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Discontinuous solutions in problems of optimization

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Discontinuous Solutions in Problems of Optimization (*)

L. CESARI - P. BRANDI - A. SALVADORI

First we mention (§1) a few points of the theory of functions of $\nu \geq 1$ independent variables, which are of class L_1 and of bounded variation (BV) in a bounded domain, hence, possibly discontinuous and not of Sobolev. Calculus of variations, for such functions as state variables, has been initiated in two different directions, both very promising, based on the use of the Weierstrass and of the Serrin integrals respectively. In Sections 3 and 4 we state results concerning the use of the Serrin integral, as recently obtained by Cesari, Brandi and Salvadori [10abc] for simple (§3) and multiple (§4) integrals respectively and BV possibly discontinuous solutions. In Section 5 we briefly summarize results concerning the Weierstrass approach, as obtained by Cesari [8ab], Warner [23ab], Brandi and Salvadori, first for continuous state variables [2abc], and recently for BV possibly discontinuous solutions [2defgh].

1. - BV functions of $\nu \geq 1$ indepedent variables

In 1936 Cesari [5] introduced a concept of BV real valued functions $z : G \to \mathbb{R}$, or $z(t)$, or $z(t^1, \cdots, t^\nu)$, from any bounded open subset G of $\mathbb{R}^\nu$ into $\mathbb{R}$. For the case $\nu = 2$, G the rectangle $(a, b; c, d)$ the definition is very simple: we say that z if BV in $G = (a, b; c, d)$ provided $z \in L_1(G)$ and there is a set E of measure zero in G such that the total variation $V_x(y)$ of $z(\cdot, y)$ in (a, b) is of class $L_1(c, d)$, and the total variation $V_y(x)$ of $z(x, \cdot)$ in (c, d) is of class $L_1(a, b)$, where these total variations are computed completely disregarding the

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values taken by z in E. The number

$$V_0 = V_0(z, G) = \int_a^b V_y(x)dx + \int_c^d V_x(y)dy$$

may well be taken as a definition of total variation of z in $G = (a, b; c, d)$, (with respect to such a set $E \subset G$ of measure zero). Analogous definitions hold for BV functions $z(t^1, \cdots, t^\nu)$ in an interval G of $\mathbb{R}^\nu$.

We shall state below the more involved definition of BV functions in a general bounded open subset G of $\mathbb{R}^\nu$.

If z is continuous in G, then no set E need be considered and the concept reduces to Tonelli's concept of BV continuous functions. For discontinuous functions, examples show how essential it is to disregard sets E of measure zero in G. On the other hand, the concept obviously concerns equivalent classes in $L_1(G)$.

We may think of $z(t)$, $t \in G \subset \mathbb{R}^\nu$, as defining a nonparametric discontinuous surface $S : z = z(t)$, $t \in G$, in $\mathbb{R}^{\nu+1}$, and we may take as generalized Lebesgue area $L(S)$ of S the lower limit of the elementary areas $a(\Sigma)$ of the polyhedral surfaces $\Sigma : z = Z(t)$, $t \in G$, converging to z pointwise a.e., in G (or in $L_1(G)$). More precisely, if (Σ_k) denotes any sequence of polyhedral surfaces $\Sigma_k : z = z_k(t)$, $t \in G$, converging to z pointwise a.e. in G (or in $L_1(G)$), we take for $L(S)$ the number, $0 \leq L(S) \leq +\infty$, defined by

$$L(S) = \operatorname*{Inf}_{(\Sigma_k)} \varliminf_{k\to\infty} a(\Sigma_k).$$

Cesari proved [5] that $L(S)$ is finite if and only if z is BV in G. This shows that the concept of BV functions is independent of the direction of the axes in $\mathbb{R}^\nu$. More than that, the concept of BV functions is actually invariant with respect to $1-1$ continuous transformations in $\mathbb{R}^\nu$ which are Lipschitzian in both directions.

In 1937 Cesari [6a] proved that for $\nu = 2$, $G = (0, 2\pi; 0, 2\pi)$ and z BV in G, then the double Fourier series of z converges to z (by rectangles, by lines, and by columns) a.e. in G. Comparable, though weaker, results hold for BV functions of $\nu > 2$ independent variables and their multiple Fourier series [6b].

In 1950 Cafiero [4] and later in 1957 Fleming [15] proved the relevant compactness theorem: any sequence (z_k) of BV functions with equibounded total variations, say $V_0(z_k, G) \leq C$, and equibounded mean values in G, possesses a subsequence (z_{k_s}) which is pointwise convergent a.e. in G as well as strongly convergent in $L_1(G)$ toward a BV function z.

In 1966 Conway and Smoller [12] used these BV functions in connection with the weak solutions (shock waves) of conservation laws, a class of nonlinear hyperbolic partial differential equations in $\mathbb{R}^+ \times \mathbb{R}^\nu$. Indeed, they proved that, if the Cauchy data on $(0) \times \mathbb{R}^\nu$ are locally BV, then there is a unique weak solution on $\mathbb{R}^+ \times \mathbb{R}^\nu$, also locally BV and satisfying an entropy condition. Without any entropy condition there are in general infinitely many weak solutions. Analogous results for $\nu = 1$ had been obtained before by Oleinik [18]. Later Dafermos [13] and Di Perna [14] characterized the properties of the BV weak solutions of conservation laws.

Meanwhile, in the fifties, distribution theory became known, and in 1957 Krickeberg [17] proved that the BV functions are exactly those $L_1(G)$ functions whose first order partial derivatives in the sense of distributions are finite measures in G.

