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Discontinuous stabilization of nonlinear systems: Quantized and switching controls

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Abstract

In this paper we consider the classical problem of stabilizing nonlinear systems in the case the control laws take values in a discrete set. First, we present a robust control approach to the problem. Then, we focus on the class of dissipative systems and rephrase classical results available for this class taking into account the constraint on the control values. In this setting, feedback laws are necessarily discontinuous and solutions of the implemented system must be considered in some generalized sense. The relations with the problems of quantized and switching control are discussed.

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1. Introduction

Recently, the literature about switched, quantized and hybrid systems [28,14,7,38,36] has given a new perspective to the classical problem of stabilization. In fact, on one hand, since systems considered are more general, there is a wider choice of control strategies (see, e.g. [33,34,39]). On the other hand, the new models often take into account some constraints which are important for applications. In this paper the basic assumption is that control laws take values only in a discrete set U . It can be useful to distinguish two situations, which may be related. In the first situation, the appropriate choice of the discrete set U may be part of the stabilization problem. In the second situation the set U is *a priori* fixed. In any case, since a vector field is naturally associated with each admissible control value, the system can be seen as a family of vector fields with a rule which governs the switching among them. Switching rules can be of different types: they can depend on time, on initial conditions, on the state variables, or they can be completely arbitrary. Here we consider switching rules which depend only on the state

variable (or on functions of the state variable, such as the output function) so that they can also be interpreted as discontinuous feedback laws. Controlling with a discrete set of input values has been deeply explored in the literature on quantized control. As in [14,21] for linear systems, our design of the control values follows a logarithmic law, so that the resulting control law is simpler to implement than in other approaches [27,29,13] and it does not introduce an exceedingly large number of quantization levels (cf. [11] for a different approach to stabilization of nonlinear systems using a minimal number of “quantization levels”). On the other hand, differently from Ishii–Francis in [21] and Cepeda–Astolfi in [8], we do not couple our switched controller with a dwell-time logic, the latter being an approach which turns out useful to avoid chattering-like phenomena.

One of the aims of the paper is in the spirit of the situation in which U needs to be chosen appropriately. We study conditions which guarantee that, given a continuous stabilizing feedback law, the celebrated logarithmic quantization does not cancel the stabilizing effect. A first proposition can be viewed as a discontinuous version of the results about stability under vanishing perturbations (e.g. [25]). A second proposition is a nonlinear version of a result in [17] (see also [14,21]). Previous results for nonlinear systems have appeared in [30]. Our contribution

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differs from the latter in two ways. First, we put a special emphasis on how the solution of the closed-loop system should be intended. Second, a connection between the coarseness of the quantizer and a finite L_2 -gain problem is obtained in the robust control setting pointed out by [17]. Further, it is shown how, trading off global asymptotic stability against semi-global practical stability, it is possible to overcome the limitation on the coarseness of the quantizer by appropriate redesign of the control law. A logarithmic quantizer requires an infinite number of quantization levels to guarantee asymptotic stability. Nevertheless, it is possible to cope with a finite number of quantization levels and obtain semi-global practical stability without affecting the coarseness of the quantizer. This is discussed as well.

As clearly pointed out in [17], in problems of stabilization under logarithmic quantization, the uncertainty introduced by the quantizer is a sector bounded uncertainty. An effective way to deal with stability of nonlinear systems in the presence of sector bounded uncertainties is to rely on the theory of dissipative systems. This simple observation motivates the second aim of the paper, namely to show how some classical results on feedback stabilization of nonlinear dissipative systems can be restated in this setting. The idea of extending stabilization results which use dissipativity to “non-classical” systems is not new, but there is still not a wide literature on the subject. To the best of the authors’ knowledge, the most complete paper on the subject is [18]. In this paper hybrid systems which generate left continuous dynamical systems are considered: our approach is quite different, since we do not assume uniqueness of solutions of the implemented systems. Relations with the paper [31] are discussed as well. Despite what the robust control approach allows to do, characterizing the coarsest quantizer in the dissipativity framework is harder. Nevertheless, for a special class of dissipative systems, namely the passive ones, we give conditions under which asymptotic stabilizability can be achieved with a finite and “minimal” number of quantization levels.

We remark that, once the classical control laws have been “quantized”, i.e. approximated by new control laws taking values in U , the new feedbacks are necessarily discontinuous, and solutions of the implemented systems must be intended in some generalized sense. A number of different notions of generalized solutions have been proposed (see [16,19,10,32]). Here we focus on Krasowskii and Carathéodory solutions. There are several reasons for studying Krasowskii solutions. In the literature, there are many handy results concerning existence and continuation of Krasowskii solutions and a complete Lyapunov theory. Moreover, the set of Krasowskii solutions include Filippov and Carathéodory solutions, then results which use Krasowskii solutions also hold for Filippov and Carathéodory solutions. On the other hand, the set of Krasowskii solutions may be “too large”. In particular, it may contain “unfeasible” sliding modes. Krasowskii solutions may not be completely satisfactory from another point of view: it has been argued in [3] that, in general, they may not be reproduced by any open loop control taking values in the set U . Concerning Carathéodory solutions, it is interesting to recall that they have been recently used in order to achieve a number of powerful results about discontinuous stabilization [1,2,35]. Moreover, they fit very well in the context

of switched systems as discussed in [5,9]. The major difficulty in their use is establishing their existence and continuation (see [1,26,9]). Nevertheless, in the case we treat, some reasonable conditions for existence and uniqueness can be given.

The plan of the paper is the following. In Section 2, the robust control approach to the quantized stabilization of nonlinear systems is pursued. Section 3 deals with the dissipativity approach. A special notable case (passive systems) is dealt with in Section 4. Conclusions are drawn in Section 5.

1.1. Preliminaries

We recall here some notations and definitions. We denote by $|\cdot|$ the norm in \mathbb{R}^n , $n \geq 1$ and, if $x_0 \in \mathbb{R}^n$, we use the notation $B_\varepsilon(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}$. Given a set $S \subset \mathbb{R}^n$, the symbols $\overline{\text{co}} S$, $\overset{\circ}{S}$, \bar{S} denote, respectively, the convex closure of S , i.e. the smallest closed set containing the convex hull of S , the interior of the set S and its closure.

Consider a system of ordinary differential equations

$$\dot{x} = G(t, x), \quad (1)$$

where $G : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Definition 1. A curve $\varphi : [0, +\infty) \rightarrow \mathbb{R}^n$ is a

- Carathéodory solution of (1) if it is absolutely continuous and satisfies (1) for almost every $t \geq 0$.
- Krasowskii solution of (1) if it is absolutely continuous and for almost every $t \geq 0$ it satisfies the differential inclusion $\dot{x} \in K(G(t, x))$, where $K(G(t, x)) = \bigcap_{\delta > 0} \overline{\text{co}} G(t, B_\delta(x))$.

Clearly, Carathéodory solutions are Krasowskii solutions.

Definition 2. System (1) is said to be *globally asymptotically stable* at the origin with respect to Carathéodory (Krasowskii) solutions if

- for any $x_0 \in \mathbb{R}^n$ there exists a Carathéodory (Krasowskii) solution of (1), and all Carathéodory (Krasowskii) solutions of (1) are defined on $[0, +\infty)$;
- the origin is Lyapunov stable with respect to Carathéodory (Krasowskii) solutions, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_0 \in \mathbb{R}^n$ with $|x_0| < \delta$ and for any Carathéodory (Krasowskii) solution φ of (1) with $\varphi(0) = x_0$, we have $|\varphi(t)| < \varepsilon$ for all $t \in [0, +\infty)$;
- the origin is attractive, i.e. there exists $\eta > 0$ such that for any $x_0 \in \mathbb{R}^n$ with $|x_0| < \eta$ and for any Carathéodory (Krasowskii) solution φ of (1), we have $\lim_{t \rightarrow +\infty} \varphi(t) = 0$.

We introduce here the input-affine systems we will consider in the paper:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned} \quad (2)$$

where $x \in \mathbb{R}^n$ is the state, $u \in U$ is the input variable, with U a subset of \mathbb{R} to be specified below, $y \in \mathbb{R}$ is the output variable.

In the following we make the following assumptions:

- $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are vector fields of class C^1 , $f(0) = 0$;
- $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is of class C^1 , with $h(0) = 0$;
- $U \subset \mathbb{R}, 0 \in U, U$ symmetric, i.e. if $u \in U$ then also $-u \in U$.

The set \mathcal{U} of admissible inputs is formed by all measurable and locally bounded functions $u : [0, +\infty) \rightarrow U$. For each initial state x_0 and each admissible input $u \in \mathcal{U}$, system (2) has a unique local Carathéodory solution.

2. Robust control approach

2.1. Logarithmic quantizer and global stabilizability results

An important part of the literature about quantized control focuses on techniques which allow to approximate stabilizing feedback by means of control laws which take values in a properly chosen discrete set. In the context of linear systems, the logarithmic quantizer [14] had a great success.

Let $u_0 > 0$ and $0 < \rho < 1$ be fixed, let $u_i = \rho^i u_0$ and $U = \{0, \pm u_i, i \in \mathbb{Z}\}$. Let $\delta = (1 - \rho)/(1 + \rho)$ and [14,17]

$$\Psi(y) = \begin{cases} u_i, & \frac{1}{1+\delta}u_i < y \leq \frac{1}{1-\delta}u_i, \\ 0, & y = 0, \\ -\Psi(-y), & y < 0. \end{cases} \quad (3)$$

The map Ψ is represented in Fig. 1.

