DISCONTINUOUS GALERKIN METHODS FOR ANISOTROPIC SEMIDEFINITE DIFFUSION WITH ADVECTION[*]

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Abstract. We construct and analyze a discontinuous Galerkin method to solve advection-diffusion-reaction PDEs with anisotropic and semidefinite diffusion. The method is designed to automatically detect the so-called elliptic/hyperbolic interface on fitted meshes. The key idea is to use consistent weighted average and jump operators. Optimal estimates in the broken graph norm are proven. These are consistent with well-known results when the problem is either hyperbolic or uniformly elliptic. The theoretical results are supported by numerical evidence.

Key words. discontinuous Galerkin, advection-diffusion-reaction, discontinuous coefficients, anisotropic diffusion, coupled elliptic-hyperbolic, weighted averages

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1. Introduction. Discontinuous Galerkin (DG) methods were originally introduced to solve transport equations in [23, 24, 25] and later extended to problems involving second-order elliptic operators in [1, 4, 26]. For many years, the development and analysis of DG methods have followed two somewhat parallel routes according to the hyperbolic or elliptic nature of the problem at hand. A unifying viewpoint has recently been proposed in a series of papers [10, 11, 13], where the authors rely on the Friedrichs framework originally proposed in [17] to perform an abstract analysis valid for a variety of (linear) PDE systems.

The goal of the present work is to merge even further the hyperbolic and elliptic points of view by considering advection-diffusion-reaction problems with discontinuous, anisotropic, and semidefinite diffusivity (or, using the terminology from [22], scalar-valued second-order PDEs with nonnegative characteristic form). One major difficulty associated with this class of problems is that only the conservative flux is continuous and, at variance with uniformly elliptic problems, the scalar field can be discontinuous across hyperbolic/elliptic interfaces (see (2.4) for the definition of these interfaces). The literature in numerical analysis dedicated to this topic is scarce. This issue has been investigated in one space dimension by Gastaldi and Quarteroni in [18], where interface conditions are derived using asymptotic analysis. In [22], Houston, Schwab, and Süli propose and analyze a DG method for PDEs with nonnegative characteristic form in higher space dimensions. They treat hyperbolic/elliptic interfaces in a numerical example at the end of the paper (see [22, section 6.4]). Another example can be found in [14]. Both proposed techniques are ad hoc in the sense that they require removing a priori some penalty terms at the hyperbolic/elliptic interface. Our

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objective is to go beyond [14, 18, 22] by proposing a DG method that automatically detects the so-called hyperbolic/elliptic interface under the mild assumption that the mesh fits the discontinuities of the diffusion tensor.

The material is organized as follows. We introduce the model problem in section 2. It is a multidimensional scalar-valued second-order PDE with nonnegative characteristic from. We derive the multidimensional counterpart of the one-dimensional interface condition introduced in [18]. We reformulate the problem as a symmetric Friedrichs system (in the spirit of [10, 13, 17]) by identifying appropriate interface and boundary operators. A well-posedness result is proven under some assumptions. In section 3 we focus our attention on the discrete problem. We propose a DG approximation based on the weak formulation of the continuous problem with boundary and interface conditions enforced weakly. The discrete bilinear form is designed so that the correct interface conditions are automatically enforced on fitted meshes without identifying a priori the hyperbolic/elliptic interfaces. The basic requirements on the bilinear form are that it be strongly consistent, continuous, and (in the spirit of Friedrichs) L^2-coercive. The convergence of the method is proven in section 4, and the main results are stated in Theorems 4.5 and 4.7. The error estimates are optimal in the broken graph norm and compatible with those presented in [10, 11] when the problem is either hyperbolic or uniformly elliptic. Implementation issues are addressed in section 5, and variants of the method are discussed. Section 6 is devoted to numerical experiments illustrating the performance of the proposed method. Concluding remarks are contained in section 7.

2. The continuous problem. In this section we introduce the model problem, and we recast it as a first-order PDE system endowed with a Friedrichs-like structure. We then derive a weak formulation where boundary and interface conditions are enforced naturally. This formulation will be the starting point for the design of the DG method constructed in section 3.

2.1. The PDE setting. Let $\Omega \subset \mathbb{R}^d$ be a bounded, open, and connected Lipschitz domain with boundary $\partial\Omega$ and outward normal n. The problem investigated in the present work consists of the following scalar-valued PDE:

$$(2.1) \qquad \nabla\cdot(-\nu\nabla u + \beta u) + \mu u = f,$$

with data $f \in L^2(\Omega)$. Boundary conditions are specified later in this section. The following assumptions are made on the coefficients:

(i) $\nu \in [L^\infty(\Omega)]^{d,d}$ is a symmetric positive *semi*definite tensor field, meaning that for all $r \in \mathbb{R}^d$ and for a.e. (almost every) $x \in \Omega$, $r^t\nu(x)r \geq 0$;

(ii) $\beta \in [\mathfrak{C}^1(\overline{\Omega})]^d$ and $\mu \in L^\infty(\Omega)$;

(iii) there is some $\mu_0 > 0$ so that $\mu + \frac{1}{2}\nabla\cdot\beta \geq \mu_0$ for a.e. $x \in \Omega$.

Throughout the rest of this work, the symbols $\lesssim$ and $\gtrsim$ are used for inequalities that hold up to a real positive multiplicative constant that is independent of ν (and discretization parameters like the meshsize) but may depend on β and μ (and regularity parameters of the mesh family considered later on).

We finally assume that the field ν is discontinuous and that we know the location of the discontinuities. More precisely, we assume that we are given a partition of Ω into Lipschitz connected subdomains $P_\Omega \overset{\text{def}}{=} \{\Omega_i\}_{i=1}^N$ so that the following holds:

(iv) ν is piecewise constant on the partition P_Ω, and the problem is normalized so that $\|\nu\|_{[L^\infty(\Omega)]^{d,d}} \leq 1$.

We introduce the symbol Γ to denote the union of the inner boundaries of the subdomains Ω_i, i.e.,

$$(2.2) \qquad \Gamma \stackrel{\text{def}}{=} \{x \in \Omega; \; \exists i_1, i_2 \in \{1, \ldots, N\}, \; i_1 \neq i_2, \; x \in \partial\Omega_{i_1} \cap \partial\Omega_{i_2}\}.$$

The unit outward normals to Ω_{i_1} and Ω_{i_2} are denoted by n_{i_1} and n_{i_2}. For a.e. $x \in \Gamma$, the two indices i_1, i_2 are chosen such that $(n^t \nu n)(x)|_{\Omega_{i_1}} \geq (n^t \nu n)(x)|_{\Omega_{i_2}}$. We also denote by n the two-valued field on Γ such that, for $x \in \partial\Omega_{i_1} \cap \partial\Omega_{i_2}$, $n(x) = \{n_{i_1}, n_{i_2}\}$.

For every two-valued function φ on Γ, we denote by φ_1 the value of φ which is defined on the side of Ω_{i_1} and by φ_2 the value of φ which is defined on the side of Ω_{i_2}. For example, when applying this convention to the normal vector n, we have $n_1 = n_{i_1}$ and $n_2 = n_{i_2}$. Mean values and jumps across Γ are defined as follows:

$$(2.3) \qquad \{\varphi\} \stackrel{\text{def}}{=} \tfrac{1}{2}(\varphi_1 + \varphi_2), \qquad [\![\varphi]\!] \stackrel{\text{def}}{=} \varphi_1 - \varphi_2.$$

2.2. The interface/transmission conditions. Since we are not requiring that ν be uniformly positive definite, the mathematical nature of the PDE can change over the domain. To account for this, we define the subset $I \subset \Gamma$ that we subsequently refer to as the elliptic/hyperbolic interface:

$$(2.4) \qquad I \stackrel{\text{def}}{=} \{x \in \Gamma; \; (n^t \nu n)(x)|_{\Omega_{i_1}} > 0 \; \text{ and } \; (n^t \nu n)(x)|_{\Omega_{i_2}} = 0\}.$$

For any x in I, we refer to Ω_{i_1} as the elliptic side of I at x, and we refer to Ω_{i_2} as the hyperbolic side of I at x. Note that the terms elliptic and hyperbolic are not volumic properties but interface properties. Ω_{i_1} being the elliptic side of I at x does not mean that the diffusivity is positive definite in Ω_{i_1}. The diffusivity may not be positive definite in Ω_{i_1}, but still have a nonzero component in the normal direction at x and vice versa for the hyperbolic side. We now define

$$(2.5) \qquad \Omega^\dagger \stackrel{\text{def}}{=} \Omega \setminus I, \qquad I^+ \stackrel{\text{def}}{=} \{x \in I; \; \beta \cdot n_1 > 0\}, \qquad I^- \stackrel{\text{def}}{=} \{x \in I; \; \beta \cdot n_1 < 0\}.$$

I^+ (resp., I^-) is the subset of I where the advection field flows from the elliptic side to the hyperbolic side (resp., hyperbolic side to the elliptic side). The reader is referred to Figures 2(a)–2(b) for examples illustrating these definitions.

Let $\kappa \stackrel{\text{def}}{=} \nu^{1/2}$. The assumptions on ν imply that κ is bounded and positive semidefinite. We now rewrite (2.1) in mixed form by introducing the auxiliary unknown σ so that

$$(2.6) \qquad \begin{cases} \sigma + \kappa \nabla u = 0 & \text{in } \Omega^\dagger, \\ \nabla \cdot (\kappa \sigma + \beta u) + \mu u = f & \text{in } \Omega, \end{cases}$$

and we require the following continuity property to hold:

$$(\text{INT1}) \qquad\qquad\qquad [\![u]\!] = 0 \quad \text{on } I^+.$$

Observe that σ is defined in $\Omega^\dagger$ only. Indeed, u may be discontinuous across I, in which case $\kappa \nabla u$ cannot be defined in the distributional sense, since the product of a discontinuous function and a distribution is not legitimate within the standard distribution theory. Note also that (INT1) demands that u be continuous only on the portion of I where the advection field flows from the elliptic to the hyperbolic side.

The above formulation implies the following formal property:

$$(\text{INT2}) \qquad \{(\kappa\sigma + \beta u)\cdot n\} = 0 \quad \text{on } \Gamma,$$

since the second equation in (2.6) is expected to hold in the whole domain Ω. Similarly, the first equation in (2.6) formally implies that

$$(2.7) \qquad [\![u]\!] = 0 \quad \text{across } \Gamma \setminus I.$$

Finally by combining (INT1)–(INT2) on I^+ and using Lemma 2.1 together with the continuity of β, we observe that (INT1) amounts to enforcing

$$(2.8) \qquad n_1^t \kappa_1 \nabla u_1 = 0 \quad \text{on } I^+.$$

LEMMA 2.1. *Let ν be a $d\times d$ positive semidefinite matrix; then*

$$\forall r \in \mathbb{R}^d, \quad (\nu r = 0) \Leftrightarrow (r^t \nu r = 0).$$

This simple lemma will be frequently invoked in the paper.

2.3. Asymptotic justification. In one space dimension, (INT1)–(INT2) are the interface conditions derived by Gastaldi and Quarteroni in [18], and these are the transmission conditions used in [8, 14]. These conditions are deduced by considering the following regularized problem supplemented with suitable boundary conditions:

$$(-\nu u_\epsilon' + \beta u_\epsilon)' + \mu u_\epsilon - \epsilon u_\epsilon'' = f.$$

Under the hypothesis that β is a nonzero constant, it can be proved that, as $\epsilon \to 0$, u_ϵ converges in $L^2(\Omega)$ to the so-called viscosity solution of (2.6) which satisfies (INT1)–(INT2).

As an example, consider $\Omega = (0,1)$ partitioned into $\Omega_1 \overset{\text{def}}{=} (0,\frac{1}{3})$, $\Omega_2 \overset{\text{def}}{=} (\frac{1}{3},\frac{2}{3})$, $\Omega_3 \overset{\text{def}}{=} (\frac{2}{3},1)$. Take $f = 0$, $\mu = 0$, $\beta = 1$, and set $\nu|_{\Omega_1 \cup \Omega_3} = 1$ and $\nu|_{\Omega_2} = 0$. The viscosity solution of (2.6) corresponding to the Dirichlet boundary conditions $u(0) = 1$, $u(1) = 0$ is

$$(2.9) \qquad u|_{\Omega_1} = 1, \quad u|_{\Omega_2} = 1, \quad u|_{\Omega_3} = 1 - e^{(x-1)}.$$

It can be verified that this solution satisfies (INT1)–(INT2), so that u is continuous at $x = \frac{1}{3}$ and discontinuous at $x = \frac{2}{3}$.

Let us mention at this point that there is a theoretical difficulty in the above regularization process if the advection field is zero and $\mu = 0$. In this case, the limit solution can be shown to be

$$(2.10) \qquad u|_{\Omega_1} = 1, \quad u|_{\Omega_2} = 2 - 3x, \quad u|_{\Omega_3} = 0.$$

Comparing (2.10) with (2.9), we conclude that the limit process $\lim_{\epsilon \to 0,\, \beta \to 0}$ is not uniform.

We hereafter assume that in higher space dimensions (INT1)–(INT2) can be obtained by means of a regularization process and that there is no ambiguity on the limit, provided $\mu + \frac{1}{2}\nabla\cdot\beta \geq \mu_0 > 0$. The goal of the present paper is not to justify (INT1)–(INT2) but to show that these conditions yield a well-posed problem which we propose to approximate using a DG method.

