 - 2\varphi_0} - C\hat{k} \right) \|\mathbf{u}_m^h\|_{\mathcal{A}}^2 \\ & + \left(\frac{\varphi_0}{2} - 4C\hat{k} \right) \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-\frac{1}{2}*} \boldsymbol{\sigma}_{im}^h\|_0^2 \\ & + (2 - (1 + \varphi_0) \hat{h}^{(d-1)\beta-1}) k \sum_{j=1}^m J_0^{\delta,\beta} \left(\frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2}, \frac{\mathbf{z}_j^h + \mathbf{z}_{j-1}^h}{2} \right) \\ & + k \sum_{i=1}^{N_\varphi} \sum_{j=1}^m \left\| \mathbf{D}^{-\frac{1}{2}} \left(\frac{*\boldsymbol{\sigma}_{ij}^h - *\boldsymbol{\sigma}_{i,j-1}^h}{k} \right) \right\|_0^2 \\ & \leq \|\rho^{\frac{1}{2}} \bar{\mathbf{z}}\|_0^2 + C \|\bar{\mathbf{u}}\|_2^2 + Ch^{-1} \|\mathbf{g}(0)\|_{0,\Gamma_N}^2 + Ch^{-1} \|\mathbf{g}(t_m)\|_{0,\Gamma_N}^2 \\ & + Ch^{-1} k \sum_{j=1}^m \left\| \frac{\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})}{k} \right\|_{0,\Gamma_N}^2 + Ck \sum_{j=1}^m \left\| \frac{\mathbf{f}(t_j) + \mathbf{f}(t_{j-1})}{2} \right\|_0^2 \\ & + Ck \sum_{j=0}^{m-1} \left(\|\mathbf{z}_j^h\|_0^2 + \|\mathbf{u}_j^h\|_{\mathcal{A}}^2 + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-\frac{1}{2}*} \boldsymbol{\sigma}_{ij}^h\|_0^2 \right). \end{aligned}$$

Lastly, we choose \hat{k} and \hat{h} small enough, and then an application of the discrete Gronwall lemma completes the proof. \square

Since uniqueness implies existence for linear finite dimensional problems, we can assert the existence and uniqueness of the discrete solution.

Theorem 4.5 (well-posedness) *Under the conditions of Theorem 4.4, the discrete solution exists and is unique.*

Our next goal is a fully discrete error estimate and the first step toward this is to derive an error equation. For this we set,

$$\begin{aligned} \boldsymbol{\chi}_j &:= \mathbf{u}_j^h - \check{\mathbf{u}}(t_j), & \boldsymbol{\psi}_j &:= \mathbf{z}_j^h - \check{\mathbf{z}}(t_j), & \boldsymbol{\eta}_{ij} &:= *\boldsymbol{\sigma}_{ij}^h - *\check{\boldsymbol{\sigma}}_i(t_j), \\ \boldsymbol{\xi}_j &:= \mathbf{u}(t_j) - \check{\mathbf{u}}(t_j), & \boldsymbol{\phi}_j &:= \mathbf{z}(t_j) - \check{\mathbf{z}}(t_j), & \boldsymbol{\theta}_{ij} &:= *\boldsymbol{\sigma}_i(t_j) - *\check{\boldsymbol{\sigma}}_i(t_j), \end{aligned}$$

where $\{\check{\mathbf{u}}(t), {}^*\check{\boldsymbol{\sigma}}_1(t), \dots\} \subset \mathbf{D}_r(\mathcal{E}_h)$ for each t and with $\mathbf{z} := \mathbf{u}_t$ and $\check{\mathbf{z}} := \check{\mathbf{u}}_t$.

We choose $\check{\mathbf{u}}(t) \in \mathbf{D}_r(\mathcal{E}_h)$ as the continuous interpolant of $\mathbf{u}(t)$ and ${}^*\check{\boldsymbol{\sigma}}_i$ as the $\mathbf{L}_2(\Omega)$ projection of ${}^*\boldsymbol{\sigma}_i$ into $\mathbf{L}_{r-1}(\mathcal{E}_h)$. We then have, $\check{\mathbf{z}}(t) = \check{\mathbf{u}}_t(t) \in \mathbf{D}_r(\mathcal{E}_h)$ and if $\mathbf{u}(t_j)$, $\mathbf{u}_t(t_j) \in C(\bar{\Omega})^d$ it follows that,

$$[\mathbf{u}(t_j)] = \mathbf{0}, \quad [\check{\mathbf{u}}(t_j)] = \mathbf{0}, \quad [\boldsymbol{\xi}_j] = \mathbf{0}, \quad (43)$$

and

$$[\mathbf{u}_t(t_j)] = \mathbf{0}, \quad [\check{\mathbf{u}}_t(t_j)] = \mathbf{0}, \quad [\boldsymbol{\phi}_j] = \mathbf{0}. \quad (44)$$

Moreover,

$$(\boldsymbol{\theta}_i, \mathbf{w}_i) = (\dot{\boldsymbol{\theta}}_i, \mathbf{w}_i) = (\ddot{\boldsymbol{\theta}}_i, \mathbf{w}_i) = \dots = 0 \quad \forall \mathbf{w}_i \in \mathbf{L}_{r-1}(\mathcal{E}_h), \quad (45)$$

and from standard arguments we also have,

$$\left\| \frac{\partial^n \check{\boldsymbol{\sigma}}_i}{\partial t^n} \right\|_0 \leq \left\| \frac{\partial^n \boldsymbol{\sigma}}{\partial t^n} \right\|_0.$$

Furthermore, from (13) we have,

$$\|\boldsymbol{\theta}_{ij}\|_0 = \|{}^*\boldsymbol{\sigma}_i(t_j) - {}^*\check{\boldsymbol{\sigma}}_i(t_j)\|_0 \leq Ch^r \|{}^*\boldsymbol{\sigma}_i(t_j)\|_r.$$

Now, motivated by the terms that arise below, define,

$$\Delta_j \mathbf{v} := \frac{\mathbf{v}_t(t_j) + \mathbf{v}_t(t_{j-1})}{2} - \frac{\mathbf{v}(t_j) - \mathbf{v}(t_{j-1}))}{k}.$$

Then by standard estimates for the trapezoidal quadrature rule and the Cauchy-Schwarz inequality we have the following result.

Lemma 4.6 *We have,*

$$\Delta_j \mathbf{v} = \frac{1}{2k} \int_{t_{j-1}}^{t_j} \mathbf{v}_{ttt}(t)(t_j - t)(t - t_{j-1}).$$

Moreover, if $\mathbf{v}_{ttt} \in L_2((t_{j-1}, t_j); \mathbf{L}_2(\Omega))$, then,

$$\|\Delta_j \mathbf{v}\|_0^2 \leq \frac{k^3}{4} \int_{t_{j-1}}^{t_j} \|\mathbf{v}_{ttt}(t)\|_0^2.$$

We will also make use of the following ‘summation by parts’ identity,

$$\sum_{j=1}^m (\boldsymbol{\psi}_j - \boldsymbol{\psi}_{j-1}, \mathbf{p}_j) = (\boldsymbol{\psi}_m, \mathbf{p}_m) - (\boldsymbol{\psi}_0, \mathbf{p}_1) + \sum_{j=1}^{m-1} (\boldsymbol{\psi}_j, \mathbf{p}_j - \mathbf{p}_{j+1}). \quad (46)$$

We will also need the following estimate which is proven by using Taylor’s series with integral remainder.

