Discovering the structure of complex networks by minimizing cyclic bandwidth sum

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Getting a labeling of vertices close to the structure of the graph has been proved to be of interest in many applications e.g., to follow smooth signals indexed by the vertices of the network. This question can be related to a graph labeling problem known as the cyclic bandwidth sum problem. It consists in finding a labeling of the vertices of an undirected and unweighted graph with distinct integers such that the sum of (cyclic) difference of labels of adjacent vertices is minimized. Although theoretical results exist that give optimal value of cyclic bandwidth sum for standard graphs, there are neither results in the general case, nor explicit methods to reach this optimal result. In addition to this lack of theoretical knowledge, only a few methods have been proposed to approximately solve this problem. In this paper, we introduce a new heuristic to find an approximate solution for the cyclic bandwidth sum problem, by following the structure of the graph. The heuristic is a two-step algorithm: the first step consists of traversing the graph to find a set of paths which follow the structure of the graph, using a similarity criterion based on the Jaccard index to jump from one vertex to the next one. The second step is the merging of all obtained paths, based on a greedy approach that extends a partial solution by inserting a new path at the position that minimizes the cyclic bandwidth sum. The effectiveness of the proposed heuristic, both in terms of performance and time execution, is shown through experiments on graphs whose optimal value of CBS is known as well as on real-world networks, where the consistence between labeling and topology is highlighted. An extension to weighted graphs is also proposed.

cyclic bandwidth sum problem, graph labeling, complex networks, vertex labeling, graph structure, graph topology

1. Introduction

1.1. Problem statement

In many applications, the structure of a complex network gives insights into the understanding of the underlying relationships between the vertices: It is advantageous to consider the vertices in a consistent order according to the topology. A striking example of this is the huge amount of works about the detection of communities in a network [14]: finding groups of vertices highly connected between them is for instance a powerful tool to explain the structure of social networks and to characterize them. The presence of communities is only one type of organization encountered in networks, among a much large diversity of structures: in many cases, the topology of the network is unknown and cannot be fully characterized explicitly. In this situation, it is nevertheless beneficial to have a vertex ordering consistent with the structure of the network. We can point out for example those related to distributed inference over networks [23], diffusion [6] or visualization of networks [2].

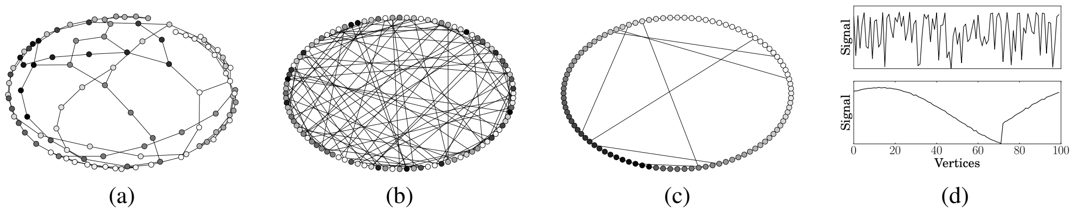
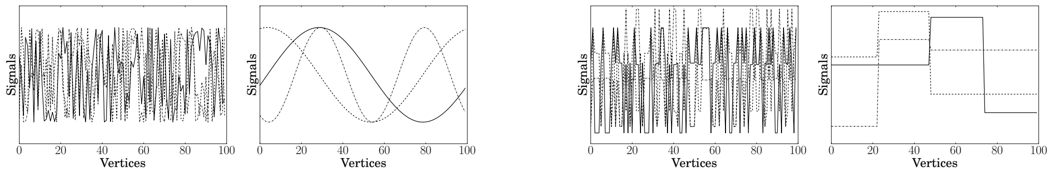


Figure 1: Example of signals over a network with 50 vertices, with a scalar value assigned to each vertex. The color codes the value of the signal on each vertex, from black to white. (a) Original network. (b) Circular representation of the network according to the index with a random ordering of vertices. (c) Circular representation of the network with a suitable ordering of vertices. (d) Representation of the signals indexed by the vertices with a random vertex ordering (top) and a proper vertex ordering (bottom).

A first motivating example comes from the field of signal processing over networks, which has been extensively developed in recent years [31]. Considering a network with an unknown topology, a value is assigned to each vertex. This situation could describe for instance a sensor network, in which a vertex represents a station which measures a quantity, and is in communication with some stations. Different issues come out from this example where a proper vertex ordering is of great interest. If we use the assumption that the signal is smooth over the network, one question which may arise is how to represent the signal according to the vertices in a two-dimensional space, to preserve its smoothness. Another point is the representation of the network itself, using a linear or circular layout, and how to order the vertices to minimize the cross of edges and hence improve the visualization. A short example shows that these two questions are linked and can be directly addressed if it is possible to obtain an ordering of vertices consistent with the topology of the network. Figure 1 gives an example of network with 50 vertices, with a scalar value is assigned to each vertex. A random vertex ordering gives both a poor circular representation of the network, and a signal with abrupt variations. Conversely, the usefulness of a vertex ordering consistent with the structure is clearly visible in the circular representation of the network, and gives a smooth representation of the signal on the vertices.



(a) Cycle graph

(b) Graph with four communities

Figure 2: Examples of transformation of a network into signals, indexed by the vertices using [30]. The resulting collection of signals is indexed by the vertices. The first three signals are displayed on each subplot (left) Random ordering of vertices (right) Suitable ordering of vertices.

A second example derives from methods of duality between signals and networks, which can be taken up using two approaches depending on the object of interest: from signals to graphs, for example for the study of time series [5], or from networks to signals, as in [33] or [30]. The latter method intends to transform a graph into a collection of signals using classical multidimensional scaling [3] and has been extended in [19, 18, 17, 20]. The aim in these works is to exhibit specific frequency patterns and link them with topological properties of the underlying network. A major issue of the transformation from graphs to signals concerns the indexation of signals, which is based on the vertex order. If two neighboring vertices in this order are not adjacent in the graph, then their values in signals might be different and the signals are blurred: It leads to abrupt variations of the signals over vertices which complicate the spectral analysis and then the monitoring of frequency patterns. To smooth the signals, the vertex order must take into account adjacency or at least geodesic distance, in other words the vertex order must reflect as much as possible the structure of the graph. Fig. 2 shows two examples of the consequence of a poor indexation to the resulting signals: transformation of a cycle graph leads to smooth harmonic oscillations if the labeling follows the cycle (2a right), but to high-frequency signals if the labeling is random (2a left). Likewise, transformation of a graph with communities leads to signals with many abrupt variations (2b left), when an indexation that is consistent with the structure in communities highlights plateaus, corresponding to each community. It is easy to observe that a spectral analysis on the obtained signals might be profitable only if the indexation is consistent with the topology. This consistency can be described as follows: Two vertices close in the indexation should be close as well in the graph. But as we may have to deal with periodic signals, the definition of the proximity (and then of the distance) has to be cyclic, i.e. if the vertex at the beginning of the indexation and the one at the end have to be close in the network.

