

*DISCOUNTING OF DELAYED REWARDS:
MODELS OF INDIVIDUAL CHOICE*

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The present paper addresses the question of the form of the mathematical relation between the time until a delayed reward and its present value. Data are presented from an experiment in which subjects chose between hypothetical amounts of money available either immediately or after a delay (Green, Fry, & Myerson, 1994). Analyses of the behavior of individual young adults demonstrated that temporal discounting is better described by hyperbola-like functions than by exponential decay functions. For most individuals, the parameter that determines the rate of discounting varied inversely with amount. Raising the denominator of the discounting function to a power resulted in better descriptions of the data from most subjects. Two possible derivations of the temporal discounting function are proposed, a repeated choice model and an expected value model. These provide theoretical interpretations for amount-dependent discounting but amount-independent exponent parameters.

Key words: temporal discounting, delayed rewards, behavioral economics, discounting function, choice, humans

Humans and other animals typically will choose more immediate rewards over delayed rewards of equal magnitude. What is surprising, at least from certain perspectives, is that often they also will choose more immediate rewards over delayed rewards of larger magnitude. Many different accounts of the latter finding have been offered (e.g., lack of impulse control, poor ego strength, conflict between multiple selves), but the dominant account in the behavioral economic literature relies on the assumption that the value of a future reward decreases with increasing length of time to its receipt (Kagel, Battalio, & Green, 1995). This decrease in value as a function of delay is termed *temporal discounting*. The temporal discounting account posits that a smaller, more immediate reward may be chosen because the present (or subjective) value of the larger, more delayed reward is discounted; hence, its present value may be less than that of the more immediate reward.

From the perspective of temporal discounting, the question of interest becomes the nature of the mathematical relation among amount, delay, and value. Economists and psychologists have typically employed two different approaches to determining this func-

tion. Economists have taken a "rational" approach to the problem and have attempted to derive a formula from theoretical assumptions, often based on a normative model of what organisms ought to do. In contrast, psychologists have taken an "empirical" approach and have attempted to find the formula that best describes what organisms are observed to do. The present work attempts to bring these two approaches together. First, we will present data from individual subjects for the purpose of evaluating several different formulas that have been proposed; second, we will present two rational derivations of what our analyses suggest is the best descriptive formula.

One formula that has been proposed to describe the temporal discounting of delayed rewards is based on the standard discounted utility model in economics (Samuelson, 1937). This model assumes that the value of a future reward is discounted because of the risk involved in waiting for it. Given a contingent relationship between an organism's choice of a delayed reward and its eventual receipt, it is further assumed that there is a constant hazard rate that this relationship will fail. In a foraging situation, for example, the constant hazard rate might correspond to a constant probability that, in any given unit of time, a predator may prevent the forager from obtaining the reward or that another organism may get the reward first. If there is a

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constant hazard rate associated with waiting, then the form of the temporal discounting function will be exponential:

$$V = Ae^{-kD}, \quad (1)$$

where V is the present value of a future reward, A is its amount, D is the delay to its receipt, and k is a parameter indicating the constant hazard rate.

Another formula that has been proposed has the form of a hyperbolic relation between value and delay (Ainslie, 1992; see also Mazur, 1987; Rachlin, 1989):

$$V = A/(1 + kD), \quad (2)$$

where k is a parameter governing the rate of decrease in value. Fit to the same data, the hyperbola will initially (at short delays) decrease faster than the exponential, but will then (at long delays) decrease more slowly than the exponential. The hyperbola has been justified primarily on empirical grounds (Ainslie, 1992; Mazur, 1987; Rachlin, Raineri, & Cross, 1991; Rodriguez & Logue, 1988), as has a variation on this formula that involves raising the denominator to a power (Green, Fry, & Myerson, 1994; Loewenstein & Prelec, 1992).

The empirical justification for using the hyperbolic model rather than the exponential model has been twofold. First, the hyperbolic model permits preference reversals in subjects' choices between a smaller reward obtainable after a brief delay and a larger reward obtainable after a longer delay. That is, although subjects may prefer the smaller to the larger reward, if an equal amount of time is then added to each delay, subjects may now prefer the larger reward (Green, Fisher, Pellow, & Sherman, 1981; Green, Fristoe, & Myerson, 1994; Navarick, 1982; Rachlin & Green, 1972). Second, the hyperbolic model predicts the slope of an indifference function (Mazur, 1987) that gives the delay to a larger reward as a function of the delay to a smaller reward of equivalent subjective value. Both exponential and hyperbolic models predict linear relations between the two delays, but only the hyperbolic model predicts that the slope will be greater than 1.0. Studies with both pigeons and humans have confirmed this prediction (Green, Fristoe, & Myerson, 1994; Mazur, 1987; Rodriguez & Logue, 1988).

It has been argued that an exponential model cannot account for either preference reversals or the slope of indifference functions. However, this argument assumes that a given delay has the same proportional effect on the value of both small and larger rewards. That is, following the discounted utility model, the parameter k in the exponential function is assumed to be the same for smaller and larger rewards, and an exponential model incorporating this assumption predicts neither preference reversals nor the slope of indifference functions.

Recently, however, the assumption of amount-independent discounting has been shown to be false. Studies of choice between delayed rewards in humans have demonstrated that larger rewards are discounted less steeply with increasing delay than are smaller rewards (Benzion, Rapoport, & Yagil, 1989; Green, Fry, & Myerson, 1994; Raineri & Rachlin, 1993; Thaler, 1981). Moreover, we have shown that if the discount parameter k is inversely related to amount, then both the exponential and hyperbolic models predict preference reversals (Green & Myerson, 1993), and both models also predict indifference functions with slopes greater than 1.0 (Green, Fristoe, & Myerson, 1994). Thus, previous arguments against exponential discounting are moot, and the correct form of the discounting function is still an unresolved issue.

