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DISCONTINUOUS GAMES
AND ENDOGENOUS SHARING RULES

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Abstract

We propose a different approach to the kinds of economic problems that lead to discontinuous games. We take the view that the underlying payoffs for these problems are only partially determined, rather than discontinuous. At points where ties occur, we propose that the sharing rule should be determined endogenously, i.e., as part of the solution to the model rather than as part of the description of the model. This leads us to define a game with an endogenous sharing rule. It consists of a strategy space for each of a finite number of players, together with a payoff correspondence, interpreted as the union of all possible sharing rules. A solution for such a game is a selection from the payoff correspondence together with a strategy profile satisfying the usual (Nash) best response criterion. Our principal result is that such a solution always exists.

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1. INTRODUCTION

A number of classical problems in economics are traditionally formulated as games with a continuum of strategies and discontinuous payoffs ("discontinuous games"). The best known of these are Bertrand's (1883) model of duopolistic price competition and Hotelling's (1929) model of duopolistic spatial competition. (Dasgupta and Maskin (1986b) provide many other examples.) In the Bertrand model, firms choose prices; the firm that charges the lower price supplies the entire market. In the Hotelling model, firms choose locations; each firm monopolizes the portion of the market that is closer to it. In each case, discontinuities arise at the points where ties occur (i.e., when the firms charge the same price or locate at the same point).

The conventional way to approach such problems is to include a particular "tie-breaking rule" in the specification of the model. (Since more generally discontinuities may arise from sources other than ties, we prefer the term *sharing rule*.) Given a sharing rule - equal division is typical - one may then investigate conditions under which the resulting discontinuous game has an equilibrium. It has been known for many years that not all such games have equilibria (Sion and Wolfe (1957)). While some existence results have been obtained since then, the conditions required are restrictive, difficult to interpret intuitively, and hard to verify. (See Dasgupta and Maskin (1986a, b) and Simon (1987).)

This paper proposes a different approach to the kinds of economic

problems that lead to discontinuous games. We propose that the underlying payoffs for these problems should be viewed as only partially determined, rather than discontinuous. At points where ties occur (or more generally, wherever the economic nature of the problem leads to indeterminacies), we propose that the sharing rule should be determined endogenously, i.e., as part of the *solution* to the model rather than as part of the *description* of the model. This leads us to define a *game with an endogenous sharing rule*. It consists of a strategy space (assumed to be compact metric) for each of a finite number of players, together with a payoff *correspondence* (assumed to be bounded and upper hemi-continuous, with convex, compact, non-empty values), interpreted as the union of all possible sharing rules. A *solution* for such a game is a selection from the payoff correspondence together with a (mixed) strategy profile satisfying the usual (Nash) best response criterion. Since any selection from the payoff correspondence amounts to a particular sharing rule, it defines a game in the ordinary sense. Hence, a solution for a game with an endogenous sharing rule may equally well be described as a particular sharing rule, together with a Nash equilibrium profile for the resulting game. Our principal result is that such a solution always exists.

As an illustration, we consider the classical Hotelling problem and a simple variant. Two psychiatrists have to choose a location on a portion of Interstate 5 running through California and Oregon. We represent the relevant portion of Interstate 5 by the interval $[0, 4]$; the California portion (which is longer) is represented by the interval $[0, 3]$, and the Oregon portion by $[3, 4]$. There is a continuum of potential clients, uniformly distributed along the Interstate; each client patronizes the

psychiatrist located closer to him. In the classical Hotelling problem, the psychiatrists are free to locate anywhere in the interval $[0, 4]$. For our variant, we assume instead that (as is the case in reality) California and Oregon have licensing rules, and that each psychiatrist is licensed only in his own state. Relicensing is assumed to be infinitely costly, so that neither psychiatrist can relocate to another state. (We maintain the assumption that clients patronize the closer doctor without regard for state lines.)

In each of these problems, an indeterminacy arises when the psychiatrists locate at the same point. In the classical problem, it is conventional to adopt equal division as the sharing rule. This rule seems "obviously" correct on several grounds (equity and focality, in particular), and yields the unique equilibrium that both psychiatrists locate at the midpoint 2. Moreover, any sharing rule that admits any equilibrium at all must reduce to equal division when both psychiatrists locate at the midpoint. For the variant, however, it seems more natural to adopt the division $3/4 - 1/4$ when both psychiatrists locate at the point 3 (the border between California and Oregon). This division gives each psychiatrist his "natural market share" (i.e., the state in which he is licensed). Also, it is the unique sharing rule that makes market shares continuous in locations. Finally, it is the unique division that is individually rational for each of the psychiatrists: by moving in from the border a little, the Californian can guarantee himself a market share of arbitrarily close to $3/4$. Similarly, the Oregonian can guarantee himself a share of arbitrarily close to $1/4$. When the market is divided $3/4 - 1/4$, the game has a unique equilibrium in which both psychiatrists locate at the border (i.e., at 3).