Lemma 4.7 *We have,*

$$\Delta_j \mathbf{v} - \Delta_{j+1} \mathbf{v} = k \left(\frac{\mathbf{v}(t_{j+1}) - 2\mathbf{v}(t_j) + \mathbf{v}(t_{j-1}))}{k^2} - \frac{\mathbf{v}_t(t_{j+1}) - \mathbf{v}_t(t_{j-1}))}{2k} \right).$$

Moreover, if, a.e. in Ω , we have $\mathbf{v}_{tttt} \in L_2(t_{j-1}, t_{j+1})$, then,

$$|\Delta_j \mathbf{v} - \Delta_{j+1} \mathbf{v}|^2 \leq Ck^5 \int_{t_{j-1}}^{t_{j+1}} |\mathbf{v}_{tttt}(t)|^2.$$

Averaging (17), (18) and (19) between t_j and t_{j-1} , and subtracting the result from the fully discrete scheme given by (35), (36) and (37) then gives three error equations,

$$\begin{aligned} & \left(\rho \frac{\psi_j - \psi_{j-1}}{k}, \mathbf{v} \right) + A \left(\frac{\chi_j + \chi_{j-1}}{2}, \mathbf{v} \right) + J_0^{\delta, \beta} \left(\frac{\psi_j + \psi_{j-1}}{2}, \mathbf{v} \right) \\ & \quad + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \gamma_i \left\{ \frac{\boldsymbol{\eta}_{ij} + \boldsymbol{\eta}_{i,j-1}}{2} \mathbf{n}_e \right\} \cdot [\mathbf{v}] \\ & \quad - \sum_{i=1}^{N_\varphi} \sum_E \int_E \gamma_i \frac{\boldsymbol{\eta}_{ij} + \boldsymbol{\eta}_{i,j-1}}{2} : \boldsymbol{\varepsilon}(\mathbf{v}) \\ & = (\rho \Delta_j \mathbf{z}, \mathbf{v}) + \left(\rho \frac{\phi_j - \phi_{j-1}}{k}, \mathbf{v} \right) + A \left(\frac{\xi_j + \xi_{j-1}}{2}, \mathbf{v} \right) \\ & + J_0^{\delta, \beta} \left(\frac{\phi_j + \phi_{j-1}}{2}, \mathbf{v} \right) + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \gamma_i \left\{ \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} \mathbf{n}_e \right\} \cdot [\mathbf{v}] \\ & \quad - \sum_{i=1}^{N_\varphi} \sum_E \int_E \gamma_i \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} : \boldsymbol{\varepsilon}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h), \end{aligned} \quad (47)$$

and,

$$\begin{aligned}
& \sum_{i=1}^{N_\varphi} \sum_E \left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} + \frac{1}{\tau_i} \frac{\boldsymbol{\eta}_{ij} + \boldsymbol{\eta}_{i,j-1}}{2}, \mathbf{w}_i \right)_E \\
& \quad - \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \left(\mathbf{D}\boldsymbol{\varepsilon} \left(\frac{\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1}}{2} \right), \mathbf{w}_i \right)_E \\
& \quad + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{ \mathbf{D}\mathbf{w}_i \mathbf{n}_e \} \cdot \left[\frac{\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1}}{2} \right] \\
= & \sum_{i=1}^{N_\varphi} \sum_E (\Delta_j^* \boldsymbol{\sigma}_i, \mathbf{w}_i)_E + \sum_{i=1}^{N_\varphi} \sum_E \left(\frac{\boldsymbol{\theta}_{ij} - \boldsymbol{\theta}_{i,j-1}}{k} + \frac{1}{\tau_i} \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2}, \mathbf{w}_i \right)_E \\
& \quad + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{ \mathbf{D}\mathbf{w}_i \mathbf{n}_e \} \cdot \left[\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right] \\
& \quad - \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \left(\mathbf{D}\boldsymbol{\varepsilon} \left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right), \mathbf{w}_i \right)_E \quad \forall \{ \mathbf{w}_i \} \in \{ \mathbf{L}_{r-1}(\mathcal{E}_h) \}, \quad (48)
\end{aligned}$$

and,

$$\begin{aligned}
& \left(\rho \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \mathbf{v} \right)_E - \left(\rho \frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k}, \mathbf{v} \right)_E = -(\rho \Delta_j \mathbf{u}, \mathbf{v})_E \\
& + \left(\rho \frac{\boldsymbol{\phi}_j + \boldsymbol{\phi}_{j-1}}{2}, \mathbf{v} \right)_E - \left(\rho \frac{\boldsymbol{\xi}_j - \boldsymbol{\xi}_{j-1}}{k}, \mathbf{v} \right)_E \quad \forall \mathbf{v} \in \mathbf{D}_r(\mathcal{E}_h). \quad (49)
\end{aligned}$$

Now, choosing $\mathbf{v} = (\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1})/k$ in (47), $\mathbf{w}_i = \mathbf{D}^{-1}(\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1})/k$ in (48) and $\mathbf{v} = (\boldsymbol{\psi}_j - \boldsymbol{\psi}_{j-1})/k$ in (49), adding the first two resulting equations together and noting from the third that,

$$\begin{aligned}
& \left(\rho \frac{\boldsymbol{\psi}_j - \boldsymbol{\psi}_{j-1}}{k}, \frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right) = \left(\rho \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j - \boldsymbol{\psi}_{j-1}}{k} \right) \\
& - \left(\rho \frac{\boldsymbol{\phi}_j + \boldsymbol{\phi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j - \boldsymbol{\psi}_{j-1}}{k} \right) + \left(\rho \frac{\boldsymbol{\xi}_j - \boldsymbol{\xi}_{j-1}}{k}, \frac{\boldsymbol{\psi}_j - \boldsymbol{\psi}_{j-1}}{k} \right) \\
& \quad + \left(\rho \Delta_j \mathbf{u}, \frac{\boldsymbol{\psi}_j - \boldsymbol{\psi}_{j-1}}{k} \right),
\end{aligned}$$

we multiply by $2k$ and sum over $j = 1, \dots, m$ to obtain,

$$\begin{aligned}
 & \|\rho^{1/2}\boldsymbol{\psi}_m\|_0^2 + \|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2 + 2k \sum_{j=1}^m J_0^{\delta,\beta} \left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right) \\
 & + 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \left\| \mathbf{D}^{-1/2} \left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_{im}\|_0^2 \\
 & = \|\rho^{1/2}\boldsymbol{\psi}_0\|_0^2 + \|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2 + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_{i0}\|_0^2 \\
 & + 2k \sum_{j=1}^m G_j \left(\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right) + 2k \sum_{j=1}^m H_j \left(\mathbf{D}^{-1} \frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \\
 & \quad + T_1 + T_2 + T_3 + T_4 + T_5 + T_6, \tag{50}
 \end{aligned}$$

where,

$$\begin{aligned}
 T_1 &:= -2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \left(\left\{ \frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \mathbf{n}_e \right\} \cdot \left[\frac{\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1}}{2} \right] \right. \\
 &\quad \left. + \left\{ \frac{\boldsymbol{\eta}_{ij} + \boldsymbol{\eta}_{i,j-1}}{2} \mathbf{n}_e \right\} \cdot \left[\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right] \right), \\
 T_2 &:= 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \int_E \left(\boldsymbol{\varepsilon} \left(\frac{\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1}}{2} \right) : \frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right. \\
 &\quad \left. + \boldsymbol{\varepsilon} \left(\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right) : \frac{\boldsymbol{\eta}_{ij} + \boldsymbol{\eta}_{i,j-1}}{2} \right), \\
 T_3 &:= 2k \sum_{j=1}^m \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \left(\left\{ \mathbf{D} \boldsymbol{\varepsilon} \left(\frac{\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1}}{2} \right) \mathbf{n}_e \right\} \cdot \left[\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right] \right. \\
 &\quad \left. - \kappa \left\{ \mathbf{D} \boldsymbol{\varepsilon} \left(\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right) \mathbf{n}_e \right\} \cdot \left[\frac{\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1}}{2} \right] \right), \\
 T_4 &:= -2k \sum_{j=1}^m J_0^{\delta, \beta} \left(\Delta_j \check{\mathbf{u}}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right), \\
 T_5 &:= -2k \sum_{j=1}^m \left(\rho \frac{\boldsymbol{\psi}_j - \boldsymbol{\psi}_{j-1}}{k}, \Delta_j \check{\mathbf{u}} \right), \\
 T_6 &:= 2k \sum_{j=1}^m \left(\rho \Delta_j \mathbf{z}, \frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right) \\
 &\quad + 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \sum_E \left(\Delta_j^* \boldsymbol{\sigma}_i, \mathbf{D}^{-1} \frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right)_E,
 \end{aligned}$$

along with,

$$\begin{aligned}
 G_j(\mathbf{v}) &:= J_0^{\delta, \beta} \left(\frac{\boldsymbol{\phi}_j + \boldsymbol{\phi}_{j-1}}{2}, \mathbf{v} \right) + \left(\rho \frac{\boldsymbol{\phi}_j - \boldsymbol{\phi}_{j-1}}{k}, \mathbf{v} \right) \\
 &\quad + A \left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2}, \mathbf{v} \right) - \sum_{i=1}^{N_\varphi} \sum_E \int_E \gamma_i \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} : \boldsymbol{\varepsilon}(\mathbf{v}) \\
 &\quad + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \left\{ \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} \mathbf{n}_e \right\} \cdot [\mathbf{v}],
 \end{aligned}$$

and,

$$\begin{aligned}
 H_j(\mathbf{v}) &:= \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\mathbf{D}v\mathbf{n}_e\} \cdot \left[\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right] \\
 &\quad - \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \left(\mathbf{D}\boldsymbol{\varepsilon} \left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right), \mathbf{v} \right)_E \\
 &\quad + \sum_{i=1}^{N_\varphi} \sum_E \left(\frac{\boldsymbol{\theta}_{ij} - \boldsymbol{\theta}_{i,j-1}}{k} + \frac{1}{\tau_i} \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2}, \mathbf{v} \right)_E.
 \end{aligned}$$

To get these we noted first that,

$$\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} - \frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} = -\Delta_j \check{\mathbf{u}}, \quad (51)$$

because $(\mathbf{z}_j^h + \mathbf{z}_{j-1}^h)/2 = (\mathbf{u}_j^h - \mathbf{u}_{j-1}^h)/k$, and secondly that,

$$\frac{\boldsymbol{\phi}_j + \boldsymbol{\phi}_{j-1}}{2} - \frac{\boldsymbol{\xi}_j - \boldsymbol{\xi}_{j-1}}{k} = \Delta_j \mathbf{u} - \Delta_j \check{\mathbf{u}}.$$

We now start estimating. These lemmas are all subject to the assumptions made later in Theorem 4.13.