Following [17], one can consider both state and output feedback and, in the latter case, one can further distinguish between the cases of quantized input or quantized measurement. When the full state is measured, the controller is $u = \Psi(k(x))$. On the other hand, in the presence of input quantization, the dynamic output feedback takes the form

$$\begin{aligned} \dot{\xi} &= f_c(\xi, h(x)) + g_c(\xi, h(x))\Psi(k_c(\xi, h(x))), \\ u &= \Psi(k_c(\xi, h(x))), \end{aligned} \quad (4)$$

whereas, in the presence of output quantization,

$$\begin{aligned} \dot{\xi} &= f_c(\xi) + g_c(\xi)\Psi(h(x)), \\ u &= k_c(\xi) + \ell_c(\xi)\Psi(h(x)), \end{aligned} \quad (5)$$

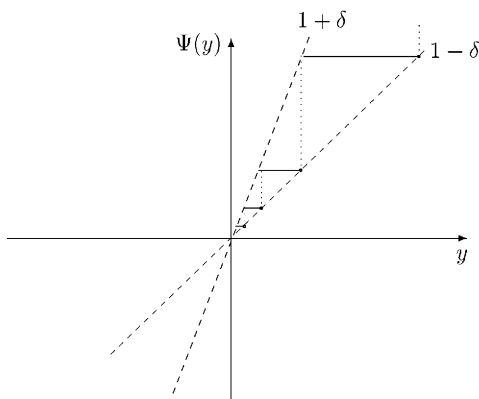


Fig. 1. Quantized control and sector bound.

with $\xi \in \mathbb{R}^{n_c}$. The closed-loop system turns out to be

$$\dot{\mathcal{X}} = \mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\Psi(\mathcal{K}(\mathcal{X})), \quad \mathcal{X} \in \mathbb{R}^N, \quad (6)$$

where the actual expressions of $\mathcal{X}, N, \mathcal{F}, \mathcal{G}, \mathcal{K}$ depend on the feedback employed and are understood from the context. Clearly, the “nominal”, i.e. with no quantization, system writes as

$$\dot{\mathcal{X}} = \mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X}), \quad \mathcal{X} \in \mathbb{R}^N. \quad (7)$$

We now give two propositions which state sufficient conditions for a stabilizing feedback law to be “quantizable” by means of the logarithmic quantizer (3). As a first step we consider Krasowskii solutions of the system in which the quantized feedback law is implemented. More precisely, we remark that Krasowskii solutions of (6) are absolutely continuous functions which satisfy the following differential inclusion:

$$\begin{aligned} \dot{\mathcal{X}} &\in \mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})K(\Psi(\mathcal{K}(\mathcal{X}))) \\ &\subseteq \begin{cases} \mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\{1 + \lambda\delta, & \mathcal{X} \neq 0, \\ \lambda \in [-1, 1]\}\mathcal{K}(\mathcal{X}), \\ 0, & \mathcal{X} = 0. \end{cases} \end{aligned} \quad (8)$$

In fact, let \mathcal{X} be such that $\mathcal{K}(\mathcal{X}) > 0$ (analogous considerations can be repeated for $\mathcal{K}(\mathcal{X}) < 0$). Since $(1 - \delta)\mathcal{K}(\mathcal{X}) \leq \Psi(\mathcal{K}(\mathcal{X})) \leq (1 + \delta)\mathcal{K}(\mathcal{X})$ then for all $v \in K(\Psi(\mathcal{K}(\mathcal{X})))$, $(1 - \delta)\mathcal{K}(\mathcal{X}) \leq v \leq (1 + \delta)\mathcal{K}(\mathcal{X})$, i.e. $v \in \{\mathcal{K}(\mathcal{X})(1 + \lambda\delta), \lambda \in [-1, 1]\}$. Hence, given $\mathcal{V} : \mathbb{R}^N \rightarrow \mathbb{R}$, for any $\mathcal{X} \in \mathbb{R}^N$ and any $v \in K(\Psi(\mathcal{K}(\mathcal{X})))$, we will need to study $\nabla\mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v)$. We can rewrite $\mathcal{K}(\mathcal{X}) - v = \lambda\delta\mathcal{K}(\mathcal{X})$, for some $\lambda \in [-1, 1]$, so that

$$\begin{aligned} \nabla\mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v) &\leq \nabla\mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X}) \\ &\quad - \mathcal{G}(\mathcal{X})\lambda\delta\mathcal{K}(\mathcal{X})). \end{aligned} \quad (9)$$

Proposition 1. Assume that there exist $\mathcal{V}, \mathcal{W} : \mathbb{R}^N \rightarrow \mathbb{R}$ continuous positive definite, \mathcal{V} is of class C^1 and radially unbounded, $\mathcal{K} : \mathbb{R}^n \rightarrow \mathbb{R}$ continuous such that for all $\mathcal{X} \in \mathbb{R}^N$,

$$\nabla\mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X})) \leq -\mathcal{W}(\mathcal{X}) \quad (10)$$

and assume moreover that there exists $\alpha > 0$ such that for all $\mathcal{X} \in \mathbb{R}^N$

$$\alpha|\nabla\mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X})| \leq \mathcal{W}(\mathcal{X}). \quad (11)$$

Then for every $\delta < \alpha$ the closed-loop system (6) is globally asymptotically stable at the origin with respect to Krasowskii solutions.

Proof. We prove that $\nabla\mathcal{V}(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v) < 0$ for every $\mathcal{X} \neq 0$ and for every $v \in K(\Psi(\mathcal{K}(\mathcal{X})))$. Let us first consider the case $\mathcal{X} \neq 0$ is such that $\nabla\mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X}) \neq 0$. Thanks to

(8) and (11) we have

$$\begin{aligned}
& \nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v) \\
&= \nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X})) \\
&\quad + \lambda \delta \nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X}) \\
&\leq -W(\mathcal{X}) + \delta |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X})| \\
&< -W(\mathcal{X}) + \alpha |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X})| \leq 0.
\end{aligned}$$

Let us now consider $\mathcal{X} \neq 0$ such that $\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X}) = 0$. From (10) we deduce $\nabla \mathcal{V}(\mathcal{X})\mathcal{F}(\mathcal{X}) < 0$ and then also

$$\nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v) = \nabla \mathcal{V}(\mathcal{X})\mathcal{F}(\mathcal{X}) < 0. \quad \square$$

Remark. Proposition 1 recalls the results about stability under vanishing perturbations which are collected in [25, Section 5.1, p. 204]. In fact, the term $\lambda \delta \mathcal{K}(\mathcal{X})$, $\lambda \in [-1, 1]$, can be seen as a discontinuous vanishing perturbation affecting system (7). Bearing in mind this, it is not difficult to realize that, if (7) is exponentially stable, then system (6) is exponentially stable as well.

In case a stabilizing feedback and a Lyapunov function are known, but condition (11) is not satisfied, one can turn to the following proposition, which is inspired by the linear discrete-time scenario studied in [17], and shows a connection between the coarseness of the quantizer and a finite L_2 -gain problem.

Proposition 2. Assume that system (7) is globally asymptotically stable and there exist $\mathcal{V} : \mathbb{R}^N \rightarrow \mathbb{R}$ of class C^1 , positive definite and radially unbounded and $\gamma > 0$ such that for all $\mathcal{X} \in \mathbb{R}^N$,

$$\begin{aligned}
& \nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X})) \\
&+ \frac{1}{4\gamma^2} |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|^2 + \mathcal{K}^2(\mathcal{X}) \leq 0. \quad (12)
\end{aligned}$$

Then for any $\delta \leq 1/\gamma$ the closed-loop system (6) is globally asymptotically stable at the origin with respect to Krasowskii solutions.

Proof. For any $\mathcal{X} \in \mathbb{R}^N$ and for any $v \in K(\Psi(\mathcal{K}(\mathcal{X})))$, consider the equality (9). Completion of the squares yields

$$\begin{aligned}
& \nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v) \\
&\leq \nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X})) \\
&\quad + \frac{\delta^2}{4} |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|^2 + \mathcal{K}^2(\mathcal{X}) \\
&\leq \nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\mathcal{K}(\mathcal{X})) \\
&\quad + \frac{1}{4\gamma^2} |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|^2 + \mathcal{K}^2(\mathcal{X}) \leq 0, \quad (13)
\end{aligned}$$

where the latter inequality holds thanks to (12). Let now \mathcal{X} be such that $\mathcal{K}(\mathcal{X}) = 0$. From (12) we get that

$$\begin{aligned}
& \nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v) \leq \nabla \mathcal{V}(\mathcal{X})\mathcal{F}(\mathcal{X}) \\
&\leq -\frac{1}{4\gamma^2} |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|^2 \leq 0. \quad (14)
\end{aligned}$$

Thanks to LaSalle invariance principle for differential inclusions (see [4]), we obtain that all solutions of (6) tend to the largest weakly invariant subset of

$$\begin{aligned}
& \{\mathcal{X} \in \mathbb{R}^N : \exists v \in K(\Psi(\mathcal{K}(\mathcal{X}))) \text{ such that } \nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) \\
&\quad + \mathcal{G}(\mathcal{X})v) = 0\} \subseteq \{\mathcal{X} \in \mathbb{R}^N : \mathcal{K}(\mathcal{X}) = 0\},
\end{aligned}$$

the inclusion being deduced from (13), (14). The thesis is then deduced by recalling that, since Ψ is continuous at 0, solutions of (12) which lie on $\{\mathcal{X} \in \mathbb{R}^N : \mathcal{K}(\mathcal{X}) = 0\}$ coincide with solutions of (7). \square

Remark. The same result could be given using nonlogarithmic quantizers. For this class of quantizers, in fact, the map Ψ satisfies $(1 - \delta^-)y \leq \Psi(y) \leq (1 + \delta^+)y$, with δ^-, δ^+ a given pair of positive constants. Then, Proposition 2 continues to hold provided that the constant δ in the statement is replaced by $\max\{|\delta^-|, |\delta^+|\}$.