Thus a BV function $z(t)$, $t \in G$, G a bounded domain in $\mathbb{R}^\nu$, possesses first order partial derivatives in the sense of distributions which are finite measures μ_j, $j = 1, \cdots, \nu$. On the other hand, if we think of the initial definition of z, we see that the set E of measure zero in G has intersection $E \cap \ell$ of linear measure zero on almost all lines ℓ parallel to the axes. Hence, z is BV on almost all such straight lines when we disregard the values taken by z on E, and has therefore "usual" partial derivatives $D^j z$ a.e. in G, and these derivatives are functions in G of class $L_1(G)$. We call these $D^j z(t)$, $t \in G$, $j = 1, \cdots, \nu$, computed by usual incremental quotients disregarding the values taken by z on E, the generalized first order partial derivatives of z in G.

Much work followed on BV functions in terms of the new definition, that is, thought of as those $L_1(G)$ functions whose first order derivatives are finite measures. We mention here Fleming [15], Volpert [22], Gagliardo [16], Anzellotti and Giaquinta ([1]) and also De Giorgi, Da Prato, Giusti, Miranda, Ferro, Caligaris, Oliva, Fusco, Temam. However, there are advantages in using both view points.

Great many properties of BV functions have been proved. To begin with, a "total variation" $V(z, G)$ can be defined globally in terms of functional analysis,

$$V(z, G) = \text{Sup} \left[\left(\int_G f_1 d\mu_1\right)^2 + \cdots + \left(\int_G f_\nu d\mu_\nu\right)^2 \right]^{1/2},$$

where the Sup is taken for all $f_1, \cdots, f_\nu \in C_0^1(G)$ with $f_1^2 + \cdots + f_\nu^2 \leq 1$.

If (z_k) is a sequence of BV functions on G with equibounded total variations, say $V(z, G) \leq C$, and $z_k \to z$ in $L_1(G)$, then z is BV and $V(z, G) \leq \varliminf_{k \to \infty} V(z_k, G)$.

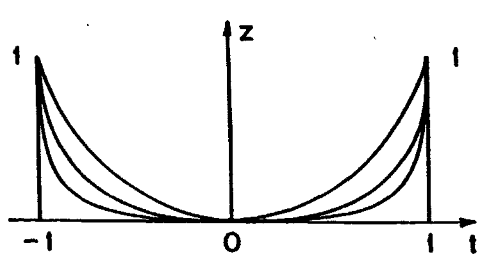
The question of the existence of traces $\gamma : \partial G \to \mathbb{R}$ for BV functions $z : G \to \mathbb{R}$ has been discussed under both view points. Note that for a BV function z in an interval $G = (a, b; c, d)$ it is trivial that the generalized limits $z(a+, y)$ and $z(b-, y)$, $z(x; c+)$ and $z(x, d-)$, exist a.e., and are L_1 functions, i.e., the trace $\gamma(z)$ of z on ∂G exists and is $L_1(\partial G)$. For general domains G in $\mathbb{R}^\nu$ possessing the cone property everywhere on ∂G, a theorem of Gagliardo [16] characterizes the properties of ∂G, and one can prove that any BV function z in a bounded domain G with the cone property and $\mathcal{H}^{\nu-1}(\partial G) < \infty$, possesses a trace $\gamma(z)$ on ∂G with $\gamma(z) \in L_1(\partial G)$.

We mention here the following theorem by Gagliardo on bounded domains G with the cone property: If G is a bounded open domain in $\mathbb{R}^\nu$ having the cone property, then there is a finite system $(G_1, \cdots, G_m)$ of open subsets of G with max diam G_s as small as we want, each G_s has the cone property, and has locally Lipschitzian boundary ∂G_s.

From this result, and trace properties for Lipschitzian domains, it is possible to define the trace $\gamma(z)$ of a BV function on ∂G, for G bounded and with the cone property. An equivalent definition of traces of BV functions in terms of the distributional definition is also well known.

We come now to the delicate question of the continuity of the traces $\gamma(z)$ of BV functions z in a domain G, in other words whether $z_k \to z$, say in $L_1(G)$, may actually imply - under assumptions - that $\gamma(z_k) \to \gamma(z)$ in $L_1(\partial G)$. A number of devices have been proposed to this effect. For instance, Anzellotti and Giaquinta ([1]) have recently proved the following statement in terms of the distributional definition of BV functions: If G has the cone property at every point of ∂G, if $\mathcal{H}^{\nu-1}(\partial G) < \infty$, if the functions z_k are BV with $V(z_k) \leq C$, if $z_k \to z$ in $L_1(G)$ with $V(z_k) \to V(z)$, then $\gamma(z_k) \to \gamma(z)$ in $L_1(\partial G)$. A parallel proof of this statement is available in terms of the original definition of BV functions. We mention here that it is well known that any BV function $z(t)$, $t \in G \subset \mathbb{R}^\nu$, can be approximated in $L_1(G)$ by BV smooth functions z_k with $V(z_k) \to V(z)$, hence $V(z_k) \leq C$.

The following example shows a simple situation in which the trace operator is not continuous:



$$z_k(t) = t^{2k}, \quad -1 \leq t \leq 1, \quad k = 1, 2, \cdots,$$

$$z(t) = 0, \quad -1 < t < 1, \quad z(-1) = z(1) = 1,$$

$$z_k(-1+0) = 1, \quad z_k(1-0) = 1, \quad z(-1+0) = 0, \quad z(1-0) = 0.$$

Of course the condition of Giaquinta and Anzellotti is not satisfied here since

$$V(z_k, \ (-1,1)) = 2, \quad V(z, \ (-1,1)) = 0.$$

Another relevant statement concerning the continuity of the trace operator is as follows: For z, z_k all BV in $G \subset \mathbb{R}^\nu$, let μ, μ_k denote the systems of their first order partial derivatives in the sense of distributions, namely, ν-vector valued finite measures in G. Let Z, Z_k denote the mean values of z, z_k in G. If $Z_k \to Z$ and if $\mu_k \to \mu$ weakly, then $\gamma(z_k) \to \gamma(z)$ in $L_1(\partial G)$.