Rachlin et al. (1991) have shown that a hyperbola (Equation 2) provides a better fit to group data than does an exponential decay function (Equation 1). However, it should be recalled that the form of the function describing aggregate (e.g., group) data is not necessarily the same as the form of the function describing unaggregated (e.g., individual) data (Estes, 1956; Sidman, 1952). Therefore, the Rachlin et al. finding does not demonstrate that the hyperbolic model provides a more accurate description of data from individual subjects than an exponential model does. In order to compare hyperbolic and exponential accounts of individual behavior, we now reanalyze the data from a previous study of human choice behavior (Green, Fry, & Myerson, 1994). In addition to considering simple one-parameter exponential and hyperbolic models (Equations 1 and 2), we also examine several slightly more

complicated versions of these models. Finally, we provide rational derivations for the models that best describe the individual data.

METHOD

Subjects

The present study reanalyzes the data from the 12 college students whose group median data were reported by Green, Fry, and Myerson (1994).

Procedure

Participants were tested individually in a quiet room. Hypothetical amounts of money were printed on sets of cards (4 in. by 6 in.). Two sets of cards were placed on a table in front of the participant. One set indicated the delayed, fixed-amount reward (i.e., \$1,000 or \$10,000), and the other set indicated the immediate variable amount (i.e., 30 values ranging from 0.1% to 100% of the fixed amount). Participants were told:

In this experiment, you will be asked to make a series of hypothetical decisions between monetary alternatives. As you can see, there are two sets of cards in front of you. The cards on your left will offer you an amount of money to be paid right now. This amount will vary from card to card. On the cards on your right, the amount will be either \$1,000 or \$10,000, but its payment will be delayed. Please look at the sample cards at this time. It will be your job to point to the card you would prefer. You will be given four practice trials before you begin, and the experimenter will turn the cards for you.

Participants were shown two cards at a time and made a series of choices between the fixed-amount reward that could be obtained after a delay (shown on the right card) and an immediately obtainable reward that varied in amount (shown on the left card). For example, the participant had to make a choice between \$10,000 in 5 years or \$6,000 now. The eight delays at which the \$1,000 and \$10,000 fixed amounts could be obtained were 1 week, 1 month, 6 months, 1 year, 3 years, 5 years, 10 years, and 25 years.

Participants were studied first with one of the two values of the fixed, delayed amount paired with immediate reward amounts presented in both ascending and descending order. This procedure was followed with the

same fixed amount at each of the eight delays before the other fixed amount was presented. The order of presentation of the fixed (i.e., \$1,000 or \$10,000) rewards and the corresponding titration (i.e., either ascending or descending) of the immediate rewards were counterbalanced. However, the fixed-amount delays were always presented from the shortest delay (1 week) to the longest delay (25 years).

For each fixed amount at each delay, we calculated the average of the immediate amount at which the participant switched preference from the immediate to the delayed reward on the descending titration and the amount at which the participant switched preference from the delayed to the immediate reward on the ascending titration. This average immediate amount, termed the present value of the delayed reward, corresponds to V in Equations 1 and 2.

RESULTS

Figure 1 shows temporal discounting functions (i.e., present value as a function of delay) for group data in the \$1,000 and \$10,000 delayed-reward conditions. Distributions of the present value of delayed rewards typically are skewed due to the limits imposed on subjects' choices (i.e., the present value of an amount of money available after a brief delay can never be greater than the amount itself, and the present value of an amount available after a long delay can never be less than zero; Rachlin et al., 1991), hence the median is the appropriate measure of central tendency. The hyperbola (Equation 2) provides a better fit to the data than an exponential decay function (Equation 1). Note that the exponential overestimates the present value of a delayed reward at briefer delays and underestimates the values at longer delays. Although there is a tendency for the hyperbola to also do this for the \$10,000 reward, the error is clearly much smaller. For the \$1,000 delayed reward, the proportions of variance accounted for by the hyperbolic and exponential models were .992 and .923, respectively. For \$10,000, the corresponding proportions of variance accounted for were .938 and .810.

The present value of the \$1,000 delayed reward decreased more sharply as a function of

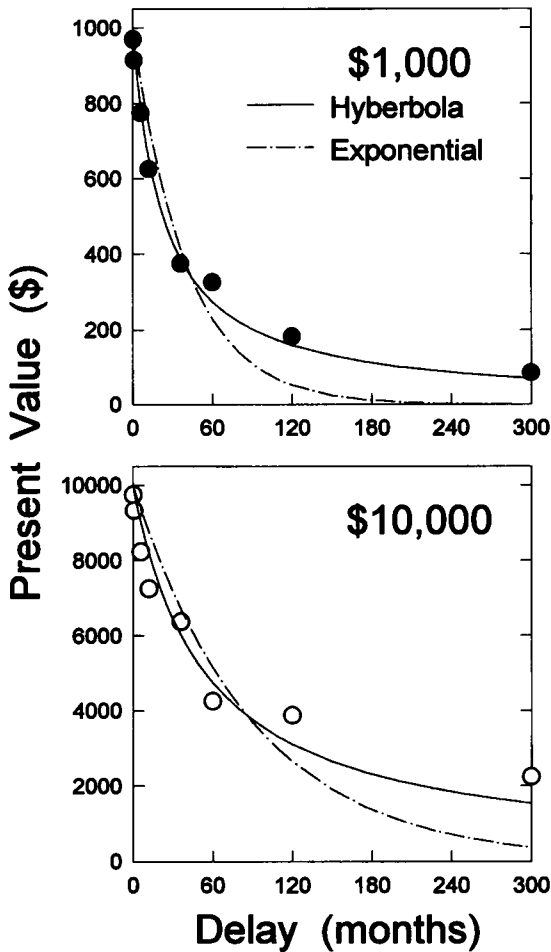


Fig. 1. Temporal discounting functions (i.e., present value as a function of delay) for the \$1,000 (top panel) and \$10,000 (bottom panel) delayed rewards. For each delay, the data point represents the median amount of the immediate reward judged to be equal in value to the delayed reward. The best fitting hyperbola (Equation 2) and exponential decay (Equation 1) are represented by solid and broken curves, respectively.

delay than did the present value of the \$10,000 reward. This is reflected in the k parameters that govern the rates of discounting predicted by both the hyperbolic and exponential models. For the hyperbola (Equation 2), the estimates of the k parameters were .044 and .018 for the \$1,000 and \$10,000 rewards, respectively; for the exponential model (Equation 1), the corresponding estimates of the k parameter were .025 and .011.