Remark. Roughly speaking, Proposition 2 states that, if we know a stabilizing static or dynamic, state or output feedback controller such that we can solve (12) for some \mathcal{V} , then any quantization of the control input or of the measured output by means of a function Ψ in a sector bound whose ‘‘amplitude’’ is smaller than $1/\gamma$ does not cancel the stabilizing effect of the controller. In the case of a static state-feedback control law, the fulfillment of inequality (12) implies the existence of a map k which renders the L_2 -gain from w to z of the system

$$\begin{aligned}
\dot{x} &= f(x) + g(x)u + g(x)w, \\
z &= u
\end{aligned}$$

less than or equal to γ . Observe that, in [17], it is shown how, for linear discrete-time systems, the inverse of the smallest γ for which the above L_2 -gain attenuation problem is solvable gives the coarsest quantizer for which (quadratic) stabilization via quantized state feedback is achievable. Related results are also found in [24, Lemma 1 and Remark 4]. On the other hand, in the case of a dynamic output feedback controller with quantized output, the fulfillment of (12) implies the existence of a controller of the form

$$\begin{aligned}
\dot{\xi} &= f_c(\xi) + g_c(\xi)y, \\
u &= k_c(\xi) + \ell_c(\xi)y,
\end{aligned}$$

which renders the L_2 -gain from w to z of

$$\begin{aligned}
\dot{x} &= f(x) + g(x)u, \\
y &= h(x) + w, \\
z &= h(x)
\end{aligned}$$

less than or equal to γ . Again, in the case of linear discrete-time systems, the smallest value of γ for which this attenuation problem is solvable corresponds to the coarsest quantizer for which (quadratic) stabilization via dynamic quantized-output feedback is solvable (see [17, Theorem 3.2]).

Until now we have considered Krasowskii solutions. On the other hand, as already discussed in the Introduction, Carathéodory solutions seem to be appropriate in order to deal with problems in which the control takes values in a discrete set. Their use is limited by the difficulty in giving sufficient conditions for existence and continuation of solutions for general discontinuous vector fields. Nevertheless, in some particular cases of great interest for applications, such conditions can be actually given in a reasonably easy form. Here we consider the case of system (6).

Lemma 1. *Assume that \mathcal{K} is of class C^1 , $\nabla\mathcal{K}(\mathcal{X}) \neq 0$ for $\mathcal{X} \neq 0$ and*

- (i) *for all $i \in \mathbb{Z}$, for all $\mathcal{X} \neq 0$ such that $\mathcal{K}(\mathcal{X}) = u_i/(1 - \delta)$ one of the following alternative conditions holds:*
 - (i.a) $\nabla\mathcal{K}(\mathcal{X}) \cdot (\mathcal{F}(\mathcal{X}) + (1/\rho)\mathcal{G}(\mathcal{X})u_i) > 0$,
 - (i.b) $\nabla\mathcal{K}(\mathcal{X}) \cdot (\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})u_i) < 0$,
 - (i.c) *there exists $\varepsilon > 0$ such that if $\tilde{\mathcal{X}} \in B_\varepsilon(\mathcal{X})$ and $\mathcal{K}(\tilde{\mathcal{X}}) = u_i/(1 - \delta)$ then $\nabla\mathcal{K}(\tilde{\mathcal{X}}) \cdot (\mathcal{F}(\tilde{\mathcal{X}}) + \mathcal{G}(\tilde{\mathcal{X}})u_i) = 0$,*
- (ii) *for all $i \in \mathbb{Z}$, for all $\mathcal{X} \neq 0$ such that $\mathcal{K}(\mathcal{X}) = -u_i/(1 - \delta)$ one of the following alternative conditions holds:*
 - (ii.a) $\nabla\mathcal{K}(\mathcal{X}) \cdot (\mathcal{F}(\mathcal{X}) - (1/\rho)\mathcal{G}(\mathcal{X})u_i) < 0$,
 - (ii.b) $\nabla\mathcal{K}(\mathcal{X}) \cdot (\mathcal{F}(\mathcal{X}) - \mathcal{G}(\mathcal{X})u_i) > 0$,
 - (ii.c) *there exists $\varepsilon > 0$ such that if $\tilde{\mathcal{X}} \in B_\varepsilon(\mathcal{X})$ and $\mathcal{K}(\tilde{\mathcal{X}}) = -u_i/(1 - \delta)$ then $\nabla\mathcal{K}(\tilde{\mathcal{X}}) \cdot (\mathcal{F}(\tilde{\mathcal{X}}) - \mathcal{G}(\tilde{\mathcal{X}})u_i) = 0$,*
- (iii) *for all $\mathcal{X} \neq 0$ such that $\mathcal{K}(\mathcal{X}) = 0$ one of the following alternative conditions holds:*
 - (iii.a) $\nabla\mathcal{K}(\mathcal{X}) \cdot \mathcal{F}(\mathcal{X}) \neq 0$,
 - (iii.b) *there exists $\varepsilon > 0$ such that if $\tilde{\mathcal{X}} \in B_\varepsilon(\mathcal{X})$ and $\mathcal{K}(\tilde{\mathcal{X}}) = 0$ then $\nabla\mathcal{K}(\tilde{\mathcal{X}}) \cdot \mathcal{F}(\tilde{\mathcal{X}}) = 0$.*

Then for any initial condition at least one local Carathéodory solution of (6) exists.

The proof of the previous lemma is postponed to the Appendix. We use the lemma in order to give results analogous to Propositions 1 and 2 in terms of Carathéodory solutions.

Proposition 3. *Let the assumptions of Proposition 1 and Lemma 1 be satisfied. Then, for any $\delta < 1/\alpha$, system (6) is globally asymptotically stable with respect to Carathéodory solutions.*

Proof. Local existence of Carathéodory solutions is obtained thanks to Lemma 1. We recall that Carathéodory solutions are also Krasowskii solutions. Note that from the proof of Proposition 1 we deduce that (6) is Lyapunov stable with respect to Krasowskii solutions and then also with respect to Carathéodory

solutions. This implies that Carathéodory solutions are bounded. Using the fact that the right-hand side of (6) is locally bounded it can be easily deduced that Carathéodory solutions are right continuable up to $+\infty$. By recalling again that Carathéodory solutions of (6) are also Krasowskii solutions, we conclude, as in the proof of Proposition 1, that (6) is globally asymptotically stable with respect to Carathéodory solutions. \square

Proposition 4. *Assume that the conditions of Proposition 2 and Lemma 1 are satisfied. Then, for any $\delta \leq 1/\gamma$, system (6) is globally asymptotically stable with respect to Carathéodory solutions.*

The proof follows exactly the same line as the proof of Proposition 3.

2.2. Overcoming the limitation of the quantization density

We have seen so far that, unless a solution is found to inequality (12), it may be difficult to asymptotically stabilize a nonlinear system using a coarse quantization. To overcome this limitation, we resort here to a different approach. Rather than investigating the conditions under which the quantization error can be tolerated by the controlled system, we pose the problem in the following terms. Given the uncertainty due to the quantization, is it possible to devise a control law that, besides stabilizing, is able to actively counteract the quantization error? The (positive) answer is provided by the following statement.

Proposition 5. *Assume there exist $\mathcal{V}, \mathcal{W} : \mathbb{R}^N \rightarrow \mathbb{R}$ continuous positive definite, \mathcal{V} of class C^1 and radially unbounded, $\mathcal{K} : \mathbb{R}^n \rightarrow \mathbb{R}$ continuous such that (10) holds for all $\mathcal{X} \in \mathbb{R}^N$. For any pair $0 < r < R$, for any $\delta \in (0, 1)$, there exist $u_0 \geq 0$ and a continuous function $\tilde{\mathcal{K}} : \mathbb{R}^N \rightarrow \mathbb{R}$ such that for any Krasowskii solution φ of*

$$\dot{\mathcal{X}} = \mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\Psi(\tilde{\mathcal{K}}(\mathcal{X})), \quad (15)$$

if $\varphi(0) \in \overline{B_R(0)}$, then there exists $T > 0$ such that $\varphi(t) \in \overline{B_r(0)}$ for all $t \geq T$.

Remark. Compared with Proposition 2, the result states that, even for those δ which are not smaller than $1/\gamma$, it is possible to stabilize, although not asymptotically, the system under feedback quantization.

Proof. Let $M = \max_{\mathcal{X} \in \overline{B_R(0)}} \mathcal{V}(\mathcal{X})$, $\Omega_M = \{\mathcal{X} \in \mathbb{R}^N : \mathcal{V}(\mathcal{X}) \leq M\}$, m be such that $\overset{\circ}{\Omega}_m = \{\mathcal{X} \in \mathbb{R}^N : \mathcal{V}(\mathcal{X}) < m\} \subset B_r(0)$, $\kappa = \max_{\mathcal{X} \in \Omega_M} |\mathcal{K}(\mathcal{X})|$ and $2\varepsilon \leq \min_{\mathcal{X} \in \Omega_M \setminus \overset{\circ}{\Omega}_m} \mathcal{W}(\mathcal{X})$. Then define

$$\tilde{\mathcal{K}}(\mathcal{X}) = \mathcal{K}(\mathcal{X}) - \alpha \nabla \mathcal{V}(\mathcal{X}) \mathcal{G}(\mathcal{X}),$$

with $\alpha \geq (\delta\kappa)^2 / (4\varepsilon(1 - \delta))$ a fixed constant. Finally, let $u_0 = \max_{\mathcal{X} \in \Omega_M \setminus \overset{\circ}{\Omega}_m} |\mathcal{K}(\mathcal{X})|$ in the definition of the map Ψ .