Our next illustration is a Bertrand model. There are two firms. One can produce at a constant marginal cost of \$1 per unit; the other's marginal cost is \$2 per unit. A strategy for a firm is to announce a price. If the firms announce different prices, the one whose price is lower supplies the market. If both firms announce the same price, market shares are indeterminate. It is customary to resolve this indeterminacy by dividing the market equally. With this sharing rule, however, the resulting game violates some assumption of any of the known existence theorems for discontinuous games. (For example, both Dasgupta and Maskin (1986a) and Simon (1987) require that the *sum* of payoffs be an upper semi-continuous function. This condition is violated at the point where both firms announce a price of \$2.) There is, however, an alternative division that seems completely natural: when both firms announce a price of \$2, assign the entire market to the more efficient firm. As in the previous example, this is the unique division of the market that is individually rational for each player. The more efficient firm would not agree to any other division, since by cutting his price slightly, he can capture the entire market and increase his profit. (The less efficient firm is indifferent between all rules at this point.) When the market is divided in this way, the game has a pure strategy equilibrium in which the price charged is \$2.

Thus, in each of these problems, it seems that there is a unique and compelling "right answer" to the question "What should the sharing rule be?" *But these answers are different*. The moral is, simply, that the choice of an appropriate sharing rule for a game must surely take into account the particular context; the same sharing rule should not be expected to work

for all problems. Indeed, even within the same problem, different shares may be appropriate at different points in the strategy space. The main result of our paper is that for any game in which indeterminacies arise because of an unspecified sharing rule, there will always be at least *one* rule that is consistent with the existence of equilibrium.

What does it mean to endogenize the sharing rule? One alternative has already been suggested. We can view the determination of the sharing rule as part of a preplay agreement between the players themselves. In each of our examples, there was a unique division that was individually rational for each player. At least in these cases, it is hard to imagine the players agreeing upon any *other* division. Thus, once we view the sharing rule as determined by preplay agreement, there is no reason to expect that equal division will be the chosen sharing rule in every instance.

A second alternative is to view the sharing rule as a statistic, summarizing the actions taken by unseen agents whose behavior is not modelled explicitly. In each of the examples above, these agents are the consumers. The sharing rule that is part of our solution can be thought of as a proxy for consumers' equilibrium behavior in an appropriately specified extensive form game. In our variant of the Hotelling model, for example, this game would have a continuum of consumers. The psychiatrists would first choose their locations, then each consumer would choose which doctor to patronize. When this view is taken, there is a one-to-one relationship between the set of all possible sharing rules (which is the payoff correspondence for the game with an endogenous sharing rule) and the correspondence mapping first stage locations into second stage equilibrium

choices by consumers. If both psychiatrists choose the same location in the first stage, then any division of the market can be implemented as a Nash equilibrium of the associated second-stage subgame. The unique division that is consistent with Nash equilibrium for the game as a whole is the division $3/4 - 1/4$.¹ A very similar extensive form game can be specified for the Bertrand example.

There is, in the treatment of indeterminacy here, an analogy with general equilibrium theory. An equilibrium allocation is by definition a family of choices from individual excess demand sets that clears the market. When preferences are not strictly convex, excess demand will not generally be single-valued. In this case, there will generally be prices for which some families of choices clear the market (and hence are equilibrium allocations), while others do not. In our game-theoretic context, choosing a sharing rule *in advance* is analogous to specifying a selection from the excess demand correspondences *in advance*. This hardly seems like a sensible approach, in either the general equilibrium context or in ours. One cannot push this analogy too far, however. It is well known that generically, market excess demand is single valued in continuum economies. (See Mas-Colell and Neufeind (1977).) Therefore, the selection issue arises only rarely in general equilibrium theory, while in our context, it is the typical case.

1 - To solve this game, we would have to confront exactly the problems that are addressed in this paper. That is, optimizing behavior by these players would generate discontinuities in firms' payoff functions of a kind to which extant equilibrium existence results would not apply.

One could ask *why* consumers behave in the way summarized by our endogenously determined sharing rule. The answer is as it always is: equilibrium theory *never* explains why *any* agents act in *any* particular way. The concern of equilibrium theory is with how the *economy* behaves, given a particular specification of agents' characteristics, not with *why* consumers behave in a certain way.

A final comment: Games with infinitely many strategies are sometimes viewed as proxies for games with a large finite number of strategies. From this point of view it is the equilibria (or approximate-equilibria) of the finite games which are of real interest; equilibria of the infinite games are merely convenient proxies. Ideally, it would be the case that (1) every limit of equilibria of finite games is an equilibrium of the infinite game, and (2) every equilibrium of the infinite game is a limit of equilibria (or approximate-equilibria) of nearby finite games. (See Fudenberg and Levine (1986).) Our approach is not inconsistent with this view. Indeed, the proof of our existence theorem shows that (1) is true, in the sense that every limit of equilibria of finite games is an equilibrium for the infinite game defined by *some* sharing rule. (2) remains an open (and apparently difficult) problem.