We propose in this paper to find such an order that reflects the topology of the underlying network. The core of the method consists of the study of a related labeling problem, which seek for a mapping from vertices to integers, in such a way that an objective function is minimized. This approach is widely made explicit in the following.

1.2. General framework of graph labeling

Graph labeling consists of assigning labels to vertices or edges of a graph. The way in which these labels are classically assigned are driven by the minimization of a certain objective function, defined for the purposes of a specific application. There exists a wide variety of labeling problems that are related to distinctive applications, as described by Diaz et al [11]. We focus in this article on the labeling of vertices of an unweighted and undirected graph, with the objective to find a graph labeling which reflects the topology of the graph. As described above, such a labeling shall minimize the distances between labels of adjacent vertices, which could be a challenge with high stake for many applications. We propose in the following to traverse the graph by tackling the cyclic bandwidth sum problem, which consists in minimizing the distance between the labels of pairs of connected vertices in the graph.

Let $G = (V, E)$ be a simple connected, unweighted and undirected graph with V the set of vertices, and E the set of edges. The number of vertices is noted $n = \#V$. Chung [8] proposed a framework which encompasses many graph labeling problems. It is based on a mapping between V and the set of vertices of a host graph $H = (N, E_H)$ with $N = \{0, \cdots, n-1\}$. Graph labeling problems are then defined as finding the best mapping π from V to N, according to minimization or maximization of an objective function often using distances between labels taken between pairs of adjacent vertices of G. This distance, noted d_H, is defined as the length of the shortest path between the corresponding vertices in the host graph H. Two possible objective functions are often considered:

1. the maximum distance d_H between the labels of two adjacent vertices of G is minimized, i.e. it amounts to find a labeling $\hat{\pi}$ such that:

$$\hat{\pi} = \arg\min_{\pi} \max_{\{u,v\} \in E} d_H(\pi[u], \pi[v]) \tag{1}$$

2. the sum of distances d_H between all pairs of adjacent vertices of G is minimized, i.e. it amounts to find a labeling $\hat{\pi}$ such that:

$$\hat{\pi} = \arg\min_{\pi} \sum_{\{u,v\} \in E} d_H(\pi[u], \pi[v]) \tag{2}$$

The resulting graph labeling problems have been extensively studied in the case where the host graph is a path graph P, where $E_P = \{\{i, i+1\} \mid i = 0 \ldots n-2\}$. The length of the shortest path between two vertices u and v in this graph is given by:

$$d_P(\pi[u], \pi[v]) = |\pi[u] - \pi[v]| \tag{3}$$

These problems are called bandwidth problem (condition 1) and bandwidth sum problem (condition 2).

Lin [24] and Jianxiu [22] introduced the problems where the host graph is a cycle C, where $E_C = \{\{i, i+1\} \mid i = 0 \ldots n-2\} \bigcup \{n-1, 0\}$. In this case, the distance between two vertices $u, v \in V$ is given by:

$$d_C(\pi[u], \pi[v]) = \min\{|\pi[u] - \pi[v]|, n - |\pi[u] - \pi[v]|\} \tag{4}$$

The resulting problems are called cyclic bandwidth problem (condition 1) and cyclic bandwidth sum problem (condition 2). We focus in this paper on the cyclic bandwidth sum problem (CBSP). It is thus defined as the minimization of a quantity called cyclic bandwidth sum (CBS):

$$\min_{\pi} CBS(G) = \min_{\pi} \sum_{\{u,v\}\in E} d_C(\pi[u], \pi[v]) \tag{5}$$

Examples of optimal labeling solving Eq. (5) are shown in Fig. 3 for some standard graphs. We can see that the labeling closely follows the structure.

These problems are generally NP-hard problems, as shown by Papadimitriou for the bandwidth problem [26] and Lin for the cyclic bandwidth problem [24].

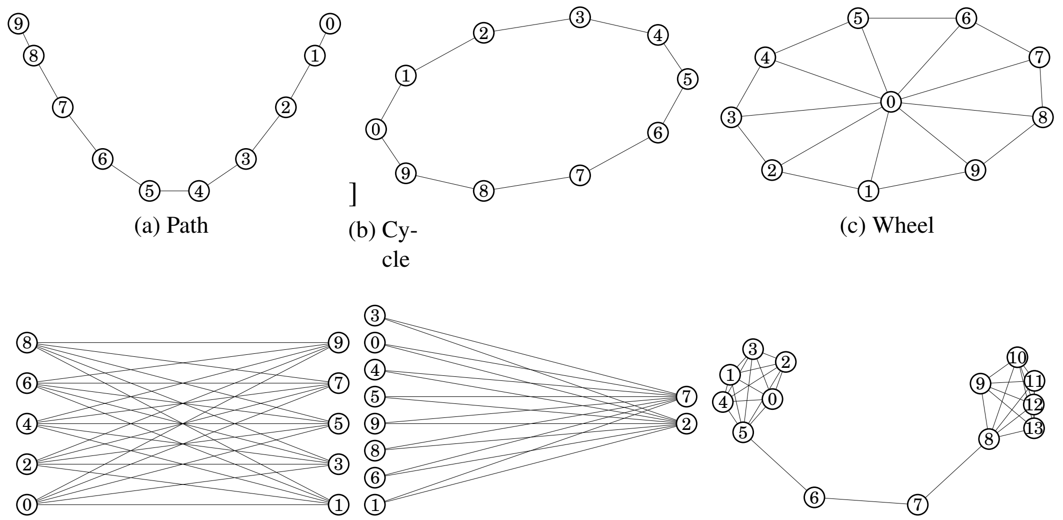
1.3. Related works

Many works have been done on the study of labeling graph problems: as mentioned previously, Díaz [11] proposed a review of several graph labeling problems from an algorithmic point of view. Among these problems, the bandwidth problem and bandwidth sum problem have been extensively studied: Papadimitriou [26] proves the NP-Complenetess of the bandwidth problem, highlighting the necessity of heuristics, as the one developed by Cuthill et al [9], to find efficiently a good labeling for these problems. Some studies have also been performed on other graph labeling problems, such that cyclic bandwidth problem [24, 27], antibandwidth problem [4] or cyclic antibandwidth problem [25], both in terms of theoretical results and algorithms. Conversely, only few results are available in the literature for solving the cyclic bandwidth sum problem. Two articles focus on the mathematical aspects of this problem: Jianxiu [22] introduced cyclic bandwidth sum problem and proposed theoretical results for some standard graphs, such as wheel or k-regular graphs, in terms of optimal value of CBS or upper bounds for this value. Later on, Chen et al. [7] extend this work by adding some results, for instance for complete bipartite graphs. Whereas these theoretical results do not help to get the optimal labeling of a graph, they are nonetheless useful to evaluate the quality of a solution of the CBS problem, especially when it is obtained thanks to heuristic algorithms. To the best of our knowledge, only one heuristic was proposed to solve the cyclic bandwidth sum problem, published in [29] and extended in [28]. The heuristic is based on a general variable neighborhood search (GVNS). The idea of GVNS is to change the labeling both globally and locally to descent to local minima of CBS, using two distinct phases: A shaking phase in which the labeling is changed by applying several operations where the vertices are either shifted, reversed, flipped or swapped without taking into account the proximity of vertices. This operation enables the algorithm to escape from valleys and to browse the solution space. A local search is then performed to descent in a valley to a local minima and is performed by switching consecutive vertex or swapping adjacent vertices whose edge distance (see Eq. 4) is the highest.