For any $\mathcal{X} \in \Omega_M$, for each $v \in K(\Psi(\tilde{\mathcal{K}}(\mathcal{X})))$, we have

$$\begin{aligned}
& \nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v) \\
& \leq -\mathcal{W}(\mathcal{X}) - \alpha |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|^2 + \nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})(v - \tilde{\mathcal{K}}(\mathcal{X})) \\
& \leq -\mathcal{W}(\mathcal{X}) - \alpha |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|^2 + \delta |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})| |\tilde{\mathcal{K}}(\mathcal{X})| \\
& \leq -\mathcal{W}(\mathcal{X}) - \alpha(1 - \delta) |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|^2 \\
& \quad + \delta |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})| \cdot |\tilde{\mathcal{K}}(\mathcal{X})| \\
& \leq -\mathcal{W}(\mathcal{X}) - \alpha(1 - \delta) |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|^2 + \delta \kappa |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})| \\
& \leq -\frac{\mathcal{W}(\mathcal{X})}{2} - \frac{\mathcal{W}(\mathcal{X})}{2} - \alpha(1 - \delta) |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|^2 \\
& \quad + \delta \kappa |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|.
\end{aligned}$$

In particular, for $\mathcal{X} \in \Omega_M \setminus \overset{\circ}{\Omega}_m$, by definition of ε and α , it is easily seen that the latter inequality becomes $\nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v) \leq -\mathcal{W}(\mathcal{X})/2 \leq -\varepsilon$, which shows the thesis, by standard arguments. \square

Remark. In the case of output feedback, the actual implementation of this result may require additional hypotheses.

It is seen from the latter proposition that we must trade off global asymptotic stability against semi-global practical stability in order to stabilize the system without posing any constraint on the quantization density. Nevertheless, in a special noticeable case pointed out below, it is possible to recover asymptotic stability.

Corollary 1. *Let the hypothesis of Proposition 5 hold, with \mathcal{K} continuously differentiable, and additionally assume that \mathcal{K} renders the closed-loop system locally exponentially stable. Then, for any $R > 0$ and any $\delta \in (0, 1)$, there exists a continuous function $\tilde{\mathcal{K}} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that the closed-loop system (15) is locally asymptotically stable at the origin with respect to Krasowskii solutions, and for any Krasowskii solution φ , $\varphi(0) \in \overline{B_R(0)}$ implies $\lim_{t \rightarrow \infty} \varphi(t) = 0$.*

Proof. Note that, without loss of generality (cf. e.g. [23, Lemma 10.1.5]), one can always assume that, for r sufficiently small, $\mathcal{W}(\mathcal{X}) = c|\mathcal{X}|^2$, for some $c > 0$, over the set $\overline{B_r(0)}$. Take $\tilde{\mathcal{K}}$ as in the proof of Proposition 5, with

$$\alpha \geq \frac{\delta^2}{1 - \delta} \max \left\{ \frac{\kappa^2}{2\varepsilon}, \frac{k^2}{c} \right\},$$

k being the Lipschitz constant of \mathcal{K} on $\overline{B_r(0)}$. Any Krasowskii solution converges in finite time to $\overline{B_r(0)}$, over which, from the proof of Proposition 5, the following is known to hold:

$$\begin{aligned}
& \nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v) \\
& \leq -c|\mathcal{X}|^2 - \alpha(1 - \delta) |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|^2 \\
& \quad + \delta k |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})| |\mathcal{X}|.
\end{aligned}$$

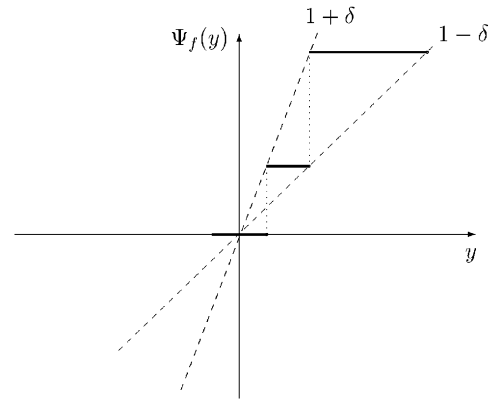


Fig. 2. Finitely quantized control.

By the definition of α , a trivial completion-of-the-squares argument shows that the right-hand side of the inequality is bounded from above by $-c|\mathcal{X}|^2/2$, and asymptotic convergence is immediately inferred. \square

As in the previous subsection, it is possible to restate the results in terms of Carathéodory solutions.

2.3. Semi-global practical stabilization by means of finite valued feedback laws

The results of the previous section require an infinite number of quantization levels. Here we investigate the case in which only a finite number of quantization levels can be used. This problem has been deeply investigated in the case of linear discrete-time systems in [14,15]. When a continuous stabilizing feedback law is known, it is relatively easy for nonlinear continuous-time systems to obtain semi-global practical stabilization by quantizing such feedback law. We introduce the truncated version of (3) (Fig. 2):

$$\Psi_f(y) = \begin{cases} u_0, & \frac{1}{1 + \delta} u_0 < y, \\ u_i, & \frac{1}{1 + \delta} u_i < y \leq \frac{1}{1 - \delta} u_i, \quad 1 \leq i \leq j, \\ 0, & 0 \leq y \leq \frac{1}{1 + \delta} u_j, \\ -\Psi_f(-y), & y < 0, \end{cases} \quad (16)$$

with j to determine.

Proposition 6. *Assume that there exist $\mathcal{V}, \mathcal{W} : \mathbb{R}^N \rightarrow \mathbb{R}$ continuous positive definite, \mathcal{V} of class C^1 and radially unbounded, $\mathcal{K} : \mathbb{R}^n \rightarrow \mathbb{R}$ continuous such that (10) holds for all $\mathcal{X} \in \mathbb{R}^N$. For any pair $0 < r < R$, for any $\delta \in (0, 1)$, there exist $u_0 \geq 0$, $j \in \mathbb{N}$, and a continuous function $\tilde{\mathcal{K}} : \mathbb{R}^N \rightarrow \mathbb{R}$ such that for any Krasowskii solution φ of*

$$\dot{\mathcal{X}} = \mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})\Psi_f(\tilde{\mathcal{K}}(\mathcal{X})), \quad (17)$$

if $\varphi(0) \in \overline{B_R(0)}$ then there exists $T > 0$ such that $\varphi(t) \in \overline{B_r(0)}$ for all $t \geq T$.

Proof. Let $M, \Omega_M, \overset{\circ}{\Omega}_m, \kappa, \varepsilon, \alpha, \tilde{\mathcal{K}}$ and u_0 , be as in the proof of Proposition 5. We prove that for an appropriate choice of j , we have $\nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v) \leq -\varepsilon$ for any $\mathcal{X} \in \Omega_M \setminus \overset{\circ}{\Omega}_m$ and for any $v \in K(\Psi_f(\tilde{\mathcal{K}}(\mathcal{X})))$. As $\overset{\circ}{\Omega}_m \subseteq \overline{B_r(0)} \subset \overline{B_R(0)} \subseteq \Omega_M$, the thesis is inferred in a standard way.

We distinguish the cases $\max_{\mathcal{X} \in \Omega_M \setminus \overset{\circ}{\Omega}_m} |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})| = 0$ and $\max_{\mathcal{X} \in \Omega_M \setminus \overset{\circ}{\Omega}_m} |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})| > 0$. In the former case, we have $\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X}) \equiv 0$ on $\Omega_M \setminus \overset{\circ}{\Omega}_m$, and it follows from (10) that

$$\begin{aligned} & \nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v) \\ &= \nabla \mathcal{V}(\mathcal{X})\mathcal{F}(\mathcal{X}) \leq -\mathcal{W}(\mathcal{X}) \leq -\frac{1}{2}\mathcal{W}(\mathcal{X}) \leq -\varepsilon. \end{aligned}$$

In the latter case, let

$$\mu = \frac{\min_{\mathcal{X} \in \Omega_M \setminus \overset{\circ}{\Omega}_m} [\mathcal{W}(\mathcal{X})/2 + \alpha|\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|^2]}{\max_{\mathcal{X} \in \Omega_M \setminus \overset{\circ}{\Omega}_m} |\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|},$$

and $j \in \mathbb{N}$ satisfy

$$j \geq \left(\log \frac{\mu(1+\delta)}{u_0} \right) \left(\log \frac{1-\delta}{1+\delta} \right)^{-1}. \quad (18)$$

From the proof of Proposition 5, we have that for any $\mathcal{X} \in \Omega_M \setminus \overset{\circ}{\Omega}_m$, for each $v \in K(\Psi_f(\tilde{\mathcal{K}}(\mathcal{X})))$,

$$\begin{aligned} & \nabla \mathcal{V}(\mathcal{X})(\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})v) \\ & \leq -\varepsilon - \left[\frac{\mathcal{W}(\mathcal{X})}{2} + \alpha|\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|^2 \right] \\ & \quad + \nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})(v - \tilde{\mathcal{K}}(\mathcal{X})). \end{aligned}$$

Hence, it is sufficient to prove that for any $\mathcal{X} \in \Omega_M \setminus \overset{\circ}{\Omega}_m$

$$|v - \tilde{\mathcal{K}}(\mathcal{X})| \leq \mu \leq \frac{\mathcal{W}(\mathcal{X})/2 + \alpha|\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|^2}{|\nabla \mathcal{V}(\mathcal{X})\mathcal{G}(\mathcal{X})|}. \quad (19)$$

If $u_j/(1+\delta) < |\tilde{\mathcal{K}}(\mathcal{X})| \leq u_0$, then $|v - \tilde{\mathcal{K}}(\mathcal{X})| \leq \delta|\tilde{\mathcal{K}}(\mathcal{X})|$, and that (19) is satisfied has already been proven in the proof of Proposition 5. If $|\tilde{\mathcal{K}}(\mathcal{X})| \leq u_j/(1+\delta)$, then

$$|v - \tilde{\mathcal{K}}(\mathcal{X})| \leq |\tilde{\mathcal{K}}(\mathcal{X})| \leq \frac{u_j}{1+\delta} = \frac{1}{1+\delta} \left(\frac{1-\delta}{1+\delta} \right)^j u_0,$$

and (19) is satisfied because of (18). \square

Remark. Employing the “zooming-in” technique of [27] (see also [22,12]), it is not difficult to modify the previous arguments to obtain asymptotic stability. The difference with [27] is that we do not assume input-to-state stability.