These conclusions regarding the form of the temporal discounting function and whether or not its parameters are amount de-

pendent should be considered provisional because they are based on analyses of group data. With respect to the issue of the form of the function (e.g., is the relation exponential or hyperbolic?), it is well known that group functions may not reflect the form of individual functions (e.g., Estes, 1956; Sidman, 1952). In addition, the question of whether discounting is steeper for the smaller reward is also best answered based on analyses of individual data. Even if group functions are similar in form to individual functions, the Type I error rate for decisions regarding parameter estimates (e.g., does the k for \$1,000 differ significantly from the k for \$10,000?) is inflated when those decisions are based on aggregate data from repeated measures designs (Lorch & Myers, 1990).

Figures 2 and 3 show the data from each of the 12 subjects. For ease of comparison, the present value of the delayed reward is expressed as a proportion of its nominal value (i.e., either \$1,000 or \$10,000). All subjects showed fairly orderly temporal discounting. Moreover, most subjects discounted the value of the smaller reward more steeply, although the opposite was seen in 2 subjects (S4 and S12). In addition, there was a tendency for present value to level off without reaching zero in some subjects (e.g., S3 and S8).

Both exponential and hyperbolic models (Equations 1 and 2) were fit to the data from the individual subjects. Table 1 presents the values of k and the proportions of variance explained by exponential and hyperbolic discounting functions for each subject as well as the median of these k values and the median proportion of explained variance for the \$1,000 and \$10,000 delayed rewards. Because the distributions of these values are skewed, the median (rather than the mean) is the appropriate measure of central tendency, and nonparametric tests provide the appropriate basis for statistical inferences. For the \$1,000 condition, the median of the R^2 s for fits to individual subjects' data was .952 for the hyperbola versus .852 for the exponential decay. When the R^2 s obtained with Equations 1 and 2 were compared using a Wilcoxon matched-pairs signed-ranks test, the hyperbolic fits proved to be significantly better, $T = 8$, $p < .05$. A similar comparison based on the data from the \$10,000 condition again revealed that the median of the R^2 s for the hyperbola

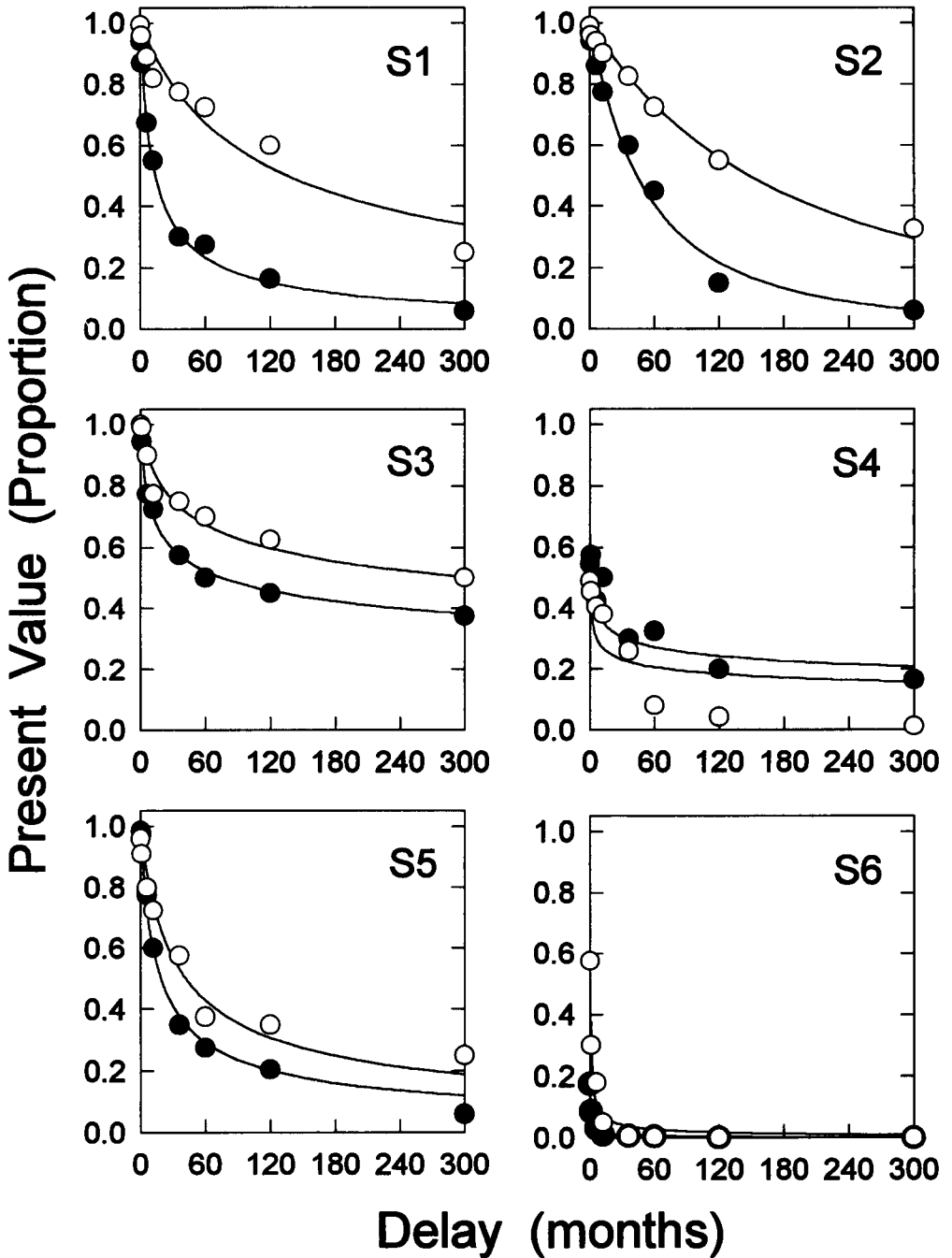


Fig. 2. Temporal discounting functions for Subjects 1 through 6. For each delay, the data points represent the amounts of the immediate reward (expressed as a proportion of the delayed reward) judged to be equal in value to the delayed rewards. Solid symbols represent the present (proportional) value of the \$1,000 delayed reward, and open symbols represent the present (proportional) value of the \$10,000 delayed reward. The curves represent the fit of a theoretical model of temporal discounting (Equation 6).