2.4. The functional setting. We now reformulate the above problem in an appropriate functional setting. To this end, we set

$$L_u \stackrel{\text{def}}{=} L^2(\Omega), \quad L_\sigma \stackrel{\text{def}}{=} [L^2(\Omega^\dagger)]^d, \quad L \stackrel{\text{def}}{=} L_\sigma \times L_u,$$

so that $L = \{(z^\sigma, z^u); z^\sigma \in L_\sigma, z^u \in L_u\}$. L is a Hilbert space when equipped with the product norm. We define the so-called graph space:

$$W \stackrel{\text{def}}{=} \{z \in L; \kappa \nabla z^u \in L_\sigma, \nabla \cdot (\kappa z^\sigma + \beta z^u) \in L_u\},$$

where all the derivatives are understood in the weak sense. Then we consider the following operators:

$$\begin{aligned}
K &: L \ni z \mapsto (z^\sigma, \mu z^u) \in L, \\
A &: W \ni z \mapsto (\kappa \nabla z^u, \nabla \cdot \Phi(z)) \in L, \\
\tilde{A} &: W \ni z \mapsto (-\kappa \nabla z^u, (\nabla \cdot \beta) z^u - \nabla \cdot \Phi(z)) \in L,
\end{aligned}$$

where, for all $y \in W$, $\Phi(y) \stackrel{\text{def}}{=} \kappa y^\sigma + \beta y^u$. A and $\tilde{A} \in \mathcal{L}(W; L)$ are formal adjoints of each other. W is clearly a Hilbert space when equipped with the following norm:

$$\|y\|_W^2 \stackrel{\text{def}}{=} \|y\|_L^2 + \|Ay\|_L^2.$$

Moreover, K and A are bounded operators; i.e., $K \in \mathcal{L}(L; L)$ and $A \in \mathcal{L}(W; L)$. We refer to W as the graph space of A, and the norm of W is called the graph norm. Note that functions in W satisfy (INT2) but not necessarily (INT1).

2.5. Boundary operators. Following [10, 13], we consider the operator $D : W \longrightarrow W'$ defined by

(2.11) $$\langle Dz, y \rangle_{W',W} \stackrel{\text{def}}{=} (Az, y)_L - (z, \tilde{A}y)_L.$$

Clearly $D \in \mathcal{L}(W; W')$, and D is a boundary operator in the following sense.

LEMMA 2.2. *The following holds:*

(2.12) $$\langle Dz, y \rangle_{W',W} = \int_{\partial\Omega} [\Phi(z) \cdot ny^u + \Phi(y) \cdot nz^u - (\beta \cdot n) z^u y^u] - \int_I (\beta \cdot n_1) [\![z^u]\!] [\![y^u]\!]$$

for all $(z, y) \in W \times W$ smooth enough for the integrals to make sense.

Proof. Integrating by parts over $\Omega^\dagger$ in (2.11) yields

$$\begin{aligned}
\langle Dz, y \rangle_{W',W} = \int_I 2 \left\{ z^u n^t \kappa y^\sigma + y^u n^t \kappa z^\sigma + (\beta \cdot n) z^u y^u \right\} \\
+ \int_{\partial\Omega} \left[z^u n^t \kappa y^\sigma + y^u n^t \kappa z^\sigma + (\beta \cdot n) z^u y^u \right].
\end{aligned}$$

We conclude using the fact that on I, $n_1^t \kappa_1 z_1^\sigma = -\beta \cdot n_1 [\![z^u]\!]$ and $n_2^t \kappa_2 = 0$, so that $2 \{ z^u n^t \kappa y^\sigma + y^u n^t \kappa z^\sigma + \beta \cdot n z^u y^u \} = -(\beta \cdot n_1) [\![y^u]\!] z_1^u - (\beta \cdot n_1) [\![z^u]\!] y_1^u + (\beta \cdot n_1) y_1^u z_1^u + (\beta \cdot n_2) y_2^u z_2^u = -(\beta \cdot n_1) [\![z^u]\!] [\![y^u]\!]$. $\square$

In other words, if z and y are smooth enough, D admits the following integral representation:

(2.13) $$\langle Dz, y \rangle_{W',W} = \int_{\partial\Omega} y^t \mathcal{D}z - \int_I (\beta \cdot n_1) [\![z^u]\!] [\![y^u]\!], \quad \text{where} \quad \mathcal{D} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & \kappa n \\ (\kappa n)^t & \beta \cdot n \end{bmatrix}.$$

When the traces of z and y are not in $L^2(\partial\Omega \cup I)$, the above integrals have to be understood in some duality sense that we do not try to identify here.

Let $\alpha \in \{-1, +1\}$. Still following [10, 13], we assume that there is a second boundary operator $M \in \mathcal{L}(W; W')$ such that we have the following:

(i) M is monotone, meaning that $\langle My, y\rangle_{W', W} \geq 0$ for all $y \in W$.

(ii) Letting

$$(2.14) \qquad V \stackrel{\text{def}}{=} \operatorname{Ker}(M - D), \qquad V^* \stackrel{\text{def}}{=} \operatorname{Ker}(M^* + D),$$

where M^* is the adjoint of M, the following holds:

$$(2.15) \qquad D(V)^\perp = V^*, \qquad D(V^*)^\perp = V,$$

where, for all $E \subset W'$, $E^\perp$ denotes the polar set of E; i.e., $E^\perp$ is composed of those linear forms on W that vanish on E.

(iii) If $(z, y) \in W \times W$ are smooth enough for the integrals to make sense, M admits the following integral representation:
(2.16)

$$\langle Mz, y\rangle_{W', W} = \int_{\partial\Omega} y^t \mathcal{M}z + \int_I |\beta{\cdot}n|[\![z^u]\!][\![y^u]\!], \quad \text{where} \quad \mathcal{M} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -\alpha\kappa n \\ \alpha(\kappa n)^t & |\beta{\cdot}n| \end{bmatrix}.$$

The above hypotheses essentially assert the existence of surjective trace operators on I and $\partial\Omega$. The purpose of the operator M is to impose boundary conditions and to enforce (INT1). The choice $\alpha = +1$ (resp., $\alpha = -1$) is used to enforce Dirichlet (resp., Neumann) boundary conditions. Indeed, one can verify that if $z \in W$ is smooth enough, then

(i) if $\alpha = +1$,

$$(2.17) \qquad \{z \in V\} \iff \{[\![z^u]\!]|_{I^+} = 0 \text{ and } z^u|_{\{x \in \partial\Omega; \ \kappa n \neq 0 \text{ or } \beta{\cdot}n < 0\}} = 0\},$$

$$(2.18) \qquad \{z \in V^*\} \iff \{[\![z^u]\!]|_{I^-} = 0 \text{ and } z^u|_{\{x \in \partial\Omega; \ \kappa n \neq 0 \text{ or } \beta{\cdot}n > 0\}} = 0\};$$

(ii) if $\alpha = -1$,

$$(2.19) \qquad \{z \in V\} \iff \{[\![z^u]\!]|_{I^+} = 0 \text{ and } \Phi(z){\cdot}n = \tfrac{1}{2}(\beta{\cdot}n + |\beta{\cdot}n|)z^u\},$$

$$(2.20) \qquad \{z \in V^*\} \iff \{[\![z^u]\!]|_{I^-} = 0 \text{ and } \Phi(z){\cdot}n = \tfrac{1}{2}(\beta{\cdot}n - |\beta{\cdot}n|)z^u\}.$$

For instance, taking $\alpha = +1$, if $z \in V$ is smooth enough, then for all $y \in W$, $\langle (M - D)z, y\rangle_{W', W} = 0$, so that if y is smooth enough,

$$(2.21) \qquad -\int_{\partial\Omega} [(n^t \kappa y^\sigma)z^u + (\beta{\cdot}n)^- y^u z^u] + \int_{I^+} (\beta{\cdot}n_1)[\![y^u]\!][\![z^u]\!] = 0,$$

whence (2.17) is inferred.

2.6. Well-posedness. Let $a_0 \in \mathcal{L}(W \times L; \mathbb{R})$, $a_0^* \in \mathcal{L}(W \times L; \mathbb{R})$ be defined by

$$(2.22) \qquad a_0(z, y) \stackrel{\text{def}}{=} (Kz, y)_L + (Az, y)_L \qquad \forall(z, y) \in W \times L,$$

$$(2.23) \qquad a_0^*(z, y) \stackrel{\text{def}}{=} (Kz, y)_L + (\tilde{A}z, y)_L \qquad \forall(z, y) \in W \times L.$$

LEMMA 2.3 (L-coercivity). a_0 and a_0^* are L-coercive on V and V^*, respectively, in the following sense:

(2.24)
$$\forall y \in V, \quad a_0(y,y) \geq \|y^\sigma\|_{L_\sigma}^2 + \mu_0\|y^u\|_{L_u}^2 + \tfrac{1}{2}\|[\![y^u]\!]\|_{L^2(|\beta\cdot n|;I)}^2 + \tfrac{1}{2}\|y^u\|_{L^2(|\beta\cdot n|;\partial\Omega)}^2,$$

(2.25)
$$\forall y \in V^*, \quad a_0^*(y,y) \geq \|y^\sigma\|_{L_\sigma}^2 + \mu_0\|y^u\|_{L_u}^2 + \tfrac{1}{2}\|[\![y^u]\!]\|_{L^2(|\beta\cdot n|;I)}^2 + \tfrac{1}{2}\|y^u\|_{L^2(|\beta\cdot n|;\partial\Omega)}^2.$$

Proof. Using the definition of D and V, we infer, for all $y \in V$,

$$((K+A)y,y)_L = ((K+\tfrac{1}{2}(A+\tilde{A}))y,y)_L + (\tfrac{1}{2}(A-\tilde{A})y,y)_L$$
$$= \|y^\sigma\|_{L_\sigma}^2 + ((\mu + \tfrac{1}{2}\nabla\cdot\beta)y^u, y^u)_{L_u} + \tfrac{1}{2}\langle My, y\rangle_{W',W}.$$

The desired result then follows from the construction of M. Proceed similarly to prove (2.25). □

Consider the following problem: For $f \in L_u$,

(2.26)
$$\begin{cases} \text{find } z \in V \text{ such that, } \forall y \in L, \\ a_0(z,y) = (f, y^u)_{L_u}. \end{cases}$$

PROPOSITION 2.4. *Assume that there is an $M \in \mathcal{L}(W;W')$ satisfying (2.15). Then, problem (2.26) is well-posed.*

Proof. See the appendix. □

Remark 2.1. Note that the boundary and interface conditions in problem (2.26) are strongly enforced by requiring the solution to be a member of V.

Having in mind that boundary conditions are weakly enforced in DG methods, we introduce the following bilinear form:

(2.27)
$$a(z,y) \stackrel{\text{def}}{=} a_0(z,y) + \tfrac{1}{2}\langle (M-D)z, y\rangle_{W',W} \qquad \forall(z,y) \in W \times W.$$

Clearly, all the terms above are well defined and $a \in \mathcal{L}(W \times W; \mathbb{R})$.

LEMMA 2.5 (L-coercivity). *a is L-coercive on W.*

Proof. Clearly for all $y \in W$,

$$a(y,y) = ((K+A)y,y)_L + \tfrac{1}{2}\langle (M-D)y, y\rangle_{W',W}$$
$$= ((K+\tfrac{1}{2}(A+\tilde{A}))y,y)_L + \tfrac{1}{2}((A-\tilde{A})y,y)_L + \tfrac{1}{2}\langle (M-D)y,y\rangle_{W',W}$$
$$\geq \|y^\sigma\|_{L_\sigma}^2 + \mu_0\|y^u\|_{L_u}^2 + \tfrac{1}{2}\|[\![y^u]\!]\|_{L^2(|\beta\cdot n|;I)}^2 + \tfrac{1}{2}\|y^u\|_{L^2(|\beta\cdot n|;\partial\Omega)}^2;$$

that is to say, a is L-coercive. □

We henceforth consider the following reformulation of problem (2.26): For $f \in L_u$,

(2.28)
$$\begin{cases} \text{find } z \in W \text{ such that, } \forall y \in W, \\ a(z,y) = (f, y^u)_{L_u}. \end{cases}$$

PROPOSITION 2.6 (well-posedness). *Assume that there is an $M \in \mathcal{L}(W;W')$ satisfying (2.15). Then, problem (2.28) is well-posed, and the solutions to (2.26) and (2.28) coincide.*

Proof. The unique solution to (2.26) solves (2.28). Moreover, coercivity (see Lemma 2.5) immediately implies that the solution to (2.28) is unique. □

3. The discrete problem. The goal of this section is to construct a DG approximation of the model problem (2.28). After introducing the DG setting in section 3.1, we explain in detail in section 3.2 how the DG bilinear form a_h (see (3.17)) is constructed.

3.1. The discrete setting. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of affine meshes of Ω compatible with the partition P_Ω, which, for simplicity, is supposed to be composed of polyhedra. The elements are not necessarily simplices, and the interfaces are not required to match. We denote by $\mathcal{F}_h^i$ the set of element interfaces, i.e., $F \in \mathcal{F}_h^i$ if F is a $(d-1)$-manifold and there are $T_1, T_2 \in \mathcal{T}_h$ such that $F = \partial T_1 \cap \partial T_2$. The set of the faces that separate the mesh from the exterior of Ω is denoted by $\mathcal{F}_h^\partial$; i.e., $F \in \mathcal{F}_h^\partial$ if F is a $(d-1)$-manifold and there is a $T \in \mathcal{T}_h$ such that $F = \partial T \cap \partial\Omega$. The set of all the faces is denoted by $\mathcal{F}_h$; i.e., $\mathcal{F}_h \overset{\text{def}}{=} \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$. Moreover, for every face $F \in \mathcal{F}_h$ we introduce the set $\mathcal{T}_h(F) \overset{\text{def}}{=} \{T \in \mathcal{T}_h;\ F \subset \partial T\}$. The diameters of $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_h$ are denoted by h_T and h_F, respectively. Without loss of generality, we assume that $h \leq 1$.