2. GAMES WITH ENDOGENOUS SHARING RULES

An N -player *game with an endogenous sharing rule* is an $N+1$ -tuple $\Gamma = (S_1, \dots, S_N, Q)$ consisting of a *strategy space* S_i for each player and a *payoff correspondence* $Q: S = S_1 \times S_2 \times \dots \times S_N \rightarrow \mathbb{R}^N$. An element $s \in S$ is a *strategy profile*. We interpret $Q(s)$ as the universe of utility possibilities given the strategy profile s . Throughout, we assume that each of the strategy spaces S_i is a compact metric space and that the payoff correspondence Q is bounded and upper hemi-continuous, with non-empty, convex, compact values. Upper hemi-continuity of the correspondence Q means that the set of utility possibilities for each strategy profile is at least as large as the set of limits of utility possibilities of nearby profiles. Convexity of each $Q(s)$ means that we allow for randomization. A *sharing rule* is a Borel measurable selection from the correspondence Q ; i.e., a Borel measurable function $q: S \rightarrow \mathbb{R}^N$ such that $q(s) \in Q(s)$ for each $s \in S$. Since $Q(s)$ is the universe of utility possibilities given the strategy profile s , a sharing rule is just a particular choice of payoff at each point of the space of strategy profiles. The reader should keep in mind that the payoff correspondence Q will not generally admit *continuous* selections.

As the Introduction suggests, many games with endogenous sharing rules arise from economic situations for which payoff functions are well-defined and continuous on a large set, but indeterminate elsewhere. In the classical Hotelling problem, for instance, payoffs are well-defined and continuous everywhere except on the diagonal. More generally, suppose we are given strategy spaces S_i , a dense subset S^* of S , and

a bounded continuous function $\varphi : S^* \rightarrow \mathbb{R}^N$. Let $C_\varphi : S \rightarrow \mathbb{R}^N$ be the correspondence whose graph is the closure of the graph of φ , and define $Q_\varphi(s)$ to be the convex hull of $C_\varphi(s)$ for each $s \in S$; we call the correspondence Q_φ the *convex completion* of φ . It is not hard to see that Q_φ is bounded and upper hemi-continuous, that it has non-empty, convex, compact values, and that $Q_\varphi(s) = \varphi(s)$ for each $s \in S^*$ (because φ is continuous on S^*). (The correspondence Q_φ may also be described as the smallest upper hemi-continuous, compact and convex-valued correspondence from S into \mathbb{R}^N which agrees with φ on S^* .) Note that any selection q from the correspondence Q_φ agrees with φ on S^* , and hence every sharing rule is an extension of the given payoff function φ on S^* to the entire space S of strategy profiles. The failure of φ to be specified at points of $S - S^*$ is an indeterminacy, and a sharing rule resolves the indeterminacy. Of course, as is the case in the classical Hotelling problem, the function φ need not admit any *continuous* extension to all of S .

There are games with endogenous sharing rules that do not arise in this way. However, it is probably true that most of the economically interesting examples are of this form, and the reader should think of them as typical.

As usual a *mixed strategy* for player i is a probability measure on S_i , and a *mixed strategy profile* is an N -tuple $(\alpha_1, \dots, \alpha_N)$ of mixed strategies. We write $S_{-i} = S_1 \times \dots \times S_{i-1} \times S_{i+1} \times \dots \times S_N$, and $\alpha_{-i} = \alpha_1 \times \dots \times \alpha_{i-1} \times \alpha_{i+1} \times \dots \times \alpha_N$. We abuse notation to identify $S_i \times S_{-i}$ with S and $\alpha_i \times \alpha_{-i}$ with $\alpha = \alpha_1 \times \alpha_2 \times \dots \times \alpha_N$.

A *solution* for Γ is a sharing rule and a mixed strategy profile with the property that, given the sharing rule, each player's action is a best response to the actions of other players. More precisely, a solution for Γ is a pair $(q, (\alpha_1, \dots, \alpha_N))$, where q is a Borel measurable selection from the payoff correspondence Q and $(\alpha_1, \dots, \alpha_N)$ is a profile of mixed strategies, with the property that, for each i , and for each probability measure β_i on S_i ,

$$\int q_i(s) d(\alpha_i \times \alpha_{-i}) \geq \int q_i(s) d(\beta_i \times \alpha_{-i}) .$$

We could also adopt an alternative view. Each sharing rule q defines a game Γ_q in the usual sense (perhaps with discontinuous payoffs), with strategy spaces S_i and payoff function q . A solution for Γ is simply a sharing rule q , together with a profile of mixed strategies which constitutes a Nash equilibrium for the game Γ_q . Once again, the reader should keep in mind that the typical payoff correspondence Q will not admit selections which are continuous, or even nearly so. In particular, none of the known equilibrium existence results may apply to any of the games Γ_q .

Our central result is:

THEOREM: Every game with an endogenous sharing rule has a solution.

3. EXAMPLE

In this Section, we discuss a simple game with an endogenous sharing rule. (This particular example was chosen for illustrative purposes and is not intended to represent an economic situation.) In conjunction with the proof of the Theorem, the example is intended to point out that the values of the solution sharing rule q must be chosen carefully at the equilibrium strategy profile *and* at all profiles in which only one player is deviating.