1.4. Outline

The following sections present the heuristic we developed to address the cyclic bandwidth sum problem efficiently, that will be called MaCH for MinimizAtion of Cbs Heuristic. Section 2 sketches the principles of the proposed method. Detailed algorithms are presented in Section 3, while a worst-case complexity study is given in Section 4. The performance of the algorithm is investigated in Section 5 through the comparison of the solution of our algorithm with the theoretical results when available, or with GVNS. A study is then performed in Section 6 on graphs exhibiting common properties encountered in real-world networks, with a qualitative approach to visually validate the performance of the heuristic to discover the structure of complex networks. Section 7 discusses extensions of the proposed method to handle weighted graphs and Section 8 concludes the paper.

2. Heuristic to minimize the Cyclic Bandwidth Sum of a graph



(a) Path] (b) Cycle (c) Wheel

(d) Complete bipartite (5, 5) (e) Complete bipartite (8, 2) (f) Graph with cliques

Figure 3: Examples of standard graphs with optimal labeling minimizing the CBS.

The aim of the heuristic is to build a labeling by traversing the graph to discover its structure. The vertex labels are then constrained by the regularities of the structure, which may have multiple forms. For instance, in the simple case of a cycle (see Fig. 3b), the correct behavior of the algorithm should be as follows: Starting from a random vertex, it will label it and recursively jump to one of its unlabeled adjacent vertices, label it with the next integer, and so forth until all vertices are labeled. In the less trivial case where the graph is organized by several cliques (see Fig. 3f), the algorithm should browse all the vertices inside a clique before jumping to another one. More generally, the algorithm has to adapt its search to the structure of the graph, whatever the structure is.

One solution to achieve this goal is to perform a self-avoiding random walk on the graph that successively numbers the vertices when they are reached. However, this approach has one drawback: the choice of the next vertex depends only on the neighborhood of the current vertex, and not on a more extended neighborhood. It implies that the random walk has to be controlled to avoid going to vertices already numbered, and the walk can stop before visiting all the vertices.

The heuristic we propose below fills in the gaps of a random walk and consists of a two-step algorithm. The first step performs local searches in order to find a collection of independent paths with respect to the local structure of the graph, while the second step determines the best way to arrange the paths such that the objective function of the CBSP is minimized.

2.1. Step 1: Guiding the search towards locally similar vertices

The heuristic starts taking as input a graph where each vertex can be referred by a unique identifier. The first step consists in finding a collection of paths in the graph, that is to say some sequences of vertices consecutively connected. The algorithm performs a depth-first search in which the next vertex is chosen based on its similarity to the current vertex. This similarity depends on the intersection of the two vertex neighborhoods: the more the neighborhoods of the two vertices intersect, the closer their labels are. Concretely the search is executed as follows: Starting from a vertex, the algorithm jumps to the most similar neighbor not yet labeled, and so on until there is no more accessible vertices. Then, the algorithm starts a new path from a vertex which has not been yet inserted in a path, and then continues to build paths until all the vertices are in a path. At the end of this step, a collection of paths is obtained that partitions the graph vertex set.

Initialization Any vertex not yet inserted in a path can be used as starting node. However, to favor the computation of longer paths, vertices that are at the periphery of the graph are preferred. The incentive behind this choice lies on the fact that the path should start at one of the extremity of the graph if there is one. For example, let us consider a simple path graph: Starting from a vertex in the middle of the path will generate two paths, although it is obvious that the graph can be traversed using a single path. There are several measures to determine the centrality of a vertex, that can also be used to find vertices that are outer of the graph. We chose the simplest one, to minimize the computational cost, by namely using the degree of the vertices: the vertex with the smallest degree is selected to start the path.

Construction of a path A path is obtained by performing a depth-first search where the next adjacent vertex is the one that (1) is not labeled and (2) has a neighborhood that is the most similar to the one of the current vertex. The neighborhood similarity of two vertices is evaluated based on the Jaccard index [21], a quantity used to compare the similarity between two sets by looking at the total number of common elements (including the considered vertices) over the total number of elements. Let $\texttt{Adj}(u)$ returns the adjacent vertices of the vertex u, i.e. the neighborhood of u. The similarity index between the vertex u and v, noted $J(u,v)$, is defined by:

$$J(u,v) = \frac{\#\Big((\texttt{Adj}(u) \cap \texttt{Adj}(v)) \cup \{u,v\}\Big)}{\#\Big(\texttt{Adj}(u) \cup \texttt{Adj}(v)\Big)} \tag{6}$$

This measure is equal to 1 if the two vertices have the same adjacent vertices, otherwise it is strictly lower than 1. A value close to 0 means that the total number of vertices in the two neighborhoods is much higher than the number of common neighbors.

It may happen that two neighbors of the current vertex u have the same similarity index with u. In this case, the selected vertex is the first vertex encountered by the algorithm.

It is preferable that the adjacent vertices of degree 1 that are only adjacent, to the current vertex, are not chosen as the following vertices, despite their high similarity, because it would end up the path. These vertices are immediately inserted after their unique neighbor to guarantee that the vertices are as close in the labeling as they are in the graph. However we let the traversal continue.

End of the search The search for a path ends when all the neighbors of the current vertex have been inserted in a path. The algorithm starts a new path using the remaining vertices, until all the vertices belong to a path.

2.2. Step 2: Greedy merge of paths

The second step aims at aggregating the paths obtained in Step 1 in a unique labeling in such a way that the CBS is minimized. The results of this step is a list of vertices, where the position of the vertex in the list gives its label. We perform a greedy search that takes the locally optimal choice while merging a new path in the partial labeling under construction: The algorithm computes the CBS value for the insertion of the path and the reverse of the path at each position of the current partial solution and retains the argument that minimizes the CBS. The paths are selected in turns according to their length, the largest one being selected first. The rational behind this choice is so that to broadly explore the space of solutions.