Lemma 4.8 *We have,*

$$\begin{aligned}
 |T_4 + T_5 + T_6| &\leq 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \frac{\epsilon'_6}{2} \left\| \mathbf{D}^{-1/2} \left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right\|_0^2 \\
 &+ \frac{k}{\epsilon_6} \sum_{j=0}^m \|\rho^{1/2} \boldsymbol{\psi}_j\|_0^2 + \epsilon_5 \|\rho^{1/2} \boldsymbol{\psi}_m\|_0^2 + \|\rho^{1/2} \boldsymbol{\psi}_0\|_0^2 + k \sum_{j=1}^{m-1} \|\rho^{1/2} \boldsymbol{\psi}_j\|_0^2 \\
 &+ Ck^3 \int_0^{t_1} \|\check{\mathbf{u}}_{ttt}(t)\|_0^2 + \frac{Ck^3}{\epsilon_5} \int_{t_{m-1}}^{t_m} \|\check{\mathbf{u}}_{ttt}(t)\|_0^2 + Ck^4 \int_0^{t_m} \|\check{\mathbf{u}}_{tttt}(t)\|_0^2 \\
 &+ Ck^4 \int_0^{t_m} \left((\epsilon_6 + 1) \|\mathbf{z}_{ttt}(t)\|_0^2 + \|\check{\mathbf{u}}_{ttt}(t)\|_0^2 + \frac{1}{\epsilon'_6} \sum_{i=1}^{N_\varphi} \|\check{\boldsymbol{\sigma}}_i(t)\|_0^2 \right).
 \end{aligned}$$

Proof. Firstly, $T_4 = 0$ by (43) and (44). Now, using (46), we can write T_5 as,

$$T_5 = -2(\rho \boldsymbol{\psi}_m, \Delta_m \check{\mathbf{u}}) + 2(\rho \boldsymbol{\psi}_0, \Delta_1 \check{\mathbf{u}}) - 2 \sum_{j=1}^{m-1} (\rho \boldsymbol{\psi}_j, \Delta_j \check{\mathbf{u}} - \Delta_{j+1} \check{\mathbf{u}}).$$

From this we estimate with the Cauchy-Schwarz and Young's inequalities as follows,

$$\begin{aligned} |T_5| &\leq \epsilon_5 \|\rho^{1/2} \boldsymbol{\psi}_m\|_0^2 + \|\rho^{1/2} \boldsymbol{\psi}_0\|_0^2 + k \sum_{j=1}^{m-1} \|\rho^{1/2} \boldsymbol{\psi}_j\|_0^2 \\ &+ \frac{C(\rho)}{\epsilon_5} \|\Delta_m \ddot{\mathbf{u}}\|_0^2 + C(\rho) \|\Delta_1 \ddot{\mathbf{u}}\|_0^2 + \frac{C(\rho)}{k} \sum_{j=1}^{m-1} \|\Delta_j \ddot{\mathbf{u}} - \Delta_{j+1} \ddot{\mathbf{u}}\|_0^2. \end{aligned}$$

Using Lemma 4.6 results in,

$$\frac{C(\rho)}{\epsilon_5} \|\Delta_m \ddot{\mathbf{u}}\|_0^2 \leq \frac{Ck^3}{\epsilon_5} \int_{t_{m-1}}^{t_m} \|\ddot{\mathbf{u}}_{ttt}(t)\|_0^2,$$

and

$$C(\rho) \|\Delta_1 \ddot{\mathbf{u}}\|_0^2 \leq Ck^3 \int_0^{t_1} \|\ddot{\mathbf{u}}_{ttt}(t)\|_0^2,$$

while Lemma 4.7 leads to,

$$\frac{C(\rho)}{k} \|\Delta_j \ddot{\mathbf{u}} - \Delta_{j+1} \ddot{\mathbf{u}}\|_0^2 \leq Ck^4 \int_{t_{j-1}}^{t_{j+1}} \|\ddot{\mathbf{u}}_{tttt}(t)\|_0^2.$$

For T_6 we have by the Cauchy-Schwarz and Young's inequalities, Lemma 4.6 and the triangle inequality with (51), that,

$$\begin{aligned} |T_6| &\leq \frac{k}{\epsilon_6} \sum_{j=0}^m \|\rho^{1/2} \boldsymbol{\psi}_j\|_0^2 + 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \frac{\epsilon'_6}{2} \left\| \mathbf{D}^{-1/2} \left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right\|_0^2 \\ &+ Ck^4 \int_0^{t_m} \left((\epsilon_6 + 1) \|\mathbf{z}_{ttt}(t)\|_0^2 + \|\ddot{\mathbf{u}}_{ttt}(t)\|_0^2 + \frac{1}{\epsilon'_6} \sum_{i=1}^{N_\varphi} \|\dot{\boldsymbol{\sigma}}_i^*(t)\|_0^2 \right). \end{aligned}$$

These estimates complete the proof. \square

The terms T_1 , T_2 and T_3 can be handled in much the same way as in the proof of Theorem 4.4, although some modifications are necessary.

Lemma 4.9 *We have*

$$\begin{aligned}
 |T_1 + T_2 + T_3| &\leq 2Ch^{(d-1)\beta/2-1/2}\|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2 \\
 &+ 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \frac{\hat{\epsilon}_i}{2} \left\| \mathbf{D}^{-1/2} \left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right\|_0^2 \\
 &\quad + \frac{2Ck}{\check{\epsilon}} \sum_{j=0}^m \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_{ij}\|_0^2 \\
 &+ \sum_{i=1}^{N_\varphi} \frac{\bar{\epsilon}}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_{im}\|_0^2 + \frac{1-\varphi_0}{\bar{\epsilon}} |\boldsymbol{\chi}_m|_{\mathcal{E}}^2 + \frac{2C^2 h^{(d-1)\beta-1}}{\epsilon''} J_0^{1,\beta}(\boldsymbol{\chi}_m, \boldsymbol{\chi}_m) \\
 &+ Ck \sum_{j=0}^m \frac{1}{\epsilon'} \|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2 + \frac{\epsilon''}{2} \|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2 + Ck \sum_{j=0}^m \sum_{i=1}^{N_\varphi} \frac{\gamma_i^2 h^{(d-1)\beta-1}}{\hat{\epsilon}_i} J_0^{1,\beta}(\boldsymbol{\chi}_j, \boldsymbol{\chi}_j) \\
 &+ 2h^{(d-1)\beta-1} k (2\check{\epsilon}(1-\varphi_0) + \epsilon') \sum_{j=1}^m J_0^{\delta,\beta} \left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right).
 \end{aligned}$$

Proof. First, using (51), (43) and (44), we have,

$$J_0^{\delta,\beta} \left(\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k}, \frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right) = J_0^{\delta,\beta} \left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right).$$

With this we refer back to the proof of Theorem 4.4 and obtain,

$$\begin{aligned}
 |T_1| &\leq 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \frac{\hat{\epsilon}_i}{2} \left\| \mathbf{D}^{-1/2} \left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right\|_0^2 \\
 &\quad + Ck \sum_{j=0}^m \sum_{i=1}^{N_\varphi} \frac{\gamma_i^2 h^{(d-1)\beta-1}}{\hat{\epsilon}_i} J_0^{1,\beta}(\boldsymbol{\chi}_j, \boldsymbol{\chi}_j) \\
 &\quad + \frac{2Ck}{\check{\epsilon}} \sum_{j=0}^m \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_{ij}\|_0^2 \\
 &\quad + 2h^{(d-1)\beta-1} k \check{\epsilon} (1-\varphi_0) \sum_{j=1}^m J_0^{\delta,\beta} \left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right).
 \end{aligned}$$