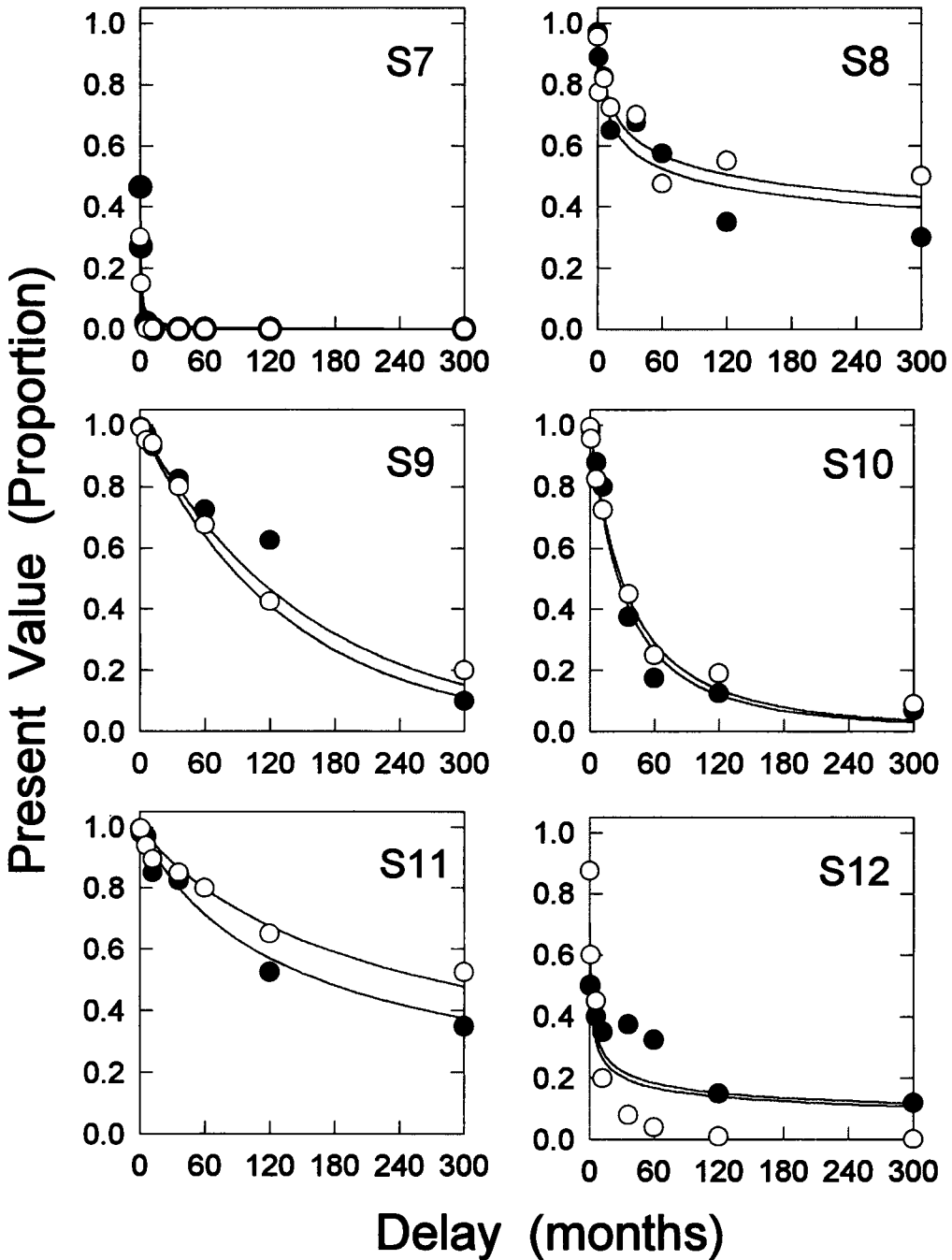


Fig. 3. Temporal discounting functions for Subjects 7 through 12. See Figure 2 for details.

was greater than that for the exponential decay, .946 versus .796. Moreover, the hyperbolic fits were again significantly better based on a Wilcoxon matched-pairs signed-ranks test, $T = 3$, $p < .01$.

Inspection of the aggregated data as well as data from some individual subjects suggests that the present value of a delayed reward decreases less sharply at long delays than is predicted by either Equation 1 or Equation 2

Table 1

Individual values of *k* and proportions of explained variance (*R*²) for exponential (Equation 1) and hyperbolic (Equation 2) discounting functions.

Subject	\$1,000				\$10,000			
	Hyperbola		Exponential		Hyperbola		Exponential	
	<i>k</i> _{hyp}	<i>R</i> ²	<i>k</i> _{exp}	<i>R</i> ²	<i>k</i> _{hyp}	<i>R</i> ²	<i>k</i> _{exp}	<i>R</i> ²
1	0.065	.978	0.035	.879	0.008	.937	0.005	.926
2	0.025	.973	0.015	.984	0.007	.992	0.005	.968
3	0.015	.706	0.009	.471	0.006	.738	0.003	.570
4	— ^a	.000 ^b	— ^a	.000 ^b	— ^a	.000 ^b	— ^a	.000 ^b
5	0.048	.994	0.027	.934	0.022	.922	0.014	.764
6	17.402	.941	6.930	.696	2.480	.947	1.531	.822
7	3.941	.968	2.124	.824	8.580	.968	4.581	.769
8	0.015	.818	0.009	.648	0.010	.055	— ^a	.000 ^b
9	0.008	.913	0.005	.969	0.009	.981	0.006	.992
10	0.040	.966	0.025	.984	0.037	.994	0.022	.970
11	0.006	.963	0.004	.940	0.004	.945	0.003	.874
12	— ^a	.000 ^b	— ^a	.000 ^b	0.355	.948	0.157	.884
Median	0.033	.952	0.020	.852	0.010	.946	0.010	.796

^a The value of *k* is omitted because the poor fit made the parameter estimate meaningless (see below).