Incremental computing of the CBS The evaluating of the CBS, as given in Eq. (5) is demanding computationally, as it requires considering every edge of the graph. For each insertion of path, the current CBS is computed twice (ordered and reverse ordered) for each possible insertion index of the current labeling. It is thus very costly, but can be largely alleviated by observing that, from an index to the next one, many edges have the same contribution in the total CBS value. From this perspective, we propose an incremental update of the CBS to take into account the state before the shift: At each update, only the edges whose adjacent vertex labels have been modified are considered.

To explain the incremental computation of the CBS, let us consider the insertion of a path, noted P, into a labeling, noted O, at the index i. The labeling can be decomposed into three parts: the first part is noted O_1 and is made of the vertices located before i. The vertex right after the index of insertion i is noted k, while the remaining vertices compose the third part called O_2. The path P is inserted into the labeling between O_1 and k when the index of insertion is i, and between k and O_2 when the index of insertion is $i+1$. This is schematically represented in Fig. 4: Line 1 represents the current labeling made of a sequence of vertices O_1 followed by the vertex k at position i and ended by the sequence of vertices O_2. P (line 2) is the sequence of vertices that is currently inserted at the index i (line 3), i.e. just before vertex k. Thus, the current labeling begins by the path O_1, is followed by P, then comes the vertex k and the path O_2. Line 4 gives the current labeling when P is inserted at the position $i+1$, where the vertex k has been shifted from right to left.

labeling O	$\vert - - - O_1 - - - \vert - k - \vert - - - O_2 - - - \vert$
path P	$\vert - - - P - - - \vert$
insertion of P at index i	$\vert - - - O_1 - - - \vert - - - P - - - \vert - k - \vert - - - O_2 - - - \vert$
insertion of P at index $i+1$	$\vert - - - O_1 - - - \vert - k - \vert - - - P - - - \vert - - - O_2 - - - \vert$

Figure 4: Schema of the insertion of path P in the current labeling O

As the label of vertices are given by the position of the vertices in the labeling, it is clear that from an index to the next one, only the vertices in P and k will have different labels. Let $\#P = p$ the number of vertices in the path P, and $\pi_i[u]$ the label of vertex u when P is inserted at index i. The changes in the labels for each group of vertices are the following:

$$\pi_{i+1}[k] = \pi_i[k] - p \tag{7}$$

$$\forall u \in P,\, \pi_{i+1}[u] = \pi_i[u] + 1 \tag{8}$$

$$\forall u \in O_1,\, \pi_{i+1}[u] = \pi_i[u] \tag{9}$$

$$\forall u \in O_2,\, \pi_{i+1}[u] = \pi_i[u] \tag{10}$$

We note $CBS^{(i)}$ the value of the cyclic bandwidth sum when P is inserted at index i. The computation of $CBS^{(i)}$ can be decomposed according to the different groups of vertices defined above:

$$\begin{aligned} CBS^{(i)} = {} & CBS^{(i)}(O_1,O_1) + CBS^{(i)}(O_2,O_2) + CBS^{(i)}(O_1,O_2) + CBS^{(i)}(P,P) \\ & + CBS^{(i)}(k,O_1) + CBS^{(i)}(k,O_2) + CBS^{(i)}(k,P) \\ & + CBS^{(i)}(P,O_1) + CBS^{(i)}(P,O_2) \end{aligned} \tag{11}$$

where $CBS^{(i)}(X,Y) = \sum_{u\in X, v\in Y, \{u,v\}\in E} d_C(\pi_i[u], \pi_i[v])$ is the value of CBS when only the edges of the graph between the two sets X and Y are considered, with $d_C(\pi[u], \pi[v])$ defined in Eq. 4. The definition of the distance d_C shows trivially that if the labels of the endpoint vertices of an edge are not shifted or are shifted equally, then the value of d_C remains the same:

$$CBS^{(i+1)}(O_1,O_1) = CBS^{(i)}(O_1,O_1) \tag{12}$$

$$CBS^{(i+1)}(O_2,O_2) = CBS^{(i)}(O_2,O_2) \tag{13}$$

$$CBS^{(i+1)}(O_1,O_2) = CBS^{(i)}(O_1,O_2) \tag{14}$$

$$CBS^{(i+1)}(P,P) = CBS^{(i)}(P,P) \tag{15}$$

When the labels of endpoint vertices are not shifted equally, it is necessary to consider not only the changes induced by the shift, but also which terms between $|\pi[u] - \pi[v]|$ and $n - |\pi[u] - \pi[v]|$ is the minimum, both at index i and $i+1$, as it can vary. We prove in the following the results when the endpoint vertices are k and a vertex in O_1. The other results are given in Appendix A.

Theorem 2.1. *Edges between k and the vertices of* O_1
Let $u \in O_1$ *and* $\Delta = \pi_i[k] - \pi_i[u]$*. We have:*

1. *if* $\Delta \leq \frac{n}{2}$ *then* $CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u) - p$.

2. *if* $\Delta \geq \frac{n}{2}+p$ *then* $CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u)+p$.

3. *if* $\frac{n}{2} < \Delta < \frac{n}{2}+p$ *then* $CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u)+2\Delta-(n+p)$

Proof. For all $u \in O_1$, we have $\pi_{i+1}[u] = \pi_i[u] < \pi_{i+1}[k] < \pi_i[k]$ from Eqs 7 and 9. Thus $0 < \pi_{i+1}[k]-\pi_{i+1}[u] < \Delta$, allowing for the removal of the absolute value in Eq. 4.

Let us consider the case where the minimum term retained to compute $CBS_i(u,k)$ in Eq. (4) is the first term. It means that:

$$\Delta \leq n-\Delta \Leftrightarrow \Delta \leq \frac{n}{2} \tag{16}$$

When $CBS_{i+1}(u,k)$ is considered, the first term is retained in Eq. (4) if:

$$\begin{aligned} \pi_{i+1}[k]-\pi_{i+1}[u] &\leq n-(\pi_{i+1}[k]-\pi_{i+1}[u]) \\ \Delta-p &\leq n-(\Delta-p) \\ 2(\Delta-p) &\leq n \\ \Delta &\leq \frac{n}{2}+p \end{aligned} \tag{17}$$

Symmetrically the second term in the minimum function in Eq. (4) is used for $CBS^{(i)}(u,k)$ if $\Delta \geq \frac{n}{2}$ and for $CBS^{(i+1)}(u,k)$ if $\Delta \geq \frac{n}{2}+p$.

