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DISCONTINUOUS GROUPS OF AFFINE TRANSFORMATIONS OF C^3

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1. **Introduction.** Let G be a group of affine transformations acting freely and properly discontinuously on C^n. Suppose that C^n/G is compact. Let G_0 be the subgroup of G consisting of translations, which is a normal subgroup of G. Moreover we assume that $H = G/G_0$ is a finite group. Enriques and Severi show that in the case of surfaces i.e., $n = 2$, H is a cyclic group of order d, $d = 1, 2, 3, 4, 6$, [1]. In this paper in the case of $n = 3$ we shall prove the following

THEOREM. *If H is cyclic, then* $H \cong Z/d$, $d = 1, 2, 3, 4, 5, 6, 8, 10, 12$. *If H is not cyclic but abelian, then* $H \cong Z/d_1 \oplus Z/d_2$, $(d_1, d_2) = (2, 2)$, $(2, 4)$, $(2, 6)$, $(2, 12)$, $(3, 3)$, $(3, 6)$, $(4, 4)$, $(6, 6)$. *Finally, if H is not abelian, then H is D_4: a dihedral group of order* 8.

2. Let g be an affine transformation of C^n i.e., $gx = A(g)x + a(g)$ where $x \in C^n$, $A(g) \in GL(n, C)$, $a(g) \in C^n$. If g has no fixed points, then at least one eigenvalue of $A(g)$ has to be 1. It is easy to see that if g has no fixed points, then g^m has no fixed points. We call $A(g)$ the holonomy part of g and A a holonomy representation.

PROPOSITION 1. *Let G be the group in Introduction. If K is an abelian subgroup of G with finite index, then G_0 contains K i.e., G_0 is the largest abelian subgroup of G with finite index.*

PROOF. As K is commutative, all the elements of K can be diagonalized simultaneously. Suppose $K - G_0 \neq \emptyset$ and choose $g \in K - G_0$. Then $gx_j = \alpha_j x_j + a_j$, where $\alpha_1 = 1$, $\alpha_n \neq 1$. May assume $a_n = 0$, because otherwise we consider hgh^{-1} instead of g, h being a translation defined by ${}^t(0, \cdots, 0, a_n/(\alpha_n - 1))$. Owing to the commutativity of K this implies that any $g' \in K$ acts like $g'x_n = \beta_n x_n$. Hence C^n/K is not compact, which contradicts the assumption $|G:K| < \infty$.

COROLLARY 1. *Let G' be the group similar to G. If $G \simeq G'$ by an isomorphism φ, then $\varphi G_0 = G'_0$. Hence $H = G/G_0 \simeq H' = G'/G'_0$.*

PROOF. $\varphi(G_0) \subset G'_0$, and $\varphi^{-1}(G'_0) \subset G_0$, by Proposition 1.

Thus, G_0 and H depend only on the group structure of G.

3. In what follows we assume $n = 3$.

PROPOSITION 2. *The order of any element* $\bar{g} \in H$ *is one of* 1, 2, 3, 4, 5, 6, 8, 10, 12. *Hence the first part of Theorem is proved.*

PROOF. Let Ω denote the period matrix of the torus C^3/G_0. Since $gG_0g^{-1} = G_0$, it follows $A\Omega = \Omega N$, where A is the holonomy part of g and N an integral matrix. Eigenvalues of N are m-th roots of 1. Since $N \in GL(6, \boldsymbol{Z})$, $\varphi(m) \leqq 4$. Hence $m = 1, 2, 3, 4, 5, 6, 8, 10, 12$.

REMARK 1. Det $G \subset \boldsymbol{C}^*$ is a cyclic group isomorphic to $\boldsymbol{Z}/d$, $d = 1, 2, 3, 4, 5, 6, 12$. Actually, any element $\bar{g} = A(g)$ of order 10 is mapped to $\det A(g)$ whose order is 5. The similar argument is available to exclude the case of order 8.

Let $G_1 = \{g \in G; \det A(g) = 1\}$. Then the order m of $\bar{g}_1 \in H_1 = G_1/G_0$ is 1, 2, 3, 4, 6 because $\varphi(m) \leqq 2$. Hence the order of H_1 is 2^a3^b. Since H is an extention of H_1 by a cyclic group $\det G \subset \boldsymbol{C}^*$, we have

PROPOSITION 3. *H is a solvable group.*

LEMMA 1. *If* $^{\#}H = |H: 1|$ *is a multiple of* 5, *then* $^{\#}H = 2^a3^b5$.