Similarly, for T_3 we obtain,

$$\begin{aligned} |T_3| \leq & \frac{2Ck}{\epsilon'} \sum_{j=0}^m \|\chi_j\|_{\mathcal{A}}^2 + \frac{\epsilon''}{2} \|\chi_m\|_{\mathcal{A}}^2 + \frac{2C^2 h^{(d-1)\beta-1}}{\epsilon''} J_0^{1,\beta}(\chi_m, \chi_m) \\ & + 2Ch^{(d-1)\beta/2-1/2} \|\chi_0\|_{\mathcal{A}}^2 \\ & + 2k\epsilon' h^{(d-1)\beta-1} \sum_{j=1}^m J_0^{\delta,\beta} \left(\frac{\psi_j + \psi_{j-1}}{2}, \frac{\psi_j + \psi_{j-1}}{2} \right). \end{aligned}$$

Finally,

$$|T_2| \leq \sum_{i=1}^{N_\varphi} \frac{\bar{\epsilon}}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_{im}\|_0^2 + \frac{1-\varphi_0}{\bar{\epsilon}} |\chi_m|_{\mathcal{E}}^2,$$

and this completes the proof. \square

Lemma 4.10 *Assuming that $h \leq \hat{h}$, $\beta \geq (d-1)^{-1}$, $|e| \leq Ch^{d-1}$ and $\|\mathbf{v}\|_{0,e} \leq Ch^{-1/2} \|\mathbf{v}\|_{0,E}$ if e is an edge of E , we have,*

$$\begin{aligned} & \left| 2k \sum_{j=1}^m H_j \left(\mathbf{D}^{-1} \frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right| \\ \leq & 2k \sum_{j=1}^m \left(\sum_{i=1}^{N_\varphi} \epsilon'_H \left\| \mathbf{D}^{-1/2} \left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right\|_0^2 + \frac{C}{\epsilon'_H} \left| \frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right|_{\mathcal{E}}^2 \right). \end{aligned}$$

Proof. Using (45) we have,

$$H_j \left(\mathbf{D}^{-1/2} \frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) = T_1 + T_2$$

where,

$$T_1 := \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \left\{ \frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \mathbf{n}_e \right\} \cdot \left[\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right],$$

and

$$T_2 := - \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \left(\boldsymbol{\varepsilon} \left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right), \frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right)_E.$$

Now, $T_1 = 0$ due to (43), and for T_2 we have,

$$\begin{aligned} |T_2| \leq & \sum_{i=1}^{N_\varphi} \gamma_i \left\| \mathbf{D}^{1/2} \boldsymbol{\varepsilon} \left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right) \right\|_0 \left\| \mathbf{D}^{-1/2} \frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right\|_0, \\ \leq & \sum_{i=1}^{N_\varphi} \epsilon'_H \left\| \mathbf{D}^{-1/2} \frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{\gamma_i^2}{4\epsilon'_H} \left| \frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right|_{\mathcal{E}}^2. \end{aligned}$$

This completes the proof. \square

Lemma 4.11 *We have,*

$$\begin{aligned}
 & \left| 2k \sum_{j=1}^m G_j \left(\frac{\chi_j - \chi_{j-1}}{k} \right) \right| \leq 2k \sum_{j=1}^m \frac{1}{2} \|\rho^{1/2} \Delta_j \check{\mathbf{u}}\|_0^2 \\
 & + 2k \sum_{j=1}^m \frac{1 + \epsilon_G''}{2} \left\| \rho^{1/2} \frac{\phi_j - \phi_{j-1}}{k} \right\|_0^2 + 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} C \epsilon_G' \left\| \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} \right\|_0^2 \\
 & + \frac{k}{\epsilon_G'} \sum_{j=1}^m J_0^{\delta,\beta} \left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right) + 2k \sum_{j=0}^m \frac{1}{2\epsilon_G''} \|\rho^{1/2} \boldsymbol{\psi}_j\|_0^2 \\
 & + Ch^{2r} \left((1 + \epsilon_G''') \|\mathbf{u}\|_{L^\infty(0,t_m; \mathbf{H}^{r+1}(\Omega))}^2 + \|\mathbf{u}_t\|_{L^2(0,t_m; \mathbf{H}^{r+1}(\Omega))}^2 \right) \\
 & + \frac{1}{2} \|\chi_0\|_{\mathcal{A}}^2 + \frac{1}{2\epsilon_G'''} \|\chi_m\|_{\mathcal{A}}^2 + \frac{k}{2} \sum_{j=1}^{m-1} \|\chi_j\|_{\mathcal{A}}^2,
 \end{aligned}$$

for all positive ϵ_G' , ϵ_G'' and ϵ_G''' .

Proof. Using (45) we have,

$$\begin{aligned}
 G_j \left(\frac{\chi_j - \chi_{j-1}}{k} \right) &= J_0^{\delta,\beta} \left(\frac{\phi_j + \phi_{j-1}}{2}, \frac{\chi_j - \chi_{j-1}}{k} \right) \\
 &+ \left(\rho \frac{\phi_j - \phi_{j-1}}{k}, \frac{\chi_j - \chi_{j-1}}{k} \right) \\
 &+ \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \left\{ \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} \mathbf{n}_e \right\} \cdot \left[\frac{\chi_j - \chi_{j-1}}{k} \right] \\
 &+ A \left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2}, \frac{\chi_j - \chi_{j-1}}{k} \right) \\
 &= T_1 + T_2 + T_3 + T_4,
 \end{aligned}$$

but, firstly, $T_1 = 0$ by (44). Secondly, for T_2 ,

$$\begin{aligned}
 |T_2| &\leq \left| \left(\rho \frac{\phi_j - \phi_{j-1}}{k}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right) \right| + \left| \left(\rho \frac{\phi_j - \phi_{j-1}}{k}, \Delta_j \check{\mathbf{u}} \right) \right|, \\
 &\leq \left\| \rho^{1/2} \frac{\phi_j - \phi_{j-1}}{k} \right\|_0 \left(\left\| \rho^{1/2} \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right\|_0 + \|\rho^{1/2} \Delta_j \check{\mathbf{u}}\|_0 \right), \\
 &\leq \frac{\epsilon_G'' + 1}{2} \left\| \rho^{1/2} \frac{\phi_j - \phi_{j-1}}{k} \right\|_0^2 + \frac{1}{2} \|\rho^{1/2} \Delta_j \check{\mathbf{u}}\|_0^2 \\
 &+ \frac{1}{4\epsilon_G''} \left(\|\rho^{1/2} \boldsymbol{\psi}_j\|_0^2 + \|\rho^{1/2} \boldsymbol{\psi}_{j-1}\|_0^2 \right).
 \end{aligned}$$

Thirdly, for T_3 we use (51) and (43) and get,

$$\begin{aligned}
 |T_3| &\leq \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e \in \Gamma_h \cup \Gamma_D} \left\| \left\{ \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} \right\} \right\|_{0,e} \left\| \left[\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right] \right\|_{0,e}, \\
 &\leq \sum_{i=1}^{N_\varphi} \gamma_i \left(\sum_e \left(\frac{|e|^\beta}{\delta} \left\| \left\{ \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} \right\} \right\|_{0,e}^2 \right) \right)^{1/2} \\
 &\quad \times \left(\sum_e \left(\frac{\delta}{|e|^\beta} \left\| \left[\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right] \right\|_{0,e}^2 \right) \right)^{1/2}, \\
 &\leq \sum_{i=1}^{N_\varphi} C \gamma_i h^{(d-1)\beta/2-1/2} \left\| \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} \right\|_0 \\
 &\quad \times J_0^{\delta,\beta} \left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right)^{1/2}, \\
 &\leq \sum_{i=1}^{N_\varphi} C \epsilon'_G \left\| \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} \right\|_0^2 + \frac{1}{2\epsilon'_G} J_0^{\delta,\beta} \left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right).
 \end{aligned}$$

Turning to T_4 and noting (43) we have,

$$\begin{aligned}
 A \left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2}, \frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right) &= \sum_E \int_E \mathbf{D}\boldsymbol{\varepsilon} \left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right) : \boldsymbol{\varepsilon} \left(\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right) \\
 &\quad - \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \left\{ \mathbf{D}\boldsymbol{\varepsilon} \left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right) \mathbf{n}_e \right\} \cdot \left[\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right].
 \end{aligned}$$