For every nonnegative integer p, we define

$$(3.1) \qquad P_{h,p} \overset{\text{def}}{=} \{v_h \in L^2(\Omega);\ \forall T \in \mathcal{T}_h,\ v_h|_T \in \mathbb{P}_p(T)\},$$

where $\mathbb{P}_p(T)$ denotes the set of d-variate polynomials of total degree at most p on T. The mesh family $\{\mathcal{T}_h\}_{h>0}$ is assumed regular in the sense that

$$(3.2) \qquad h_T \lesssim h_F, \quad F \subset \partial T,$$

$$(3.3) \qquad \|\nabla v_h\|_{[L^2(T)]^d} \lesssim h_T^{-1}\|v_h\|_{L^2(T)} \quad \forall T \in \mathcal{T}_h,\ \forall v_h \in P_{h,p},$$

$$(3.4) \qquad \|v_h\|_{L^2(F)} \lesssim h_F^{-1/2}\|v_h\|_{L^2(\mathcal{T}_h(F))} \quad \forall F \in \mathcal{F}_h,\ \forall v_h \in P_{h,p}.$$

Let p_u and p_σ be two nonnegative integers such that $p_u - 1 \leq p_\sigma \leq p_u$, and define the following spaces:

$$\Sigma_h = [P_{h,p_\sigma}]^d, \qquad U_h = P_{h,p_u}, \qquad W_h = \Sigma_h \times U_h.$$

Observe that the inverse and trace inequalities (3.3) and (3.4) are local and that they can be applied componentwise to the functions in Σ_h. Let $H^s(\mathcal{T}_h) \overset{\text{def}}{=} \{v \in L^2(\Omega);\ v \in H^s(T)\ \forall T \in \mathcal{T}_h\}$ be equipped with the usual broken Sobolev norm denoted by $\|\cdot\|_{H^s(\mathcal{T}_h)}$, and define

$$W(h) \overset{\text{def}}{=} W \cap [H^1(\mathcal{T}_h)]^{d+1} + W_h.$$

According to the assumptions listed in section 2.1 and since $\{\mathcal{T}_h\}_{h>0}$ is compatible with the partition P_Ω, we have that

$$(3.5) \qquad \nu \in [P_{h,0}]^{d,d} \quad \text{and} \quad \kappa \in [P_{h,0}]^{d,d}.$$

Remark 3.1. (i) The hypotheses on ν could be slightly weakened by assuming that ν is Lipschitz in each subdomain, but to avoid unnecessary technicalities we restrict ourselves to piecewise constant diffusivity. (ii) The mesh being compatible with the partition P_Ω means that the meshes fit the discontinuities of the diffusivity.

3.2. Design of the DG bilinear form. We now construct a discrete DG counterpart of the bilinear form a defined in (2.27) under the following constraints: (i) The discrete bilinear form should satisfy a discrete version of Lemma 2.5 (L-coercivity) and be strongly consistent. (ii) It should not require that the elliptic-hyperbolic interface I be identified a priori. (Since computers work in finite precision arithmetic, it may happen in practice that $n^t \nu n$ takes small values instead of being exactly zero, so that I is possibly difficult to identify.) (iii) It should include stabilizing terms suitable to weakly enforcing boundary and interface conditions.

3.2.1. Step 0: Discrete analogue of a. We start by localizing the volume and boundary/interface integrals in (2.27) and by deriving a discrete counterpart of the operator M defined by (2.16).

The localization of a_0 and D does not pose any problem and is done by splitting the volume and boundary/interface integrals over the mesh elements and faces. Now let us now focus our attention on the discretization of M.

We first construct a discrete counterpart of the boundary integral over $\partial\Omega$ in (2.16). To this end, we take inspiration from [11, 12] and we proceed as follows. For every $F \in \mathcal{F}_h^\partial$ we define the operator $M_F \in \mathcal{L}([L^2(F)]^{d+1}; [L^2(F)]^{d+1})$ such that, for all $(z, y) \in W(h) \times W(h)$,

$$(M_F(z), y)_{L,F} \stackrel{\text{def}}{=} -\alpha(z^u, \kappa y^\sigma \cdot n)_{L_u, F} + \alpha(\kappa z^\sigma \cdot n, y^u)_{L_u, F} + (M_F^{uu}(z^u), y^u)_{L_u, F},$$

where the boundary operator $M_F^{uu} \in \mathcal{L}(L^2(F); L^2(F))$ is defined by

$$(3.6) \qquad M_F^{uu}(y^u) \stackrel{\text{def}}{=} \left(|\beta \cdot n| + \tfrac{1}{2}(\alpha + 1)\lambda^2 h_F^{-1} \right) y^u,$$

and we have set $\lambda \stackrel{\text{def}}{=} \sqrt{n^t \nu n}$. Observe that M_F satisfies the following consistency conditions:

$$(3.7) \qquad \text{Ker}(\mathcal{M} - \mathcal{D}) \subset \text{Ker}(M_F - \mathcal{D}), \qquad \text{Ker}(\mathcal{M} + \mathcal{D}) \subset \text{Ker}(M_F + \mathcal{D}).$$

Since M_F^{uu} is nonnegative, we define

$$|y^u|_{M,F}^2 \stackrel{\text{def}}{=} (M_F^{uu}(y^u), y^u)_{L_u, F}, \qquad |y^u|_M^2 \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h^\partial} |y^u|_{M,F}^2.$$

The $|\cdot|_M$-seminorm will be used to measure the error due to the weak enforcement of boundary conditions.

Second, we construct a discrete counterpart of the boundary integral over the interface I in (2.16). To this end we extend the definition of λ to interior faces. For every $F \in \mathcal{F}_h^i$, the two-valued field $\lambda|_F = \{\lambda_1, \lambda_2\}$ is defined by

$$(3.8) \qquad \lambda_i \stackrel{\text{def}}{=} \sqrt{n_i^t \nu n_i}, \quad i \in \{1, 2\}.$$

Recall that the indexing is defined such that $\lambda_1 \geq \lambda_2$. A discrete counterpart of the manifold I is defined by

$$\mathcal{I}_h \stackrel{\text{def}}{=} \{F \in \mathcal{F}_h^i; \ \lambda_1 > 0 \text{ and } \lambda_2 = 0\}.$$

To avoid unnecessary extra technicalities we suppose for the time being that the sign of $\beta \cdot n$ is constant on every interface F in $\mathcal{F}_h^i$. Then we identify two subsets of $\mathcal{I}_h$, say $\mathcal{I}_h^\pm$, representing the discrete versions of $I^\pm$. The boundary integral over the interface

I in (2.16) is then discretized by splitting it over $\mathcal{I}_h$. The sets $\mathcal{I}_h, \mathcal{I}_h^\pm$ and the above assumption on the sign of $\beta \cdot n$ are introduced solely to explicate the design of the DG bilinear form. These sets are not required in the final definition (3.17).

Combining all the steps above, we obtain the discrete bilinear form $a_h^{(0)}$ such that, for all $(z, y) \in W(h) \times W(h)$,

(3.9)
$$
\begin{aligned}
a_h^{(0)}(z, y) &\overset{\text{def}}{=} \sum_{T \in \mathcal{T}_h} [(Kz, y)_{L,T} + (Az, y)_{L,T}] + \tfrac{1}{2} \sum_{F \in \mathcal{F}_h^\partial} ((M_F - \mathcal{D})z, y)_{L,F} \\
&\quad + \sum_{F \in \mathcal{I}_h^+} ((\beta \cdot n_1)[\![z^u]\!], [\![y^u]\!])_{L_u, F}.
\end{aligned}
$$

3.2.2. Step 1: L-coercivity and consistency. The bilinear form $a_h^{(0)}$ is not appropriate since it does not satisfy the design criteria (i) (L-coercivity) and (ii) (the interface $\mathcal{I}_h^+$ is explicitly identified), as stated in the introduction of section 3.2.

To remedy the above two problems, we introduce weighted averages. Let $\Gamma_h \overset{\text{def}}{=} \bigcup_{F \in \mathcal{F}_h^i} F$, and let ω be a two-valued weight function such that $[L^2(\Gamma_h)]^2 \ni \omega = (\omega_1, \omega_2)$ with $\omega_1 + \omega_2 = 1$ for a.e. $x \in \Gamma_h$. For all $y \in W(h)$ and every $F \in \mathcal{F}_h^i$, we define the weighted average and weighted jump of y^u across F as follows:

(3.10)
$$
\{y^u\}_\omega \overset{\text{def}}{=} \omega_1 y_1^u + \omega_2 y_2^u, \qquad [\![y^u]\!]_\omega \overset{\text{def}}{=} 2(\omega_2 y_1^u - \omega_1 y_2^u),
$$

where, for a.e. $x \in F$, $y_i^u(x) = \lim_{y \to x} y^u(y)|_{T_i}$, $i \in \{1, 2\}$. We adopt similar definitions for $\{\Phi(y) \cdot n\}_\omega$ and $[\![\Phi(y) \cdot n]\!]_\omega$. To alleviate notation we set $\{\cdot\} \overset{\text{def}}{=} \{\cdot\}_{(\frac{1}{2}, \frac{1}{2})}$ and $[\![\cdot]\!] \overset{\text{def}}{=} [\![\cdot]\!]_{(\frac{1}{2}, \frac{1}{2})}$. The following algebraic formula will be used repeatedly in the rest of the paper:

(3.11)
$$
\{ab\} = \{a\}\{b\}_\omega + \tfrac{1}{4}[\![a]\!]_\omega [\![b]\!].
$$

LEMMA 3.1. *The following integration by parts formula holds:*
(3.12)
$$
\sum_{T \in \mathcal{T}_h} [(Az, y)_{L,T} - (z, \tilde{A}y)_{L,T}] = \sum_{F \in \mathcal{F}_h^\partial} (\mathcal{D}z, y)_{L,F} + \sum_{F \in \mathcal{F}_h^i} 2\left[\chi_{F,\omega}(z, y) + \chi_{F,\omega}(y, z)\right],
$$

where

(3.13) $\chi_{F,\omega}(z, y) \overset{\text{def}}{=} (\{\Phi(z) \cdot n\}, \{y^u\}_\omega)_{L_u, F} + \left([\![z^u]\!], \dfrac{1}{4}[\![\Phi(y) \cdot n]\!]_\omega - \dfrac{\beta \cdot n_1}{2}\{y^u\}\right)_{L_u, F}.$

Proof. Indeed, let *LHS* be the left-hand side of (3.12), and observe that

$$
LHS = \sum_{F \in \mathcal{F}_h^\partial} (\mathcal{D}z, y)_{L,F} + 2 \sum_{F \in \mathcal{F}_h^i} \int_F [\{\Phi(z) \cdot n y^u\} + \{\Phi(y) \cdot n z^u\} - \{(\beta \cdot n)z^u y^u\}].
$$

Apply (3.11) to the averages involving $\Phi(z)$ and $\Phi(y)$, and observe that $\{(\beta \cdot n)z^u y^u\} = \frac{\beta \cdot n_1}{2}[\![z^u]\!]\{y^u\} + \frac{\beta \cdot n_1}{2}[\![y^u]\!]\{z^u\}$. $\square$

Applying (3.12) in (3.9), we infer that

$$
\begin{aligned}
a_h^{(0)}(y, y) &= \sum_{T \in \mathcal{T}_h} (K + \tfrac{1}{2}(A + \tilde{A})y, y)_{L,T} + \tfrac{1}{2} \sum_{F \in \mathcal{F}_h^\partial} (M_F(y), y)_{L,F} \\
&\quad + \sum_{F \in \mathcal{I}_h^+} (|\beta \cdot n|[\![y^u]\!], [\![y^u]\!])_{L_u, F} + 2 \sum_{F \in \mathcal{F}_h^i} \chi_{F,\omega}(y, y).
\end{aligned}
$$

This expression suggests that the following bilinear form can be considered in order to restore L-coercivity:

$$(3.14) \qquad \hat{a}_h^{(1)}(z,y) \overset{\text{def}}{=} a_h^{(0)}(z,y) - 2 \sum_{F \in \mathcal{F}_h^i} \chi_{F,\omega}(z,y).$$

Let us now examine the consistency of the new bilinear form $\hat{a}_h^{(1)}$. Let $z \in V \cap W(h)$ and let $y_h \in W_h$. Since $a_h^{(0)}$ is consistent, we just have to verify the consistency of the sum involving $\chi_{F,\omega}(z,y)$. For all $F \in \mathcal{F}_h^i \setminus \mathcal{I}_h$, it is clear that $\chi_{F,\omega}(z,y_h) = 0$ owing to (INT1)–(INT2). Let us now consider a face F in $\mathcal{I}_h$. From this point on, we design ω so that

$$(3.15) \qquad \forall F \in \mathcal{I}_h, \ \forall x \in F, \quad \omega(x) = (1,0).$$

This implies that $\{y_h^u\}_\omega = y_h^u|_{T_1}$ and, since $z \in V$ and $F \in \mathcal{I}_h$,

$$\frac{1}{4} \llbracket \Phi(y_h) \cdot n \rrbracket_\omega - \frac{\beta \cdot n_1}{2} \{y_h^u\} = \frac{\beta \cdot n_1}{2} y_h^u|_{T_2(F)} - \frac{\beta \cdot n_1}{2} \{y_h^u\} = -\frac{\beta \cdot n_1}{4} \llbracket y_h^u \rrbracket.$$