The game has two players, with strategy spaces $S_1 = S_2 = [-1, 1]$. Write $S = [-1, 1] \times [-1, 1]$ for the set of strategy pairs, and let $S^* = \{(s_1, s_2) : s_1 s_2 \neq 0\}$; i.e., S^* is the set of strategy pairs for which neither player plays 0. As in Section 2, we consider a payoff function which is defined and continuous on S^* and indeterminate elsewhere; the particular payoff function ϕ we have in mind is defined in Figure 1.

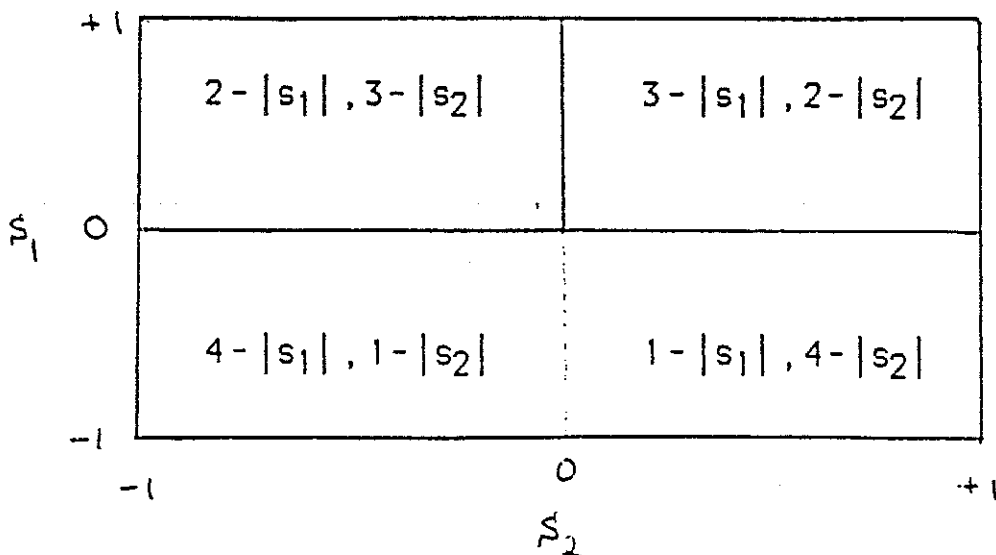


FIGURE 1

(In keeping with the usual tradition that the first player plays "rows" we have labelled axes so that the first coordinate is on the vertical axis.) The convex completion Q of this function is obtained by closing the graph of ϕ and convexifying the values of the resulting correspondence. Hence:

$$\begin{aligned} Q(s_1, 0) &= \{(2 - |s_1| + t, 3 - t) : 0 \leq t \leq 1\} \text{ for } s_1 > 0; \\ Q(s_1, 0) &= \{(1 - |s_1| + t, 4 + t) : 0 \leq t \leq 3\} \text{ for } s_1 < 0; \\ Q(0, s_2) &= \{(3 - t, 2 - |s_2| + t) : 0 \leq t \leq 1\} \text{ for } s_2 > 0; \\ Q(0, s_2) &= \{(4 - t, 1 - |s_2| + t) : 0 \leq t \leq 3\} \text{ for } s_2 < 0; \\ Q(0, 0) &= \{(1 + t, 4 - t) : 0 \leq t \leq 3\}; \\ Q(s_1, s_2) &= \{\phi(s_1, s_2)\} \text{ for } (s_1, s_2) \in S^* . \end{aligned}$$

As noted before, every selection from the correspondence Q (i.e., every sharing rule) gives rise to a game in the ordinary sense. In this case, there is no continuous selection from Q . In fact, no selection from Q is sufficiently well-behaved to satisfy the requirements of any of the known existence theorems for discontinuous games. However, we shall see that there are many selections which admit equilibria.

Let $(q, (\alpha_1, \alpha_2))$ be a solution for this game, so that q is a sharing rule and (α_1, α_2) is a mixed strategy profile. Note that, so long as neither player plays 0, each player's payoff is strictly decreasing in the absolute value of his own strategy. Simple dominance arguments imply that at least one player assigns probability 1 to the strategy 0. In fact, the equilibrium strategy profiles of this game fall into two distinct classes: (1) the pure strategy equilibria in which both players assign probability 1 to the strategy 0 (as we shall see, many different sharing rules lead to such equilibrium strategy profiles), and (2) equilibria in

which exactly one player assigns probability 1 to the strategy 0 and the other player randomizes. We focus on the pure strategy equilibria, and leave the others for the reader to investigate.

For the pure strategy equilibria, the first issue is the value $q(0, 0)$ of the sharing rule q at $(0, 0)$. By definition, $q(0, 0)$ belongs to the set $Q(0, 0) = \{(1+t, 4-t) : 0 \leq t \leq 3\}$, but we can in fact say more. If player 1 is known to play 0 with probability 1, player 2 can *guarantee* himself a payoff of at least $2-\epsilon$ for $\epsilon > 0$ as small as he likes (because the sharing rule is a selection from the correspondence Q and player 2 obtains a payoff of at least $2-\epsilon$ at every point of $Q(0, \epsilon)$). Similarly, if player 2 is known to play 0 with probability 1, player 1 can guarantee *himself* a payoff of at least $2-\epsilon$ for $\epsilon > 0$ as small as *he* likes. This constrains $q(0, 0)$ to lie in the set $E = \{(2+t, 3-t) : 0 \leq t \leq 1\}$.