As for the case of the logarithmic quantizer, Proposition 6 can be reformulated in terms of Carathéodory solutions provided that sufficient conditions for local existence are given.

3. Stabilization of dissipative systems

As the conditions of Propositions 1 and 2 may be quite demanding, whereas Proposition 5 guarantees semi-global practical stabilizability (but see Corollary 1 for a result on semi-global asymptotic stabilizability), we are led to consider special classes of control systems for which classical asymptotic stabilization results are known, and whose features allow for extensions in case discontinuous control is employed. The remaining part of the paper focuses on a special class of nonlinear single input systems (2), namely those which are *dissipative* with respect to a quadratic supply rate [20,23].

We consider Krasowskii solutions (but a comment on Carathéodory solutions is provided later) and prove a result on asymptotic stability of interconnected dissipative systems where one of the systems is a memoryless dissipative function, possibly discontinuous. In particular, we examine the case this function is sector bounded and give a nonlinear discontinuous version of the circle criterion.

The technique used in the proofs does not differ very much from the classical one, but in the novel context further technical assumptions are needed. The study performed until now is not comprehensive: some possible developments are outlined in the conclusions.

The class of dissipative systems is recalled below. In order to deal with possibly discontinuous systems, we need to slightly extend the notion of dissipativity.

Definition 3. System (2) is said to be *dissipative*, respectively *co-dissipative*, with respect to a *quadratic supply rate*

$$q(u, y) = uRu + 2uSy + yQy \quad (20)$$

with $R, S, Q \in \mathbb{R}$, if there exists a C^1 , positive definite and radially unbounded function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ such that, for all $x \in \mathbb{R}^n$, for all $u \in U$, for all $y = h(x)$,

$$\nabla V(x) \cdot (f(x) + g(x)u) \leq q(u, y), \quad (21)$$

respectively, for all $x \in \mathbb{R}^n$, for all $v \in \overline{\text{co}}U$, for all $y = h(x)$,

$$\nabla V(x) \cdot (f(x) + g(x)v) \leq q(v, y). \quad (22)$$

Any function V which verifies either (21) or (22) is said to be a *storage function* for (2).

Remark. Important classes of nonlinear systems are dissipative. If $q(u, y) = uy$ the system is said to be *passive*. In the latter case, since we have assumed $0 \in U$, by taking $u = 0$ in (21), we get that for all $x \in \mathbb{R}^n$, $\nabla V(x) \cdot f(x) \leq 0$, which implies that the unforced system

$$\dot{x} = f(x) \quad (23)$$

is Lyapunov stable. Since system (2) is affine in the input variable u , dissipativity with respect to the supply rate $q(u, y) = uy$ (i.e. passivity) is equivalent to co-dissipativity (that we may call *co-passivity*).

Analogously we introduce the notion of dissipativity for a static memoryless system. Since we are interested in the

negative interconnection of such systems with (2), we restrict to functions taking values in U .

Definition 4. A system with input $\tilde{u} \in \mathbb{R}$ and output $\tilde{y} \in U$ related by the function $\tilde{y} = \psi(\tilde{u})$, $\psi : \mathbb{R} \rightarrow U$, is said to be *dissipative*, respectively *co-dissipative*, with respect to a supply rate $\tilde{q} : \mathbb{R} \times U \rightarrow \mathbb{R}$, if for all $\tilde{u} \in \mathbb{R}$, for all $\tilde{y} = \psi(\tilde{u})$, $\tilde{q}(\tilde{u}, \tilde{y}) \geq 0$, respectively, for all $\tilde{u} \in \mathbb{R}$, for all $\tilde{y} \in K(\psi(\tilde{u}))$, $\tilde{q}(\tilde{u}, \tilde{y}) \geq 0$. Equivalently the function ψ is said to be dissipative, respectively co-dissipative.

Remark. An important class of static memoryless systems are sector bounded systems. More precisely the system is said to be *sector bounded* if it is dissipative with respect to

$$\tilde{q}(\tilde{u}, \tilde{y}) = (\tilde{y} - \alpha\tilde{u})(\beta\tilde{u} - \tilde{y}) \quad (24)$$

for some pair of real numbers $\beta > \alpha > 0$.

Note that if $\psi : \mathbb{R} \rightarrow U$ is dissipative with respect to the previous \tilde{q} , i.e. it is sector bounded, then it is locally bounded, continuous at zero, $\psi(\tilde{u}) = 0$ if and only if $\tilde{u} = 0$, and moreover it is also co-dissipative, i.e. we may say it is also *co-sector bounded*.

To infer *asymptotic* stabilizability for dissipative systems, the following property is relevant.

Definition 5. Let $Z_h = \{x \in \mathbb{R}^n : h(x) = 0\}$. System (2) is said to be *zero-state detectable* if for any Carathéodory solution $\varphi : [0, +\infty) \rightarrow \mathbb{R}^n$ of system (23) such that $\varphi(t) \in Z_h$ for all $t \geq 0$, it holds $\lim_{t \rightarrow +\infty} \varphi(t) = 0$.

The previous notion of zero-state detectability may be strengthened when dealing with discontinuous systems. The following definition was given in [31].

Definition 6. System (2) is said to be *strongly zero-state detectable* if for any measurable function $v : [0, +\infty) \rightarrow \mathbb{R}$ taking values in a closed subset of \mathbb{R} and for any Carathéodory solution $\phi : [0, +\infty) \rightarrow \mathbb{R}^n$ of the system $\dot{x} = f(x) + v(t)$ such that $\phi(t) \in Z_h$ for all $t \geq 0$, it holds $\lim_{t \rightarrow +\infty} \phi(t) = 0$.

It is not hard to characterize conditions under which dissipative systems can be asymptotically stabilized. This descends from a result about asymptotic stability of interconnected dissipative systems (see e.g. [23]), where one of the systems is a *memoryless* dissipative function. The usual assumption on the function ψ is to be sufficiently regular (for instance, locally Lipschitz). Our first result shows that such assumption can be removed. On the other hand, the assumption of co-dissipativity must be adopted.

Proposition 7. Let $U \subset \mathbb{R}$ be given. Assume that

- (i) system (2) is co-dissipative with respect to the quadratic supply rate (20) and zero-state detectable;
- (ii) $\psi : \mathbb{R} \rightarrow U$ is measurable, locally bounded, continuous at 0 with $\psi(\tilde{u}) = 0$ if and only if $\tilde{u} = 0$, and ψ is co-dissipative

with respect to the quadratic supply rate

$$\tilde{q}(\tilde{u}, \tilde{y}) = \tilde{u}\tilde{R}\tilde{u} + 2\tilde{y}\tilde{S}\tilde{u} + \tilde{y}\tilde{Q}\tilde{y}; \quad (25)$$

- (iii) there exists $a > 0$ such that the matrix

$$M = \begin{bmatrix} Q + a\tilde{R} & -S + a\tilde{S} \\ -S + a\tilde{S} & R + a\tilde{Q} \end{bmatrix}$$

is definite negative.

Then the closed-loop system

$$\dot{x} = f(x) - g(x)\psi(h(x)) \quad (26)$$

is globally asymptotically stable at the origin with respect to Krasowskii solutions.

Proof. We consider Krasowskii solutions of system (26), i.e. solutions of the differential inclusion

$$\dot{x} \in K(f(x) - g(x)\psi(h(x))). \quad (27)$$

Note that

$$K(f(x) - g(x)\psi(h(x))) = f(x) - g(x)K(\psi(h(x))).$$

By co-dissipativity of ψ , we have that $\tilde{q}(y, v) \geq 0$ for all $y \in \mathbb{R}$ and $v \in K(\psi(y))$. Using co-dissipativity of (2), one can show (see e.g. [23]) that, for any $v \in K(\psi(y))$, with $y = h(x)$,

$$\nabla V(x)(f(x) - g(x)v) \leq [h(x) \ v] M \begin{bmatrix} h(x) \\ v \end{bmatrix}.$$

As $\psi(0) = 0$ and ψ is continuous at 0 we have that, for any $v \in K(\psi(h(x)))$, $x \in \mathbb{R}^n$,

$$\nabla V(f(x) - g(x)v) \begin{cases} < 0 & \text{if } h(x) \neq 0, \\ = 0 & \text{if } h(x) = 0. \end{cases} \quad (28)$$

Hence, for each Krasowskii solution φ of (26), φ is right continuous up to $+\infty$, V is decreasing along φ and system (26) is Lyapunov stable with respect to Krasowskii solutions (see, e.g. [6, p. 131]). Moreover, thanks to LaSalle's invariance principle (see e.g. [3]), we get that Krasowskii solutions of (26) tend to the largest weakly invariant subset M of $\{x \in \mathbb{R}^n : \exists v \in K(f(x) - g(x)\psi(h(x))) : \nabla V(x) \cdot v = 0\} \subset Z_h$.