We only mention here that by $\mu_k \to \mu$ weakly we mean, in terms of duality, that $\int_G < f, d\mu_k > \longrightarrow \int_G < f, d\mu >$ for all continuous functions f on $\overline{G}$, or $f \in (C(\overline{G}))^\nu$, and in both integrals we mean that G is (bounded) and open in $\mathbb{R}^\nu$.

For $\nu = 1$, i.e., for functions $z(t)$, $a \le t \le b$, of a real variable t, say of bounded variation in (a, b), then the generalized limits $z(a+)$ and $z(b-)$ (the traces) obviously exist and are finite. If $z_k(t)$, $a \le t \le b$, is a sequence of BV functions, say equibounded and with equibounded total variations, then by Helly's theorem, there is a subsequence z_{k_s} which converges pointwise everywhere in $[a, b]$ (as well as in $L_1(G)$) toward a BV function $z(t)$, $a \le t \le b$, with $V(z) \le \underline{\lim}\, V(z_{k_s})$, and in particular, we may require that $z_k(a) \to z(a)$, $z_k(b) \to z(b)$. Thus, for $\nu = 1$, the question of the continuity of the end values is trivially answered in the affirmative by everywhere pointwise convergence and Helly's theorem.

We shall see now how these ideas have been used in questions of optimization.

2. - Calculus of variations in classes of *BV* functions

When the state variable z, or $z(t) = (z^1, \cdots, z^m)$, $t \in G \subset \mathbb{R}^\nu$, is only BV, the usual Lebesgue integral of the calculus of variations

$$I(z) = \int_G f_0(t, z(t), Dz(t))dt,$$

$$t = (t^1, \cdots, t^\nu) \in G \subset \mathbb{R}^\nu, \;\; z(t) = (z^1, \cdots, z^m), \;\; \nu \ge 1, \;\; m \ge 1,$$

may not give a true, or stable value for the functional of interest. There are two basic processes to determine a true, or stable value for the underlying functional, and both have generated a great deal of recent work.

One is the limit process already proposed by Weierstrass, leading to a functional, or Weierstrass integral, $W(z)$. Tonelli made use of it in his early work (1914) on the direct method in the calculus of variations for parametric continuous curves C, or $z(t) = (z^1, \cdots, z^m)$, $a \le t \le b$, of finite length, hence, all z^i are BV and continuous. Recently, Cesari [8ab] presented an abstract formulation of the Weierstrass integral as a Burkill type limit on "quasi additive" set functions $\varphi(I) = (\varphi_1, \cdots, \varphi_n), I \subset G$, and state functions $z(t) = (z^1, \cdots, z^m)$, $t \in G \subset \mathbb{R}^\nu$. Cesari also proved [8b] that $W(z)$ has a representation as a Lebesgue-Stieltjes integral in terms of measures and

Radon-Nikodym derivatives derived from the set function φ. Warner [23] then proved lower semicontinuity theorems for continuous varieties, and very recently Brandi and Salvadori [2defgh] extended further the abstract formulation, proved further representation properties, and lower semicontinuity theorems, both in the parametric and in the nonparametric case, and for vector functions z (state variables) only BV, possibly discontinuous, possibly not Sobolev (see §5 below).

Another approach was proposed by Serrin [20a] leading to a functional, or Serrin integral, $\Im(z)$, in classes of BV vector functions $z(t)$, $t \in G \subset \mathbb{R}^\nu$. The Serrin functional $\Im(z)$ is obtained by taking lower limits on the value of I on AC, or $W^{1,1}(G)$ functions, a process which is similar to the one with which Lebesgue area is defined. Recently, Cesari, Brandi and Salvadori [10ab] proved closure and lower closure theorems, hence theorems of lower semicontinuity in the L_1-topology, and finally theorems of existence of the absolute minimum of $\Im(z)$ in classes of BV vector functions whose total variations $V(z)$ are equibounded [10ab]. (See also [19]). We proved also that $I(z) \leq \Im(z)$, and that $\Im$ is a proper extension of I in the sense that $\Im(z) = I(z)$ for all z which are AC, or $W^{1,1}(G)$ (see §§3, 4 below). A number of applications of this approach has been announced [9abc, 11ab].

3. - Problems of optimization for simple integrals, $\nu = 1$, by the use of Serrin's functional

We may be interested either in problems of the classical calculus of variations involving a vector valued state variable $z(t) = (z^1, \cdots, z^n)$, $t_1 \leq t \leq t_2$, or in problems of optimal control involving an analogous state variable $z(t) = (z^1, \cdots, z^n)$ and a control variable $u(t) = (u^1, \cdots, u^m)$, $t_1 \leq t \leq t_2$, with given control space $U(t,z)$ and constraint $u(t) \in U(t, z(t))$.

It is more general, and more satisfactory, (cfr. [7]), to deparametrize the problems of optimal control, and concern ourselves exclusively with generalized problems of the calculus of variations with constraints on the derivatives, say

$$(1)\qquad \begin{gathered} I(z) = \int_{t_1}^{t_2} f_0(t, z(t), z'(t))dt = \text{minimum}, \\ (t, z(t)) \in A \subset \mathbb{R}^{n+1},\ z'(t) \in Q(t, z(t)), \end{gathered}$$

where $t \in [t_1, t_2] \subset \mathbb{R}$ (a.e.), where A is a subset of $\mathbb{R}^{n+1}$ whose projection on the t-axis contains $[t_1, t_2]$, and where, for every $(t, z) \in A$, a set $Q(t, z)$ is given constraining the direction $z'(t)$ of the tangent to the state variable z a.e. in $[t_1, t_2]$.