Thus, the following simplification occurs:

$$-2\chi_{F,\omega}(z,y_h) = -2(\{\Phi(z) \cdot n\}, y_h^u|_{T_1(F)})_{L_u,F} + \left(\frac{\beta \cdot n_1}{2} \llbracket z^u \rrbracket, \llbracket y_h^u \rrbracket \right)_{L_u,F}$$

$$= \left(\frac{\beta \cdot n_1}{2} \llbracket z^u \rrbracket, \llbracket y_h^u \rrbracket \right)_{L_u,F}.$$

In this form it is now clear that $\chi_{F,\omega}(z,y_h) \neq 0$ over $\mathcal{I}_h$ (actually over $\mathcal{I}_h^-$ only); i.e., $\hat{a}_h^{(1)}$ is not consistent. To make the bilinear form consistent, we remove the trouble-making term by defining the new bilinear form

$$a_h^{(1)}(z,y) \overset{\text{def}}{=} \hat{a}_h^{(1)}(z,y) - \sum_{F \in \mathcal{I}_h} \left(\frac{\beta \cdot n_1}{2} \llbracket z^u \rrbracket, \llbracket y^u \rrbracket \right)_{L_u,F}$$

$$(3.16) \qquad = \sum_{T \in \mathcal{T}_h} [(Kz,y)_{L,T} + (Az,y)_{L,T}] + \frac{1}{2} \sum_{F \in \mathcal{F}_h^\partial} ((M_F - \mathcal{D})z,y)_{L,F}$$

$$- 2 \sum_{F \in \mathcal{F}_h^i} \chi_{F,\omega}(z,y) + \sum_{F \in \mathcal{I}_h} \left(\frac{|\beta \cdot n|}{2} \llbracket z^u \rrbracket, \llbracket y^u \rrbracket \right)_{L_u,F}.$$

Then, the above arguments imply that $a_h^{(1)}$ is strongly consistent and satisfies the following coercivity property: For all $y \in W(h)$,

$$a_h^{(1)}(y,y) \geq \|y^\sigma\|_{L_\sigma}^2 + \mu_0 \|y^u\|_{L_u}^2 + \tfrac{1}{2} |y^u|_M^2 + \tfrac{1}{2} \|\llbracket y^u \rrbracket\|_{L^2(|\beta \cdot n|;\mathcal{I}_h)}^2.$$

3.2.3. Step 2: Elimination of $\mathcal{I}_h$ and stability. Note that $a_h^{(1)}$ is not yet satisfactory since it still requires $\mathcal{I}_h$ be explicitly identified by the user, thus violating our design constraint (ii).

The key idea to eliminate any reference to $\mathcal{I}_h$ in (3.16) is to observe that the bilinear form $(|\beta \cdot n| \llbracket z^u \rrbracket, \llbracket y^u \rrbracket)_{L_u,F}$ is positive and consistent on every face which is not in $\mathcal{I}_h$. In other words, L-coercivity and consistency are preserved by extending the sum over $\mathcal{I}_h$ to $\mathcal{F}_h^i$. We thus replace $a_h^{(1)}$ by

(3.17)

$$a_h(z,y) \overset{\text{def}}{=} \sum_{T \in \mathcal{T}_h} [(Kz,y)_{L,T} + (Az,y)_{L,T}] + \frac{1}{2} \sum_{F \in \mathcal{F}_h^\partial} (M_F(z) - \mathcal{D}z, y)_{L,F}$$

$$- 2 \sum_{F \in \mathcal{F}_h^i} \left[(\{\Phi(z) \cdot n\}, \{y^u\}_\omega)_{L_u,F} + \left([\![z^u]\!], \frac{1}{4}[\![\Phi(y) \cdot n]\!]_\omega - \frac{\beta \cdot n_1}{2} \{y^u\} \right)_{L_u,F} \right]$$

$$+ \sum_{F \in \mathcal{F}_h^i} (S_F([\![z^u]\!]), [\![y^u]\!])_{L,F},$$

where the interface operator $S_F \in \mathcal{L}(L^2(F); L^2(F))$ is defined for all $F \in \mathcal{F}_h^i$ by

(3.18)
$$S_F(v) \overset{\text{def}}{=} \left(\frac{|\beta \cdot n|}{2} + \frac{\lambda_2^2}{h_F} \right) v.$$

Observe that λ_2 is by definition the minimum of λ_1 and λ_2. Hence, automatically

(3.19)
$$S_F(v) = \frac{|\beta \cdot n|}{2} v \qquad \forall F \in \mathcal{I}_h.$$

Finally, to avoid any reference to $\mathcal{I}_h$ in the definition of the weighting function ω while satisfying condition (3.15), we define ω as follows:

(3.20)
$$\omega \overset{\text{def}}{=} \begin{cases} (\frac{\lambda_1}{2\{\lambda\}}, \frac{\lambda_2}{2\{\lambda\}}) & \text{if } \lambda_1 > 0, \\ (\frac{1}{2}, \frac{1}{2}) & \text{otherwise.} \end{cases}$$

Although other expressions for ω are possible, this one is simple and yields robust error estimates. A similar choice is made in [6, 15].

The discrete problem is now formulated as follows:

(3.21)
$$\begin{cases} \text{Seek } z_h \in W_h \text{ such that} \\ a_h(z_h, y_h) = (f, y_h^u)_{L_u} \quad \forall y_h \in W_h. \end{cases}$$

Observe that the σ-component of the unknown can be eliminated locally since the jumps of this quantity across element interfaces are not penalized; see, e.g., [11, section 4.4].

Remark 3.2. The use of weights in DG methods has been highlighted in several articles (see, e.g., [16, 20, 19, 21]). Although some of the above references point out that using weights may yield higher accuracy, they do not connect the weights with the coefficients of the problem. The weight/diffusivity dependence has recently been investigated in [6, 15], where the authors show that the use of a particular weighted average improves the stability of the numerical scheme in problems with high diffusivity contrasts, the diffusivity being still positive definite. In the present case, resorting to weighted average and jump operators is required for the method to select the proper interface conditions automatically, i.e., the design constraint (ii).

4. Convergence analysis. In this section we carry out the convergence analysis of the discrete problem (3.21). The main results are Theorems 4.5 and 4.7.

4.1. Basic convergence estimates. For all $y \in W(h)$ we introduce the following seminorm:

(4.1)
$$|y^u|_{J,F}^2 \overset{\text{def}}{=} (S_F([\![y^u]\!]), [\![y^u]\!])_{L_u,F}, \qquad |y^u|_J^2 \overset{\text{def}}{=} \sum_{F \in \mathcal{F}_h^i} |y^u|_{J,F}^2.$$

The space $W(h)$ is equipped with the following discrete norm:

$$(4.2) \qquad \|y\|_{h,\kappa}^2 \overset{\mathrm{def}}{=} \|y\|_L^2 + |y^u|_J^2 + |y^u|_M^2 + \sum_{T \in \mathcal{T}_h} \|\kappa \nabla y^u\|_{L_\sigma,T}^2.$$

The following two lemmata follow from the design procedure outlined in section 3.2.

LEMMA 4.1 (consistency). *Let z solve (2.6) and z_h solve (3.21). Assume, moreover, that $z \in [H^1(\mathcal{T}_h)]^{d+1}$. Then,*

$$\forall y_h \in W_h, \quad a_h(z - z_h, y_h) = 0.$$

LEMMA 4.2 (*L*-coercivity). *For all h and for all y in $W(h)$,*

$$a_h(y, y) \gtrsim \|y\|_L^2 + |y^u|_J^2 + |y^u|_M^2.$$

In order to estimate the L^2-norm of the diffusive derivative $\kappa \nabla z^u$ we need the following result.

LEMMA 4.3 (stability). *The following bound holds:*

$$\forall z_h \in W_h, \quad \|z_h\|_{h,\kappa} \lesssim \sup_{y_h \in W_h \setminus \{0\}} \frac{a_h(z_h, y_h)}{\|y_h\|_{h,\kappa}}.$$

Proof. Let $z_h \in W_h$ and set $\mathbb{S} \overset{\mathrm{def}}{=} \sup_{y_h \in W_h \setminus \{0\}} \frac{a_h(z_h, y_h)}{\|y_h\|_{h,\kappa}}$.

(1) Owing to Lemma 4.2,

$$(4.3) \qquad \|z_h\|_L^2 + |z_h^u|_J^2 + |z_h^u|_M^2 \lesssim a_h(z_h, z_h) \lesssim \mathbb{S}\|z_h\|_{h,\kappa}.$$

(2) Control of $\mathtt{B} \overset{\mathrm{def}}{=} \sum_{T \in \mathcal{T}_h} \|\kappa \nabla z_h^u\|_{L_\sigma,T}^2$. Let $\pi_h^\sigma \in \Sigma_h$ be the field such that, for all $T \in \mathcal{T}_h$, $\pi_h^\sigma|_T \overset{\mathrm{def}}{=} \kappa \nabla z_h^u|_T$. From the definition of a_h it follows that

$$\mathtt{B} = a_h(z_h, (\pi_h^\sigma, 0)) - (z_h^\sigma, \pi_h^\sigma)_{L_\sigma} + \sum_{F \in \mathcal{F}_h^\partial} \frac{1+\alpha}{2} (\kappa n z_h^u, \pi_h^\sigma)_{L_\sigma, F}$$

$$+ \frac{1}{2} \sum_{F \in \mathcal{F}_h^i} ([\![z_h^u]\!], [\![n^t \kappa \pi_h^\sigma]\!]_\omega)_{L_u, F}.$$

Let R_i, $i \in \{1, 2, 3\}$ denote the last three terms in the right-hand side. The first term is bounded from above as follows:

$$|R_1| \leq \|z_h^\sigma\|_{L_\sigma} \|\pi_h^\sigma\|_{L_\sigma} \lesssim \|z_h^\sigma\|_{L_\sigma}^2 + \gamma \mathtt{B},$$

where γ can be chosen as small as needed.

The second term vanishes if $\alpha = -1$. If $\alpha = +1$, use trace inequality (3.4) together with (3.6) to get

$$|R_2| \lesssim \sum_{F \in \mathcal{F}_h^\partial} h_F^{-\frac{1}{2}} |(n^t \nu n z_h^u, z_h^u)_{L_u, F}|^{\frac{1}{2}} \|\pi_h^\sigma\|_{L_\sigma, \mathcal{T}_h(F)} \lesssim \sum_{F \in \mathcal{F}_h^\partial} |z_h^u|_{M, F} \|\pi_h^\sigma\|_{L_\sigma, \mathcal{T}_h(F)}.$$

Consequently, $|R_2| \lesssim |z_h^u|_M^2 + \gamma \mathtt{B}$. According to (3.20), for all $\mathcal{F}_h^i \ni F = \partial T_1 \cap \partial T_2$,

$$\|[\![n^t \kappa \pi_h^\sigma]\!]_\omega\|_{L_u, F} = \frac{\lambda_1 \lambda_2}{\{\lambda\}} \|[\![(n^t \kappa/\lambda) \pi_h^\sigma]\!]\|_{L_u, F} \lesssim h_F^{-\frac{1}{2}} \frac{\lambda_1 \lambda_2}{\{\lambda\}} \|\pi_h^\sigma\|_{L_\sigma, \mathcal{T}_h(F)}.$$

Using the above relation together with (3.18) yields

$$|R_3| \lesssim \sum_{F \in \mathcal{F}_h^i} \frac{\lambda_1}{\{\lambda\}} \left(\frac{\lambda_2^2}{h_F} \|[\![z_h^u]\!]\|_{L_u,F}^2 \right)^{\frac{1}{2}} \|\pi_h^\sigma\|_{L_\sigma, \mathcal{T}_h(F)} \lesssim |z_h^u|_J^2 + \gamma \mathsf{B}.$$

The above bounds with $\gamma = \frac{1}{6}$ together with Lemma 4.2 give

$$(4.4) \qquad \tfrac{1}{2} \sum_{T \in \mathcal{T}_h} \|\kappa \nabla z_h^\sigma\|_{L_\sigma,T}^2 \lesssim a_h(z_h, (\pi_h^\sigma, 0)) + a_h(z_h, z_h) \lesssim \mathsf{S} \|z_h\|_{h,\kappa},$$

where we used the fact that, by definition, $\|(\pi_h^\sigma, 0)\|_{h,\kappa} = \|\pi_h^\sigma\|_{L_\sigma} \leq \|z_h\|_{h,\kappa}$. Observe that, owing to the choice of the weight function ω, the above estimate is robust with respect to the possible discontinuity and anisotropy of ν.

(3) Equations (4.3)–(4.4) yield $\|z_h\|_{h,\kappa}^2 \lesssim \mathsf{S} \|z_h\|_{h,\kappa}$, i.e., the desired result.   □

Let us now introduce

$$(4.5) \qquad W_h^\perp \stackrel{\text{def}}{=} \{y \in W(h); \ \forall w_h \in W_h, \ (y, w_h)_L = 0\}.$$

Moreover, we define the following norm on $W(h)$:

$$(4.6) \quad [\![y]\!]^2 \stackrel{\text{def}}{=} \|y\|_{h,\kappa}^2 + \sum_{T \in \mathcal{T}_h} \left[\frac{\mathfrak{h}_T}{h_T^2} \|y^u\|_{L_u,T}^2 + h_T \|y^\sigma\|_{L_\sigma, \partial T}^2 + \sum_{F \subset \partial T} \frac{\mathfrak{h}_F}{h_F} \|y^u\|_{L_u,F}^2 \right],$$

where, for all $T \in \mathcal{T}_h$ and for all $F \in \mathcal{F}_h$, we have defined

$$(4.7) \qquad \mathfrak{h}_T \stackrel{\text{def}}{=} \max(\|\nu\|_{[L^\infty(T)]^{d,d}}, h_T), \qquad \mathfrak{h}_F \stackrel{\text{def}}{=} \begin{cases} \max(\lambda_1^2, h_F) & \text{if } F \in \mathcal{F}_h^i, \\ \max(\lambda^2, h_F) & \text{if } F \in \mathcal{F}_h^\partial. \end{cases}$$

The last property needed to prove convergence is stated in the following.