Then, using Eq. 7 and Eq. 9, there are 3 possible cases :

1. If $\Delta \leq \frac{n}{2}$, then the first term is retained for $CBS^{(i)}(u,k)$ and $CBS^{(i+1)}(u,k)$:

$$\begin{aligned} CBS^{(i+1)}(k,u)-CBS^{(i)}(k,u) &= (\pi_{i+1}[k]-\pi_{i+1}[u])-(\pi_i[k]-\pi_i[u]) \\ &= (\pi_i[k]-p-\pi_i[u])-(\pi_i[k]-\pi_i[u]) \\ &= -p \end{aligned} \tag{18}$$

2. If $\Delta \geq \frac{n}{2}+p$, then the second term is retained for $CBS^{(i)}(u,k)$ and $CBS^{(i+1)}(u,k)$:

$$\begin{aligned} CBS^{(i+1)}(k,u)-CBS^{(i)}(k,u) &= (n-(\pi_{i+1}[k]-\pi_{i+1}[u]))-(n-(\pi_i[k]-\pi_i[u]) \\ &= -(\pi_i[k]-p-\pi_i[u])+(\pi_i[k]-\pi_i[u]) \\ &= p \end{aligned} \tag{19}$$

3. $\frac{n}{2} < \Delta < \frac{n}{2}+p$, then the second term is retained for $CBS^{(i)}(u,k)$ and the first term for $CBS^{(i+1)}(u,k)$:

$$\begin{aligned} CBS^{(i+1)}(k,u)-CBS^{(i)}(k,u) &= (\pi_{i+1}[k]-\pi_{i+1}[u])-(n-(\pi_i[k]-\pi_i[u])) \\ &= (\pi_i[k]-p-\pi_i[u])-n+(\pi_i[k]-\pi_i[u]) \\ &= 2\Delta-(n+p) \end{aligned} \tag{20}$$

□

2.3. Comments

2.3.1. Influence of the initialization

The algorithm is completely deterministic and several executions will lead to the same solution with a similar input. The algorithm can nevertheless return different solutions for a same graph by changing the initial identifiers of the vertices: three steps of the heuristic produce a stochastic behavior and all of them originate from the same statement: When a sort is performed, whatever the criterion of sorting, if several elements have the same value, then the first element encountered by the algorithm is selected before the other ones. This happens when (1) the vertices are sorting according to their degree to select the first vertex of a path, (2) when several paths have the same length and (3) when the path insertion at several positions leads to the same CBS value. The stochasticity induced by the initial vertex order is studied in Section 5, by randomly ordering the vertices k times and selecting the minimal value of CBS over the k repetitions, for different values of k. It shows that when the number of repetitions is high, the solution obtained is little bit better. However, the improvement of the performance is not really high and indicates the good robustness of the heuristic, even with a moderate number of repetitions.

2.3.2. Local search against global search

A drawback of the heuristic is that it relies on local searches in the graph. Therefore, the algorithm cannot go to a vertex which is not a neighbor of the previous one. The labeling is hence really tailored to the structure of the graph, as required. Nevertheless, the optimal labeling is sometimes either not consistent with the topology of the network for instance if high jumps should appear, or it is consistent but using a different organization, not reachable by the heuristic. As our main motivation is to follow closely the network structure, the obtained labeling can lead to bad results in terms of optimal CBS, while finding a path will follow the network.

3. Detailed algorithm

The whole algorithm MaCH comprises the consecutive execution of two steps, introduced in Section 2. For readability, the algorithm of each step is described separately, respectively in Algorithms 1 and 2.

From a connected, unweighted and undirected graph $G = (V, E)$ with n vertices, the algorithm outputs a one-to-one mapping π from V to $\{0, \cdots, n-1\}$. We consider in the following a `List` as a list of elements with the associated functions `List-Insert`(A, a, idx) which inserts the element a in the list A at the index idx (if idx is not given, the element a is inserted at the end of the list), `List-Remove`(A, a) which removes the element a from the list A, `Length`(A) which returns the number of elements of the list A, and the function `Reverse`(A), which returns the list A in the reverse order. The function `Degree`(u) returns the degree of the vertex u in the graph G, i.e. the number of vertices adjacent to the vertex u. Finally `Adj`(u) returns the adjacent vertices of u.

Algorithm 1 computes the first step of the heuristic as described in Section 2.1. It consists in building a collection of paths containing the vertices of the graph, each path traversing the graph

Algorithm 1 Step 1: Guiding the search towards locally similar vertices

```
Require: G = (V, E)
Ensure: Paths
1: S = List(V)
2: Paths ← List()
3: while S is not empty do
4:    u_0 ← arg min_{u∈S} Degree(u)
5:    List-Remove(S, u_0)
6:    exist_successors ← True
7:    P ← List()
8:    while exist_successors do
9:       List-Insert(P, u_0)
10:      H ← List()
11:      for all v ∈ Adj(u_0) ∩ S do
12:         if Degree(v) = 1 then
13:            List-Insert(P, v)
14:            List-Remove(S, v)
15:         else
16:            List-Insert(H, v)
17:         end if
18:      end for
19:      if H is not empty then
20:         u_0 ← arg max_{w∈H} Similarity_Index(u, w)
21:      else
22:         exist_successors ← False
23:      end if
24:   end while
25:   List-Insert(Paths, P)
26: end while
```

following its structure. Line 1 initializes a list S containing all the vertices of the graph, while Line 2 initializes an empty list which will contain the paths . The search of paths (Lines 3 to 26) is then performed until all vertices are included in a path. A vertex of S minimum degree value is considered (Line 4). The selected vertex, noted u_0, is removed from S (Line 5) and is the starting vertex of the search from Line 8 to Line 24. The path is defined as a sequence of vertex added in a list P (Line 7), and is closed when there are no more successor available to extend the path or when the depth-first search ends. The first step of this loop consists of adding the vertex u_0 to the path P (Line 9). A new list H is then initialized (Line 10) and will contain the potential successors of u_0. These successors are selected among the adjacent vertices of u_0 which are still in the list S, i.e. which have not been included in a path beforehand (Line 11). For each of the successors, noted v, if the degree of v is equal to 1, i.e. the vertex v has only the vertex u_0 as adjacent vertex, then v is directly added in the path (Lines 12 to 14). Otherwise, it is added to the list H (Line 16). When all the potential successors have been either added to P or H, the next vertex to be considered is chosen among the elements of H, as the one with the highest similarity with the current u_0 according to Eq 6 and given by the function `Similarity_Index`. The heuristic then loops using the updated value of u_0. If H is empty, `exist_successors` is set to `False` (Line 22) and the path is inserted in the list of paths (Line 25). If S is not empty, then u_0 is updated using the procedure described in Line 4 and the search of a path from this vertex is repeated. When S is empty, the first step is completed.