The process of deparametrization mentioned above can be summarized as follows. Given a problem of optimal control:

$$I(z,u) = \int_{t_1}^{t_2} F(t, z(t), u(t))dt,$$

with ordinary differential system and constraints

$$z'(t) = f(t, z(t), u(t)), \ \ t \in [t_1, t_2] \text{ (a.e.)},$$
$$(t, z(t)) \in A \subset \mathbb{R}^{n+1}, \ \ u(t) \in U(t, z(t)) \subset \mathbb{R}^m,$$

let us take

$$f_0(t, z, \xi) = \text{Inf}\,[\eta \geq F(t, z, u) \,|\, \xi = f(t, z, u), u \in U(t, z)]\,,$$
$$Q(t, z) = [\xi = f(t, z, u) \,|\, u \in U(t, z)]\,.$$

Then, the corresponding problem of the calculus of variations with constraints on the derivatives is as follows:

$$J(z) = \int_{t_1}^{t_2} f_0(t, z(t), z'(t))dt,$$
$$(t, z(t)) \in A, \quad z'(t) \in Q(t, z(t)).$$

For what concerns boundary conditions for problem (1), we restrict ourselves here to Dirichlet type boundary conditions

(2) $$z(t_1) = z_1, \quad z(t_2) = z_2.$$

Above, let M denote the set $M = [(t, z, \xi) \,|(t, z) \in A, \xi \in Q(t, z)] \subset \mathbb{R}^{1+2n}$, and let $f_0(t, z, \xi)$ be a real valued function on M. Let Ω be a class of admissible functions, i.e., functions $z : [t_1, t_2] \to \mathbb{R}^n$, or $z(t) = (z^1, \cdots, z^n)$, such that

(i) z is BV in $[t_1, t_2]$;

(ii) $(t, z(t)) \in A$, $z'(t) \in Q(t, z(t))$ a.e. in $[t_1, t_2]$;

(iii) $f_0(\cdot, z(\cdot), z'(\cdot)) \in L_1[t_1, t_2]$.

It is easy to see that the Lebesgue integral definition (1) of the functional I does not yield stable and realistic values for I, and one may use a Serrin type integral. To this effect, for every $z \in \Omega$ we denote by $\Gamma(z)$ the class of all sequences (z_k) of elements $z_k \in \Omega$ with

(a) z_k is AC in $[t_1, t_2]$;

(b) $z_k \to z$ pointwise a.e. in $[t_1, t_2]$.

If $\Gamma(z)$ is empty we take $\Im(z) = +\infty$. If $\Gamma(z)$ is not empty, then we take

$$\Im(z) = \inf \varliminf \int_{t_1}^{t_2} f_0(t, z_k(t), z_k'(t))dt \tag{3}$$
$$= \inf_{\Gamma(z)} \varliminf_{k\to\infty} I(z_k).$$

This is the Serrin type definition of the functional which was inspired to the Lebesgue area of nonparametric surfaces.

If problem (1) has assigned boundary conditions, say of the Dirichlet type (2), then let $\Gamma(z)$ denote the class of all sequences (z_k) of elements z_k in Ω with

(a) z_k is AC and satisfies the boundary conditions;

(b′) $z_k \to z$ pointwise a.e. in $[t_1, t_2]$, in particular $z_k(t_i) \to z(t_i)$, $i = 1, 2$.

Then the analogous integral defined by (3) could be denoted by $\Im^*$ and obviously $\Im \leq \Im^*$.

We can state now a lower semicontinuity theorem and an existence theorem for integrals on BV functions. To this purpose we have first to define as usual the "augmented" sets $\tilde{Q}(t, z)$ as follows:

$$\tilde{Q}(t, z) = [(\tau, \xi) \mid \tau \geq f_0(t, z, \xi), \xi \in Q(t, z)] \subset \mathbb{R}^{n+1}.$$

A lower semicontinuity theorem

Let us assume that

(i) A is closed;

(ii) the sets $\tilde{Q}(t, z)$ are closed, convex, and satisfy property (Q) with respect to (t, z) at every $(t, z) \in A$;

(iii) $f_0(t, z, \xi)$ is lower semicontinuous in M, and there exists some function $\lambda \in L_1[t_1, t_2]$ such that $f_0(t, z, \xi) \geq \lambda(t)$ for all $(t, z, \xi) \in M$.

Let $z(t)$, $t \in [t_1, t_2]$, be BV, and let $z_k(t)$, $t \in [t_1, t_2]$, $k = 1, 2, \cdots$, be a sequence of AC functions z_k such that $z_k \to z$ pointwise a.e. in $[t_1, t_2], (t, z_k(t)) \in A$, $z_k'(t) \in Q(t, z_k(t))$ a.e. in $[t_1, t_2]$, and $V(z_k) \leq C$. Then, $(t, z(t)) \in A$, $z'(t) \in Q(t, z(t))$ a.e. in $[t_1, t_2]$, and $I(z) \leq \underline{\lim}_{k\to\infty} I(z_k)$ [10a].

A fundamental consequence of this lower semicontinuity theorem is that if (z_k) is any of the sequences of AC elements in $\Gamma(z)$, with $V(z_k) \leq C$, and we take $j = \underline{\lim}_{k\to\infty} I(z_k)$, then

$$I(z) \leq \Im(z) \leq j = \underline{\lim}_{k\to\infty} I(z_k).$$

Furthermore, the Serrin integral $\Im$ is actually an extension of the integral I. Indeed, if $z \in \Omega \cap AC$, then, by taking $z_k = z$ we conclude that $I(z) \leq \Im(z) \leq \underline{\lim}_{k\to\infty} I(z_k) = I(z)$.