We now prove that every Krasowskii solution φ actually tends to 0. Since $V \circ \varphi$ is monotone, there exists $\lim_{t \rightarrow +\infty} V(\varphi(t)) = c$, for some $c \geq 0$. We now prove that $c = 0$. Since V is assumed to be continuous and positive definite, this will imply that $\lim_{t \rightarrow +\infty} \varphi(t) = 0$. Let Γ be the ω -limit set associated with φ . Γ is nonempty, compact, weakly invariant and is contained in the set Z_h .

By definition, $V(x) = c$ for all $x \in \Gamma$. Let φ_Γ be a Krasowskii solution of (26) such that $\varphi_\Gamma(t) \in \Gamma$ for all $t \geq 0$. As $\Gamma \subseteq Z_h$, $h(x) = 0$, $\psi(0) = 0$, and ψ is continuous at 0, we have that $K(f(x) - g(x)\psi(h(x))) = f(x)$ for $x \in \Gamma \subseteq Z_h$. Hence, $h(\varphi_\Gamma(t)) = 0$ for all $t \geq 0$ implies $\varphi_\Gamma(t) \rightarrow 0$ as $t \rightarrow \infty$, as a consequence of the zero-state detectability. Since $c = V(\varphi_\Gamma(t))$ for all $t \geq 0$ we conclude that $c = \lim_{t \rightarrow \infty} V(\varphi_\Gamma(t)) = V(\lim_{t \rightarrow \infty} \varphi_\Gamma(t)) = 0$. \square

Remark. One of the interests of this result lies in the possibility of stabilizing nonlinear systems using quantized feedback. In fact, consider a *discontinuous sector-bounded* static nonlinearity ψ satisfying (24). A possible function of this kind is illustrated in Fig. 1 in the case $0 < \alpha = 1 - \delta$, $\beta = 1 + \delta$. It is straightforward to determine conditions on Q, R, S under which any nonlinear system (2) co-dissipative with respect to a supply rate (20) in negative feedback interconnection with a sector bounded memoryless nonlinearity is such that $M < 0$, and, as such, is asymptotically stabilizable under the hypothesis of zero-state detectability. Note that dissipative systems which satisfy conditions as discussed above can be well unstable when $u = 0$, as it is the case for the system in [37, Example 2.35].

In some cases, when it is not possible to find $a > 0$ such that $M < 0$, similar arguments can be applied: a result analogous to the previous one can be given the form of the *nonlinear discontinuous* version of the celebrated *circle criterion* [23,37].

Proposition 8. *Let $U \subset \mathbb{R}$ be given. Assume that*

- (i) *system (2) is zero-state detectable;*
- (ii) *there exists a pair of real numbers $\beta > \alpha > 0$ such that system (2) is co-dissipative with respect to the supply rate*

$$q(u, y) = \left(y + \frac{1}{\beta}u \right) \left(u + k \left(y + \frac{1}{\beta}u \right) \right),$$

with $k = \beta\alpha/(\beta - \alpha)$;

- (iii) *the function $\psi : \mathbb{R} \rightarrow U$ is measurable and dissipative with respect to (24), with $\tilde{q}(\tilde{u}, \tilde{y}) = 0$ if and only if $\tilde{u} = 0$.*

Then, system (26) is globally asymptotically stable at the origin with respect to Krasowskii solutions.

Proof. (Sketch) One can repeat the arguments used in [37]. Co-dissipativity of (2) implies that there exist a C^1 positive definite and radially unbounded function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that for any $x \in \mathbb{R}^n$, for any $v \in K(\psi(h(x)))$, for any $\tilde{y} = h(x) - v/\beta$

$$\nabla V(x)(f(x) - g(x)v) \leq -v\tilde{y} + k\tilde{y}^2.$$

Then one shows that, for any $x \in \mathbb{R}^n$, for any $v \in K(\psi(h(x)))$, for any $y = h(x)$

$$\nabla V(x)(f(x) - g(x)v) \leq -\frac{1}{\beta - \alpha}(\beta y - v)(v - \alpha y) \leq 0,$$

where the latter inequality holds by (24) and Remark 3, and that the equality holds if and only if $v = y = 0$. Then, it is possible to apply the same machinery as in the proof of Proposition 7 to infer the result. \square

Remark. In case ψ is a sector bound nonlinearity of the form (3), conditions of Lemma 1 can be used in order to get stabilization results for dissipative systems in terms of Carathéodory solutions.

4. Stabilization of passive systems via small inputs

The results of the previous section require continuity of ψ at the origin. Investigating the case in which this requirement is not met is particularly interesting if the sector bound is such that $\alpha=0$ and $\beta=+\infty$. In the case of passive systems, we derive the following propositions in which stabilization is obtained by means of feedback laws taking values in a finite set U . We recall here that a previous result on stabilization of passive systems which makes use of Krasowskii solutions was proved in [31].

Proposition 9. *Let $U = \{0, \varepsilon, -\varepsilon\}$, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Assume that system (2) is passive. Moreover, assume that one of the following condition holds:*

- (i) *for each $x \in Z_h$, $x \neq 0$,*

$$\nabla h(x) \cdot (f(x) + \lambda g(x)) \neq 0, \tag{29}$$

for any $\lambda \in [-\varepsilon, \varepsilon]$;

- (ii) *system (2) is zero-state detectable and for each $x \in Z_h$, $x \neq 0$, for which there exists $\lambda \in [-\varepsilon, \varepsilon]$ such that*

$$\nabla h(x) \cdot (f(x) + \lambda g(x)) = 0, \tag{30}$$

either $\lambda = 0$ or $g(x) = 0$;

- (iii) *system (2) is strongly zero-state detectable.*

Then, the closed-loop system

$$\dot{x} = f(x) - \varepsilon g(x) \operatorname{sgn} h(x) \tag{31}$$

is globally asymptotically stable at the origin with respect to Krasowskii solutions.

Proof. First of all we remark that $K(f(x) - \varepsilon g(x) \operatorname{sgn} h(x))$ is included in

$$\begin{cases} \{f(x) - \varepsilon g(x) \operatorname{sgn} h(x)\} & \text{if } h(x) \neq 0, \\ \{f(x) + \lambda g(x), \lambda \in [-\varepsilon, \varepsilon]\} & \text{if } h(x) = 0. \end{cases}$$

Let $v \in K(f(x) - \varepsilon g(x) \operatorname{sgn} h(x))$. From the fact that system (2) is passive and the Remark following Definition 3 it follows that

$$\nabla V(x) \cdot v = \begin{cases} -|h(x)| & \text{if } h(x) \neq 0, \\ 0 & \text{if } h(x) = 0. \end{cases}$$

Then, as in the proof of Proposition 7, the system is proven to be Lyapunov stable and that Krasowskii solutions of (31) tend to the largest weakly invariant subset of $\{x \in \mathbb{R}^n : \exists v \in K(f(x) - \varepsilon g(x) \operatorname{sgn} h(x)) : \nabla V(x) \cdot v = 0\} \subset Z_h$. To prove that every Krasowskii solution φ actually tends to 0, it suffices to show that $V(x) = 0$ for all $x \in \Gamma$, with $\Gamma \subseteq Z_h$ the ω -limit set associated with φ . Let φ_Γ be a Krasowskii solution of (31) such that $\varphi_\Gamma(t) \in \Gamma$ for all $t \geq 0$. Since $\Gamma \subseteq Z_h$, we have that $h(\varphi_\Gamma(t)) = 0$ for all $t \geq 0$ and then $\nabla h(\varphi_\Gamma(t))\dot{\varphi}_\Gamma(t) = 0$ for almost every $t \geq 0$.

Assume that (i) holds. We show that φ_Γ is indeed the *trivial* solution. Suppose on the contrary this is not the case.

Then, there must exist a time t_0 , with $\varphi_\Gamma(t_0) \neq 0$, where φ_Γ is differentiable, such that $\nabla h(\varphi_\Gamma(t_0))\dot{\varphi}_\Gamma(t_0) = 0$. The former equality clearly contradicts (29), for $\dot{\varphi}_\Gamma(t_0) \in K(f(x) - \varepsilon g(x) \operatorname{sgn} h(x))$.

Now, assume that (ii) holds. Then, thanks to (30), $\dot{\varphi}_\Gamma(t) = f(\varphi_\Gamma(t))$ for almost every $t \geq 0$ and $\varphi_\Gamma(t) \rightarrow 0$ as $t \rightarrow \infty$, as a consequence of the zero-state detectability. Since $c = V(\varphi_\Gamma(t))$ for all $t \geq 0$ we conclude that $c = \lim_{t \rightarrow \infty} V(\varphi_\Gamma(t)) = V(\lim_{t \rightarrow \infty} \varphi_\Gamma(t)) = 0$.