Algorithm 2 Step 2: Greedy merge of paths

Require: *Paths*

1: $Order \leftarrow \arg\max_{P \in Paths}$ `Length`(P)
2: `List-Remove`$(Paths, Order)$
3: **while** *Paths* is not empty **do**
4: $\quad P_0 \leftarrow \arg\max_{P \in Paths}$ `Length`(P)
5: $\quad idx, reverse \leftarrow$ `Incremental_CBS`$(Order, P_0)$
6: $\quad$ **if** *reverse* is `True` **then**
7: $\qquad$ `Insert-List`$(Order,$ `Reverse`$(P_0), idx)$
8: $\quad$ **else**
9: $\qquad$ `Insert-List`$(Order, P_0, idx)$
10: $\quad$ **end if**
11: $\quad$ `List-Remove`$(Paths, P_0)$
12: **end while**
13:
14: $i \leftarrow 0$
15: **for** $i = 0 : (n-1)$ **do**
16: $\quad \pi[Order[i]] \leftarrow i$
17: **end for**
18: **return** π

Algorithm 2 computes the second step of the heuristic as described in Section 2.2. A list *Order* is first initialized as the path in *Paths* with the highest number of elements (Line 1). This path is then removed from the list *Paths*, and the algorithm inserts all the remaining path in the list *Order* using a loop from Line 3 to Line 12. The path with the highest number of elements is selected (Line 4). The function `Incremental_CBS` returns the index and the direction of insertion of P that minimizes the CBS. Depending on the value of the boolean variable *reverse*, the path P_0 is inserted reversed (Line 7) or not (Line 9). The path is then removed from the list of paths *Paths* and the heuristic loops until all the paths have been inserted in *Order*. As the result, the mapping π is built using the vertices as keys and the index of the vertices in the list *Order* as values.

4. Worst-case complexity of the algorithm

We now examine the worst-case time complexity of the algorithm MaCH described in the previous section, when applied on a graph $G(V,E)$ with $\#V = n$ and $\#E = m$.

We first examine the complexity of Algorithm 1. The set S initialized Line 1 can be implemented as a min-priority queue with a binary min-heap. The time to build the binary min-heap is $O(n)$. Lines 4 and 5 can be done using the EXTRACT-MIN function that takes time $O(\log n)$. Similarly, the set *Paths* can be implemented as a max-priority queue with a binary max-heap and Line 13 takes in the worst case a time proportional to the logarithm of the number of paths, that is in the worst case $O(\log n)$. Using aggregate analysis, the `while` loop in Line 8 is executed at most once for each vertex of V, since the vertex u_0 is removed from S (Line 5). The function `List-Insert` is in constant time. The set H of vertices that are adjacent to u and in S is implemented as a max-priority queue using a binary heap data structure that makes possible to run MAX_HEAP_INSERT, that inserts a new element into H (Line 16) while maintaining the heap property of H in $O(\log(\#H))$, that is in the worst case in $O(\log(\#\mathtt{Adj}[u]))$. Thus, the loop on Lines 8-18 in is executed $\#\mathtt{Adj}[u]$ times and at each iteration (1) the similarity index computation takes time $\Theta(\min(\#\mathtt{Adj}[u],\#\mathtt{Adj}[v]))$ and (2) MAX_HEAP_INSERT takes time $O(\log(\#\mathtt{Adj}[u]))$. Therefore, loop Lines 8-18 is in $O(\#\mathtt{Adj}[u]^2)$. Line 20 can be done using EXTRACT_MAX in time $O(\log(\#\mathtt{Adj}[u]))$ and the total complexity of Lines 8-24 is in $O(\#\mathtt{Adj}[u]^2)$. Consequently, the total time is $O(\sum_{u\in V}(\#\mathtt{Adj}[u]^2))$. As K. Das established in [10] that

$$\sum_{u\in V} \#\mathtt{Adj}[u]^2 \leq m\left(\frac{2m}{n-1}+n-2\right) \tag{21}$$

we can conclude that the total cost of Algorithm 1 is in $O(n\log n+mn)=O(mn)$

We can also use an aggregate analysis to evaluate the time taken by the Algorithm 2. Lines 2 and 5 are in $O(n)$, when almost all the vertices have already been merged in `Order`. `Incremental_CBS` runs through (1) all the edges between the vertices of the current path P and the ones of `Order` and (2) between the vertex of `Order` at position `Position` and the other vertices of $\mathtt{Order}\cup P$.

- Step (1) takes $O(mn)$ since all the edges of the graph are examined when aggregating the analysis over the all paths: the adjacency list of each vertex is examined once. Furthermore, for each of these m edges, the n positions of `Order` are evaluated.;
- Step (2) is also in $O(mn)$ since aggregating the adjacency lists of the vertices of `Order` leads to the m edges of the graph that are evaluated for each of the at most n paths.

The other instructions of the loop are executed in constant time. Therefore, the total time spent in Algorithm 2 is $O(mn)$.

Finally, we have that the whole algorithm has a worst case complexity in $O(mn)$.

5. Computational experiments

This section describes the computational experiments that we carried out to assess the performance of the heuristic MaCH discussed in the previous sections. The aim of this part is to test the ability of the algorithm to obtain a good approximate solution for the CBSP, in a reasonable amount of time.

5.1. Experimental setup

The assessment of the heuristic is performed on three aspects:

Performance The value of CBS obtained using MaCH is compared with a reference value, chosen as the theoretical results when available, or as the value of CBS achieved using an existing method. For each instance of graph, random identifiers are assigned to the vertices, and the heuristic MaCH is executed, returning the labeling and the value of CBS achieved at the end. 30 repetitions of this process are performed to obtain 30 values of *CBS* for each instance, in order to check the robustness of the results;

Robustness The value of CBS obtained using MaCH is compared for different numbers of repetitions of the heuristic, to assess the stochasticity of the heuristic and its influence on the results. The process described above is repeated k times, and the minimal value over these k repetitions is retained. The algorithm is tested for $k \in \{10, 20, 50\}$: For each value of k, 30 repetitions are performed as above, to obtain 30 values of CBS;

Execution time The average execution time of the algorithm is observed to assess the speed of the heuristic. For each instance, the time in seconds for the 30 repetitions of MaCH is used to obtain an average time execution for one repetition.

A comparison is performed between the median value of CBS over the 30 repetitions, noted *median CBS*, with a reference value, noted *ref*, which depends on the type of graphs, by computing the relative distance *rd*:

$$rd = \frac{median\ CBS - ref}{ref} \tag{22}$$

The sign of *rd* indicates if *median CBS* is greater ($rd > 0$) or lower ($rd < 0$) than the reference value, while its value gives how far *median CBS* is far from *ref*. For example, $rd = 0.80$ indicates that *median CBS* is 1.80 times higher than the reference value while $rd = -0.25$ means that the median of the value of CBS is 1.25 times lower than the reference value. Raw data of *median CBS* and *ref* are available in Appendix B.

MaCH algorithm is implemented in Python 2.7 using the module *Networkx* [16]. All the experiments were conducted on a 2.60 GHz Intel Core i7 with 8 GB of RAM.

5.2. Datasets

Eight types of graphs have been considered for the experiments: a detailed description of each data set follows.

5.2.1. Graphs with known CBS optimal value

Path graphs A path graph is defined as a sequence of vertices such that each vertex, except the first and the last ones, are linked with its previous and next vertices in the sequence. The optimal value for a path P_n of size n is $CBS_{\text{opt}}(P_n) = n - 1$. The data set consists of all the paths up to 448 vertices.