^b *R*² of .000 indicates that the function accounted for less of the variance than did the mean.

(for a clear example, see the bottom panel of Figure 1). A two-parameter exponential model that may capture this property of the data is given by Equation 3:

$$V = (A - s) e^{-kD} + s. \tag{3}$$

Rather than decaying to zero as *D* increases, value approaches an asymptote of *s*. (In Equation 1 the expression *e*^{-*kD*} was multiplied by *A*, but in Equation 3 the exponential expression is multiplied by *A* - *s* so that, in both cases, when *D* is zero, *V* equals *A*.) A two-parameter hyperbola-like model that may also capture the form of the decrease in present value has been proposed (Green, Fry, & Myerson, 1994; Loewenstein & Prelec, 1992):

$$V = A / (1 + kD)^s. \tag{4}$$

Here, *s* modifies the form of the hyperbola so that when *s* is less than 1.0, it flattens the curve causing it to level off as *D* increases.

For the data from the \$1,000 condition, the median of the *R*²s for fits to individual subjects' data was .979 for Equation 4 and .956 for Equation 3. When the *R*²s obtained with Equations 3 and 4 were compared using a Wilcoxon matched-pairs signed-ranks test, the fits based on Equation 4 proved to be significantly better, *T* = 11, *p* < .05. For the data from the \$10,000 condition, the median of the *R*²s for Equation 4 was again greater than that for Equation 3, .976 versus .919. More-

over, the fits based on Equation 4 were again significantly better according to a Wilcoxon matched-pairs signed-ranks test, *T* = 3, *p* < .01. Thus, regardless of whether one compares Equations 1 and 2 or Equations 3 and 4, a hyperbola-like model describes the temporal discounting by individual subjects better than an exponential model does.¹

Up to this point, we have been comparing models with different forms but of similar complexity (i.e., Equation 1 vs. Equation 2 and Equation 3 vs. Equation 4). We now compare models of related form that differ in complexity (e.g., models with an amount-independent *k* parameter versus models in which *k* varies with amount, and models without an exponent versus models with an exponent). How can one decide between simpler and more complicated models? Given that the addition of a free parameter will generally improve the fit of a model to data, simple comparisons of the proportion of vari-

¹ Strictly speaking, an equation with an exponent such as Equation 4 is not a hyperbola. Hence, we have used the term *hyperbola-like* when referring to Equation 4 and similar formulations (e.g., Equation 6). In addition, it may be noted that the term *hyperbolic function* refers to a member of a class of trigonometric functions, and thus Equation 2 is not a hyperbolic function. However, the term *hyperbolic* may be used as an adjective in other expressions (as in hyperbolic model and hyperbolic relation, and even hyperbolic discounting function) that refer to a hyperbola such as Equation 2.

ance accounted for (such as those we used to compare models with the same number of free parameters) will not suffice. When the question is whether an additional parameter is needed in the sense that a particular parameter takes on different values in different situations (as in the case in which the question is whether a single k can be used for different amounts of delayed reward), parameter estimates can be treated just like any other measure, and standard statistical tests apply (e.g., Lorch & Myers, 1990). When the question is whether an additional parameter is needed or not (as in the case in which the question is whether the denominator of the hyperbolic model must be raised to a power), one standard approach (e.g., Gallant, 1987) is to test whether estimates of the value of the parameter differ significantly from the value predicted by the null hypothesis (i.e., that the value of the exponent is 1.0).

We first addressed the question of whether separate k parameters are necessary for different amounts of delayed reward. That is, does an equation that has only a single amount-independent k parameter accurately describe the data, or is a model with two k parameters necessary because the smaller amount is discounted more steeply? Consistent with the latter view, the median of the individuals' discount parameters for the smaller (\$1,000) amount was .033, compared with .010 for the larger (\$10,000) amount, but is this difference statistically significant?

For 2 subjects (S4 and S12), the question of whether there was a difference in the discount parameters for the two delayed reward amounts is moot because at least one of their discount parameters could not be reliably estimated (i.e., a hyperbola accounted for less of the variance than the mean). For 8 of the remaining 10 subjects, the discount parameter for the \$1,000 reward was larger than that for the \$10,000 reward (see Table 1), and a Wilcoxon matched-pairs signed-ranks test revealed that, as predicted, the k parameter was significantly greater for the smaller amount, $T = 10$, $p < .05$.

Given that an adequate model of the present data appears to require separate discount parameters for \$1,000 and \$10,000 delayed rewards, we next addressed the question of whether it is necessary to raise the denominator to a power, as in Equation 4. That is, if

an exponent is added to an equation with two amount-dependent discount parameters, will the estimate of the exponent differ significantly from 1.0?

In order to answer this question, Equation 4 was first reformulated in terms of proportions to facilitate fitting the data for the two delayed amounts with one equation. Recall that V , the present value of the delayed reward, is operationally defined as the amount, A_i , of an immediate reward judged to be of equivalent value. Substituting A_i for V and dividing both sides of Equation 4 by the nominal amount, A_d , of the delayed reward yields

$$A_i/A_d = 1/(1 + kD)^s \quad (5)$$

Because the preceding analysis revealed that the rate of temporal discounting is amount dependent, separate k parameters for each amount were incorporated into Equation 5 as follows:

$$A_i/A_d = 1/[1 + (k' + a\Delta k)D]^s \quad (6)$$

where k' is the discount parameter for \$10,000 delayed rewards, Δk is the difference between the discount parameters for \$1,000 and \$10,000 delayed rewards, and a is a dummy or indicator variable whose value is 0 when the delayed reward is \$10,000 and 1 when the delayed reward is \$1,000. Thus, when the amount of the delayed reward is \$10,000 (and $a = 0$), Equation 6 is equivalent to

$$A_i/A_d = 1/(1 + k'D)^s,$$

and when the delayed reward is \$1,000 (and $a = 1$), Equation 6 is equivalent to

$$A_i/A_d = 1/[1 + (k' + \Delta k)D]^s.$$

Note that for the \$1,000 amount, the discount parameter is equal to the sum of k' and Δk .