PROOF. By Remark 1, the cyclic group det G is $\boldsymbol{Z}/5$. Hence by $^{\#}H = {}^{\#}(G/G_1) \cdot {}^{\#}H_1$, we obtain the result.

PROPOSITION 4. *H has no abelian subgroup of type* (p, p, p). *Moreover H_1 has no abelian subgroup of type* (q, q), $q = 3, 4, 6$.

Proof. Let K be an abelian subgroup of H. Then $K = C_1 \times C_2 \times C_3$, where each C_i is a cyclic group acting on $\boldsymbol{C}^3$. If K is of type (p, p, p), then each $C_i \cong \boldsymbol{Z}/p$. Hence, a general element of K has not 1 as its eigenvalue. If $K \subset H_1$ is of type (q, q), then we arrive at a contradiction by the similar consideration.

COROLLARY 2. *If H is an abelian group, it is a cyclic group or a product of two cyclic groups.*

PROPOSITION 5. *The 3-Sylow group of H is* $\boldsymbol{Z}/3 \oplus \boldsymbol{Z}/3$ *or* $\boldsymbol{Z}/3$ *or* 1.

PROOF. Let Q be the 3-Sylow group of H. Suppose Q is not an abelian group, then the holonomy representation $Q \subset GL(3, C)$ is irreducible. Take $A \in Z(Q) - \{1\}$. Then by Schur's lemma A is a scalar matrix $\lambda 1$ and hence any eigenvalue of A is not 1, a contradiction. Thus Q is abelian. By Propositions 2 and 4 we obtain the result.

LEMMA 2. *If the* 5-*Sylow group of* H *is not trivial, the* 3-*Sylow group of* H *is trivial.*

PROOF. Since $^{\#}H = 5 \cdot 2^a \cdot 3^b$, $b = 0, 1, 2$, we have only to consider the two cases: (i) $^{\#}H = 5 \cdot 2^a \cdot 9$ and (ii) $^{\#}H = 5 \cdot 2^a \cdot 3$. In (i), $^{\#}H_1 = 2^a 9$. Hence, H_1 has a subgroup of order 9, which is isomorphic to $Z/3 \oplus Z/3$. This contradicts Proposition 4. In (ii), recalling that H is solvable, there exists a subgroup of order 15 by Hall's theorem, which is $Z/15$. This contradicts Proposition 2.

LEMMA 3. *Suppose that* H *is a non-abelian group and is generated by* 2 *elements* $A(g)$, $A(h)$ *satisfying* $A(g)A(h)A(g)^{-1} = A(h)^{-1}$. *Then any element* $A(k)$ *of* H *can be represented as* $\alpha(k) \dotplus \beta(k)$ *where* $\alpha(k) \in C^*$, $\beta(k) \in GL(2, C)$, *by choosing a suitable base. Moreover we have* $A(g)^2 = 1$, $\alpha(g) \neq 1$.

PROOF. As the abelian group generated by $A(g)^2$, $A(h)$ has the index 2 in H, the degree of the irreducible representation of H is one or two. Hence H can be represented as above. Since H is non-abelian, $A(h)^2 \neq 1$. On the other hand, $\beta(h)$ and $\beta(h)^{-1}$ have the same eigenvalue. Hence $\beta(h)$ does not have eigenvalue 1 and so $\alpha(h) = 1$. Suppose $\alpha(g) = 1$. Since $gx_1 = x_1 + a_1$ and $hx_1 = x_1 + b_1$ we have $(gh)^2 g^{-2} x_1 = x_1 + 2b_1$. Hence $(gh)^2 g^{-2} h^{-2} x_1 = x_1$. The eigenvalue of $\beta((gh)^2 g^{-2} h^{-2}) = \beta(h^{-2})$ is not 1, so $(gh)^2 g^{-2} h^{-2}$ has a fixed point. Thus $\alpha(g) \neq 1$. Since $A(g)^2 \in Z(H)$, $\beta(g)^2$ is a scalar matrix. If $\alpha(g)^2 \neq 1$ and $\beta(g)^2 = 1$, then $A(g^2 h) - 1$ is non-degenerate. If $\beta(g)^2 \neq 1$, then $A(g) - 1$ is non-degenerate. Hence $\alpha(g)^2 = \beta(g)^2 = 1$ so $A(g)^2 = 1$.

LEMMA 4. *If* H *is a non-abelian* 2-*group, it is* D_4.