Finally, assume that (iii) holds. Then there exists a measurable function v taking values in the closed set $\{\lambda g(\varphi_\Gamma(t)), t \in [0, +\infty), \lambda \in [-\varepsilon, \varepsilon]\}$, for which $\dot{\varphi}_\Gamma(t) = f(\varphi_\Gamma(t)) + v(t)$ for almost every $t \geq 0$. Since $\varphi_\Gamma(t) \in Z_h$ for all $t \geq 0$, strong zero-state detectability implies that $\varphi_\Gamma(t) \rightarrow 0$ as $t \rightarrow +\infty$. The conclusion is obtained as in the previous case. \square

Remark. A few observations are in order:

- It is worth comparing this result with Proposition 7, from the point of view of switched systems. Proposition 7, when ψ is a function of the kind illustrated in Fig. 1, allows to conclude stabilizability of a nonlinear dissipative system using a countable number of control values. On the other hand, Proposition 9 proves stabilizability using a *finite* number of control values. This is possible because of the additional condition (i), (ii) or (iii). If the latter are not fulfilled, it is still possible to stabilize the system, but at the expenses of employing “more” control values.
- Proposition 9 can be seen as the quantized counterpart of a well-known consequence of passivity: the possibility of stabilizing systems via small inputs (see e.g. [23, Chapter 14]).
- To the purpose of establishing a deeper connection with the literature in switched control, Proposition 9(i) can be given an alternative form. Define the vector fields

$$f_1(x) = f(x) - \varepsilon g(x), \quad f_2(x) = f(x) + \varepsilon g(x),$$

and correspondingly the system

$$\dot{x} = u f_1(x) + (1 - u) f_2(x),$$

with u taking values in the set $\{0, 1\}$. Suppose there exists a C^1 positive definite and radially unbounded function V and a real number $\alpha \in (0, 1)$ such that $\nabla V(x)(\alpha f_1(x) + (1 - \alpha) f_2(x)) \leq 0$ for all $x \in \mathbb{R}^n$. Assume additionally that, for any $x \neq 0$ such that $\nabla V(x) f_1(x) = \nabla V(x) f_2(x)$, and for any $\beta \in [0, 1]$, $\nabla(\nabla V(x)(\beta f_1(x) - f_2(x)))(\beta f_1(x) + (1 - \beta) f_2(x)) \neq 0$. Then, there exists a static state (discontinuous) feedback $u = k(x)$ with values in the set $\{0, 1\}$ such that the closed-loop system is globally asymptotically stable at the origin with respect to Krasowskii solutions. When both f_1 and f_2 are linear, the result is Theorem 1 in [3]. The reader is referred to the latter reference for a thorough discussion of the result within the framework of switched control.

- The main difference of the latter proof (ii) with respect to the one of Proposition 7 lies in guaranteeing that the solution φ_Γ asymptotically tends to zero as $t \rightarrow \infty$ notwithstanding the

fact that it is evolving on the *discontinuity* manifold $h(x) = 0$. To this purpose, condition (ii) plays a fundamental role. In fact, the second one of the examples below shows that zero-state detectability alone is not enough in order to get asymptotic stability of system (31).

- Strong zero-state detectability may seem a very conservative condition. On the other hand it is satisfied in some very interesting cases (see [31]).

Of course, Proposition 9 can be reformulated in terms of Carathéodory solutions. In this case, conditions for local existence become particularly easy.

Proposition 10. Let $U = \{0, \varepsilon, -\varepsilon\}$, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Assume that

- (i) system (2) is passive and zero-state detectable;
- (ii) if $x \in Z_h$ is such that $\nabla h(x) = 0$ then $x = 0$;
- (iii) if $x \in Z_h$, $x \neq 0$, one of the following conditions holds:
 - (iii.a) $\nabla h(x) \cdot (f(x) - \varepsilon g(x)) > 0$,
 - (iii.b) $\nabla h(x) \cdot (f(x) + \varepsilon g(x)) < 0$,
 - (iii.c) there exists $\varepsilon > 0$ such that for all $z \in B_\varepsilon(x)$, such that $h(z) = 0$, it holds $\nabla h(z) \cdot f(z) = 0$.

Then the closed-loop system (31) is globally asymptotically stable with respect to Carathéodory solutions.

Proof. First of all we remark that the feedback law is discontinuous on Z_h . Conditions (ii), (iii) guarantee local existence of solutions issuing from Z_h (the proof is analogous to the proof of Proposition 2 in [9] under conditions (H2) and (H3)). Since Carathéodory solutions are also Krasowskii solutions, we conclude, as in the proof of Proposition 9, that (26) is Lyapunov stable with respect to Krasowskii and Carathéodory solutions. This implies that Carathéodory solutions are bounded. Using the fact that the right-hand side of (31) is locally bounded it can be easily deduced that Carathéodory solutions are right continuous up to $+\infty$. By recalling again that Carathéodory solutions are Krasowskii solutions, we deduce, as in the proof of Proposition 9, that they tend to a subset of the largest weakly invariant subset of Z_h . Next we remark that Carathéodory solutions of (31) lying in Z_h satisfy $\dot{x} = f(x)$, and we can argue as in the proof of Proposition 7 and conclude that all Carathéodory solutions of (31) actually tend to the origin. \square

Remark. A desirable property for the closed-loop system is that switches among the different vector fields do not arise too quickly. If in Proposition 10 we make the additional assumption

- (iv) if $x \in Z_h$ is such that $\nabla h(x) \cdot f(x) = \nabla h(x) \cdot g(x) = 0$ then $g(x) = 0$,

we get that Carathéodory solutions are piecewise differentiable and that for any Carathéodory solution φ of (31) the times (also called *switching times*) where the control law $-\varepsilon \operatorname{sgn}(h \circ \varphi)$ changes value do not accumulate in finite time, i.e. the so called Zeno phenomenon does not occur. This is a consequence of Theorems 1 and 2 in [9].

We now discuss the results about passive systems by means of some examples.

Example. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + x_1 u, \\ y &= x_1 x_2. \end{aligned} \tag{32}$$

Let $U = \{0, \varepsilon, -\varepsilon\}$, with $\varepsilon < 1$. Condition (i) of Proposition 9 holds, then the feedback law $u = k(x) = -\varepsilon \operatorname{sgn}(x_1 x_2)$ stabilizes the system asymptotically with respect to Krasowskii solutions. In this case if Carathéodory and Krasowskii solutions actually coincide then the closed-loop system is asymptotically stable also with respect to Carathéodory solutions.

The following example shows that the assumption of zero-state detectability is not sufficient in order to prove stabilizability of passive systems with respect to Krasowskii solutions.

Example. Consider again system (32) with $U = \{0, -1, 1\}$. Condition (i) of Proposition 9 does not hold on the x_1 -axis. It is immediate to verify that the system is passive with storage function $V(x) = x^T x / 2$. The system is also zero-state detectable. As a matter of fact, let $u(t) = 0$ and $y(t) = 0$ for all $t \geq 0$. Then $x_1(t)x_2(t) = 0$ for all $t \geq 0$. Then $\dot{x}_1(t)x_2(t) + x_1(t)\dot{x}_2(t) = x_2^2(t) - x_1^2(t) = 0$ which implies $x_2(t) = \pm x_1(t)$, then $x_1(t) = 0, x_2(t) = 0$ for all $t \geq 0$. In this case the feedback controller is given by $u = -\operatorname{sgn}(x_1 x_2)$. Note that $K(f(\cdot) - g(\cdot) \operatorname{sgn} h(\cdot))(x_1, 0) = \{(0, -(1 - \lambda)x_1), \lambda \in [-1, 1]\}$, then, by taking $\lambda = 1$, we get that the points on the x_1 -axis are equilibrium positions for the closed-loop system, which is not asymptotically stable. In this example the set of Krasowskii solutions of the closed-loop system is larger than the set of Carathéodory solutions. Let us then consider Carathéodory solutions of the closed-loop system. Condition (iii.c) of Proposition 10 is verified on the points of Z_h , then Carathéodory solutions exist and the system is asymptotically stable with respect to these solutions. Note that also condition (iv) of Remark 4 holds. This implies that any Carathéodory solutions are piecewise differentiable and that switching times do not accumulate in finite time. Actually it could also be seen that there is a lower bound on the distance between any two switching times [5,9].

In the following example condition (ii) of Proposition 9 applies.

Example. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + x_1^k x_2^\ell u, \\ y &= x_1^k x_2^{\ell+1}, \end{aligned} \tag{33}$$

where k, ℓ are integers not smaller than 1. Choosing as storage function $V(x) = x^T x / 2$, it is immediately shown that the system

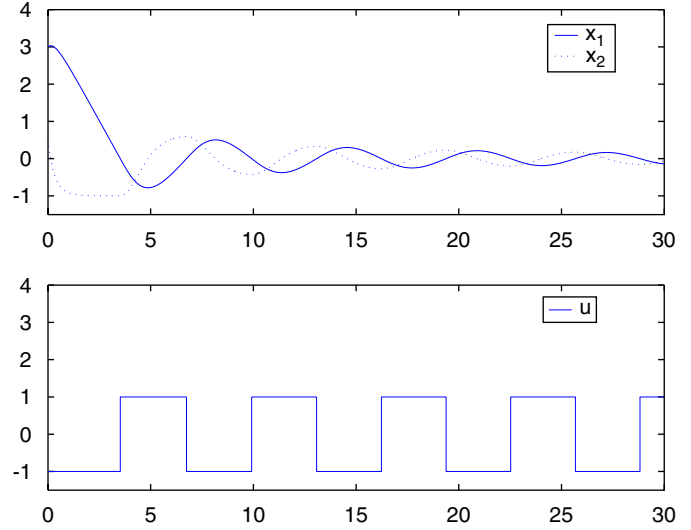


Fig. 3. Switched control of the system $\dot{x}_1 = x_2, \dot{x}_2 = -x_1 + x_1 x_2 u, y = x_1 x_2^2$. The graph at the top illustrates the state response of the system under the action of the control law $u = -\operatorname{sgn} x_1 x_2^2$ depicted in the figure at the bottom. Note that in this case for every initial condition there exists a unique Krasowskii solution.