Note that, for sequences (z_k) as above with $V(z_k)$ unbounded, it may well occur that $\Im(z) < I(z)$ as it has been proved by examples (cfr. [10a]).

We mention here that Kuratowski's property (K) at a point (t_0, z_0) is expressed by the relation

$$\tilde{Q}(t_0, z_0) = \bigcap_{\delta>0} cl \left[\cup\ \tilde{Q}(t, z),\ (t - t_0)^2 + |z - z_0|^2 \leq \delta^2\right].$$

The analogous condition (Q) at the point (t_0, z_0) is expressed by the relation

$$\tilde{Q}(t_0, z_0) = \bigcap_{\delta>0} cl\ co \left[\cup\ \tilde{Q}(t, z),\ (t - t_0)^2 + |z - z_0|^2 \leq \delta^2\right].$$

If problem (1) has assigned boundary conditions of the type (2), then in the theorem above we assume that $z_k \to z$ a.e. in $[t_1, t_2]$, in particular $z_k(t_i) \to z(t_i)$, $i = 1, 2$, and the same statement holds for $\Im^*$.

An existence theorem for the integral $\Im$

Let us assume that

(i) A is compact and M is closed;

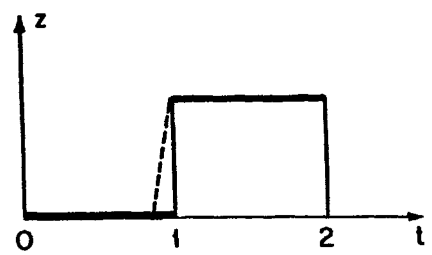
(ii) the sets $\tilde{Q}(t, z)$ are closed, convex, and satisfy property (Q) with respect to (t, z) at every point (t, z) of A;

(iii) $f_0(t, z, \xi)$ is lower semicontinuous in M.

Assume that the class Ω is nonempty and closed, $V(z) \leq C$ for all $z \in \Omega$, and $\Gamma(z)$ is nonempty for at least one z. Then the functional $\Im$ has an absolute minimum $z \in BV$ in Ω [10a].

In other words, let i denote the infimum of $I(z)$ for $z \in AC \cap \Omega$, let (z_k) denote a sequence of elements $z_k \in AC \cap \Omega$ with $I(z_k) \to i$. Then, there is an element $z \in \Omega$, $z \in BV$, such that $I(z) \leq \Im(z) = i$.

EXAMPLE 1. Let



$$I(z) = \int_0^2 |1-t||z'(t)| \, dt,$$

$$z(0) = 0, \quad z(2) = 1,$$

with

$$A = [0,2] \times [0,1], \ n = 1, \ Q(t,z) = [0,+\infty), \ f_0(t,z,\xi) = |1-t||\xi| \geq 0.$$

If we take

$$\begin{aligned} z_k(t) &= 0 && \text{for } 0 \leq t \leq 1 - k^{-1}, \\ z_k(t) &= 1 && \text{for } 1 \leq t \leq 2, \\ z_k(t) &= 1 - k + kt && \text{for } 1 - k^{-1} \leq t \leq 1, \end{aligned}$$

we have

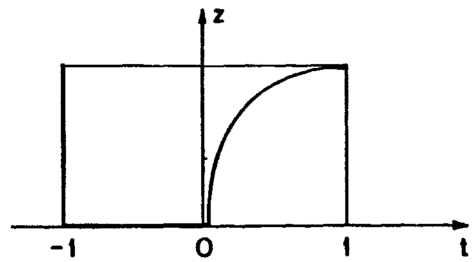
$$0 \leq \Im(z_k) = I(z_k) = \int_{1-k^{-1}}^{1} (1-t)k dt = (2k)^{-1} \to 0.$$

Thus $i = 0$, (z_k) is a minimizing sequence. The minimum of $\Im$ is attained by the discontinuous function

$$z(t) = 0 \text{ for } 0 \leq t \leq 1, \qquad z(t) = 1 \text{ for } 1 < t \leq 2,$$

and $I(z) = \Im(z) = i = 0$.

EXAMPLE 2. Let



$$I(z) = \int_{-1}^{1} |t| \; z'^2(t)dt,$$

$$z(-1) = 0, \quad z(1) = 1,$$

with

$$A = [-1, 1] \times [0, 1], \;\; n = 1, \;\; Q(t, z) = [-1, +\infty),$$

$$f_0(t, z, \xi) = |t|\xi^2 \geq 0, \;\; I(z) \geq 0, \;\; \mathfrak{I}(z) \geq 0.$$

If we take

$$z_k(t) = 0 \qquad \text{for } -1 \leq t \leq k^{-1},$$

$$z_k(t) = (\log\; k)^{-1} \log\; t + 1 \qquad \text{for } k^{-1} \leq t \leq 1,$$

we have

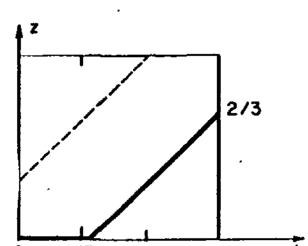
$$0 \leq \mathfrak{I}(z_k) = I(z_k) = \int_{k^{-1}}^{1} (\log\; k)^{-2} \; t^{-1}dt = (\log\; k)^{-1} \to 0.$$

Thus, $i = 0$, (z_k) is a minimizing sequence. The minimum is attained by the discontinuous function

$$z(t) = 0 \text{ for } -1 \leq t \leq 0, \qquad z(t) = 1 \text{ for } 0 < t \leq 1,$$

and $I(z) = \mathfrak{I}(z) = i = 0$.