PROOF. By choosing an appropriate base, $A(h) \in H$ can be represented as a direct sum of $\alpha(h) \in C^*$ and $\beta(h) \in GL(2, C)$. The representation β is faithfull. In fact otherwise we have $A(h_1) = \alpha \dotplus 1_2$, $\alpha \neq 1$ and $A(h_2) = 1 \dotplus \beta 1_2$, $\beta \neq 1$ where $A(h_2) \in Z(H) - 1$. Then $A(h_1 h_2) - 1$ is non-degenerate. Let $N = \{A(h) \in H; \det \beta(h) = 1\}$. Then N is a normal subgroup of H and the element $A(h)$ of order 2 in N satisfies $\beta(h) = -1_2$. Hence such an $A(h)$ is unique. In addition N does not contain the elements of order 8. It follows that N is either a quaternion group or a cyclic group of order at most 4. (Hall [2], Theorem 12.5.2). By Lemma 3, N is cyclic. Let $N = \langle y \rangle$. As H/N is cyclic, let x be the element of H which generates H/N. Then $H = \langle x, y \rangle$. Since N is a cyclic group of order at most 4 and H is a non-abelian group, we have a relation $xyx^{-1} = y^{-1}$, $y^2 \neq 1$. Hence $y^4 = 1$ and by Lemma 3 we have $x^2 = 1$. Thus H is D_4.

LEMMA 5. *If H contains an element of order* 5, *it is a cyclic group of order* 5 *or* 10.

PROOF. At first note that $^{\#}H$ is $5\cdot 2^a$ and $^{\#}H_1$ is 2^a. By Lemmas 1 and 4 we have $a \leqq 3$. Hence the 5-Sylow group of H is a normal subgroup $\langle x\rangle$. For any $y\in H_1$, $yxy^{-1}=x^k$. Hence $\det x = (\det x)^k$, so $k=1$. Consequently H is an abelian group. Since H cannot have an abelian subgroup of type (2, 10), it turns out to be $\boldsymbol{Z}/d$, $d=5, 10$.

PROPOSITION 6. *H cannot contain a subgroup which is isomorphic to* S_3.

PROOF. Suppose that H contains such a group K. Since there is one and only one irreducible representation of degree two of S_3, we may assume that K is generated by

$$A(g)=\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\qquad A(h)=\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix}$$

ω: a primitive cubic root of 1.

Then $hg^2h^{-1}g(x)=x$ has a solution $x_1=a_1/2$, $x_2=\lambda$, $x_3=\lambda+\omega^2a_2-\omega a_3$ where $a(g)={}^t(a_1, a_2, a_3)$, $\lambda\in\boldsymbol{C}$.

PROPOSITION 7. *H cannot contain a subgroup K which is isomorphic to* A_4.

PROOF. Suppose H contains such a group K. Since A_4 has the only one irreducible representation of degree 3 and three representations of degree 1, we may assume that K can be generated by

$$A(g)=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\qquad A(h)=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}.$$

Then $gh^3g^{-1}h$ has a fixed point.

LEMMA 6. *If* $^{\#}H=2^a3^b$, *then H is a product of the* 2-*Sylow group and the* 3-*Sylow group.*

PROOF. Use the induction on $^{\#}H$. Take a normal subgroup K such that $|H\colon K|=2$ or 3. By induction hypothesis we have $K=M\times N$ where M is a 2-group and N is a 3-group, in which M and N are normal subgroups of H. In case $|H\colon K|=2$, choose $x\in H-K$ such that $x^{2^m}=1$ for some m. Then $\langle x, M\rangle$ is the 2-Sylow group of H. If $[x, N]=1$, then $H=\langle x, M\rangle\times N$. If $[x, N]\neq 1$, then we have an element $y\in N$ such that $y^3=1$ and $xyx^{-1}=y^{-1}$. By Lemma 3, we have $x^2=1$. Hence

$\langle x, y\rangle \cong S_3$. In case $|H: K| = 3$, choose $x \in H - K$ such that $x^3 = 1$. If $[x, M] = 1$, then $H = M \times \langle x, N\rangle$. If $[x, M] \neq 1$, then M is abelian, since $\mathrm{Aut}\, D_4 \cong D_4$. Hence $\langle x, M\rangle$ has a subgroup which is isomorphic to A_4.