A Wilcoxon matched-pairs signed-ranks test revealed that the median value of the exponent s did not differ significantly from 1.0. Nevertheless, inspection of data from individual subjects strongly suggested that some individuals' behavior might be much better described by Equation 6 than by a similar model without an exponent. After all, if the necessary exponent were greater than 1.0 in some cases but less than 1.0 in others, then the median value of the exponent need not differ from 1.0. In order to determine whether an exponent was needed in individual cases, we

Table 2

Individual values of k , s , and proportions of explained variance (R^2) for the three-parameter model (Equation 6).

Subject	$k_{\$1,000}$	$k_{\$10,000}$	s	R^2
1	0.126	0.013	0.677*	.976
2	0.009	0.003	2.138**	.987
3	0.428	0.105	0.197**	.985
4	39.963	210.823	0.167**	.738
5	0.118	0.053	0.589**	.982
6	62.762	5.774	0.614**	.980
7	5.617	12.806	0.815	.974
8	0.635	0.393	0.176**	.847
9	0.0002	0.0001	46.564	.980
10	0.016	0.014	2.007	.985
11	0.011	0.007	0.680	.960
12	9.857	7.219	0.278**	.752
Median	0.277	0.079	0.645	.978

* $t(13) = 2.36, p < .05$. ** all $t(13) > 4.76, ps < .01$.

subtracted the estimate of the s parameter for each subject from 1.0 and divided the difference by the standard error of the estimate of the exponent for that subject. The resulting ratio follows the t distribution with $(n - p)$ degrees of freedom, where n is the number of data points and p is the number of parameters in the model, and thus a simple t test can be used to determine whether the exponent for an individual subject differs from 1.0 (Gallant, 1987).

For 8 of the 12 subjects, s was significantly different from 1.0, and the median of the individual R^2 s increased to .978; for 7 of these 8 subjects, s was significantly less than 1.0 (see Table 2). Although the improvement in the median R^2 was small, the addition of the s parameter allowed the model to fit the data from 1 subject (S4) whose data were not described by simple hyperbolas, and improved the proportion of explained variance by more than .20 for 3 other subjects (S3, S8, and S12). Moreover, the number of subjects for whom the discount parameter for \$1,000 was greater than that for \$10,000 increased from 8 to 10 when their data were described by Equation 6 rather than Equation 2 (cf. Tables 1 and 2).

Having concluded that a hyperbolic-like model of individual decisions about delayed rewards may require a separate discount parameter for each amount and at least one exponent, our final question concerned whether, as was the case with the discount

parameters, amount-specific exponents are necessary. That is, if two exponent parameters are estimated, one for each delayed reward, will they differ significantly? To address this question, Equation 6 was modified to

$$A_i/A_d = 1/[1 + (k' + a\Delta k)D]^{s+a\Delta s}. \quad (7)$$

Again, a is a dummy variable whose value is either 0 or 1 depending on whether the amount of the delayed reward is \$10,000 or \$1,000. Thus, when the delayed reward is \$10,000 (and $a = 0$), Equation 7 simplifies to

$$A_i/A_d = 1/(1 + k' D)^s,$$

whereas when the delayed reward is \$1,000 (and $a = 1$), Equation 7 becomes

$$A_i/A_d = 1/[1 + (k' + \Delta k)D]^{s+\Delta s}.$$

Simple t tests based on the ratio of Δs to its standard error (Gallant, 1987) were used to evaluate whether, for individual subjects, estimates of Δs differed significantly from the null hypothesis (i.e., $\Delta s = 0$). The estimated value of Δs differed significantly from zero in only two of the 12 cases. Moreover, the median proportion of explained variance increased by less than .001. Thus, amount-specific exponent parameters added little or nothing to a description of temporal discounting by individual subjects, and an equation such as Equation 6 with a discount parameter for each delayed amount but only a single exponent appears to be the most appropriate model.

Fits of the three-parameter model (Equation 6) to the data from each individual subject are represented by the curves in Figures 2 and 3. For each subject, the values of k and s and the proportions of variance explained by the three-parameter model are presented in Table 2. The three-parameter model also accurately described the group data for both delayed rewards fit simultaneously ($R^2 = .984$), as shown in Figure 4. The estimate of the exponent parameter was 0.583, consistent with the fact that most subjects had exponents less than 1.0. The discount parameters of the equation that best described the group data shown in Figure 4 were 0.111 and 0.045 for the \$1,000 and \$10,000 rewards, respectively, consistent with amount-dependent discounting.

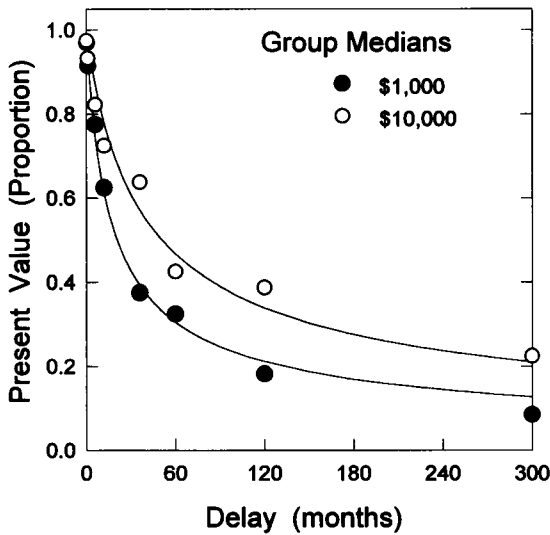


Fig. 4. Temporal discounting functions (i.e., present value as a function of delay) for the \$1,000 (solid symbols) and \$10,000 (open symbols) delayed rewards. For each delay, the data points represent the median amounts of the immediate reward (expressed as a proportion of the delayed reward) judged to be equal in value to the delayed rewards. The curves represent the best fitting function based on a three-parameter model (Equation 6) that includes an amount-dependent discount parameter (k) and a single amount-independent exponent.

DISCUSSION

At the group level, orderly temporal discounting of delayed rewards was observed, and the smaller delayed amount was discounted more steeply. Hyperbolas provided much better fits to the group data than did the exponential decay functions. This same pattern was also observed at the individual level. That is, all subjects showed orderly discounting that in most cases was better described by hyperbolas than exponentials (although in some subjects, the present value of delayed rewards tended to level off without reaching zero, a characteristic not captured by either a simple hyperbola or an exponential model). Most subjects discounted the \$1,000 reward more steeply than the \$10,000 reward, resulting in a larger discount parameter (k) for the smaller reward.