Cycle graphs A cycle graph is a path whose first and last vertices are linked, forming a circular sequence of vertices. The optimal value for a cycle C_n of size n is $CBS_{\text{opt}}(C_n) = n$. The data set consists of all the cycles up to 448 vertices.

Wheel graphs A wheel graph is defined as a cycle whose all vertices are also linked to a single vertex called *hub*. The optimal value of *CBS* for a wheel W_n with n vertices is $CBS_{\text{opt}}(W_n) = n + \lfloor \frac{1}{4}n^2 \rfloor$ as proved in [22]. The data set consists of all the cycles up to 448 vertices.

Power graphs of cycles (PGC) The kth power of a cycle graph C_n is a graph with n vertices and edges such that u and v are linked if and only if the length of the shortest path between u and v in C_n is equal or lower than k. The optimal value for the kth power of cycle graph C_n^k is $CBS_{\text{opt}}(C_n^k) = \frac{1}{2}nk(k+1)$ as proved in [22]. The data set consists of all the kth power of cycles up to 448 vertices, with $k \in \{2, 10\}$.

Complete bipartite graphs (CBG) A complete bipartite graph is composed of two sets with respectively n_1 and n_2 vertices: each vertex of the first set is linked with all the vertices of the second set and there is no link between two vertices of the same set. Chen et al. [7] proved that the optimal value of CBS for a complete bipartite graph $K_{n_1 n_2}$ is given by:

$$CBS_{\text{opt}}(K_{n_1 n_2}) = \begin{cases} \frac{n_1 n_2^2 + n_1^2 n_2}{4} & \text{if } n_1 \text{ and } n_2 \text{ are even} \\ \frac{n_1 n_2^2 + n_1^2 n_2 + n_1}{4} & \text{if } n_1 \text{ is even and } n_2 \text{ is odd} \\ \frac{n_1 n_2^2 + n_1^2 n_2 + n_1 + n_2}{4} & \text{if } n_1 \text{ and } n_2 \text{ are odd} \\ \frac{n_1 n_2^2 + n_1^2 n_2 + n_2}{4} & \text{if } n_1 \text{ is odd and } n_2 \text{ is even} \end{cases}$$

The data set consists of all the complete bipartite graphs up to 448 vertices with three different ratios between n_1 and n_2: 1, 3 and 7. Only the values of n_1 and n_2 eligible for the desired ratios have been retained.

5.2.2. Graphs with upper bound of the CBS optimal value

Cartesian products The Cartesian product of two graphs $G = (V_G, E_G)$, with $\#V_G = m$, and $H = (V_H, E_H)$, with $\#V_H = n$, noted $G \times H$, is the graph with vertex set $V_G \times V_H = \{(u, v) | u \in V_G, v \in V_H\}$. The vertices (u_G, u_H) and (v_G, v_H) are adjacent if and only if $u_G = v_G$ and $(u_H, v_H) \in E_H$ or $u_H = v_H$ and $(u_G, v_G) \in E_G$.

Jianxiu [22] proved upper bounds for the optimal value of CBS when G and H are either a path, a cycle or a complete graph. A complete graph with n vertices, noted K_n, is the graph where all pairs of vertices are linked. Using the notations given above, we have:

$$CBS_{\text{opt}}(P_m \times P_n) \leq m(n-1) + n^2(m-1), \quad m \geq n \tag{23}$$

$$CBS_{\text{opt}}(C_m \times C_n) \leq m(n^2 + 2n - 2), \quad m \geq n \geq 3 \tag{24}$$

$$CBS_{\text{opt}}(K_m \times K_n) \leq \frac{1}{6}mn\left(n^2 + 3n\left\lfloor\frac{m}{2}\right\rfloor\left\lceil\frac{m}{2}\right\rceil - 1\right), \quad m \geq n \tag{25}$$

$$CBS_{\text{opt}}(P_m \times C_n) \leq n(m^2 + m - 1) \tag{26}$$

$$CBS_{\text{opt}}(P_m \times K_n) \leq \frac{1}{2}m^2 n\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil + n(m-1) \tag{27}$$

$$CBS_{\text{opt}}(C_m \times K_n) \leq n\left(\frac{1}{2}m^2\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil + 2m - 2\right) \tag{28}$$

The data set consists of the Cartesian products of graphs cited above, with m and n up to 25, with the specific constraints on m and n if necessary.

5.2.3. Graphs with unknown CBS optimal value

Random connected graphs A random graph [13] is a graph where the edges between the vertices are randomly drawn. The Erdös-Rényi model has been used to generate random graphs: for each pair of vertices, an edge between the two vertices has a probability p to appear. The data sets consists of 50 random graphs built as follows: for each value of $p \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$, 10 instances of the Erdös-Rényi model are generated with a fixed number of vertices set to 100.

Harwell-Boeing collection The Harwell-Boeing Sparse Matrix Collection Graphs [12] consists of a set of standard matrices arising from various problems in engineering and scientific fields. Graphs are derived from these matrices as follows: Let M_{ij} be the element of the ith row and the jth column of a sparse matrix M of size $n \times n$, the resulting graph has n vertices such that there is an edge between vertices i and j if and only if $M_{ij} \neq 0$ and $i \neq j$. We selected 27 matrices from this collection, from small graphs (24 vertices) to large graphs (1454 vertices), representing a wide variety of structures. Table 12 gives the list of matrices used with the number of vertices and the number of edges of the resulting graphs.

5.3. Performances of the heuristic MaCH

5.3.1. Comparison with known CBS optimal value

The heuristic MaCH achieves the optimal value of CBS given by the theoretical results for all instances from the following datasets: paths, cycles, wheels, power graphs of cycles and complete bipartite graphs.

5.3.2. Comparison with the upper bound of the CBS optimal value

The values of CBS obtained using the heuristic MaCH have been compared with the theoretical upper bound given by Jianxiu [22]. Table 1 shows a summary of the results: each line concerns one type of Cartesian products. The first two columns give the two graphs used in the Cartesian products and the third column the number of graph in the collection, corresponding for all couple of values *m* and *n*. The fourth column gives the relative distance averaged over all graphs of the collection. Finally, the fifth column refers to the table of detailed results in the Appendix.

G	*H*	*# graphs*	*mean rd*	*Table*
Path	Complete	400	-0.84	6
Cycle	Complete	400	-0.83	7
Complete	Complete	210	-0.20	8
Path	Cycle	400	0.50	9
Path	Path	210	0.76	10
Cycle	Cycle	210	0.83	11

Table 1: Summary of the performance of the heuristic MaCH on the Cartesian products of paths, cycles and complete graphs. The first two columns give the two graphs used in the Cartesian products and the third column the number of graph in the collection. The fourth column gives the averaged relative distance. The fifth column refers to the table of detailed results in the Appendix.