In fact, any vector in E can be supported as the value of $q(0, 0)$ in a pure strategy equilibrium, but only if we are careful about the values of q elsewhere. Note that player 1 can reach any point on the *vertical* axis by deviating, while player 2 can reach any point on the *horizontal* axis by deviating. The values of q along these axes must therefore satisfy the constraints: $q_1(s_1, 0) \leq q_1(0, 0)$ and $q_2(0, s_2) \leq q_2(0, 0)$. This will be the case if we choose $q(s_1, 0)$ to minimize the payoff of player 1 and choose $q(0, s_2)$ to minimize the payoff of player 2 (but other choices may also be possible).

4. PROOFS

Before commencing the proof of the Theorem, we collect some preliminaries. The first of these is a simple lemma concerning weak convergence of measures and integration of lower semi-continuous functions. (Recall that $f : S \rightarrow \mathbb{R}$ is *lower semi-continuous* if $\liminf_{s' \rightarrow s} f(s') \geq f(s)$ for each $s \in S$.)

LEMMA 1: Let S be a compact metric space, let $\{\mu_n\}$ be a sequence of positive measures on S converging weakly to μ , and let f be a lower semi-continuous real-valued function on S . Then

$$\liminf \int f d\mu_n \geq \int f d\mu .$$

(Note: If f were *continuous*, the definition of weak convergence of measures would of course imply that $\lim \int f d\mu_n = \int f d\mu$.)

PROOF: Since f is lower semi-continuous, there is a sequence $\{f_k\}$ of continuous functions such that $f_k(s) \leq f_{k+1}(s)$ and $f_k(s) \rightarrow f(s)$ for each $s \in S$. Lebesgue's monotone convergence theorem implies that

$$(*) \quad \int f_k d\mu \rightarrow \int f d\mu .$$

Each of the measures μ_n is positive, each of the functions f_k is continuous, and $f_k(s) \leq f(s)$ for each $s \in S$; hence

$$(**) \quad \liminf \int f d\mu_n \geq \liminf \int f_k d\mu_n = \int f_k d\mu$$

for each k . Combining (*) and (**) yields the desired result. ■

We now recall some facts about absolute continuity of measures and the Radon-Nikodym theorem. Consider two real-valued Borel measures μ, ν defined on the same measure space, with μ positive. (In our applications μ and ν will be Borel measures on a compact metric space.) By definition, ν is *absolutely continuous with respect to* μ if $\nu(E) = 0$ whenever $\mu(E) = 0$. The *Radon-Nikodym theorem* asserts that ν is absolutely continuous with respect to μ if and only if there is a measurable, μ -integrable, real-valued function f (called the *Radon-Nikodym derivative of ν with respect to μ*) with the property that $\nu(F) = \int_F f d\mu$ for every measurable set F ; it is customary to abbreviate this by writing $\nu = f\mu$. (When such a function f exists, it is uniquely determined except for its values on a set of μ -measure zero.) It follows that, for every measurable, ν -integrable function g , $\int g d\nu = \int gf d\mu$. (For further discussion, see Royden (1968) for example.)

If ν takes its values in \mathbb{R}^N instead of in \mathbb{R} (but μ remains real-valued and positive), we can obtain entirely analogous results simply by treating the components of ν individually, but interpreting the results in vector notation. Thus, to say that ν is absolutely continuous with respect to μ means that $\nu(E) = 0$ (the zero vector in \mathbb{R}^N) whenever $\mu(E) = 0$. As in the scalar case, this will be so exactly when there is a measurable, μ -integrable function f taking values in \mathbb{R}^N such that $\nu(F) = \int_F f d\mu$ for every measurable set F ; we continue to abbreviate this by writing $\nu = f\mu$.

The second lemma deals with weak convergence of absolutely continuous measures and selections from a correspondence; it is an analog of a lemma of Artstein (1979).

LEMMA 2: Let S be a compact metric space, let $\{\mu_n\}$ be a sequence of positive measures on S converging weakly to μ , and let $Q : S \rightarrow \mathbb{R}^N$ be a bounded, upper hemi-continuous correspondence on S with compact, convex, non-empty values. For each n , let q_n be a Borel measurable selection from Q . If the sequence $\{q_n \mu_n\}$ of vector-valued measures converges weakly to the vector-valued measure ν , then ν is absolutely continuous with respect to μ and there is a Borel measurable selection q from Q such that $\nu = q\mu$.

PROOF: Since Q is bounded, there is a constant C such that $\sum |x_i| \leq C$ for every $x \in Q(s)$ and every $s \in S$. If \hat{q}_n is the sum of the absolute values of the components of q_n , then $0 \leq \hat{q}_n(s) \leq C$ for every $s \in S$. Hence, $0 \leq \hat{q}_n \mu_n \leq C \mu_n$, so that, if $\hat{\nu}$ is the sum of the components of ν , we obtain that $0 \leq \hat{\nu} \leq C \mu$. In particular, this means that ν is absolutely continuous with respect to μ , so the Radon-Nikodym theorem provides a Borel measurable function $q : S \rightarrow \mathbb{R}^N$ such that $\nu = q\mu$. (Note: this of course depends crucially on the boundedness of Q .) Since q is determined almost everywhere (with respect to μ), this means we need only show that $q(s) \in Q(s)$ almost everywhere (with respect to μ), and then correct q on the set of measure zero where $q(s) \notin Q(s)$.