Taking the sum over j as needed by the lemma we use a variant of (46) and get for the first term that,

$$\begin{aligned}
 &2k \sum_{j=1}^m \sum_E \int_E \mathbf{D}\boldsymbol{\varepsilon} \left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right) : \boldsymbol{\varepsilon} \left(\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right) \\
 &= - \sum_{j=1}^{m-1} \sum_E \int_E \mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_{j+1} - \boldsymbol{\xi}_{j-1}) : \boldsymbol{\varepsilon}(\boldsymbol{\chi}_j) \\
 &+ \sum_E \int_E \mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_m + \boldsymbol{\xi}_{m-1}) : \boldsymbol{\varepsilon}(\boldsymbol{\chi}_m) - \sum_E \int_E \mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_0) : \boldsymbol{\varepsilon}(\boldsymbol{\chi}_0).
 \end{aligned}$$

Hence,

$$\begin{aligned} & \left| \sum_E \int_E \mathbf{D}\varepsilon(\boldsymbol{\xi}_m + \boldsymbol{\xi}_{m-1}) : \varepsilon(\boldsymbol{\chi}_m) \right| \\ & \leq \epsilon_G''' C \|\boldsymbol{\xi}_m + \boldsymbol{\xi}_{m-1}\|_{1,\Omega}^2 + \frac{1}{2\epsilon_G'''} |\boldsymbol{\chi}_m|_{\mathcal{E}}^2, \\ & \leq \epsilon_G''' C h^{2r} \|\mathbf{u}\|_{L^\infty(0,t_m; \mathbf{H}^{r+1}(\Omega))}^2 + \frac{1}{2\epsilon_G'''} |\boldsymbol{\chi}_m|_{\mathcal{E}}^2, \end{aligned}$$

because,

$$\|\boldsymbol{\xi}_m + \boldsymbol{\xi}_{m-1}\|_{1,\Omega} \leq Ch^r \|\mathbf{u}(t_m) + \mathbf{u}(t_{m-1})\|_{r+1}.$$

Similarly,

$$\left| \sum_E \int_E \mathbf{D}\varepsilon(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_0) : \varepsilon(\boldsymbol{\chi}_0) \right| \leq Ch^{2r} \|\mathbf{u}\|_{L^\infty(0,t_m; \mathbf{H}^{r+1}(\Omega))}^2 + \frac{1}{2} |\boldsymbol{\chi}_0|_{\mathcal{E}}^2.$$

And,

$$\begin{aligned} & \left| \sum_{j=1}^{m-1} \sum_E \int_E \mathbf{D}\varepsilon(\boldsymbol{\xi}_{j+1} - \boldsymbol{\xi}_{j-1}) : \varepsilon(\boldsymbol{\chi}_j) \right| \leq Ck \sum_{j=1}^{m-1} \left\| \frac{\boldsymbol{\xi}_{j+1} - \boldsymbol{\xi}_{j-1}}{k} \right\|_1 |\boldsymbol{\chi}_j|_{\mathcal{E}}, \\ & \leq Ch^{2r} \|\mathbf{u}_t\|_{L_2(0,t_m; \mathbf{H}^{r+1}(\Omega))}^2 + \frac{k}{2} \sum_{j=1}^{m-1} |\boldsymbol{\chi}_j|_{\mathcal{E}}^2, \end{aligned}$$

where we used,

$$\begin{aligned} \left\| \frac{\boldsymbol{\xi}_{j+1} - \boldsymbol{\xi}_{j-1}}{k} \right\|_1 & = \left\| \frac{\mathbf{u}(t_{j+1}) - \mathbf{u}(t_{j-1})}{k} - \frac{\check{\mathbf{u}}(t_{j+1}) - \check{\mathbf{u}}(t_{j-1})}{k} \right\|_1, \\ & \leq Ch^r \left\| \frac{\mathbf{u}(t_{j+1}) - \mathbf{u}(t_{j-1})}{k} \right\|_{r+1}, \end{aligned}$$

and (by the fundamental theorem of calculus),

$$k \sum_{j=1}^{m-1} \left\| \frac{\mathbf{u}(t_{j+1}) - \mathbf{u}(t_{j-1})}{k} \right\|_{r+1} \leq C \|\mathbf{u}_t\|_{L_2(0,t_m; \mathbf{H}^{r+1}(\Omega))}.$$

For the second term in T_4 we proceed similarly:

$$\begin{aligned}
 & -2k \sum_{j=1}^m \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \left\{ \mathbf{D}\varepsilon \left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right) \mathbf{n}_e \right\} \cdot \left[\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right] \\
 & = - \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{ \mathbf{D}\varepsilon(\boldsymbol{\xi}_m + \boldsymbol{\xi}_{m-1}) \mathbf{n}_e \} \cdot [\boldsymbol{\chi}_m] \\
 & \quad + \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{ \mathbf{D}\varepsilon(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_0) \mathbf{n}_e \} \cdot [\boldsymbol{\chi}_0] \\
 & \quad + \sum_{j=1}^{m-1} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{ \mathbf{D}\varepsilon(\boldsymbol{\xi}_{j+1} - \boldsymbol{\xi}_{j-1}) \mathbf{n}_e \} \cdot [\boldsymbol{\chi}_j].
 \end{aligned}$$

Now,

$$\begin{aligned}
 & \left| \sum_e \int_e \{ \mathbf{D}\varepsilon(\boldsymbol{\xi}_m + \boldsymbol{\xi}_{m-1}) \mathbf{n}_e \} \cdot [\boldsymbol{\chi}_m] \right| \\
 & \leq \frac{\epsilon_G''' Ch^{(d-1)\beta}}{2\delta} \sum_e \|\mathbf{D}\varepsilon(\boldsymbol{\xi}_m + \boldsymbol{\xi}_{m-1}) \mathbf{n}_e\|_{0,e}^2 + \frac{1}{2\epsilon_G'''} J_0^{\delta,\beta}(\boldsymbol{\chi}_m, \boldsymbol{\chi}_m), \\
 & \leq \frac{\epsilon_G''' Ch^{2r}}{2\delta} \|\mathbf{u}(t_m) + \mathbf{u}(t_{m-1})\|_{r+1}^2 + \frac{1}{2\epsilon_G'''} J_0^{\delta,\beta}(\boldsymbol{\chi}_m, \boldsymbol{\chi}_m),
 \end{aligned}$$

since $(d-1)\beta - 1 \geq 0$ and,

$$\|\mathbf{D}\varepsilon(\boldsymbol{\xi}_m + \boldsymbol{\xi}_{m-1}) \mathbf{n}_e\|_{0,e} \leq Ch^{r-1/2} \|\mathbf{u}(t_m) + \mathbf{u}(t_{m-1})\|_{r+1}.$$

Similarly,

$$\begin{aligned}
 & \left| \sum_e \int_e \{ \mathbf{D}\varepsilon(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_0) \mathbf{n}_e \} \cdot [\boldsymbol{\chi}_0] \right| \\
 & \leq \frac{Ch^{2r}}{2\delta} \|\mathbf{u}(t_1) + \mathbf{u}(t_0)\|_{r+1}^2 + \frac{1}{2} J_0^{\delta,\beta}(\boldsymbol{\chi}_0, \boldsymbol{\chi}_0).
 \end{aligned}$$

We also have,

$$\begin{aligned}
 & \left| \sum_{j=1}^{m-1} \sum_e \int_e \{ \mathbf{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_{j+1} - \boldsymbol{\xi}_{j-1}) \mathbf{n}_e \} \cdot [\boldsymbol{\chi}_j] \right| \\
 & \leq k \sum_{j=1}^{m-1} \sum_e \left\| \mathbf{D}\boldsymbol{\varepsilon} \left(\frac{\mathbf{u}(t_{j+1}) - \mathbf{u}(t_{j-1})}{k} - \frac{\check{\mathbf{u}}(t_{j+1}) - \check{\mathbf{u}}(t_{j-1})}{k} \right) \mathbf{n}_e \right\|_{0,e} \\
 & \quad \times \| [\boldsymbol{\chi}_j] \|_{0,e}, \\
 & \leq k \sum_{j=1}^{m-1} \frac{Ch^{2r-1+(d-1)\beta}}{2\delta} \left\| \frac{\mathbf{u}(t_{j+1}) - \mathbf{u}(t_{j-1})}{k} \right\|_{r+1}^2 + \frac{k}{2} \sum_{j=1}^{m-1} J_0^{\delta,\beta}(\boldsymbol{\chi}_j, \boldsymbol{\chi}_j), \\
 & \leq \frac{Ch^{2r}}{\delta} \|\mathbf{u}_t\|_{L_2(0,t_m;\mathbf{H}^{r+1}(\Omega))}^2 + \frac{k}{2} \sum_{j=1}^{m-1} J_0^{\delta,\beta}(\boldsymbol{\chi}_j, \boldsymbol{\chi}_j),
 \end{aligned}$$

where we used,

$$\begin{aligned}
 & \left\| \mathbf{D}\boldsymbol{\varepsilon} \left(\frac{\mathbf{u}(t_{j+1}) - \mathbf{u}(t_{j-1})}{k} - \frac{\check{\mathbf{u}}(t_{j+1}) - \check{\mathbf{u}}(t_{j-1})}{k} \right) \mathbf{n}_e \right\|_{0,e} \\
 & \leq Ch^{r-1/2} \left\| \frac{\mathbf{u}(t_{j+1}) - \mathbf{u}(t_{j-1})}{k} \right\|_{r+1,E}.
 \end{aligned}$$

Assembling these estimates then yields,

$$\begin{aligned}
 \left\| 2k \sum_{j=1}^m T_4 \right\| & \leq Ch^{2r} \left((1 + \epsilon_G'') \|\mathbf{u}\|_{L_\infty(0,t_m;\mathbf{H}^{r+1}(\Omega))}^2 + \|\mathbf{u}_t\|_{L_2(0,t_m;\mathbf{H}^{r+1}(\Omega))}^2 \right) \\
 & \quad + \frac{1}{2} \|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2 + \frac{1}{2\epsilon_G'''} \|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2 + \frac{k}{2} \sum_{j=1}^{m-1} \|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2,
 \end{aligned}$$

which completes the proof. \square

Before stating the error estimate we need one more estimate—connected with a term in the previous lemma.