LEMMA 4.4 (continuity). *The following holds:*

$$\forall (z, y_h) \in W_h^\perp \times W_h, \quad a_h(z, y_h) \lesssim [\![z]\!] \|y_h\|_{h,\kappa}.$$

Proof. Let $(z, y_h) \in W_h^\perp \times W_h$. Using the integration by parts formula (3.12), we obtain

$$(4.8) \quad a_h(z, y_h) = \sum_{T \in \mathcal{T}_h} (z, (K + \tilde{A})y_h)_{L,T} + 2 \sum_{F \in \mathcal{F}_h^i} \chi_{F,\omega}(y_h, z)$$

$$+ \tfrac{1}{2} \sum_{F \in \mathcal{F}_h^\partial} (M_F(z) + \mathcal{D}z, y_h)_{L,F} + \sum_{F \in \mathcal{F}_h^i} (S_F([\![z^u]\!]), [\![y_h^u]\!])_{L_u,F}.$$

We now derive bounds for the four terms in the right-hand side, say $R_1, \ldots, R_4$. For the first one we have

$$(z, (K + \tilde{A})y_h)_{L,T}$$
$$= (z^\sigma, y_h^\sigma - \kappa \nabla y_h^u)_{L_\sigma,T} + (z^u, \mu y_h^u - \nabla \cdot (\kappa y_h^\sigma) - \overline{\beta} \cdot \nabla y_h^u - (\beta - \overline{\beta}) \cdot \nabla y_h^u)_{L_u,T},$$

where, for all $T \in \mathcal{T}_h$, $\overline{\beta}|_T$ is the mean value of the field β over T. Observe that, since $\kappa \nabla y_h^u \in \Sigma_h$, $\overline{\beta} \cdot \nabla y_h^u \in U_h$, and $z \in W_h^\perp$, $(z^\sigma, \kappa \nabla y_h^u)_{L_\sigma} = 0$ and $(z^u, \overline{\beta} \cdot \nabla y_h^u)_{L_u} = 0$. As a result,

$$|R_1| \lesssim \|z\|_L \|y_h\|_L + \sum_{T \in \mathcal{T}_h} \left[\|z^u\|_{L_u,T} h_T^{-1} \|\kappa y_h^\sigma\|_{L_\sigma,T} + (z^u, (\beta - \overline{\beta}) \cdot \nabla y_h^u)_{L_u,T} \right]$$

$$\lesssim \|z\|_L \|y_h\|_L + \sum_{T \in \mathcal{T}_h} \left[\frac{\mathfrak{h}_T^{\frac{1}{2}}}{h_T} \|z^u\|_{L_u,T} \|y_h^\sigma\|_{L_\sigma,T} + \|\beta\|_{[\mathfrak{C}^1(\Omega)]^d} \|y_h^u\|_{L_u,T} \|z^u\|_{L_u,T} \right],$$

and, therefore, $|R_1| \lesssim \|z\| \|y_h\|_L$. The second term R_2 can be simplified as follows:

$$R_2 = 2 \sum_{F \in \mathcal{F}_h^i} \left[(\{z^u\}_\omega, \{n^t \kappa y_h^\sigma\})_{L_u,F} + \frac{1}{4} (\llbracket n^t \kappa z^\sigma \rrbracket_\omega, \llbracket y_h^u \rrbracket)_{L_u,F} + \left(\frac{\beta \cdot n_1}{2} \{z^u\}, \llbracket y_h^u \rrbracket \right)_{L_u,F} \right].$$

Let $R_{2,i}$, $i = 1, 2, 3$, be the addends of R_2. Using the definition of the weight function ω, (3.20), together with the inverse trace inequality (3.4) and definition (4.7), we infer

$$|R_{2,1}| \lesssim \sum_{F \in \mathcal{F}_h^i} \mathfrak{h}_F^{\frac{1}{2}} h_F^{-\frac{1}{2}} (\|z_1^u\|_{L_u,F} + \|z_2^u\|_{L_u,F}) \|y_h^\sigma\|_{L_\sigma,\mathcal{T}_h(F)},$$

$$|R_{2,2}| \lesssim \sum_{F \in \mathcal{F}_h^i} \frac{\lambda_1}{\{\lambda\}} h_F^{\frac{1}{2}} (\|z_1^\sigma\|_{L_\sigma,F} + \|z_2^\sigma\|_{L_\sigma,F}) \lambda_2 h_F^{-\frac{1}{2}} \|\llbracket y_h^u \rrbracket\|_{L_u,F}$$

$$\lesssim \sum_{F \in \mathcal{F}_h^i} h_F^{\frac{1}{2}} (\|z_1^\sigma\|_{L_\sigma,F} + \|z_2^\sigma\|_{L_\sigma,F}) |y_h^u|_{J,F},$$

$$|R_{2,3}| \lesssim (\|z_1^u\|_{L_u,F} + \|z_2^u\|_{L_u,F}) |y_h^u|_{J,F} \lesssim \mathfrak{h}_F^{\frac{1}{2}} h_F^{-\frac{1}{2}} (\|z_1^u\|_{L_u,F} + \|z_2^u\|_{L_u,F}) |y_h^u|_{J,F}.$$

Therefore, $|R_2| \lesssim \|z\| \|y_h\|_L$. The third term is expanded as follows:

$$|R_3| = \sum_{F \in \mathcal{F}_h^\partial} \left[\frac{1-\alpha}{2} (z^u, n^t \kappa y_h^\sigma)_{L_u,F} + \frac{1+\alpha}{2} (n^t \kappa z^\sigma, y_h^u)_{L_u,F} + \frac{1}{2} ((M_F^{uu} + \beta \cdot n) z^u, y_h^u)_{L_u,F} \right].$$

Let $R_{3,i}$, $i = 1, \ldots, 3$, be the addends of R_3. If $\alpha = -1$, then $R_{3,2} = 0$, and using (4.7) and (3.4), we infer that

$$|R_{3,1}| \lesssim \sum_{F \in \mathcal{F}_h^\partial} \lambda h_F^{-\frac{1}{2}} \|z^u\|_{L_u,F} \|y_h^\sigma\|_{L_\sigma,\mathcal{T}_h(F)} \lesssim \sum_{F \in \mathcal{F}_h^\partial} \mathfrak{h}_F^{\frac{1}{2}} h_F^{-\frac{1}{2}} \|z^u\|_{L_u,F} \|y_h^\sigma\|_{L_\sigma,\mathcal{T}_h(F)},$$

whereas, if $\alpha = +1$, then $R_{3,1} = 0$, and (3.18) implies that

$$|R_{3,2}| \lesssim \sum_{F \in \mathcal{F}_h^\partial} h_F^{\frac{1}{2}} \|z^\sigma\|_{L_\sigma,F} \lambda h_F^{-\frac{1}{2}} \|y_h^u\|_{L_u,F} \lesssim \sum_{F \in \mathcal{F}_h^\partial} h_F^{\frac{1}{2}} \|z^\sigma\|_{L_\sigma,F} |y_h^u|_{M,F}.$$

Finally, (3.18) yields $|R_{3,3}| \lesssim |z^u|_M |y_h^u|_M$. Therefore, $|R_3| \lesssim \|z\| \|y_h\|_L$. For the fourth term we immediately have $|R_4| \leq |z^u|_J |y_h^u|_J$. The desired result is obtained by collecting the above bounds. $\square$

Let π_h be the L^2-projection onto W_h. Upon collecting the above results (consistency, stability, and continuity) and observing that $z - \pi_h z \in W_h^\perp$, the second Strang lemma immediately yields the following convergence result.

THEOREM 4.5 (convergence). *Let z solve (2.28) and z_h solve (3.21). Assume that $z \in [H^1(\mathcal{T}_h)]^{d+1}$. Then,*

$$\|z - z_h\|_{h,\kappa} \lesssim \|z - \pi_h z\|.$$

Owing to the regularity of the mesh family $\{\mathcal{T}_h\}_{h>0}$, the following interpolation property holds: For all $z \in [H^{r_\sigma}(\mathcal{T}_h)]^d \times H^{r_u}(\mathcal{T}_h)$,

$$
(4.9) \quad \| z - \pi_h z \| \lesssim \left(\sum_{T \in \mathcal{T}_h} h_T^{2s_\sigma} \|z^\sigma\|^2_{[H^{s_\sigma}(T)]^d} + \mathfrak{h}_T h_T^{2s_u-2} \|z^u\|^2_{H^{s_u}(T)} \right)^{\frac{1}{2}}
$$
$$
+ \left(\sum_{F \in \mathcal{F}_h} \mathfrak{h}_F h_F^{2s_u-2} \|z^u\|^2_{H^{s_u}(\mathcal{T}_h(F))} \right)^{\frac{1}{2}},
$$

where $s_\sigma \overset{\text{def}}{=} \min(r_\sigma, p_\sigma+1)$ and $s_u \overset{\text{def}}{=} \min(r_u, p_u+1)$. Since $p_u - 1 \le p_\sigma$ and provided $(r_\sigma, r_u) \ge (p_\sigma + 1, p_u + 1)$, the above interpolation error is of order h^{p_u}, i.e.,

$$
(4.10) \quad \| z - \pi_h z \| \lesssim h^{p_u} \|z\|_{[H^{p_\sigma+1}(\mathcal{T}_h)]^d \times H^{p_u+1}(\mathcal{T}_h)}.
$$

Remark 4.1. The above estimate is optimal for the $\|\cdot\|_{h,\kappa}$-norm but yields suboptimal convergence in the L^2-norm. If $p_\sigma = p_u - 1$, the error estimate is optimal in the L^2-norm for z_h^σ but is still suboptimal for the L^2-norm of z_h^u. From a theoretical viewpoint, it is optimal to set $\Sigma_h = [P_{h,p_u-1}]^d$, i.e., to work with different approximation orders for the σ- and u-components. However, working with equal-order approximation may be more convenient for implementation purposes.

Remark 4.2 (positive definite diffusivity). If the diffusivity is such that $\nu \ge \nu_0 \mathcal{I}_d$ with $\nu_0 = \mathcal{O}(1)$, the estimate (4.10) can be improved using a duality argument. More precisely, consider the mapping $L_u \ni y^u \longmapsto \psi \in V^*$ defined by

$$
(K + \tilde{A})\psi = (0, y^u),
$$

and assume that the following bound holds:

$$
(4.11) \quad \|\psi^u\|_{H^2(\mathcal{T}_h)} + \|\psi^\sigma\|_{[H^1(\mathcal{T}_h)]^d} \lesssim \|y^u\|_{L_u}.
$$

Adapting the reasoning in [11, section 5.3], if $r_u \ge p_u + 1$ and $p_u \ge 1$, it can be proved that

$$
\|z - z_h\|_{L_u} \lesssim h^{p_u+1} \|z\|_{[H^{p_\sigma+1}(\mathcal{T}_h)]^d \times H^{p_u+1}(\mathcal{T}_h)}.
$$

4.2. Improved convergence estimates. Owing to the definition of the $\|\cdot\|_{h,\kappa}$-norm, the convergence result of Theorem 4.5 does not contain an estimate involving the advective derivative. Such an estimate can be obtained, assuming that

$$
(4.12) \quad \kappa \text{ is scalar-valued,}
$$

and we still admit that κ may vanish over a portion of the domain. Define the following new discrete norm on $W(h)$:

$$
(4.13) \quad \|y\|^2_{h,\kappa,\beta} \overset{\text{def}}{=} \|y\|^2_{h,\kappa} + \|y^u\|^2_{h,\beta} \quad \text{with} \quad \|y^u\|^2_{h,\beta} \overset{\text{def}}{=} \sum_{T \in \mathcal{T}_h} h_T \|\beta \cdot \nabla y^u\|^2_{L_u,T}.
$$

LEMMA 4.6. *Assume that κ satisfies (4.12). Then the following bound holds:*

$$
\forall z_h \in W_h, \quad \|z_h\|_{h,\kappa,\beta} \lesssim \sup_{y_h \in W_h \setminus \{0\}} \frac{a_h(z_h, y_h)}{\|y_h\|_{h,\kappa,\beta}}.
$$

Proof. Let $z_h \in W_h$ and set $\mathbb{S} \overset{\text{def}}{=} \sup_{y_h \in W_h \setminus \{0\}} \frac{a_h(z_h, y_h)}{\|y_h\|_{h,\kappa,\beta}}$.