To this end, set $E = \{s \in S : q(s) \notin Q(s)\}$. Since Q is compact-valued and upper hemi-continuous, and q is a Borel measurable function, the set E is itself a Borel set; we need to show that $\mu(E) = 0$.

Since $Q(s)$ is a compact, convex set (for each s), any point of \mathbb{R}^N not belonging to $Q(s)$ can be separated from $Q(s)$ by a hyperplane; in particular, if $q(s) \notin Q(s)$ then there is a vector $\xi \in \mathbb{R}^N$ and there are real numbers a, a' such that $\xi \cdot q(s) > a > a' > \xi \cdot y$ for every $y \in Q(s)$. Since \mathbb{R} and \mathbb{R}^N are separable, we can choose countable dense subsets $\{a_j\}$ of \mathbb{R} and $\{\xi_k\}$ of \mathbb{R}^N ; for each j, j' and k , and write:

$$E(j, j', k) = \{s \in S : \xi_k \cdot q(s) > a_j > a_{j'} > \xi_k \cdot y \text{ for every } y \in Q(s)\}.$$

Since $\{a_j\}$ is a dense subset of \mathbb{R} and $\{\xi_k\}$ is a dense subset of \mathbb{R}^N , the union of all the sets $E(j, j', k)$ is E ; hence at least one of them has positive μ -measure. After relabeling, we conclude that there is a vector $\eta \in \mathbb{R}^N$ and real numbers b, c , so that the set

$$F = \{s \in S : \eta \cdot q(s) > b > c > \eta \cdot y \text{ for every } y \in Q(s)\}$$

has positive μ -measure.

Since the measure μ is regular, we can find a compact set $K \subset F$ which also has positive μ -measure. Since Q is upper hemi-continuous, there is an open subset U of S which contains K and has the property that $c > \eta \cdot y$ for every $y \in Q(u)$ and every $u \in U$; we may also choose U so that $\mu(U) - \mu(K) = \mu(U - K)$ is as small as we like. Applying Urysohn's lemma yields a continuous function $f : S \rightarrow [0, 1]$ which is

identically 1 on K and identically 0 on the complement of U .

We now consider the integral $\int f(s)\eta \cdot q(s) d\mu$. On the one hand, our construction, together with weak convergence of $q_n\mu_n$ to $\nu = q\mu$ and continuity of $f\eta$, yields:

$$\begin{aligned}
 (*) \quad \int f(s)\eta \cdot q(s) d\mu &= \lim \int f(s)\eta \cdot q_n(s) d\mu_n \\
 &= \lim \int_U f(s)\eta \cdot q_n(s) d\mu_n \\
 &\leq \lim \int_U f(s)c d\mu_n \\
 &= \lim \int_S f(s)c d\mu_n \\
 &= \int f(s)c d\mu \\
 &< c\mu(U).
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 (**) \quad \int f(s)\eta \cdot q(s) d\mu &= \int_K f(s)\eta \cdot q(s) d\mu + \int_{U-K} f(s)\eta \cdot q(s) d\mu \\
 &\geq b\mu(K) - \|\eta\|c\mu(U-K).
 \end{aligned}$$

Since $\mu(U-K) = \mu(U-K)$ may be made as small as we like, it follows that $\int f(s)\eta \cdot q(s) d\mu$ is almost as large as $b\mu(K)$. On the other hand, we have already seen that $\int f(s)\eta \cdot q(s) d\mu$ can be no bigger than $c\mu(U)$. When $\mu(U-K) = \mu(U-K)$ is small, $\mu(U)$ is close to $\mu(K)$; since $b > c$, this is a contradiction. The proof is now complete. \square

With these preliminaries out of the way, we are ready for the proof of the Theorem. For the guidance of the reader, we preface the proof proper with a brief overview. The proof is divided into six steps. Given a game Γ with an endogenous sharing rule, we begin (Step 1) by constructing a family $\{\Gamma^r\}$ of finite games which "approximate" Γ . Each of these finite games has a Nash equilibrium, which can be viewed as a mixed strategy profile for Γ . An appropriate subsequence of these profiles converges to a mixed strategy profile $(\alpha_1, \dots, \alpha_N)$, and an appropriate subsequence of payoff functions for the games Γ^r also converges - in the sense of Lemma 2 - to a function q (Step 2); moreover, q is a sharing rule for Γ (Step 3). The desired solution mixed strategy profile will be $(\alpha_1, \dots, \alpha_N)$, and the solution sharing rule will be a perturbation of q . This perturbation may be necessary because the limit sharing rule q is determined only up to sets of $\alpha_1 \times \dots \times \alpha_N$ -measure zero. This leaves open the possibility that there are pure strategies for the i -th player which are superior to the strategy α_i (see the Example of Section 3). However, the set of such pure strategies is of measure zero (Step 4), and they can be eliminated by perturbing q (Step 5). Finally, it only needs to be verified that the perturbed sharing rule, together with the mixed strategy profile $(\alpha_1, \dots, \alpha_N)$, constitutes a solution for the game Γ (Step 6). This follows from Lemma 1 and the specific construction of q .