Lemma 4.12 *We have,*

$$\begin{aligned}
 & 2k \sum_{j=1}^m \frac{1 + \epsilon_G''}{2} \left\| \rho^{1/2} \frac{\phi_j - \phi_{j-1}}{k} \right\|_0^2 \\
 & \leq Ch^{2r} \|\mathbf{u}_t\|_{L_\infty(0,t_m;\mathbf{H}^{r+1}(\Omega))}^2 + C(1 + \epsilon_G'') k^4 \|\mathbf{u}_{tttt}\|_{L_2(0,t_m;\mathbf{L}_2(\Omega))}^2.
 \end{aligned}$$

Proof. By the triangle and Young's inequalities we have,

$$\begin{aligned} 2k \sum_{j=1}^m \frac{1 + \epsilon_G''}{2} \left\| \rho^{1/2} \frac{\phi_j - \phi_{j-1}}{k} \right\|_0^2 &\leq 2k \sum_{j=1}^m (1 + \epsilon_G'') \|\rho^{1/2} \phi_t(t_{j-1/2})\|_0^2 \\ &\quad + 2k \sum_{j=1}^m (1 + \epsilon_G'') \left\| \rho^{1/2} \left(\phi_t(t_{j-1/2}) - \frac{\phi_j - \phi_{j-1}}{k} \right) \right\|_0^2, \\ &\leq Ct_m h^{2r} \|\mathbf{u}_t\|_{L^\infty(0, t_m; \mathbf{H}^{r+1}(\Omega))}^2 + Ck^4 (1 + \epsilon_G'') \|\phi_{ttt}\|_{L_2(0, t_m; \mathbf{L}_2(\Omega))}^2. \end{aligned}$$

Now use $\|\phi_{ttt}\|_{L_2(0, t_m; \mathbf{L}_2(\Omega))} \leq C \|\mathbf{u}_{tttt}\|_{L_2(0, t_m; \mathbf{L}_2(\Omega))}$. \square

Now we can give the error estimate.

Theorem 4.13 (fully discrete ‘energy’ error estimate) *Assume that we have $h \leq \hat{h}$, $k \leq \hat{k}$, $\beta \geq (d-1)^{-1}$, $\bar{\mathbf{u}} \in \mathbf{H}^{r+1}(\Omega)$, $\bar{\mathbf{z}} \in \mathbf{H}^r(\Omega)$ and $\mathbf{u} \in H^4(\mathbf{L}_2) \cap H^2(\mathbf{H}^1) \cap W_\infty^1(\mathbf{H}^{r+1}) \cap C^1(C(\bar{\Omega})^d)$. Then, for \hat{h} and \hat{k} small enough, and δ large enough,*

$$\begin{aligned} \|\rho^{1/2}(\mathbf{u}_t(t_m) - \mathbf{z}_m^h)\|_0 + \|\mathbf{u}(t_m) - \mathbf{u}_m^h\|_{\mathcal{A}} \\ + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2}(*\sigma_i(t_m) - *\sigma_{im}^h)\|_0 \leq C(h^r + k^2), \end{aligned}$$

where C is a ‘Gronwall’ constant independent of h and k .

Proof. We start with (50) and invoke Lemmas 4.8, 4.9, 4.10, 4.11 and 4.12.

These results collect up as follows:

$$\begin{aligned} \|\rho^{1/2} \boldsymbol{\psi}_m\|_0^2 + \|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2 + 2k \sum_{j=1}^m J_0^{\delta, \beta} \left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right) \\ + 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \left\| \mathbf{D}^{-1/2} \left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_{im}\|_0^2 \\ \leq \|\rho^{1/2} \boldsymbol{\psi}_0\|_0^2 + \|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2 + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_{i0}\|_0^2 \end{aligned}$$

$(T_1 + T_2 + T_3)$

$$\begin{aligned}
 & + 2Ch^{(d-1)\beta/2-1/2}\|\chi_0\|_{\mathcal{A}}^2 \\
 & + 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \frac{\hat{\epsilon}_i}{2} \left\| \mathbf{D}^{-1/2} \left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right\|_0^2 \\
 & + \frac{2Ck}{\bar{\epsilon}} \sum_{j=0}^m \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_{ij}\|_0^2 \\
 & + \sum_{i=1}^{N_\varphi} \frac{\bar{\epsilon}}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_{im}\|_0^2 + \frac{1-\varphi_0}{\bar{\epsilon}} |\chi_m|_{\mathcal{E}}^2 + \frac{2C^2 h^{(d-1)\beta-1}}{\epsilon''} J_0^{1,\beta}(\chi_m, \chi_m) \\
 & + Ck \sum_{j=0}^m \frac{1}{\epsilon'} \|\chi_j\|_{\mathcal{A}}^2 + \frac{\epsilon''}{2} \|\chi_m\|_{\mathcal{A}}^2 + Ck \sum_{j=0}^m \sum_{i=1}^{N_\varphi} \frac{\gamma_i^2 \hat{h}^{(d-1)\beta-1}}{\hat{\epsilon}_i} J_0^{1,\beta}(\chi_j, \chi_j) \\
 & + 2h^{(d-1)\beta-1} k (2\bar{\epsilon}(1-\varphi_0) + \epsilon') \sum_{j=1}^m J_0^{\delta,\beta} \left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right)
 \end{aligned}$$

$(T_4 + T_5 + T_6)$

$$\begin{aligned}
 & + \frac{k}{\epsilon_6} \sum_{j=0}^m \|\rho^{1/2} \boldsymbol{\psi}_j\|_0^2 + 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \frac{\epsilon'_6}{2} \left\| \mathbf{D}^{-1/2} \left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right\|_0^2 \\
 & + \epsilon_5 \|\rho^{1/2} \boldsymbol{\psi}_m\|_0^2 + \|\rho^{1/2} \boldsymbol{\psi}_0\|_0^2 + k \sum_{j=1}^{m-1} \|\rho^{1/2} \boldsymbol{\psi}_j\|_0^2 \\
 & + Ck^3 \int_0^{t_1} \|\check{\mathbf{u}}_{ttt}(t)\|_0^2 dt + \frac{Ck^3}{\epsilon_5} \int_{t_{m-1}}^{t_m} \|\check{\mathbf{u}}_{ttt}(t)\|_0^2 dt + Ck^4 \int_0^{t_m} \|\check{\mathbf{u}}_{tttt}(t)\|_0^2 dt \\
 & + Ck^4 \int_0^{t_m} (1 + \epsilon_6) \|\mathbf{z}_{ttt}(t)\|_0^2 + \|\check{\mathbf{u}}_{ttt}(t)\|_0^2 + \frac{1}{\epsilon'_6} \sum_{i=1}^{N_\varphi} \|\check{\boldsymbol{\sigma}}_i^*(t)\|_0^2 dt
 \end{aligned}$$