EXAMPLE 3. Let



$$I(z) = \int_0^1 |z'(t)|\, dt,$$

$$n = 1, \ Q(t,z) = \mathbb{R},$$

with

$$A = [(t,z)|\, 0 \leq t \leq 1, \ 0 \leq z \leq 1, \ t - 3^{-1} \leq z \leq t + 3^{-1}], \ f_0(t,z,\xi) = |\xi|.$$

Let $\phi(t)$, $0 \leq t \leq 1$, denote the usual Cantor ternary function. Then $(t, \phi(t)) \in A$, $I(\phi) = i_0 = 0$, where $i_0 = 0$ is the infimum of $I(z)$ in Ω. Let i be the infimum of $I(z)$ in $AC \cap \Omega$. Then $i = 2/3$, and the minimum of $\mathfrak{I}$ is attained by

$$z(t) = 0 \qquad \text{for } 0 \leq t \leq 1/3,$$
$$z(t) = t - 1/3 \qquad \text{for } 1/3 \leq t \leq 1.$$

Thus

$$I(\phi) = i_0 = 0 < I(z) = \mathfrak{I}(z) = i = 2/3.$$

Now consider the same problem with boundary data

$$z(0) = 0, \quad z(1) = 1.$$

The infimum of $I(z)$ in Ω is still $i_0 = 0$. The infimum of $I(z)$ in $AC \cap \Omega$ is now $i = 1$, and $i = 1$ is assumed by all non decreasing AC functions $z \in \Omega$ and by the discontinuous function $\bar{z}(t)$ defined by

$$\bar{z}(t) = 0 \qquad \text{for } 0 \leq t \leq 1/3,$$
$$\bar{z}(t) = t - 1/3 \qquad \text{for } 1/3 \leq t < 1, \ \bar{z}(1) = 1.$$

Thus

$$I(\phi) = i_0 = 0 < I(\bar{z}) = 2/3 < \mathfrak{I}(\bar{z}) = i = 1.$$

4. - Problems of optimization for multiple integrals and BV discontinuous functions, $\nu > 1$, by the use of Serrin's functional

Let G be a bounded open subset of the t-space $\mathbb{R}^\nu$, $t = (t^1, \cdots, t^\nu)$. For every $j = 1, \cdots, \nu$, let G'_j denote the projection of G on the t'_j-space $\mathbb{R}^{\nu-1}$, $t'_j = (t^1, \cdots, t^{j-1}, t^{j+1}, \cdots, t^\nu)$, and for any $\tau \in G'_j$ let r_τ denote the straight line $t'_j = \tau$. Then, the intersection $G \cap r_\tau$ is the countable union of open disjoint intervals (α_s, β_s), or $G \cap r_\tau = \cup_s (\alpha_s, \beta_s)$. We say that a function $f \in L_1(G)$ is of bounded variation in the sense of Cesari in G (BV) ([5], 1936) and [10b]) if there exists a set $E \subset G$ with $|E| = 0$ such that, for every $j = 1, \cdots, \nu$ and for almost all $\tau \in G'_j$, the total variations $V_{js} = V(f(\cdot, \tau), \ (\alpha_s, \beta_s))$, computed disregarding the values taken by f on E, are finite, $V_j(\tau) = \Sigma_s V_{j_s}$ is finite, and $V_j(\cdot) \in L_1(G'_j)$.

Let $\nu > 1$, $n \geq 1$, and let $G \subset \mathbb{R}^\nu$ be a bounded open subset in the t-space $\mathbb{R}^\nu$, $t = (t^1, \cdots, t^\nu)$, possessing the cone property at every point of its boundary ∂G. Let $A \subset \mathbb{R}^{\nu+n}$ be a compact subset of the tz-space $\mathbb{R}^{\nu+n}$, whose projection on the t-space contains G.

We shall deal with vector valued functions $z(t) = (z^1, \cdots, z^n)$, z^i BV in G, therefore possessing first order partial derivatives in the sense of distributions which are measures μ_{ij}, $j = 1, \cdots, \nu$, $i = 1, \cdots, n$, and in addition also generalized first order derivatives $D^j z^i$ a.e. in G, as functions of class $L_1(G)$, which are obtained as limits of incremental quotients when we disregard the values taken by the functions in suitable sets E of measure zero in G. We may need only a subset of such derivatives $D^j z^i$ as follows.

For every $i = 1, \cdots, n$, let $\{j\}_i$ be a system of indices $1 \leq j_1 < \cdots < j_s \leq \nu$, let $D^j z^i$, $j \in \{j\}_i$, denote the corresponding system of first order partial derivatives of the functions z^i, and let N be their total number. Then by Dz we denote the N-vector function $Dz(t) = (D^j z^i,\ j \in \{j\}_i,\ i = 1, \cdots, n)$, $t \in G$ (a.e.).

For every $(t, z) \in A$ let $Q(t, z)$ be a given subset of $\mathbb{R}^N$. Let $M \subset \mathbb{R}^{\nu+n+N}$ denote the set $M = [(t, z, \xi) |\ (t, z) \in A,\ \xi \in Q(t, z)]$, and let $f_0(t, z, \xi)$ be a given real-valued function in M. We are interested in the multiple integral problem of the calculus of variations with constraints on the derivatives

$$(1) \qquad \begin{aligned} &I(z) = \int_G f_0(t, z(t), Dz(t))dt = \text{minimum}, \\ &(t, z(t)) \in A, \quad Dz(t) \in Q(t, z(t)), \quad t \in G \text{ (a.e.)}, \end{aligned}$$

and possible Dirichlet type boundary conditions of the form $\gamma z(t) = \phi(t)$, $t \in \partial G$ ($\mathcal{H}^{\nu-1}$ – a.e.) on ∂G.

Again we introduce a Serrin type integral.