PROOF OF THE THEOREM: As outlined above, the proof is in six steps.

STEP 1: Finite approximations. For each i and each $r = 1, 2, \dots$ choose a finite subset S_i^r of S_i so that the Hausdorff distance between S_i^r and S_i is at most $1/r$. For each r , choose a Borel measurable

selection q^Γ from Q (we could take each of the selections q^Γ to be the same, but there is no *need* to do so). Let $\Gamma^\Gamma = (S_1^\Gamma, \dots, S_N^\Gamma, q^\Gamma)$, so that Γ^Γ is a finite game. Let $(\alpha_1^\Gamma, \dots, \alpha_N^\Gamma)$ be a Nash equilibrium for the game Γ^Γ . Write $\alpha^\Gamma = \alpha_1^\Gamma \times \dots \times \alpha_N^\Gamma$.

STEP 2: Limits. Each α_i^Γ is a probability measure on S_i^Γ , which is a subset of S_i , and hence may be regarded as a probability measure on S_i that is supported on S_i^Γ . Taking this point of view, we may (passing to a subsequence if necessary) assume that, for each i the sequence $\{\alpha_i^\Gamma\}$ converges weakly to a probability measure α_i on S_i . Similarly, we may regard each α^Γ as a probability measure on S which is supported on $S^\Gamma = S_1^\Gamma \times \dots \times S_N^\Gamma$, and each $q^\Gamma \alpha^\Gamma$ as an \mathbb{R}^N -valued vector measure on S which is supported on S^Γ . We may also assume that the sequence $\{q^\Gamma \alpha^\Gamma\}$ converges weakly to a vector measure ν .

STEP 3: Selections. By Lemma 2, the measure ν is absolutely continuous with respect to α and there is a Borel measurable selection q from Q such that $\nu = q\alpha$. (In an appropriate sense, the selections q^Γ converge to the selection q . We caution the reader, however, that the sense of this convergence is very weak; in general it need not be the case that $q^\Gamma(s)$ converges to $q(s)$ for any strategy profile $s \in S$. Of course, since q^Γ and q are selections from Q it will certainly be the case that $q^\Gamma(s) = q(s)$ whenever $Q(s)$ is a singleton. If Q is a convex completion (as in Section 2), this certainly means that q^Γ and q agree except at points of indeterminacy.)

STEP 4: Better responses. For each i , let q_i be the i -th component of q and set:

$$E_i = \{x \in S_i : \int q_i d(\delta_x \times \alpha_{-i}) > \int q_i d(\alpha_i \times \alpha_{-i}) \},$$

so that the set E_i consists of those pure strategies for player i which are better than α_i in response to α_{-i} . Since q_i is a Borel measurable function, the set E_i is a Borel set; we **claim** that $\alpha_i(E_i) = 0$.

Suppose this is not so; we construct a strategy for player i in some game Γ^r which is a better response to α_{-i}^r than is α_i^r . Regularity of the measure α_i means that we can find a compact set $K \subset E_i$ so that $\alpha_i(K) > 0$. Let U be an open subset of S_i which contains K ; write $\varepsilon = \alpha_i(U - K) = \alpha_i(U) - \alpha_i(K)$, and note that we can choose U so that ε is as small as we like.

We now argue as in the proof of Lemma 2. Urysohn's lemma provides a continuous function $f : S_i \rightarrow [0, 1]$ which is identically 1 on K and identically 0 on the complement of U . Weak convergence of measures and continuity of the function f yield that:

$$\int q_i d(\alpha_i \times \alpha_{-i}) = \lim \int q_i^r d(\alpha_i^r \times \alpha_{-i}^r)$$

$$\int f q_i d(\alpha_i \times \alpha_{-i}) = \lim \int f q_i^r d(\alpha_i^r \times \alpha_{-i}^r) .$$

We can use Fubini's theorem to write $\int f q_i d(\alpha_i \times \alpha_{-i}) = \int \int f q_i d\alpha_{-i} d\alpha_i$, where the inner integral is over S_i and the outer integral is over S_{-i} . Since f is identically 1 on K and identically 0 in the complement of U , we obtain:

$$\iint f q_i d\alpha_{-i} d\alpha_i = \iint_K q_i d\alpha_{-i} d\alpha_i + \iint_{U-K} f q_i d\alpha_{-i} d\alpha_i$$

The last of these double integrals can be made as small as we like by making ε small. The double integral involving integration over K can be estimated by reversing the order of integration, observing that