(1) Proceeding as in Lemma 4.3 and observing that $\|(\pi_h^\sigma, 0)\|_{h,\kappa,\beta} = \|(\pi_h^\sigma, 0)\|_{h,\kappa}$, we conclude that

$$(4.14) \qquad \|z_h\|_{h,\kappa}^2 \lesssim \mathbb{S}\|z_h\|_{h,\kappa,\beta}.$$

(2) We define the field $W_h \ni \pi_h \overset{\text{def}}{=} (0, \pi_h^u)$ in such a way that, for all $T \in \mathcal{T}_h$, $\pi_h^u|_T = h_T \overline{\beta} \cdot \nabla z_h^u$, where $\overline{\beta}$ is the mean of β over T. Using (3.3) together with the regularity of β and the fact that $h_T \le 1$, for all $T \in \mathcal{T}_h$, we have

$$(4.15) \qquad \begin{aligned} h_T^{-\frac{1}{2}}\|\pi_h^u\|_{L_u,T} &\le h_T^{\frac{1}{2}}\|(\overline{\beta} - \beta)\cdot\nabla z_h^u\|_{L_u,T} + h_T^{\frac{1}{2}}\|\beta\cdot\nabla z_h^u\|_{L_u,T} \\ &\le h_T^{\frac{1}{2}}\|\beta\|_{[\mathcal{C}^1(\Omega)]^d}\|z_h^u\|_{L_u,T} + h_T^{\frac{1}{2}}\|\beta\cdot\nabla z_h^u\|_{L_u,T}. \end{aligned}$$

(i) We first show that $\|\pi_h\|_{h,\kappa,\beta} \lesssim \|z_h\|_{h,\kappa,\beta}$. According to the above bound, it is clear that $\|\pi_h^u\|_{L_u} \lesssim \|z_h\|_{h,\kappa,\beta}$. Commuting the operators $\kappa\nabla$ and $\overline{\beta}\cdot\nabla$ and applying the inverse inequality (3.3), we infer that

$$\sum_{T\in\mathcal{T}_h}\|\kappa\nabla\pi_h^u\|_{L_u,T}^2 = \sum_{T\in\mathcal{T}_h}h_T^2\|\overline{\beta}\cdot\nabla(\kappa\nabla z_h^u)\|_{L_u,T}^2 \lesssim \sum_{T\in\mathcal{T}_h}\|\kappa\nabla z_h^u\|_{L_\sigma,T}^2.$$

Moreover, the regularity of β and again (3.3) yield

$$\begin{aligned} \|\pi_h^u\|_{h,\beta}^2 &\lesssim \sum_{T\in\mathcal{T}_h}h_T^3\left[\|\partial_{\overline{\beta}}(\beta\cdot\nabla z_h^u)\|_{L_u,T} + \|\partial_{\overline{\beta}}\beta\|_{[L^\infty(T)]^d}h_T^{-1}\|z_h^u\|_{L_u,T}\right]^2 \\ &\lesssim \sum_{T\in\mathcal{T}_h}h_T\left[\|\beta\cdot\nabla z_h^u\|_{L_u,T} + \|z_h^u\|_{L_u,T}\right]^2. \end{aligned}$$

The term $|\pi_h^u|_J$ is treated as follows:

$$|\pi_h^u|_J^2 \lesssim \sum_{F\in\mathcal{F}_h^i}\||\beta\cdot n|^{\frac{1}{2}}[\![\pi_h^u]\!]\|_{L_u,F}^2 + \sum_{F\in\mathcal{F}_h^i}\|\lambda_2 h_F^{-\frac{1}{2}}[\![\pi_h^u]\!]\|_{L_u,F}^2 \overset{\text{def}}{=} R_1 + R_2.$$

Using (3.4) together with (4.15), we immediately conclude that

$$|R_1| \lesssim \sum_{F\in\mathcal{F}_h^i}h_F^{-1}\|\pi_h^u\|_{L_u,\mathcal{T}_h(F)}^2 \lesssim \|z_h\|_{h,\kappa,\beta}^2.$$

The second term is zero if $\lambda_2 = 0$. On the other hand, by definition, if $\lambda_2 > 0$, then $\lambda_1 > 0$; i.e., κ is nonzero on both sides of the considered element interface. We proceed using the trace inequality (3.4) together with assumption (4.12) (i.e., $\kappa_1 = \lambda_1 I$, $\kappa_2 = \lambda_2 I$, and $\lambda_1 \ge \lambda_2$), to get

$$|R_2| \lesssim \sum_{F\in\mathcal{F}_h^i}\frac{\lambda_2^2}{h_F}h_F\|\overline{\beta}\cdot\nabla z_h^u\|_{L_u,\mathcal{T}_h(F)}^2 \lesssim \sum_{F\in\mathcal{F}_h^i}\lambda_2^2\|\nabla z_h^u\|_{L_\sigma,\mathcal{T}_h(F)}^2 \le \sum_{F\in\mathcal{F}_h^i}\|\kappa\nabla z_h^u\|_{L_\sigma,\mathcal{T}_h(F)}^2,$$

whence $|\pi_h^u|_J \lesssim \|z_h\|_{h,\kappa,\beta}$. In a similar way we can prove that $|\pi_h^u|_M \lesssim \|z_h\|_{h,\kappa,\beta}$.

(ii) Estimate for $\|z_h^u\|_{h,\beta}$. Integrating by parts only the diffusive terms and setting $\tilde{\mu} \stackrel{\text{def}}{=} \mu + \nabla \cdot \beta$, we obtain

$$
\begin{aligned}
\|z_h^u\|_{h,\beta}^2 = {} & a_h(z_h, \pi_h) \\
& + \sum_{T \in \mathcal{T}_h} \left[h_T(\beta \cdot \nabla z_h^u, (\beta - \overline{\beta}) \cdot \nabla z_h^u)_{L_u, T} + (z_h^\sigma, \kappa \nabla \pi_h^u)_{L_\sigma, T} - (\tilde{\mu} z_h^u, \pi_h^u)_{L_u, T} \right] \\
& + 2 \sum_{F \in \mathcal{F}_h^i} \left[-\frac{1}{4}(\llbracket n^t \kappa z_h^\sigma \rrbracket_\omega, \llbracket \pi_h^u \rrbracket)_{L_u, F} + \left(\frac{\beta \cdot n_1}{2} \llbracket z_h^u \rrbracket, \{ \pi_h^u \} \right)_{L_u, F} \right] \\
& - \frac{1}{2} \sum_{F \in \mathcal{F}_h^\partial} \left[(1 + \alpha)(n^t \kappa z_h^\sigma, \pi_h^u)_{L_u, F} + ((M_F^{uu} - \beta \cdot n) z_h^u, \pi_h^u)_{L_u, F} \right] \\
& - \sum_{F \in \mathcal{F}_h^i} (S_F(\llbracket z_h^u \rrbracket), \llbracket \pi_h^u \rrbracket)_{L_u, F}.
\end{aligned}
$$

Let R_i, $i = 1, \ldots, 9$, be the nine terms in the right-hand side, and observe that

$$
|R_1| \lesssim \mathbb{S} \|\pi_h\|_{h,\kappa,\beta} \lesssim \mathbb{S} \|z_h\|_{h,\kappa,\beta}.
$$

Furthermore,

$$
|R_2| \lesssim \sum_{T \in \mathcal{T}_h} h_T \|\beta \cdot \nabla z_h^u\|_{L_u, T} \|\overline{\beta} - \beta\|_{[L^\infty(T)]^d} h_T^{-1} \|z_h^u\|_{L_u, T} \lesssim \gamma \|z_h^u\|_{h,\beta}^2 + \|z_h\|_{h,\kappa}^2.
$$

Moreover,

$$
|R_3| + |R_4| + |R_5| + |R_7| + |R_9| \lesssim \|z_h\|_{h,\kappa} \|\pi_h\|_{h,\kappa,\beta} \lesssim \mathbb{S}^{\frac{1}{2}} \|z_h\|_{h,\kappa,\beta}^{\frac{3}{2}},
$$

$$
|R_6| \lesssim \|z_h\|_{h,\kappa} \|z_h\|_{h,\kappa,\beta} \lesssim \mathbb{S}^{\frac{1}{2}} \|z_h\|_{h,\kappa,\beta}^{\frac{3}{2}},
$$

and $|R_8| \lesssim \|z_h\|_{h,\kappa}(\|\pi_h\|_{h,\kappa,\beta} + \|z_h\|_{h,\kappa,\beta}) \lesssim \mathbb{S}^{\frac{1}{2}} \|z_h\|_{h,\kappa,\beta}^{\frac{3}{2}}$. Hence,

$$
\|z_h^u\|_{h,\beta}^2 \lesssim \mathbb{S} \|z_h\|_{h,\kappa,\beta} + \mathbb{S}^{\frac{1}{2}} \|z_h\|_{h,\kappa,\beta}^{\frac{3}{2}},
$$

whence it follows, using (4.14), that $\|z_h^u\|_{h,\beta}^2 \lesssim \mathbb{S}^2$. $\quad\square$

By using Lemma 4.6 and proceeding as usual, we infer the next result.

THEOREM 4.7 (convergence). *Let z solve* (2.28) *and z_h solve* (3.21). *Assume that $z \in [H^1(\mathcal{T}_h)]^{d+1}$ and that κ satisfies* (4.12). *Then,*

$$
\|z - z_h\|_{h,\kappa,\beta} \lesssim \|z - \pi_h z\|.
$$

Remark 4.3 (purely hyperbolic case). A special situation is obtained when the diffusivity is identically zero over the entire domain, since, for all $T \in \mathcal{T}_h$ and for all $F \in \mathcal{F}_h$, $\mathfrak{h}_T = h_T$ and $\mathfrak{h}_F = h_F$. In such a case it is readily seen that

$$
(4.16) \qquad \|z^u - z_h^u\|_{L_u} + \|z^u - z_h^u\|_{h,\beta} \lesssim h^{p_u + \frac{1}{2}} \|z^u\|_{H^{p_u+1}(\mathcal{T}_h)},
$$

which is exactly the estimate for the problem investigated in [10, section 3.1].

5. Implementation issues. In this section we discuss important implementation aspects of the method. We show how it can be interpreted in terms of so-called numerical fluxes so as to compare it with other known approximation techniques that are defined in these terms in the literature. We also present two variants of the method that yield substantial computational savings.

5.1. Flux formulation. The notion of (numerical) fluxes is widely used by engineers. This concept originally introduced in the context of finite volume methods, naturally extends to DG methods. The link between DG methods and the concept of flux has been explored in [3] for the Laplace equation and in [11] for more general cases. A number of methods have originally been presented in terms of fluxes, and it is therefore interesting to recast our formulation in this framework so as to facilitate comparisons. To this purpose, let us define

$$(5.1) \quad \phi_{\partial T}^u(z^\sigma, z^u)|_F \stackrel{\text{def}}{=} \begin{cases} \frac{1+\alpha}{2} n^t \kappa z^\sigma + \frac{1}{2}(\beta \cdot n) z^u + \frac{1}{2} M_F^{uu} z^u & \text{if } F \in \mathcal{F}_h^\partial, \\ n_T^t \{\kappa z^\sigma\}_{\overline{\omega}} + (\beta \cdot n_T) \{z^u\} + (n_T \cdot n_F) S_F(\llbracket z^u \rrbracket) & \text{if } F \in \mathcal{F}_h^i, \end{cases}$$

$$(5.2) \quad \phi_{\partial T}^\sigma(z^u)|_F \stackrel{\text{def}}{=} \begin{cases} \frac{1-\alpha}{2}(\kappa n)^t z^u & \text{if } F \in \mathcal{F}_h^\partial, \\ (\kappa n)^t|_T \{z^u\}_\omega & \text{if } F \in \mathcal{F}_h^i, \end{cases}$$

where $\overline{\omega} \stackrel{\text{def}}{=} (1,1) - \omega$ and n_T is the outward normal to the element T. It is possible to prove (see [11, section 4.3] for the details) that the discrete problem (3.21) can be equivalently reformulated in terms of the following local problems:

$$\begin{cases} \text{Seek } z_h \in W_h \text{ such that, } \forall T \in \mathcal{T}_h \text{ and } \forall q \in [\mathbb{P}_{p_\sigma}(T)]^d \times \mathbb{P}_{p_u}(T), \\ (z_h, (K + \tilde{A})q)_{L,T} + (\phi_{\partial T}(z_h), q)_{L,\partial T} = (f, q^u)_{L_u,T}. \end{cases}$$

The above form is known as the flux formulation of (3.21). Observe that the above flux definitions lead to the use of harmonic averages of the normal component of the diffusion tensor at mesh interfaces; see (3.20). Harmonic averaging of the diffusion matrix has been considered for dual-mixed formulations, e.g., in [2].

5.2. Interior penalty (IP) variant. In this section we discuss a variant of the method designed in section 3.2 which reduces the size of the local problems to be solved to eliminate the σ-component of the unknown. The advantages of such a variant are that it is easier to implement and that the associated matrix pattern is sparser. To this purpose we introduce the lifting operator defined as follows: For all $F \in \mathcal{F}_h$ and for all $\varphi \in L^2(F)$, $r_{F,\kappa}(\varphi) \in \Sigma_h$ is defined by

$$(5.3) \quad \forall \tau_h \in \Sigma_h, \quad (r_{F,\kappa}(\varphi), \tau_h)_{L_\sigma} \stackrel{\text{def}}{=} \begin{cases} \frac{\alpha+1}{2}(\varphi n, \kappa \tau_h)_{L_\sigma, F} & \text{if } F \in \mathcal{F}_h^\partial, \\ (\varphi n_1, \{\kappa \tau_h\}_{\overline{\omega}})_{L_\sigma, F} & \text{if } F \in \mathcal{F}_h^i. \end{cases}$$

Moreover, we let $R_\kappa(\varphi) \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h} r_{F,\kappa}(\varphi)$. Observe that, unlike in [3], the lifting operator depends on the diffusivity. Moreover, for a given face $F \in \mathcal{F}_h$, it is clear that $\text{supp}(r_{F,\kappa}(\varphi)) = \mathcal{T}_h(F)$. In what follows we shall extend the definition of the jump operator to boundary faces by setting

$$\llbracket y^u \rrbracket \stackrel{\text{def}}{=} y^u \quad \forall F \in \mathcal{F}_h^\partial, \ \forall y \in W(h).$$

The following result holds.