is passive. It is also zero-state detectable. We have $h(x) = x_1^k x_2^{\ell+1}$ and

$$\begin{aligned} \nabla h(x)(f(x) + \lambda g(x)) \\ = k x_1^{k-1} x_2^{\ell+2} + (\ell + 1) x_1^k x_2^\ell (-x_1 + x_1^k x_2^\ell \lambda). \end{aligned}$$

Any $x \neq 0$ such that $h(x) = 0$ can be equal either to $(x_1, 0)$ or $(0, x_2)$. In the former case, $\nabla h(x)(f(x) + \lambda g(x)) = 0$ for any λ , and $g(x) = 0$. In the latter case, if $k > 1$, then again $\nabla h(x)(f(x) + \lambda g(x)) = 0$ for any λ , and $g(x) = 0$, whereas, if $k = 1$, $\nabla h(x)(f(x) + \lambda g(x)) = k x_2^{\ell+2}$, and therefore never equal to zero. We conclude that the feedback $u = -\operatorname{sgn} x_1^k x_2^{\ell+1}$ globally asymptotically stabilizes the system with respect to Krasowskii solutions. Moreover, along the two switching manifolds $x_1 = 0$ and $x_2 = 0$, the vector field is continuous, and no sliding mode will arise. Finally, we observe that in the polar coordinates (ρ, θ) , the system satisfies the equation

$$\dot{\theta} = -1 - \frac{x_1^{k+1} x_2^\ell}{\rho^2} \operatorname{sgn} x_1^k x_2^{\ell+1},$$

which, bearing in mind that $V(\rho, \theta) = \rho^2 / 2$ and that V is monotone nonincreasing along any trajectory of the system, yields $|\dot{\theta}| \leq 1 + \rho^{k+\ell-1}(0)$.

This allows to conclude that, after a switching has occurred, a certain amount of time (*dwell time*) must elapse before a new switching can take place (see also [9]). Simulation results are reported in Fig. 3 in the case $k = \ell = 1$.

5. Conclusions

For switched and quantized systems, it is interesting to study how the powerful stabilization techniques developed within the

robust control framework and for dissipative systems can be reinterpreted. In this paper we give a contribution in this direction in the case of single input-affine systems: the multi-input case may be analogously treated. For quantized nonlinear control systems, state quantization deserves further attention. We have pointed out how the choice of solution concept gives rise to different results. When solutions of the implemented system are considered in the Carathéodory sense, it would be important to further investigate, in the case the set U is countable, the problem of guaranteeing that Carathéodory solutions are piecewise differentiable and Zeno phenomenon does not occur. Investigating the scenario in which switching occurs not too fast is also of importance.

Appendix A. Proof of Lemma 1

In the cases (i) and (ii) and if \mathcal{X}_0 is such that $\mathcal{K}(\mathcal{X}_0) \neq u$ for all $u \in U$, the proof of the existence of a local Carathéodory solution is perfectly analogous to the proof Proposition 2 in [9]. We now consider case (iii). Let $\mathcal{X}_0 \neq 0$ be such that $\mathcal{K}(\mathcal{X}_0) = 0$. Let us consider the case (iii.a) and, more precisely, let us assume $\nabla \mathcal{K}(\mathcal{X}) \cdot \mathcal{F}(\mathcal{X}) > 0$ (the case $\nabla \mathcal{K}(\mathcal{X}) \cdot \mathcal{F}(\mathcal{X}) < 0$ can be analogously treated). \mathcal{X}_0 is an accumulation point for the set of discontinuity points of $\Psi \circ \mathcal{K}$, but, thanks to the continuity of \mathcal{F} , \mathcal{G} and $\Psi \circ \mathcal{K}$ at 0, there exist $\delta, \gamma > 0$ and $i_\gamma \in \mathbb{N}$ such that for all $\mathcal{X} \in B_\gamma(\mathcal{X}_0)$, for all $i > i_\gamma$ it holds $\nabla \mathcal{K}(\mathcal{X}) \cdot (\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})u_i) > \delta$. Let $M = \max\{|\mathcal{F}(\mathcal{X})| + |\mathcal{G}(\mathcal{X})|u_i, \mathcal{X} \in \overline{B_\gamma(\mathcal{X}_0)}\}$ and let $T < \gamma/M$. We consider a sequence of points $\{\mathcal{X}_{0_i}\} \subset \mathbb{R}^N$, $i \in \mathbb{N}$, $i \geq i_\gamma$ such that $\mathcal{X}_{0_i} \rightarrow \mathcal{X}_0$ as $i \rightarrow +\infty$ and $\Psi(\mathcal{K}(\mathcal{X}_{0_i})) = u_i$. For any i let φ_i be the Carathéodory solution of (6) defined in the following way. φ_i is the Carathéodory solution of the Cauchy problem

$$\dot{\mathcal{X}} = \mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})u_i,$$

$$\mathcal{X}(T_i^i) = \mathcal{X}_{0_i}$$

on the interval $[T_i^i, T_i^{i-1}]$ where $T_i^i = 0$ and T_i^{i-1} is such that $\Psi(\mathcal{K}(\varphi_i(t))) = u_i$ for all $t \in (T_i^i, T_i^{i-1}]$ and for all $\varepsilon > 0$ there exists $y \in B_\varepsilon(\varphi_i(T_i^{i-1}))$ such that $\Psi(\mathcal{K}(y)) = u_{i-1}$. Note that the existence of T_i^{i-1} is a consequence of the fact that $\nabla \mathcal{K}(\mathcal{X}) \cdot (\mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})u_i) > \delta$ on $B_\gamma(\mathcal{X}_0)$. φ_i is continued by induction for $k \in \mathbb{N}$ such that $i - k \geq i_\gamma$: φ_i is the Carathéodory solution of the Cauchy problem

$$\dot{\mathcal{X}} = \mathcal{F}(\mathcal{X}) + \mathcal{G}(\mathcal{X})u_{i-k},$$

$$\mathcal{X}(T_i^{i-k}) = \varphi_i(T_i^{i-k})$$

on the interval $[T_i^{i-k}, T_i^{i-k-1}]$ where T_i^{i-k-1} is such that $\Psi(\mathcal{K}(\varphi_i(t))) = u_k$ for all $t \in (T_i^{i-k}, T_i^{i-k-1}]$ and for all $\varepsilon > 0$ there exists $y \in B_\varepsilon(\varphi_i(T_i^{i-k-1}))$ such that $\Psi(\mathcal{K}(y)) = u_{i-k-1}$. We remark that the solutions φ_i are defined on $[0, T]$ for i sufficiently large. Let us consider the sequence of continuous function $\{\varphi_i\}$ on the compact interval $[0, T]$. They are uniformly bounded and uniformly continuous, in fact $|\varphi_i(t)| \leq \gamma + TM$ for all $t \in [0, T]$ and, for all $t, t' \in [0, T]$, it holds $|\varphi_i(t) - \varphi_i(t')| \leq M|t - t'|$. Then, thanks to

Ascoli–Arzelà’s theorem, there exists a subsequence of $\{\varphi_i\}$ that we still call $\{\varphi_i\}$, which uniformly converges to some continuous function φ on $[0, T]$. We now prove that such φ is a Carathéodory solution of (6) on $[0, T]$. For any i let us consider the sequence in \mathbb{R} defined as follows: $T_i^j = 0$, if $j \geq i$, and T_i^j has been defined before if $j < i$. We remark that for every fixed i, j one has $T_i^j \leq T_i^{j-1}$. Let us fix j and consider the sequence $T_i^j, i \geq i_\gamma$. Since it is bounded it admits a subsequence (that we still call T_i^j) converging to some $T^j \in [0, T]$. We remark that $T^j \leq T^{j-1}$ (otherwise we would definitively have $T_i^j > T_i^{j-1}$, contradiction). We prove that φ satisfies (6) for every $\bar{t} \in [0, T] \setminus \{0, T, T^j\}$, i.e. φ is a Carathéodory solution of (6) on $[0, T]$. Let $\varepsilon > 0$ such that $[\bar{t} - \varepsilon, \bar{t} + \varepsilon] \subset (T^j, T^{j-1})$ for some j . Note that φ_i are definitively classical solutions of (6) on $[\bar{t} - \varepsilon, \bar{t} + \varepsilon]$, then, since $\varphi_i \rightarrow \varphi$ uniformly on $[0, T]$, we have that φ is a classical solution of (6) on $[\bar{t} - \varepsilon, \bar{t} + \varepsilon]$ and, in particular, φ satisfies (6) at \bar{t} . Finally, case (iii.b) is analogous to the cases (i.c), (ii.c) since there is a local solution of (6) which lies on the set $\{\mathcal{X} \in \mathbb{R}^N : \mathcal{K}(\mathcal{X}) = 0\}$.

Note added in the proof

After the paper was accepted, we became aware of Ref. [40]. In that paper, under the hypothesis of the existence of a Hamilton–Jacobi–Isaacs equality analogous to the inequality (12) below, the authors devise an adaptive quantized control strategy for unknown nonlinear systems. The arguments employed to prove the main results in the two papers are both of Lyapunov-type. There are a number of differences however. In [40], in view of the adoption of an hysteresis-like mechanism among the switchings, uniqueness of solutions is assumed. Moreover, while the uncertainty in the model is tackled by adaptive techniques in [40], we regard the uncertainty due to the quantization as a perturbation to counteract in a robust control framework. We thank H. Ishii for making the preprint [40] available to us.

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