$q(x, \cdot) = \int q_j d\delta_x$, and keeping in mind that the set K is contained in E_j :

$$\begin{aligned} \iint_K q_i d\alpha_{-i} d\alpha_i &= \int_K \int q_i(x, \cdot) d\alpha_{-i} d\alpha_i(x) \\ &= \int_K \iint q_j d\delta_x d\alpha_{-i} d\alpha_i(x) \\ &> \int_K \iint q_j d\alpha_j d\alpha_{-i} d\alpha_i(x) \\ &= \alpha_j(K) \iint q_j d\alpha_{-i} d\alpha_i . \end{aligned}$$

For ε very small (so that $\alpha_j(K)$ is nearly $\alpha_j(U)$ and $\int f d\alpha_i$ is nearly $\alpha_i(K)$), we now consider the strategy $\beta_i^\Gamma = \{1/(\int f d\alpha_i^\Gamma)\} f \alpha_i^\Gamma$ for player i in the game Γ^Γ . Note that $\int f d\alpha_i^\Gamma \rightarrow \int f d\alpha_i$, which is nearly $\alpha_i(K)$ for ε small enough. Together with the above inequalities, this implies that β_i^Γ is a better response than α_i^Γ to α_{-i}^Γ in the game Γ^Γ , provided that r is sufficiently large and ε is sufficiently small. However, this contradicts the fact that $(\alpha_1^\Gamma, \dots, \alpha_N^\Gamma)$ is an equilibrium, and this contradiction establishes our claim that $\alpha_j(E_j) = 0$.

STEP 5: Perturbation. We can now perturb q to obtain the desired sharing rule \tilde{q} . For each i , let \tilde{p}^i be any Borel measurable selection from Q which minimizes the i -th component; write $\tilde{p}_i^i(s)$ for the i -th component of $\tilde{p}^i(s)$. Let $T = \{s \in S : s_j \in E_j \text{ for at least two indices } i\}$, and define \tilde{q} as follows:

$$\begin{aligned}\bar{q}(s) &= \bar{p}_i(s) && \text{if } s \in E_i \times S_{-i} \text{ but } s \notin T, \\ \bar{q}(s) &= q(s) && \text{otherwise.}\end{aligned}$$

Note that $(\alpha_1 \times \dots \times \alpha_N)(E_i \times S_{-i}) = 0$ for each i , so that $\bar{q}(s)$ agrees with $q(s)$ except on a set of $(\alpha_1 \times \dots \times \alpha_N)$ -measure 0. In particular, given the mixed strategy profile $(\alpha_1, \dots, \alpha_N)$, the change from q to \bar{q} leaves each player's expected payoff unchanged.

STEP 6: Solution. We claim that the sharing rule \bar{q} and the mixed strategy profile $(\alpha_1, \dots, \alpha_N)$ together constitute a solution for the game Γ . To show that this is so, we ask whether player i has a better response than α_i to α_{-i} . If so, he has a better pure strategy response; call it δ_x . We distinguish two cases.

CASE 1: $x \notin E_i$. Note that

$$(1) \quad \iint \bar{q}_i d\alpha_i d\alpha_{-i} = \iint q_i d\alpha_i d\alpha_{-i}$$

$$(2) \quad \iint \bar{q}_i d\delta_x d\alpha_{-i} = \int \bar{q}_i(x, \cdot) d\alpha_{-i} = \int q_i(x, \cdot) d\alpha_{-i}$$

because we have altered $q_i(x, \cdot)$ only on the set $[x \times S_{-i}] \cap [\cup (E_j \times S_{-j})]$ (the union taken over $j \neq i$), which has α_{-i} measure 0 since $x \notin E_i$. The fact that $x \notin E_i$ also gives:

$$(3) \quad \int q_i(x, \cdot) d\alpha_{-i} \leq \iint q_i d\alpha_i d\alpha_{-i}.$$

Combining (1), (2), (3) yields that δ_x is *not* a better response than α_i

to α_{-i} ; this disposes of CASE 1.

CASE 2: $x \in E_i$. Since Q is an upper hemi-continuous correspondence, the function p_i defined in Step 5 is lower semi-continuous; moreover, $p_i(s) = \bar{q}_i(s)$ if $s \in E_i \times S_{-i}$ and $s \notin T$. Choose a sequence $\{x^r\}$ converging to x with the property that $x^r \in S_i^r$ for each r . Then:

$$\begin{aligned}
 (4) \quad \int \bar{q}_i(x, \cdot) d\alpha_{-i} &= \iint p_i d\delta_x d\alpha_{-i} \\
 &\leq \liminf \iint p_i d\delta_{x^r} d\alpha_{-i}^r \quad (\text{by Lemma 1}) \\
 &\leq \liminf \iint q_i^r d\delta_{x^r} d\alpha_{-i}^r
 \end{aligned}$$

because p_i minimizes the payoff to player i at each point. If δ_x were a better response than α_{-i} it would follow, putting (1), (2), (4) together, that, for r sufficiently large,

$$\iint q_i^r d\delta_{x^r} d\alpha_{-i}^r > \iint q_i^r d\alpha_{-i}^r d\alpha_{-i}^r,$$

which would contradict the fact that $(\alpha_1^r, \dots, \alpha_N^r)$ is a Nash equilibrium profile in the game Γ^r . This contradiction completes the argument of CASE 2, and with it the proof of the THEOREM. ■

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