Let Ω be a class of admissible functions $z(t) = (z^1, \cdots, z^n)$, $t \in G$, such that

(i) z is BV in G;

(ii) $(t, z(t)) \in A$, $Dz(t) \in Q(t, z(t))$, $t \in G$ (a.e.);

(iii) $f_0(\cdot, z(\cdot), Dz(\cdot)) \in L_1(G)$.

To simplify notations, let AC, or $AC(G)$, denote the class of functions $z(t) = (z^1, \cdots, z^n)$, $t \in G$, whose components z^i are of Sobolev class $W^{1,1}(G)$, or, briefly, Beppo Levi functions.

For any element $z \in \Omega$ let $\Gamma(z)$ denote the class of all sequences (z_k) of elements z_k in Ω with

(a) z_k is AC in G;

(b) $z_k \to z$ strongly in $L_1(G)$.

If $\Gamma(z)$ is empty we take $\Im(z) = +\infty$. If $\Gamma(z)$ is not empty then we take

$$
\begin{aligned}
\Im(z) &= \inf \varliminf \int_G f_0(t, z_k(t), Dz_k(t))\,dt \\
&= \inf_{\Gamma(z)} \varliminf_{k\to\infty} I(z_k).
\end{aligned}
\tag{2}
$$

If the given problem has assigned Dirichlet type boundary conditions, say $\gamma z(t) = \phi(t)$, $t \in \partial G$ ($\mathcal{H}^{\nu-1}$ – a.e.) on G, then let $\Gamma(z)$ denote the class of all sequences (z_k) of elements z_k in Ω with

(a′) z_k is AC in G and $\gamma(z_k) = \phi$ on ∂G;

(b′) $z_k \to z$ strongly in $L_1(G)$;

and

$$V(z_k) \to V(z).$$

Then the Serrin integral defined by (2) could be denoted by $\Im^*$, and obviously $\Im(z) \le \Im^*(z)$. By Anzellotti and Giaquinta's theorem then $\gamma(z) = \phi$ on $\partial G(\mathcal{H}^{\nu-1}$ – a.e.).

To state an existence theorem we introduce, as usual, the augmented sets $\tilde{Q}(t,z) \subset \mathbb{R}^{N+1}$ as follows:

$$\tilde{Q}(t,z) = [(\tau,\xi) \mid \tau \ge f_0(t,z,\xi), \xi \in Q(t,z)].$$

Beside property (Q) we shall require on the sets $\tilde{Q}(t,z)$ another property, or property $(\tilde{F}_1)$.

We say that the sets $\tilde{Q}(t,z)$, $(t,z) \in A$, have property $(\tilde{F}_1)$ with respect to z at the point $(t_0, z_0) \in A$ provided, given any number $\sigma > 0$, there are constants $C > 0$, $\delta > 0$ which depend on t_0, z_0, σ, such that for any set of measurable vector functions $\eta(t)$, $z(t)$, $\xi(t)$, $t \in H$, on a measurable subset H of points t of G with

$$
\begin{aligned}
&(t, z(t)) \in A,\ |z(t) - z_0| > \sigma,\ (\eta(t), \xi(t)) \in \tilde{Q}(t, z(t)) \\
&\text{for } t \in H,\ |t - t_0| \le \delta,
\end{aligned}
$$

there are other measurable vector functions $\overline{\eta}(t), \overline{z}(t), \overline{\xi}(t), t \in H$, such that

$$
\begin{aligned}
&(t, \overline{z}(t)) \in A,\ |\overline{z}(t) - z_0| \le \sigma,\ (\overline{\eta}(t), \overline{\xi}(t)) \in \tilde{Q}(t, z(t)), \\
&|\xi(t) - \overline{\xi}(t)| \le C\,[|z(t) - \overline{z}(t)| + |t - t_0|], \\
&\overline{\eta}(t) \le \eta(t) \ + C\,[|z(t) - \overline{z}(t)| + |t - t_0|] \ \text{ for } t \in H,\ |t - t_0| \le \delta.
\end{aligned}
$$

We denote by $(\tilde{F}_2)$ the same condition with $\overline{z}(t) = z_0$. These conditions are inspired to analogous ones proposed by Rothe, Berkovitz, Browder (cfr. Cesari [7], sect. 13). Conditions (Q) and (F) are sometimes called seminormality conditions (cfr. [7]). A weaker version of them, leading to some extensions of the existence theorem, is discussed in [10c].

An existence theorem

Let us assume that

(i) A is compact and M is closed;

(ii) the sets $\tilde{Q}(t, z)$ are closed, convex, and satisfy properties (Q) and $(\tilde{F}_1)$ at every point $(t, z) \in A$;

(iii) $f_0(t, z, \xi)$ is bounded below and lower semicontinuous in (t, z, ξ).

Also assume that the class Ω is nonempty and closed, and $\Gamma(z)$ is nonempty for at least one $z \in \Omega$. Then the functional $\Im$ has an absolute minimum z in Ω, $z \in BV$ in G [10b].

In other words, let i denote the infimum of $I(z)$ for $z \in AC \cap \Omega$, let (z_k) denote any sequence of elements $z_k \in AC \cap \Omega$ with $I(z_k) \to i$. Then there is at least one element $z \in \Omega$, $z \in BV$, such that $I(z) \leq \Im(z) = i$.