These findings replicate previous reports based on analyses of aggregate data (e.g., Rachlin et al., 1991; Raineri & Rachlin, 1993) and demonstrate that the same relations observed at the group level may also be ob-

served in the behavior of individuals. Comparisons of models with different numbers of free parameters highlighted the need for amount-dependent discount parameters and, in addition, suggested that raising the denominator of a hyperbola-like model to a power often significantly improves its fit to individual data. More specifically, the exponent parameter (s) is typically less than 1.0, and has the effect of causing the theoretical curve to decline much more slowly at long delays, corresponding to the leveling off in present value observed in many subjects.

Unlike the discount parameter, the exponent parameter was not amount dependent. That is, one exponent generally sufficed to describe discounting of both \$1,000 and \$10,000, although its value varied considerably between individuals. This suggests that the exponent may be viewed as an individual-difference variable, and may reflect something about the sensitivities of different individuals to variations in the magnitudes of delays and amounts. Logue, Rodriguez, Peña-Correal, and Mauro (1984) have made a similar suggestion with respect to exponents in the generalized matching law (Baum, 1974).

A precise quantitative description of behavior is obviously of considerable value even in the absence of a theoretical account of why the description takes a particular form. However, the present findings invite consideration of possible mechanisms that may underlie the form of the temporal discounting function. Although our findings involve choices between hypothetical monetary rewards, results consistent with hyperbola-like discounting functions have been obtained at the group level using real rewards with human and non-human subjects (Mazur, 1987; Rodriguez & Logue, 1988). However, discounting functions of individual subjects have not previously been examined systematically using either real or hypothetical rewards. Future research is needed to determine the generality of the present results. Nevertheless, it does not appear to be premature to speculate on their possible theoretical implications.

The standard economic view of temporal discounting, consistent with the discounted utility model of Samuelson (1937), is that discounting compensates for the risks associated with waiting for a delayed reward. Given the assumption that there is a constant hazard

rate, an exponential decrease in value over time follows directly. This derivation is theoretical in the normative sense; it assumes that a decision-making model should prescribe optimal choice behavior.

In contrast to the theoretical underpinnings for the exponential discounting function favored by economists, psychologists have favored a hyperbolic discounting function because of its ability to describe behavior, but have been less concerned with a possible underlying mechanism. Rachlin and Green (1972) and Ainslie (1975) suggested that a reciprocal relation between value and delay (i.e., $V = A/D$) could account for preference reversals, thereby taking the empirical approach characteristic of most subsequent psychological research on discounting. That is, one begins by discovering lawful empirical relations; then, one seeks a form of equation that accurately describes these relations. The reciprocal relation between value and delay has an obvious problem, however, in that it is undefined for immediate rewards. Mazur (1987) presented a related equation (Equation 2), a hyperbola (of which the reciprocal is a special case), that avoids this problem while preserving the ability to predict preference reversals. Rachlin and Raineri (1992) pointed out that the hyperbolic relations and consequent reversals are commonly found in nature and therefore suggested that the application of the hyperbola to temporal discounting may not require any special explanation.²

Mazur (1987) showed that a hyperbolic relation between value and delay predicts a linear indifference function, that is, a linear relation between the delay to a larger reward and the delay to a smaller reward when both are judged to be equal in value. Loewenstein and Prelec (1992) reversed this logic and showed that given a linear indifference function, the relation between value and delay

must be a hyperbola-like function similar in form to Equation 6, of which the hyperbola is a special case. However, neither Mazur nor Loewenstein and Prelec deal with the question of mechanism, that is, they do not speculate on the nature of the process underlying changes in value as a function of delay. In both cases, it is the data that drive the derivation, and no rationalization of the fundamental equations or their parameters is offered. Although such restraint during the initial exploration of phenomena is appropriate and even laudable, at some point phenomena become sufficiently well established that some speculation as to underlying mechanisms may be in order. We believe that the study of choice between delayed rewards may have reached such a point and present two possible explanations for the hyperbolic-like form of the temporal discounting function.

One mechanism is suggested by the notion that subjects respond as if they will have repeated choice opportunities with some time interval between receipt of a reward and the next choice opportunity (Rachlin, Logue, Gibbon, & Frankel, 1986). Assume further that value is proportional to rate (Rachlin, 1971). That is, let the value, V , of a reward be directly proportional to the ratio of the amount, A , to the interreward interval. Let m represent the amount of time that elapses between receipt of a reward and the next choice opportunity, so that the total interreward interval equals $m + D$. These assumptions may be formalized in the equation

$$V = b[A/(m + D)], \quad (8)$$

where b is the proportionality constant that converts rate into units of value. When subjects choose the amount of an immediate reward (A_i) whose value (V_i) is equal to the value (V_d) of a delayed amount (A_d), that is, when $V_i = V_d$, it follows that

$$bA_i/m = bA_d/(m + D). \quad (9)$$

Multiplying both sides of the preceding equation by (m/b) yields

$$A_i = mA_d/(m + D),$$

and dividing both the numerator and denominator by m yields

$$A_i = A_d/(1 + D/m).$$

Substituting k for $(1/m)$ results in the familiar

² It is interesting to note that Rachlin takes a position with regard to temporal discounting similar to that attributed to Isaac Newton with regard to gravitation, in that both see the precise mathematical description of empirical relations as an end in itself. Feynman (1967) writes that "Newton was originally asked about his theory [of gravitation]—'But it doesn't mean anything—it doesn't tell us anything,' to which Newton replied, 'It tells you *how* it moves. That should be enough. I have told you how it moves, not why'" (p. 37).

hyperbolic relation (Equation 2) between the present value, V , of a delayed reward (measured as the amount, A_i , of an immediate reward judged to be of equivalent value) and the nominal delay until its receipt. Thus, a possible theoretical explanation for the discount parameter k is that it represents the reciprocal of the delay from receipt of a reward until the next choice opportunity.