(G_j)

$$\begin{aligned}
& + 2k \sum_{j=1}^m \frac{1 + \epsilon_G''}{2} \left\| \rho^{1/2} \frac{\phi_j - \phi_{j-1}}{k} \right\|_0^2 + 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} C \epsilon_G' \left\| \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} \right\|_0^2 \\
& + \frac{k}{\epsilon_G'} \sum_{j=1}^m J_0^{\delta, \beta} \left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right) + 2k \sum_{j=0}^m \frac{1}{2\epsilon_G''} \|\rho^{1/2} \boldsymbol{\psi}_j\|_0^2 \\
& + Ch^{2r} \left((1 + \epsilon_G''') \|\mathbf{u}\|_{L^\infty(0, t_m; \mathbf{H}^{r+1}(\Omega))}^2 + \|\mathbf{u}_t\|_{L_2(0, t_m; \mathbf{H}^{r+1}(\Omega))}^2 \right) \\
& + \frac{1}{2} \|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2 + \frac{1}{2\epsilon_G'''} \|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2 + \frac{k}{2} \sum_{j=1}^{m-1} \|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2 + 2k \sum_{j=1}^m \frac{1}{2} \|\rho^{1/2} \Delta_j \check{\mathbf{u}}\|_0^2
\end{aligned}$$

(H_j)

$$+ 2k \sum_{j=1}^m \left(\sum_{i=1}^{N_\varphi} \epsilon_H' \left\| \mathbf{D}^{-1/2} \left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right\|_0^2 + \frac{C}{\epsilon_H'} \left| \frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right|_{\mathcal{E}}^2 \right).$$

Rearranging this gives,

$$\begin{aligned}
& \left(1 - \epsilon_5 - \frac{k}{\epsilon_G''} - \frac{k}{\epsilon_6} \right) \|\rho^{1/2} \boldsymbol{\psi}_m\|_0^2 \\
& + \left(1 - \frac{\epsilon''}{2} - \frac{1}{2\epsilon_G'''} - \frac{Ck}{\epsilon'} \right) \|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2 \\
& + 2k \left(1 - \frac{1}{2\epsilon_G'} - \hat{h}^{(d-1)\beta-1} (2\check{\epsilon}(1 - \varphi_0) + \epsilon') \right) \\
& \quad \times \sum_{j=1}^m J_0^{\delta, \beta} \left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right) \\
& + 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \left(1 - \frac{\hat{\epsilon}_i}{2} - \epsilon_H' - \frac{\epsilon_6'}{2} \right) \left\| \mathbf{D}^{-1/2} \left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right\|_0^2 \\
& \quad + \left(1 - \bar{\epsilon} - \frac{2Ck}{\check{\epsilon}} \right) \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_{im}\|_0^2 \\
& - \left(\frac{1 - \varphi_0}{\bar{\epsilon}} |\boldsymbol{\chi}_m|_{\mathcal{E}}^2 + \hat{h}^{(d-1)\beta-1} \left(\frac{2C^2}{\epsilon''} + Ck \sum_{i=1}^{N_\varphi} \frac{\gamma_i^2}{\hat{\epsilon}_i} \right) J_0^{1, \beta}(\boldsymbol{\chi}_m, \boldsymbol{\chi}_m) \right) \\
& \leq T_1 + T_2 + T_3 + T_4 + T_5,
\end{aligned}$$

where,

$$\begin{aligned}
 T_1 &:= 2\|\rho^{1/2}\boldsymbol{\psi}_0\|_0^2 + \left(\frac{3}{2} + 2C\hat{h}^{\frac{(d-1)\beta-1}{2}}\right)\|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2 + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2}\boldsymbol{\eta}_{i0}\|_0^2, \\
 T_2 &:= Ck^4 \int_0^{t_m} \|\check{\mathbf{u}}_{tttt}(t)\|_0^2 + 2k \sum_{j=1}^m \frac{1}{2} \|\rho^{1/2}\Delta_j \check{\mathbf{u}}\|_0^2, \\
 &\quad + Ck^3 \int_0^{t_1} \|\check{\mathbf{u}}_{ttt}(t)\|_0^2 + \frac{Ck^3}{\epsilon_5} \int_{t_{m-1}}^{t_m} \|\check{\mathbf{u}}_{ttt}(t)\|_0^2 \\
 &\quad + Ck^4 \int_0^{t_m} \left((1 + \epsilon_6) \|\mathbf{z}_{ttt}(t)\|_0^2 + \|\check{\mathbf{u}}_{ttt}(t)\|_0^2 + \frac{1}{\epsilon'_6} \sum_{i=1}^{N_\varphi} \|\mathbf{*\ddot{\sigma}}_i(t)\|_0^2 \right) \\
 T_3 &:= 2k \sum_{j=1}^m \frac{1 + \epsilon''_G}{2} \left\| \rho^{1/2} \frac{\phi_j - \phi_{j-1}}{k} \right\|_0^2, \\
 T_4 &:= 2k \sum_{j=1}^m \frac{C}{\epsilon'_H} \left| \frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right|_{\mathcal{E}}^2 + 2k \sum_{j=1}^m \sum_{i=1}^{N_\varphi} C\epsilon'_G \left\| \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} \right\|_0^2 \\
 &\quad + Ch^{2r} \left((1 + \epsilon'''_G) \|\mathbf{u}\|_{L^\infty(0,t_m;\mathbf{H}^{r+1}(\Omega))}^2 + \|\mathbf{u}_t\|_{L_2(0,t_m;\mathbf{H}^{r+1}(\Omega))}^2 \right), \\
 T_5 &:= \frac{2Ck}{\bar{\epsilon}} \sum_{j=0}^{m-1} \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2}\boldsymbol{\eta}_{ij}\|_0^2 + Ck \sum_{j=0}^{m-1} \left(\frac{1}{2} + \frac{1}{\epsilon'} \right) \|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2 \\
 &\quad + Ck \sum_{j=0}^{m-1} \sum_{i=1}^{N_\varphi} \frac{\gamma_i^2 \hat{h}^{(d-1)\beta-1}}{\delta \hat{\epsilon}_i} J_0^{\delta,\beta}(\boldsymbol{\chi}_j, \boldsymbol{\chi}_j) \\
 &\quad + k \sum_{j=0}^{m-1} \left(1 + \frac{1}{\epsilon''_G} + \frac{1}{\epsilon_6} \right) \|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2.
 \end{aligned}$$

We now choose,

$$\begin{aligned}
 \epsilon'' &= \frac{\varphi_0/4}{1 - \varphi_0/2}, & \epsilon' &= \varphi_0, & \hat{\epsilon}_i &= 1, & \check{\epsilon} &= \frac{1}{4}, & \epsilon'''_G &= \frac{2 - \varphi_0}{\varphi_0}, \\
 \bar{\epsilon} &= 1 - \frac{\varphi_0}{2}, & \epsilon_5 &= \frac{1}{2}, & \epsilon''_G &= \epsilon_6 = \frac{2}{C} & \text{for some } C > 0, \\
 \epsilon'_G &= \frac{1}{1 - \varphi_0}, & \epsilon'_H &= \frac{1}{6}, & \epsilon'_6 &= \frac{1}{3},
 \end{aligned}$$

and insist that,

$$\delta \geq \frac{4C^2(2 - \varphi_0)^2 \hat{h}^{(d-1)\beta-1}}{(2 - 2\varphi_0)\varphi_0}. \quad (52)$$

These lead to,

$$\begin{aligned}
 & \left(\frac{1}{2} - C\hat{k} \right) \|\rho^{1/2}\boldsymbol{\psi}_m\|_0^2 + \left(\frac{\varphi_0}{8-4\varphi_0} - C\hat{k} \right) \|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2 \\
 & + k(1+\varphi_0)(1-\hat{h}^{(d-1)\beta-1}) \sum_{j=1}^m J_0^{\delta,\beta} \left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right) \\
 & + \frac{k}{3} \sum_{j=1}^m \sum_{i=1}^{N_\varphi} \left\| \mathbf{D}^{-1/2} \left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right\|_0^2 \\
 & + \left(\frac{\varphi_0}{2} - C\hat{k} \right) \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_{im}\|_0^2 \\
 & \leq T_1 + T_2 + T_3 + T_4 + T_5.
 \end{aligned}$$

Now, for T_2 , using Lemmas 2.1, 4.7 and stability properties of the interpolant,

$$\begin{aligned}
 |T_2| \leq Ck^4 & \left(\|\mathbf{u}_{ttt}\|_{L_\infty(0,t_m;\mathbf{L}_2(\Omega))} + \|\mathbf{u}\|_{H^2(0,t_m;\mathbf{H}^1(\Omega))} \right) \\
 & + \|\mathbf{u}_{ttt}\|_{L_2(0,t_m;\mathbf{L}_2(\Omega))} + \|\mathbf{u}_{tttt}\|_{L_2(0,t_m;\mathbf{L}_2(\Omega))} \Big)^2.
 \end{aligned}$$

For T_3 , using Lemma 4.12,

$$|T_3| \leq Ch^{2r} \|\mathbf{u}_t\|_{L_\infty(0,t_m;\mathbf{H}^{r+1}(\Omega))}^2 + Ck^4 \|\mathbf{u}_{tttt}\|_{L_2(0,t_m;\mathbf{L}_2(\Omega))}^2.$$