5. - The Weierstrass integral W

In [8ab] Cesari established a very general axiomatization concerning extensions of Burkill's integral on set functions. Let $\{I\}$ be a family of subsets I of a given topological space A, subsets that we call intervals. Let $(D, \gg)$ denote a net of finite systems $D = (I_1, \cdots, I_N)$ of nonoverlapping intervals. Cesari [8a] introduced a concept of quasi-additivity for the set functions garanteeing the existence of a limit, now called the Burkill-Cesari integral

$$B(\varphi) = \lim_{(D, \gg)} \sum_{I \in D} \varphi(I).$$

About the non-linear integral $I = \int_T F(p, q)$ over a variety T, Cesari considered the set function $\Phi(I) = F\ (T(\omega(I)),\ \varphi(I))$, where $\omega(I)$ is a choice function, i.e. $\omega(I) \in I$, and φ is a set function. He proved [8a] that if T is any continuous parametric mapping and φ is quasi-additive and BV, then also Φ is quasi-additive and BV. In other words, the non-linear transformation F preserves quasi-additivity and bounded variation. Then the integral W is defined by the Burkill-Cesari process on the function Φ, and is thus defined as a Weierstrass-type integral

$$W(T, \varphi) = \lim_{(D, \gg)} \sum_{I \in D} \Phi(I).$$

Later, many authors studied this integral, both in the parametric and in the non-parametric case, for curves and for varieties, and framed in this theory many of their properties (see [21] for a survey). Note that if F does not depend on the variety, i.e., it is of the type $F(q)$, then the sole concept of quasi-additivity permits the extension of W over BV curves and surfaces, not necessarily continuous nor Sobolev's.

In the last years Brandi and Salvadori [2defg] have extended the definition of W over BV curves or varieties, not necessarily continuous nor Sobolev's, for complete integrands $F(p, q)$.

First the term $T(\omega(I))$, in the definition of $\Phi(I)$ was replaced [2d] by a set function $P(I)$ whose values are in a metric space K, while $\varphi(I)$ is a set function whose values are in a uniformly convex Banach space X, and F: $K \times X \to E$, with E real Banach space. In order to garantee the existence of the integral W for BV transformations T, a condition on the pair of set functions (P, φ) was proposed in [2d], which is of the quasi-additivity-type, and was called Γ-quasi-additivity. This condition reduces to the quasi-additivity on φ when P is the usual set function $T(\omega(I))$ and T is continuous. In this new situation, Brandi and Salvadori proved that, if (P, φ) is Γ-quasi-additive and φ is BV, then still $\Phi(I) = F(P(I), \varphi(I))$ is quasi-additive and BV. Thus the integral $W(P, \varphi)$ is still defined by the Burkill-Cesari process on the set function Φ, and $W(P, \varphi)$ is still a Weierstrass-type integral even for T only BV, possibly discontinuous.

Note that the new condition on (P, φ) is weaker than the couple of assumptions: continuity on T and quasi-additivity on φ. Moreover, it takes advantage of the power of the quasi-additivity-type properties to extend I over BV curves and varieties, for integrands of the type $F(p, q)$, both in the parametric and in the non-parametric case (see many applications in [2def]).

Even in this more general setting, the integral $W(P, \varphi)$ admits of a Lebesgue-Stieltjes integral representation ([2d])

$$W(P, \varphi) = \int_G F(\pi(s),\ (d\mu/d\|\mu\|)(s))\ d\|\mu\|,$$

in terms of a vectorial measure μ related to φ, its total variation $\|\mu\|$, and Radon-Nikodym derivative $d\mu/d\|\mu\|$, as in the previous work of Cesari [8b] in Euclidean spaces, and in the successive extensions to abstract spaces, always for continuous varieties T (see [21] for a survey).

In the non-parametric case (see [2e]) the integral $I = \int_T f(t, p, q)$ is transformed into a suitable parametric integral in the manner of Mc-Shane, with the integrand $F\ (t, p; \ell, q)$ defined by $F\ (t, p; \ell, q) = \ell f(t, p, q/\ell)$ for $\ell > 0$ and $F\ (t, p; 0, q) = \lim_{\ell \to 0+} F\ (t, p; \ell, q)$. Then the set function Φ becomes

$$\Phi(I) = \lambda(I) f(P(I), \varphi(I)/\lambda(I)) = F\ (P(I); \lambda(I), \varphi(I)).$$

Thus, the existence result is still given in terms of Γ-quasi-additivity. Now the representation of $W(P,\varphi)$ in terms of Lebesgue-Stieltjes integral becomes

$$W(P,\varphi)=\int_G f(\pi(s),\ (d(\nu,\mu)/d\|\ (\nu,\mu)\|)(s))\ d\|(\nu,\mu)\|,$$

where μ is the vectorial measure related to φ, ν is the real measure related to λ and $\|(\nu,\mu)\|$ is the total variation of the measure (ν,μ).

Furthermore, in this non-parametric situation, a Tonelli-type inequality was proved in [2e] relating $W(P,\varphi)$ to a corresponding Lebesgue-Stielties integral, namely,

$$W(P,\varphi)\geq\int_G f(\pi(s),\ (\partial\mu/\partial\nu)(s))\ d\nu,$$

where $\partial\mu/\partial\nu$ is a derivative of the Radon-Nikodym type, and the equality sign holds if and only if the set function φ is absolutely continuous with respect to the set function λ. If φ is absolutely continuous with respect to λ, then $\partial\mu/\partial\nu$ reduces to the usual Radon-Nikodym derivative $d\mu/d\nu$. In proving this last result, as in the proof of the representation theorem, use was made of a connection between the Burkill-Cesari process and the convergence of martingales, a connection which was already made in previous papers (see [21] and the quoted papers [2def]).

Finally in [2f] the authors dealt with the problem of the lower semicontinuity for the integral $W(P,\varphi)$, both in the parametric and in the non-parametric case. A first abstract lower semicontinuity theorem was proved in terms of a suitable global convergence on the sequence $(p_n\ \varphi_n)$, defined in the same spirit of the Γ-quasi additivity and therefore again inspired to Cesari's concept of quasi additivity. In a number of applications this convergence is implied by the L_1-convergence of equi BV varieties.

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