This derivation can be generalized to explain the s parameter. Assume that amount and time are nonlinearly scaled (Stevens, 1957), such that

$$V = cA^r/a(m + D)^q, \quad (10)$$

where the constants c and r govern the scaling of amount and the constants a and q govern the scaling of time. Thus, when subjects choose the amount of an immediate reward (A_i) whose value is equal to the value of a delayed amount (A_d),

$$cA_i^r/a(m)^q = cA_d^r/a(m + D)^q. \quad (11)$$

Multiplying both sides of the equation by $[a(m)^q/cA_d^r]$ and simplifying yields

$$A_i^r/A_d^r = [m/(m + D)]^q.$$

Taking the r th root of both sides, dividing both the numerator and denominator of the fraction on the right side of the equation by m , and multiplying both sides by A_d yields

$$A_i = A_d/(1 + D/m)^{q/r}.$$

This is equivalent to Equation 6 with $k = 1/m$ and $s = q/r$, and provides a theoretical explanation for the single amount-independent exponent. That is, s reflects individual differences in scaling amount and time. Thus, if the preceding derivation is correct, the exponent s might be expected to remain constant when the same individual confronts different choice situations.

Alternatively, instead of following psychologists and basing an interpretation of temporal discounting on rate of reward, as in the preceding repeated choice model, one can follow the lead of economists and assume that the discounting of delayed rewards reflects the risk involved in waiting for their receipt, which leads to the following expected value model. In order to derive a hyperbolic relation for the present value of a delayed reward, one simply assumes that as time passes, the number of possible alternative outcomes

increases at a constant rate, k . Thus, the probability that one will receive an immediate reward equals 1.0, whereas the probability that one will receive a delayed reward is given by $p = 1/(1 + kD)$. The assumption that the likelihood of an alternative outcome increases with time seems to have no less face validity than the economists' assumption that the likelihood of an alternative outcome is constant.

If the amount of an immediate reward that is judged to be equal in value to a delayed reward depends on the expected value of the latter ($EV = pA$, where p is the probability of receiving amount A), then

$$A_i = [1/(1 + kD)]A_d, \quad (12)$$

which is equivalent to Equation 2. Nonlinear scaling of amount and probability may be incorporated easily into this model. When subjects judge an immediate, and therefore certain, reward to be equal in value to a delayed reward,

$$[1.0]^{u/c}A_i^r = [1/(1 + kD)]^{u/c}A_d^r. \quad (13)$$

(It may be noted that whereas amount is scaled by raising it to a power, r , and then multiplying by a constant, c , probability is simply raised to a power, u ; otherwise, the scaled probability of a certain outcome would not equal 1.0.) Simplifying, one may obtain

$$A_i^r = [1/(1 + kD)]^{u/c}A_d^r,$$

and taking the r th root of both sides yields

$$A_i = A_d/(1 + kD)^{u/r}.$$

This is equivalent to Equation 6 with $s = u/r$, providing an alternative theoretical explanation for the single amount-independent exponent. In this case, s reflects individual differences in scaling amount and probability, and again would be expected to remain constant across choice situations.

The contribution of these derivations may be that they provide a theoretical interpretation for each parameter. Such interpretations are testable. For example, consider a third derivation of Equation 2, one based on the assumption that no reward is immediate. That is, if there is some minimum time that elapses between the act of choosing an immediate reward and the consumption of that reward, then incorporating this minimum time leads to a derivation of Equation 2 iso-

morphic to that offered for the repeated choice model but with m now representing the minimum time from choice to consumption rather than the time from reward to the next choice opportunity. The obtained estimates for $m = 1/k$ obtained by fitting Equation 2 seem to be too large to represent reasonable magnitudes for a minimum time to consumption of an "immediate" reward. For example, the estimate of m based on the medians of the individual k s for the hyperbolas fit to the data for the \$1,000 reward ($k = 0.033$; see Table 1) was approximately 30 months ($m = 1/0.033$). On the other hand, this estimate seems to be more reasonable for the interval between choice opportunities that is assumed by the repeated choice model. Thus, comparing estimated values for parameters with the values one might expect based on different interpretations provides one way of evaluating such interpretations.

It may be possible to distinguish between the repeated choice model and expected value model based on independent estimates of the scaling of amount, time, and probability. Based on the present finding that the exponent (s) in Equation 6 was generally less than 1.0, the repeated choice model predicts that if power functions are used to describe psychophysical scaling (Stevens, 1957), then the exponent in the power function describing the scaling of amount will be greater than the exponent for time in most subjects, and the expected value model predicts that the exponent for amount will be greater than that for probability. Moreover, individual differences in s should be predictable from individual differences in the scaling of amount and either time or probability. These predictions are obviously testable.

In summary, the present findings suggest that temporal discounting by individual subjects may be well described by hyperbola-like functions with amount-dependent discount parameters and amount-independent exponents, and two possible derivations of the temporal discounting function are proposed. The interpretations of the discount parameter associated with these models suggest that our subjects' behavior was controlled by certain specific characteristics of choice situations outside the laboratory. That is, the repeated choice model suggests that opportunities to make choices involving large amounts are rar-

er than opportunities involving small amounts, whereas the expected value model suggests that for equal delays, choosing large amounts involves less risk than choosing small amounts. In addition, the exponent may be interpreted as reflecting an individual's characteristic scaling of amount and time or probability.

It may not always be necessary to incorporate an exponent in quantitative descriptions of temporal discounting. Simple hyperbolas account for more than 90% of the variance in aggregate data from adult humans (e.g., Rachlin et al., 1991; Raineri & Rachlin, 1993; the present data). However, including an exponent in the discounting function may significantly improve its fit to individual data. In addition, an exponent may prove to be useful in describing aggregate data from other populations. For example, our attention was originally called to the need for an exponent when analyzing data from children. As a group, the children had a smaller exponent and larger discount parameters than the young adults did, suggesting developmental changes in both the exponent and discount parameters (Green, Fry, & Myerson, 1994). Although to date the children's data have been analyzed only in aggregate form, analysis of individual data provides a further opportunity to test the notion that for a given individual at a specific age, the value of the exponent is consistent across choice situations.

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