LEMMA 5.1. *For all $F \in \mathcal{F}_h$ and for all $v_h \in U_h$,*

$$\|r_{F,\kappa}(\llbracket v_h \rrbracket)\|_{L_\sigma} \lesssim \begin{cases} \lambda h_F^{-\frac{1}{2}} \|v_h\|_{L_u, F} & \text{if } F \in \mathcal{F}_h^\partial, \\ \lambda_2 h_F^{-\frac{1}{2}} \|\llbracket v_h \rrbracket\|_{L_u, F} & \text{if } F \in \mathcal{F}_h^i. \end{cases}$$
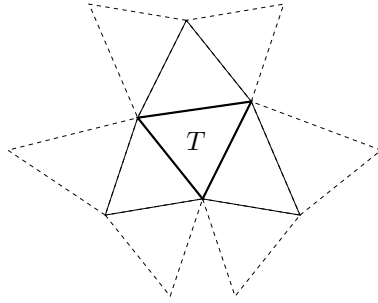
FIG. 1. *Elimination of the σ-component on element T. Stencil for the IP variant of the method (solid lines) and for the LDG variant (solid and dashed lines).*

Proof. Let $F \in \mathcal{F}_h^i$. Then, using (5.3), (3.20), and (3.4), we have that

$$\|r_{F,\kappa}(\llbracket v_h \rrbracket)\|_{L_\sigma}^2 = (\llbracket v_h \rrbracket n_1, \{\kappa r_{F,\kappa}(\llbracket v_h \rrbracket)\}_{\overline{\omega}})_{L_\sigma, F} \lesssim \|\llbracket v_h \rrbracket\|_{L_u, F} \frac{\lambda_1 \lambda_2}{2\{\lambda\}} h_F^{-\frac{1}{2}} \|r_{F,\kappa}(\llbracket v_h \rrbracket)\|_{L_\sigma},$$

from which the assertion follows readily. The proof is carried out similarly for $F \in \mathcal{F}_h^\partial$. $\square$

Proceeding in a way similar to that in [3, section 3.2] and using the fact that, owing to assumption (3.5), $\kappa\tau_h$ is in Σ_h for all $\tau_h \in \Sigma_h$, it is possible to prove that, for all $(\sigma, u) \in W(h)$ such that $\sigma = \kappa\nabla u - R_\kappa(\llbracket u \rrbracket)$ and for all $(0, v) \in W(h)$,

$$
\begin{aligned}
a_h((\sigma, u), (0, v)) = &\sum_{T \in \mathcal{T}_h} [(\kappa\nabla u - R_\kappa(\llbracket u \rrbracket), \kappa\nabla v - R_\kappa(\llbracket v \rrbracket))_{L_\sigma, T} + (\mu u, v)_{L_u, T}] \\
(5.4) \qquad &- \sum_{T \in \mathcal{T}_h} (u, \beta\cdot\nabla v)_{L_u, T} + \sum_{F \in \mathcal{F}_h^\partial} \tfrac{1}{2}(M_F^{uu}(u) + (\beta\cdot n)u, v)_{L_u, F} \\
&+ \sum_{F \in \mathcal{F}_h^i} ((\beta\cdot n_1)\{u\}, \llbracket v \rrbracket)_{L_u, F} + \sum_{F \in \mathcal{F}_h^i} (S_F(\llbracket u \rrbracket), \llbracket v \rrbracket)_{L_u, F}.
\end{aligned}
$$

Notice that σ does not appear in the expression on the right-hand side; i.e., we have found a decoupled problem for the sole primal unknown u. The expression (5.4) will henceforth be referred to as the *LDG (local discontinuous Galerkin) variant* of the discrete bilinear form because of the similarity with the method for convection-diffusion systems proposed in [7].

One can verify that, when the basis functions are defined in such a way that their support is restricted to one element of the triangulation, the stencil resulting from (5.4) is composed of all the elements shown in Figure 1 (solid and dashed lines). But by having a closer look at (5.4), one realizes that the only term involving the dashed elements in Figure 1 is the following:

$$\sum_{T \in \mathcal{T}_h} (R_\kappa(\llbracket u \rrbracket), R_\kappa(\llbracket v \rrbracket))_{L_\sigma, T} \stackrel{\text{def}}{=} -\rho_h(u, v).$$

Hence, in order to reduce the stencil, it seems reasonable to consider the following perturbation of a_h:

$$(5.5) \qquad\qquad a_h^{\text{IP}}(z, y) \stackrel{\text{def}}{=} a_h(z, y) + \rho_h(z^u, y^u).$$

Let us define the following seminorm:

$$(5.6) \quad |y^u|^2_{\mathrm{LDG}} \overset{\mathrm{def}}{=} \frac{\alpha+1}{2} \sum_{F \in \mathcal{F}_h^\partial} \|\lambda h_F^{-\frac{1}{2}} y^u\|^2_{L_u,F} + \sum_{F \in \mathcal{F}_h^i} \|\lambda_2 h_F^{-\frac{1}{2}} [\![y^u]\!]\|^2_{L_u,F} \quad \forall y \in W(h).$$

The following lemma is crucial to accommodating the proofs of Lemmata 4.2–4.4 to the new bilinear form a_h^{IP}.

LEMMA 5.2. *The following properties, uniform in h, hold:*
  (i) *For all $(z, y_h) \in (V \cap W(h)) \times W_h$ we have*

$$\forall y_h \in W_h, \quad \rho_h(z^u, y_h^u) = 0.$$

  (ii) *For all y in $W(h)$,*

$$\rho_h(y^u, y^u) \leq C N_F |y^u|^2_{\mathrm{LDG}},$$

  N_F *being the maximum number of faces of one mesh element and C a positive parameter depending only on the mesh geometry and on the polynomial order of approximation.*
  (iii) *For all $(z, y_h) \in W_h^\perp \times W_h$,*

$$(5.7) \qquad\qquad \rho_h(z, y_h) \lesssim |z^u|_{\mathrm{LDG}} |y_h^u|_{\mathrm{LDG}}.$$

Proof. (i) We know that $[\![z^u]\!] = 0$, and consequently $r_{F,\kappa}([\![z^u]\!]) = 0$, on all $F \in \mathcal{F}_h \setminus \mathcal{I}_h^-$. On the other hand, let $\mathcal{I}_h^- \ni F = \partial T_1 \cap \partial T_2$ and $\tau_h \in \Sigma_h$. Then, since $n^t \kappa|_{T_2} = 0$ entails $\lambda_2 = 0$,

$$n_1^t \{\kappa \tau_h\}_{\overline{\omega}} = \frac{\lambda_2}{2\{\lambda\}} n_1^t \cdot \kappa \tau_h|_{T_1} + \frac{\lambda_1}{2\{\lambda\}} n_1^t \cdot \kappa \tau_h|_{T_2} = 0,$$

i.e., $r_{F,\kappa}([\![z^u]\!]) = 0$, which gives the desired result.
  (ii) The second point can be proved as follows. Observe that

$$\|R_\kappa([\![y^u]\!])\|^2_{L_\sigma} \leq \sum_{F \in \mathcal{F}_h} \sum_{F' \in \mathcal{F}_h} \|r_{F,\kappa}([\![y^u]\!])\|_{L_\sigma} \|r_{F',\kappa}([\![y^u]\!])\|_{L_\sigma}.$$

Let $\mathcal{F}_h^i \ni F = \partial T_1 \cap \partial T_2$. Since $\mathrm{supp}(r_{F,\kappa}([\![y^u]\!])) = T_1 \cup T_2$, only a few products on the right-hand side are nonzero. In particular, the nonzero products are those for which $F' \in \Delta_F$, where $\Delta_F \overset{\mathrm{def}}{=} \{F' \in \mathcal{F}_h; F' \subset \partial T_1 \text{ or } F' \subset \partial T_2\}$. Therefore, the only terms involving F are

$$\sum_{F' \in \Delta_F} \|r_{F,\kappa}([\![y^u]\!])\|_{L_\sigma} \|r_{F',\kappa}([\![y^u]\!])\|_{L_\sigma} \leq \tfrac{1}{2} \sum_{F' \in \Delta_F} \left(\|r_{F,\kappa}([\![y^u]\!])\|^2_{L_\sigma} + \|r_{F',\kappa}([\![y^u]\!])\|^2_{L_\sigma}\right).$$

We realize that $\|r_{F,\kappa}([\![y^u]\!])\|^2_{L_\sigma}$ is added at most N_F times. The desired result follows by repeating this argument for the other faces and using Lemma 5.1.
  (iii) Deriving (5.7) is a simple application of Lemma 5.1.   □
  Modifying Lemmata 4.2–4.4 so as to hold for a_h^{IP} instead of a_h is now simple in view of the above result. However, observe that, according to the second point of Lemma 5.2, in order to preserve the L-coercivity, (3.6) and (3.18) should be modified as follows:

$$(5.8) \quad M_F^{uu}(v) \overset{\mathrm{def}}{=} \left(\frac{|\beta \cdot n|}{2} + N_F \eta \frac{\alpha+1}{2} \frac{\lambda^2}{h_F}\right) v, \qquad S_F(v) \overset{\mathrm{def}}{=} \left(\frac{|\beta \cdot n|}{2} + N_F \eta \frac{\lambda_2^2}{h_F}\right) v,$$

where the multiplicative factor η must be strictly greater than the constant C appearing in (5.7). The term ρ_h that has been added to simplify the elimination of the σ-component is thus counterbalanced by adding "more stabilization." The resulting method is termed the *IP variant* because of the similarity with the IP method proposed in [1, 4]. The method recently proposed in [15] also belongs to this class, although some modifications are introduced in the definition of the penalty parameter.

5.3. BRMPS variant. The parameter C in (5.7), and, consequently, η in (5.8), is possibly difficult to estimate in practical applications. To avoid this difficulty, we consider the following alternative expression for the boundary and interface operators:

$$(5.9) \qquad M_F^{uu}(v) \stackrel{\text{def}}{=} \frac{|\beta \cdot n|}{2} v + N_F \eta r_{F,\kappa}(v), \qquad S_F(v) \stackrel{\text{def}}{=} \frac{|\beta \cdot n|}{2} v + N_F \eta \{r_{F,\kappa}(v)\}_{\overline{\omega}}.$$

A closer look at the proof of the second point of Lemma 5.2 shows that it is sufficient to take $\eta > 1$ to preserve L-coercivity. Owing to the similarities with the approach first presented in [5], the resulting numerical method is termed the *BRMPS variant* (after the initials of the authors of [5]).

6. Numerical results. In this section we evaluate the performance of the proposed method on two test cases. The simulations are done using the BRMPS variant discussed in section 5.3. In both test cases, we assume that the exact solution is smooth enough elementwise.

6.1. Convergence. In order to assess the theoretical convergence estimates, we consider the problem described in Figure 2(a). Here (r, θ) denote the standard cylindrical coordinates with the angle θ measured in the anticlockwise sense starting from the positive x-axis. The domain is taken to be $(-1, 1)^2 \setminus [-0.5, 0.5]^2$, while the coefficients are set to

$$\kappa = \begin{cases} \pi & \text{if } 0 < \theta < \pi, \\ 0 & \text{if } \pi < \theta < 2\pi, \end{cases} \qquad \beta = \frac{e_\theta}{r}, \qquad \mu = 10^{-3},$$

where e_θ is the unit azimuthal vector. The exact solution for a suitable right-hand side f is

$$u = \begin{cases} (\theta - \pi)^2 & \text{if } 0 \leq \theta \leq \pi, \\ 3\pi(\theta - \pi) & \text{if } \pi < \theta < 2\pi. \end{cases}$$

Observe that, although piecewise polynomial in θ, the above solution does *not* belong to the discrete space U_h since we are solving the problem in Cartesian coordinates. Moreover, according to the interface condition (INT1), the solution is continuous across I^+, while only (INT2) is verified on I^-. We introduce the following norm:

$$\|u\|_{h,\text{BRMPS}}^2 \stackrel{\text{def}}{=} \|u\|_{L_u}^2 + |u|_J^2 + |u|_M^2 + \sum_{T \in \mathcal{T}_h} \|\kappa \nabla u\|_{L_\sigma, T}^2.$$

Let (σ_h, u_h) solve the discrete problem associated with the BRMPS variant. Then, observing that $\sigma_h = \kappa \nabla u_h + R(\llbracket u_h \rrbracket)$, it can be proved that $\|u - u_h\|_{h,\text{BRMPS}}$ is equivalent to $\|(\sigma, u) - (\sigma_h, u_h)\|_{h,\kappa}$. Coherently with the desire to avoid the additional cost coming from the computation of σ_h, $\|u - u_h\|_{h,\text{BRMPS}}$ was reported in Table 1. The convergence results confirm the sharpness of the estimates derived in sections 4.1 and 4.2. The L^2-norm is also reported for completeness, showing that convergence at order $p_u + 1$ can be expected.
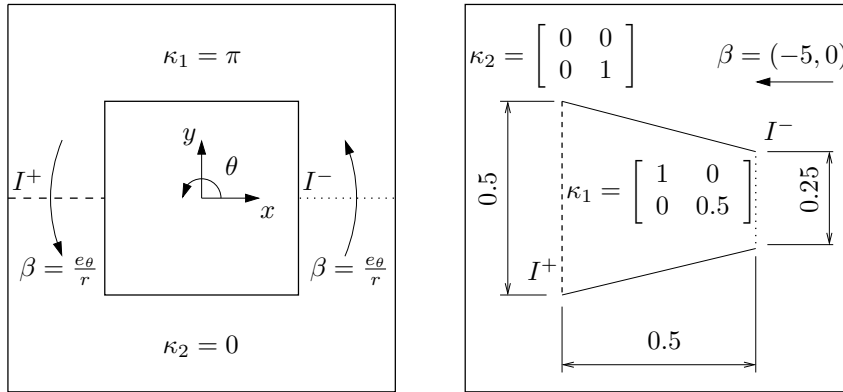
(a) Description of the test case of section 6.1.