For T_4 ,

$$\begin{aligned}
 |T_4| \leq Ch^{2r} & \left(\|\mathbf{u}\|_{L_\infty(0,t_m;\mathbf{H}^{r+1}(\Omega))} + \|\mathbf{u}_t\|_{L_2(0,t_m;\mathbf{H}^{r+1}(\Omega))} \right. \\
 & \left. + \max_{1 \leq i \leq N_\varphi} \|\boldsymbol{\sigma}_i^*\|_{L_\infty(0,t_m;\mathbf{H}^r(\Omega))} \right)^2,
 \end{aligned}$$

and use Lemma 2.2, and, for T_5 ,

$$|T_5| \leq Ck \sum_{j=0}^{m-1} \left(\|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2 + \|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2 + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_{ij}\|_0^2 \right).$$

Now select \hat{h} and \hat{k} small enough, use the initial conditions $\boldsymbol{\sigma}_i^*(0) = \mathbf{0}$ and apply the discrete Gronwall lemma to get,

$$\begin{aligned}
 & \|\rho^{1/2}\boldsymbol{\psi}_m\|_0 + \|\boldsymbol{\chi}_m\|_{\mathcal{A}} + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2} \boldsymbol{\eta}_{im}\|_0 \\
 & \leq C(h^r + k^2) + C\|\rho^{1/2}\boldsymbol{\psi}_0\|_0 + C\|\boldsymbol{\chi}_0\|_{\mathcal{A}}.
 \end{aligned}$$

Now, by (40) and (13), we have,

$$\begin{aligned} \|\rho^{\frac{1}{2}}\psi_0\|_0 &= \|\rho^{\frac{1}{2}}(\mathbf{z}_0^h - \check{\mathbf{z}}(0))\|_0, \\ &\leq \|\rho^{\frac{1}{2}}(\mathbf{z}_0^h - \bar{\mathbf{z}})\|_0 + \|\rho^{\frac{1}{2}}(\check{\mathbf{z}}(0) - \bar{\mathbf{z}})\|_0, \\ &\leq 2\|\rho^{\frac{1}{2}}(\check{\mathbf{z}}(0) - \bar{\mathbf{z}})\|_0 \leq Ch^r \|\bar{\mathbf{z}}\|_r. \end{aligned}$$

Also, using standard results for the elliptic projection (see e.g. [12]) we have,

$$\begin{aligned} \|\chi_0\|_{\mathcal{A}} &= \|\mathbf{u}_0^h - \check{\mathbf{u}}(0)\|_{\mathcal{A}}, \\ &\leq \|\bar{\mathbf{u}} - \mathbf{u}_0^h\|_{\mathcal{A}} + \|\bar{\mathbf{u}} - \check{\mathbf{u}}(0)\|_{\mathcal{A}}, \\ &\leq \|\bar{\mathbf{u}} - \mathbf{u}_0^h\|_{\mathcal{A}} + |\bar{\mathbf{u}} - \check{\mathbf{u}}(0)|_{\mathcal{E}} \leq Ch^r \|\bar{\mathbf{u}}\|_{r+1}. \end{aligned}$$

Using the triangle inequality we now obtain,

$$\begin{aligned} &\|\rho^{1/2}(\mathbf{u}_t(t_m) - \mathbf{z}_m^h)\|_0 + \|\mathbf{u}(t_m) - \mathbf{u}_m^h\|_{\mathcal{A}} \\ &+ \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2}(*\boldsymbol{\sigma}_i(t_m) - *\boldsymbol{\sigma}_{im}^h)\|_0 \\ &\leq \|\rho^{1/2}\boldsymbol{\xi}_t(t_m)\|_0 + \|\boldsymbol{\xi}(t_m)\|_{\mathcal{A}} + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2}\boldsymbol{\theta}_i(t_m)\|_0 \\ &\quad + \|\rho^{1/2}\psi_m\|_0 + \|\chi_m\|_{\mathcal{A}} + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\mathbf{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0, \\ &\leq Ch^r \|\mathbf{u}_t(t_m)\|_r + Ch^r \|\mathbf{u}(t_m)\|_{r+1} \\ &\quad + Ch^r \max_{1 \leq i \leq N_\varphi} \|\boldsymbol{\sigma}_i(t_m)\|_r + C(h^r + k^2). \end{aligned}$$

To get this we noted that, $\|\boldsymbol{\xi}(t_j)\|_{\mathcal{A}} = |\boldsymbol{\xi}(t_j)|_{\mathcal{E}} \leq Ch^r \|\mathbf{u}(t_j)\|_{r+1}$ for $j = 0, 1, \dots, m$. Finally, using Lemma 2.2 completes the proof. \square

5 Conclusion

In this article we have extended the application of the DG FEM to dynamic linear viscoelasticity problems. This builds upon the algorithm and estimates in [12] in that we have now included the inertia term. It also varies the approach in [12] in that here we have chosen to represent the viscoelastic history through evolution equations for internal stress tensors, rather than augment the momentum equation with a Volterra (hereditary) integral.

This article with [12] represent the extension of DG FEM to elliptic and (second-order) hyperbolic problems with viscoelastic memory. The

analogous parabolic problem is currently under study in [11]. Since code development for these type of problems is non-trivial, we do not present numerical results here. Instead, numerics for all three problems will be presented elsewhere at a later date when all the numerical issues have been identified.

References

- [1] J. Brilla. Error analysis for Laplace transform—finite element solution of hyperbolic equations. *Numer. Math.*, 41:55—62, 1983.
- [2] C. M. Dafermos. An abstract Volterra equation with applications to linear viscoelasticity. *J. Diff. Eqns.*, 7:554—569, 1970.
- [3] V. Girault, B. Riviere, and M. Wheeler. A discontinuous Galerkin method with non-overlapping domain decomposition for the Stokes and Navier-Stokes problems. *Mathematics of Computation*, TICAM Report 02-08 (2002), to appear.
- [4] Gregory M. Hulbert and Thomas J. R. Hughes. Space-time finite element methods for second-order hyperbolic equations. *Comp. Meth. Appl. Mech. Eng.*, 84:327—348, 1990.
- [5] A. R. Johnson. Modeling viscoelastic materials using internal variables. *The Shock and Vibration Digest*, 31:91—100, 1999.
- [6] A. R. Johnson, A. Tessler, and M. Dambach. Dynamics of thick viscoelastic beams. *Journal of Engineering Materials and Technology*, 119:273—278, 1997.
- [7] A. K. Pani, V. Thomée, and L. B. Wahlbin. Numerical methods for hyperbolic and parabolic integro-differential equations. *J. Integral Equations Appl.*, 4:533—584, 1992.
- [8] B. Rivière, M. F. Wheeler, and V. Girault. A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems. *SIAM J. Numer. Anal.*, 39:902—931, 2001.
- [9] B. Rivière and M.F. Wheeler. A discontinuous Galerkin method applied to nonlinear parabolic equations. In B. Cockburn, G.E. Karniadakis, and C.-W. Shu, editors, *Discontinuous Galerkin Methods: Theory, Computation and Applications*, volume 11 of *Lecture Notes in Computational Science and Engineering*, pages 231—244. Springer, 1999.

- [10] B. Riviere and M.F. Wheeler. Discontinuous finite element methods for acoustic and elastic wave problems. *Contemporary Mathematics*, 329:271–282, 2003.
- [11] Béatrice Rivière and Simon Shaw. Discontinuous Galerkin finite element approximation of nonlinear non-Fickian diffusion in viscoelastic polymers. Submitted to *Siam J. Numer. Anal.* See report 05/6 at www.brunel.ac.uk/bicom.
- [12] Béatrice Rivière, Simon Shaw, Mary F. Wheeler, and J.R. Whiteman. Discontinuous Galerkin finite element methods for linear elasticity and quasistatic linear viscoelasticity. *Numer. Math.*, 95:347–376, 2003.
- [13] Béatrice Rivière, Mary F. Wheeler, and Vivette Girault. Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems. Part I. *Computational Geosciences*, 3:337–360, 1999.
- [14] Simon Shaw, M. K. Warby, and J. R. Whiteman. An error bound via the Ritz-Volterra projection for a fully discrete approximation to a hyperbolic integrodifferential equation. Technical report, 94/3, BICOM, Brunel University, Uxbridge, U.K., 1994. (see www.brunel.ac.uk/bicom).
- [15] E. G. Yanik and G. Fairweather. Finite element methods for parabolic and hyperbolic partial integro-differential equations. *Nonlinear Analysis, Theory, Methods & Applications*, 12:785–809, 1988.