Now we shall prove the last part of Theorem. If ${}^{\#}H$ is a multiple of 5, then H is cyclic by Lemma 5. Hence it suffices to consider the case ${}^{\#}H = 2^a 3^b$. By Lemma 6, $H = S \times Q$ where S is a 2-group and Q a 3-group. If S is non-abelian, then $S = D_4$ by Lemma 4. Hence S is generated by

$$A(g) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } A(h) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \text{ where } i = \sqrt{-1}\ .$$

Suppose $Q \neq 1$. Hence $1 \neq A(k) \in Q$ can be written $\alpha \dotplus \beta \dotplus \beta$, $\alpha^3 = \beta^3 = 1$. If $\beta \neq 1$, $A(gk) - 1$ is non-degenerate and if $\alpha \neq 1$, $A(hk) - 1$ is non-degenerate, a contradiction.

EXAMPLE. Define $g_1, \cdots, g_7$ as follows;

$$A(g_1) = 1 \dotplus i \dotplus (-i)\ ,\quad a(g) = {}^t(1/4, 0, 0)\ ,\quad A(g_2) = -1 \dotplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$a(g_2) = {}^t(0, (1 + i)/2, 0)\ ,\quad g_3(x) = (x_1 + \alpha, x_2, x_3)\ ,\quad \mathrm{Im}\,\alpha \neq 0\ ,$$
$$g_4(x) = (x_1, x_2 + 1, x_3)\ ,\quad g_5(x) = (x_1, x_2 + i, x_3),$$
$$g_6(x) = (x_1, x_2 + (1 + i)/2, x_3 + (1 + i)/2)\ ,$$
$$g_7(x) = (x_1, x_2 + (1 + i)/2, x_3 + (1 - i)/2)\ .$$

Then the group $G = \langle g_1, \cdots, g_7\rangle$ satisfies the condition in Introduction and $H \cong D_4$.

In what follows we consider the case in which H is non-cyclic abelian. Let A and B generate H. By choosing an appropriate base we write $A = 1 \dotplus \alpha \dotplus \beta$ and $B = \gamma \dotplus \delta \dotplus \varepsilon$.

LEMMA 7. *If* $\gamma \neq 1$, *then* (1) $\alpha = \delta = 1$ *or* (2) $\beta = \varepsilon = 1$ *or* (3) $A^2 = B^2 = 1$.

PROOF. By noting one of eigenvalues of each AB, A^2B, AB^{-1} and AB^2 has to be 1, we can check this easily.

LEMMA 8. *H does not contain an element A such that the order m of its eigenvalue is* 8 *or* 12.

PROOF. Suppose that H contains such an element A. Then by Lemma 7 it is generated by A, B; $A = 1 \dotplus \alpha \dotplus \beta$, $B = 1 \dotplus \delta \dotplus \varepsilon$. Moreover we may assume $\varepsilon = 1$, because B can be chosen in the kernel of the projection $\tau: H \to C^*$, $\tau(B') = \varepsilon'$ where $B' = \gamma' \dotplus \delta' \dotplus \varepsilon'$. Then $\alpha\delta$, β, $\overline{\alpha\delta}$,

$\bar{\beta}$ turn out to be primitive m-th roots of 1. This is a contradiction.

Similarly we can prove

LEMMA 9. *If H contains an element of order* 12, *then it is an abelian group of type* $(12, 2) \cong (6, 4)$.

The group G such that G/G_0 is non-cyclic but abelian can be constructed as follows: Let ξ and η be the primitive m and n-th root of 1, respectively, where $\varphi(m) \leqq 2$ and $\varphi(n) \leqq 2$.

Set

$$\mu = \begin{cases} \xi \text{ if } \varphi(m) = 2 \\ i \text{ if } \varphi(m) = 1 \end{cases} \quad \text{and} \quad \nu = \begin{cases} \eta \text{ if } \varphi(n) = 2 \\ i \text{ if } \varphi(n) = 1 . \end{cases}$$

Define $g_1, \cdots, g_6$ as follows;

$$g_1(x) = (x_1 + 1/m, \xi x_2, x_3) , \quad g_2(x) = (x_1 + i/n, x_2, \eta x_3) ,$$
$$g_3(x) = (x_1, x_2 + 1, x_3) , \quad g_4(x) = (x_1, x_2 + \mu, x_3) ,$$
$$g_5(x) = (x_1, x_2, x_3 + 1) \quad \text{and} \quad g_6(x) = (x_1, x_2, x_3 + \nu) .$$

Then $G/G_0 \cong \boldsymbol{Z}/m \oplus \boldsymbol{Z}/n$.

Thus we have proved the whole part of Theorem.

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