(b) Description of the test case of section 6.2.
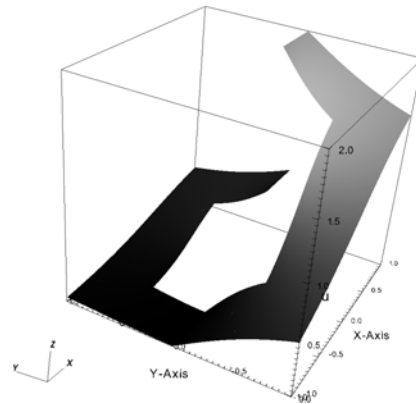
(c) Exact solution of the test case of section 6.1.

FIG. 2. *Problem setting for the numerical test cases. I^+ and I^- are plotted as dashed and dotted lines, respectively.*

6.2. Strongly anisotropic diffusivity. To demonstrate the behavior of the method in the presence of strongly anisotropic diffusivity we consider the test of Figure 2(b). The domain $\Omega = (0,1)^2$ is partitioned into two subdomains where the diffusivity takes different values; it is positive definite in one region and positive semidefinite in the other region. The advection field is $\beta = (-5,0)^t$, and the reaction coefficient is $\mu = 1$. The solution is discontinuous across the interface $I^- = \{x = 0.75; 0.375 \leq y \leq 0.625\}$. The solutions obtained for different polynomial degrees are displayed in Figure 3, showing that the predicted behavior is captured accurately.

7. Conclusion. In this work we developed and analyzed a DG method to approximate advection-diffusion-reaction equations with discontinuous, anisotropic, and semidefinite diffusivity. The proposed method is capable of treating in a robust fashion the semidefinite diffusivity case owing to our design of the boundary and penalty terms and provided the mesh fits the discontinuities of the diffusivity field. This is

TABLE 1
Convergence results.

h	$P_{h,1}$		$P_{h,2}$		$P_{h,3}$		$P_{h,4}$	
	Err	Ord	Err	Ord	Err	Ord	Err	Ord
				$\|u-u_h\|_{h,\mathrm{BRMPS}}$				
1/2	$3.15e+0$		$7.27e-1$		$1.74e-1$		$3.99e-2$	
1/4	$1.63e+0$	0.95	$2.05e-1$	1.83	$2.69e-2$	2.70	$3.51e-3$	3.51
1/8	$8.19e-1$	0.99	$5.32e-2$	1.94	$3.59e-3$	2.91	$2.51e-4$	3.81
1/16	$4.08e-1$	1.00	$1.34e-2$	1.99	$4.54e-4$	2.98	$1.63e-5$	3.95
1/32	$2.04e-1$	1.00	$3.36e-3$	2.00				
				$\|u-u_h\|_{h,\beta}$				
1/2	$1.97e-0$		$4.50e-1$		$1.13e-1$		$2.65e-2$	
1/4	$7.46e-1$	1.40	$9.87e-2$	2.18	$1.40e-2$	3.01	$1.92e-3$	3.79
1/8	$2.73e-1$	1.45	$1.90e-2$	2.38	$1.44e-3$	3.29	$1.06e-4$	4.18
1/16	$9.82e-2$	1.48	$3.44e-3$	2.46	$1.34e-4$	3.43	$5.03e-6$	4.40
1/32	$3.50e-2$	1.49	$6.08e-4$	2.50				
				$\|u-u_h\|_{L_u}$				
1/2	$2.92e-1$		$3.30e-2$		$5.79e-3$		$1.17e-3$	
1/4	$7.49e-2$	1.96	$4.75e-3$	2.80	$4.62e-4$	3.65	$5.50e-5$	4.41
1/8	$1.91e-2$	1.97	$6.09e-4$	2.96	$3.26e-5$	3.83	$2.01e-6$	4.77
1/16	$4.86e-3$	1.97	$7.76e-5$	2.97	$2.10e-6$	3.96	$6.32e-8$	4.99
1/32	$1.23e-3$	1.98	$9.82e-6$	2.98				

achieved by resorting to weighted average and jump operators. The convergence analysis yields estimates that are uniform with respect to the diffusivity. The theoretical results are supported by numerical evidence.

Appendix. Proof of Proposition 2.4.

Proof. According to the so-called Banach–Nečas–Babuška (BNB) theorem (see, e.g., [9, section 2.1.3]), the statement amounts to proving that the following conditions hold:

(BNB1) $$\forall z \in V, \quad \sup_{y \in L\backslash\{0\}} \frac{a_0(z,y)}{\|y\|_L} \gtrsim \|z\|_V,$$

(BNB2) $$\forall z \in V, \quad (\forall y \in L, \quad a_0(z,y) = 0) \implies (y = 0).$$

(i) Let us prove (BNB1). Let $z \in V$, and set $\mathbb{S} \overset{\text{def}}{=} \sup_{y \in L\backslash\{0\}} \frac{a_0(z,y)}{\|y\|_L}$. Using the definition of the L^2-norm, we deduce

$$\mathbb{S} \gtrsim \sup_{y \in L\backslash\{0\}} \frac{(Az,y)_L}{\|y\|_L} - \|z\|_L \gtrsim \|Az\|_L - \|z\|_L.$$

Then Lemma 2.3 gives

$$\|z\|_L \lesssim \frac{a_0(z,z)}{\|z\|_L} \lesssim \mathbb{S} \implies \|z\|_L + \|Az\|_L \lesssim \mathbb{S},$$

i.e., $\|z\|_V \lesssim \mathbb{S}$, which proves (BNB1).

(ii) Let us prove (BNB2). Let $y \in L$ be such that

(A.1) $$a_0(z,y) = 0 \quad \forall z \in V.$$

Let us prove that y is necessarily zero.

(a) $U_h = P_{h,1}$

(b) $U_h = P_{h,2}$

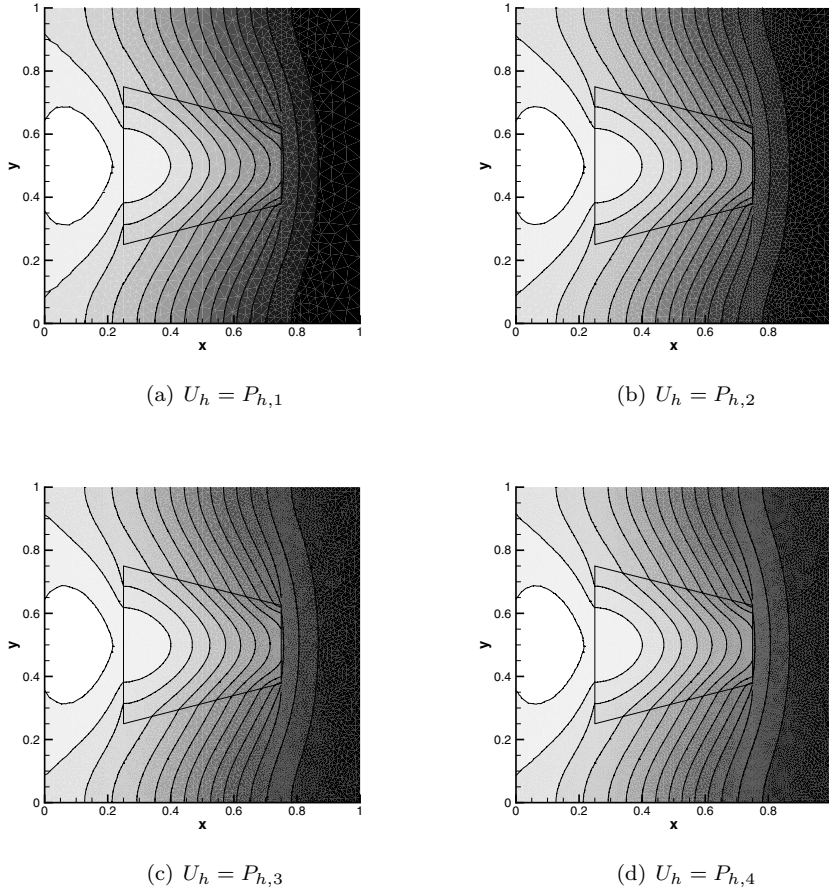(c) $U_h = P_{h,3}$

(d) $U_h = P_{h,4}$

FIG. 3. *Numerical results for the test case of section* 6.2.

(1) Let $\Omega_i \in P_\Omega$, take $z = (z^\sigma, 0)$ with $z^\sigma \in [\mathcal{D}(\Omega_i)]^d$, and observe that z is a member of V. Then using z to test (A.1), we obtain

$$a_0((z^\sigma, 0), y) = \langle y^\sigma - \kappa \nabla y^u, z^\sigma \rangle_{[\mathcal{D}(\Omega_i)]^d} = 0 \quad \forall z^\sigma \in [\mathcal{D}(\Omega_i)]^d,$$

meaning that $y^\sigma - \kappa \nabla y^u = 0$ in Ω_i, i.e., $\kappa \nabla y^u|_{\Omega_i} \in L^2(\Omega_i)$. Now let us prove that y^u is continuous across $\Gamma \setminus I$. Let x be a point in $\Gamma \setminus I$. Let Ω_i and Ω_j be the two subdomains that are on each side of the interface $\Gamma \setminus I$ at x. We assume that x is an interior point of $\overline{\Omega_i} \cap \overline{\Omega_j}$; i.e., all the points in a small neighborhood of x, say $\mathcal{V}$, belong to either $\overline{\Omega_i}$ or $\overline{\Omega_j}$. Up to a Lipschitz map, we can assume that the restriction of Γ to $\mathcal{V}$ is a hyperplane. We choose a local Cartesian coordinate system $(x_1, \ldots, x_d)$ so that the two normals n_1, n_2 are aligned with the x_1-axis. The neighborhood $\mathcal{V}$ can be chosen small enough so that $\mathcal{V} \cap I = \emptyset$. κ being piecewise constant, this means that $\kappa \cdot n_1$ is uniformly bounded away from zero in $\mathcal{V}$; in other words, the component κ_{11} is uniformly bounded away from zero in $\mathcal{V}$.

Let $\psi \in \mathcal{D}(\mathcal{V})$ and define the d-vector field $\tilde{\sigma}$ by $\tilde{\sigma}_1 = \frac{1}{\kappa_{11}} \psi$, and $\tilde{\sigma}_l = 0$ for $l > 1$. Then $\kappa \tilde{\sigma} = (1, \frac{\kappa_{21}}{\kappa_{11}}, \ldots, \frac{\kappa_{d1}}{\kappa_{11}}) \psi$. Observe that $\tilde{\sigma} \in L_\sigma$ and $\nabla \cdot (\kappa \tilde{\sigma}) \in L^2(\Omega)$; in other

words, the pair $(\tilde{\sigma}, 0)$ is in V. By testing (A.1) with $(\tilde{\sigma}, 0)$, we obtain

$$
0 = \int_{\mathcal{V}} [\tilde{\sigma} \cdot y^{\sigma} + \nabla \cdot (\kappa \tilde{\sigma}) y^u]
$$
$$
= -\langle \psi, \partial_{x_1} y^u \rangle_{\mathfrak{D}(\Omega)} + \int_{\mathcal{V}} \psi \frac{1}{\kappa_{11}} (y_1^{\sigma} - \kappa_{12} \partial_{x_2} y^u - \cdots - \kappa_{1d} \partial_{x_d} y^u),
$$

where we used that $\kappa_{12} \partial_{x_2} y^u, \ldots, \kappa_{1d} \partial_{x_d} y^u$ are in $L^2(\mathcal{V})$ from the first part of the argument. A standard distribution argument implies

$$
\partial_{x_1} y^u = \frac{1}{\kappa_{11}} (y_1^{\sigma} - \kappa_{12} \partial_{x_2} y^u - \cdots - \kappa_{1d} \partial_{x_d} y^u) \in L^2(\mathcal{V}).
$$

Then we conclude that $\partial_{x_1} y^u \in L^2(\mathcal{V})$; i.e., y^u is continuous across Γ in a neighborhood of x. Using a standard argument together with x being an interior point of $\overline{\Omega_i} \cap \overline{\Omega_j}$, we infer that $\kappa \nabla y^u$ is in L_{σ}.

(2) Use $z = (0, z^u)$ with $z^u \in \mathfrak{D}(\Omega)$ as a test function in (2.22), and observe that again z is a member of V. A distributional argument gives

$$
\langle (\mu + \nabla \cdot \beta) y^u - \nabla \cdot (\kappa y^{\sigma} + \beta y^u), z^u \rangle_{\mathfrak{D}(\Omega)} = 0.
$$

Owing to the regularity assumptions on μ and β listed in section 2.1, we conclude that $\nabla \cdot (\kappa y^{\sigma} + \beta y^u) \in L^2(\Omega)$, i.e., y is a member of W and

$$
(K + \tilde{A}) y = 0.
$$

(3) We then deduce that, for all $z \in V$,

$$
\langle Dz, y \rangle_{W', W} = ((K + A)z, y)_L - ((K + \tilde{A}) y, z)_L = 0;
$$

i.e., y is a member of $D(V)^{\perp} = V^*$. In conclusion, $a_0^*(y, w) = 0$ for all $w \in L$ and $y \in V^*$. Finally, the L-coercivity of a_0^* (see Lemma 2.3) implies that $y = 0$. $\square$

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