The results highlight the different behaviors of the heuristic MaCH according to the topology. When the graph is organized as a succession of linear well-structured subgraphs, the heuristic MaCH is highly efficient: in the case of the Cartesian product of a complete graph and a path or a cycle graph, the graph is a succession of cliques in a linear or cyclic arrangement. The heuristic MaCH is then guided by this organization to discover the underlying structure, and the value of CBS is consequently very low compared to the upper bound. However, when the structure presents regularities but not along a linear arrangements, the heuristic MaCH tends to fail to find the optimal value. This happens for the grid (Cartesian products of paths), the torus (Cartesian products of cycles) and the cylinder (Cartesian product of a path and a cycle): the heuristic MaCH has many consistent ways to traverse the graph. The discovery of the structure is nevertheless performed, but without minimizing as much as possible the value of CBS. Detailed results (Tables 8 9 10 and 11) show that when the graph is a little unbalanced, for example when the number of vertices in *G* and *H* is different, the value of CBS achieved is better, as the heuristic MaCH follows this imbalance.

5.3.3. Comparison with the heuristic GVNS

A comparison with the algorithm developed by Satsangi et al. [29] has been performed for the graphs where theoretical results for the optimal value of CBS do not exist: the dataset of random graphs and the graphs from the Harwell-Boeing collection. To assess the performance of the heuristic MaCH, the heuristic GVNS [29] has been executed using the code provided by the authors. It has been run using Matlab R2012b. The algorithm has been initially developed to start using the initial identifiers of vertices as set at the generation of the graph. These identifiers, especially for the graph from the Harwell-Boeing collection, have a topological meaning since these matrices describe engineering problems: the value of CBS is then naturally low using these identifiers. To check the algorithm in blind conditions, we added a randomization of the identifiers of the vertices before applying GVNS. For each instance, the minimum value of CBS over the 50 repetitions, as defined in [29], has been retained. Detailed results in Appendix show the value of CBS obtained using GVNS without and with this randomization, respectively noted *CBS w/o r* and *CBS w/ r*. We can note that the results without randomization *CBS w/o r* are consistent with results presented in the paper [29]. The reference value used to compute the relative distance *rd* is the value of CBS achieved with randomization (*CBS w/ r*), as we work without any prior information on the topology of the graph. Table 2a gives a summary of the results for the graphs from the Harwell-Boeing collection and Table 2b for the dataset of random graphs. The two first columns give the name of the graph and the number of graphs in the collection, while the third column provides the value of *mean rd* averaged over all the graphs.

Name	*# graphs*	*mean rd*
bcspwr	6	-0.76
dwt	9	-0.74
can	12	-0.55

(a) Harwell-Boeing collection

p	*# graphs*	*mean rd*
0.1	10	0.23
0.3	10	0.10
0.5	10	0.08
0.7	10	0.05
0.9	10	0.02

(b) Random graphs

Table 2: Summary of the performance of the heuristic MaCH on the graphs from the Harwell-Boeing collection and on random graphs. The first two columns indicate the name of the collection and the number of graphs inside. The fourth column gives the average relative distance averaged. Detailed results are given in Tables 12 and 13.

The results of the heuristic MaCH on the graphs from the Harwell-Boeing are better than those obtained using the heuristic GVNS. These graphs are indeed highly structured, and the heuristic MaCH is specially designed to traverse this type of graphs. Conversely, the results on random graphs are on average less positive compared with those obtained using the heuristic MaCH, especially when the density decreases. By definition, random graphs do not exhibit well-structured topology: the heuristic MaCH has then no support, explaining its reduced performance.

5.4. Robustness of the heuristic MaCH

The influence of the stochasticity on the results achieved using the heuristic MaCH is studied, by determining how better the results are when the heuristic MaCH is repeated several times. In this experiment, the median value *median CBS* is compared to the minimal value obtained over all the repetitions, set as the reference value. Tables 3a and 3b show the results of the tests to assess the robustness of the heuristic MaCH. For each table, the first two columns give the name and the number of graphs in the collection, while the next three columns give the relative distance averaged over all graphs for the different values of k.

The results show that in both data sets, the heuristic MaCH is robust since the increase of the number of repetitions does not highly improve the obtained results: for the graph from the

Collection		Relative distance		
Name	*# graphs*	$k = 10$	$k = 20$	$k = 50$
bcspwr	5	0.09	0.07	0.06
dwt	6	0.10	0.08	0.05
can	9	0.10	0.08	0.06

(a) Harwell-Boeing collection

Collection		Relative distance		
p	*# graphs*	$k = 10$	$k = 20$	$k = 50$
0.1	10	0.00	0.00	0.00
0.3	10	0.00	0.00	0.00
0.5	10	0.00	0.00	0.00
0.7	10	0.00	0.00	0.00
0.9	10	0.00	0.00	0.00

(b) Random graphs

Table 3: Summary of the study of the robustness of the heuristic MaCH on the graphs from the Harwell-Boeing collection and on random graphs. The first two columns indicate the name of the collection and the number of graphs in the collection. The next three columns report the relative distance between the median value of the minimal value of CBS over k repetitions and the minimal value obtained over all repetitions, for three values of k. Detailed results are given in Tables 14 and 15.

Harwell-Boeing collection, the advantage is slight, while there is no improvement in the case of random graphs. These two examples show that even if the heuristic MaCH has a stochastic behavior, it has only a slight influence on the results.

5.5. Execution time

The speed of the heuristic MaCH is assessed by looking at the average execution time for different instances of graphs. Table 4 shows the average execution time in seconds of the heuristic MaCH for different values of n on different types of graphs.

$\|V\|$	Path	Cycle	Wheel	PGC 2	PGC 10	CBG 1	CBG 3	CBG 7
8	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01
64	< 0.01	< 0.01	0.01	0.02	0.05	0.07	0.46	0.17
128	0.02	0.02	0.02	0.03	0.1	0.26	2.02	1.03
192	0.01	0.01	0.03	0.04	0.14	0.75	10.2	3.35
256	0.03	0.04	0.04	0.03	0.18	1.58	24.7	7.86
320	0.01	0.03	0.08	0.05	0.24	2.82	32.7	15.3
384	0.03	0.04	0.12	0.07	0.3	4.49	41.9	26.2
448	0.05	0.05	0.13	0.05	0.32	6.61	52.5	42.9

Table 4: Averaged execution time in seconds of the heuristic MaCH on several types of graphs for different values of n.

When the topology of the graph is simple, as for instance for the paths, the cycles or the wheels, the algorithm goes quite fast, running in less than one second. The computational cost of the algorithm hugely increases when the graph is the complete bipartite graph, which can be explained by the peculiar structure of these graphs: Indeed, the algorithm will first of all compute a first path containing all the vertices of the smaller subset and the same number of vertices in the other one. All the remaining vertices will be considered as a path of length 1 (since they are isolated when the smaller subset is removed), and the algorithm will spend a huge amount of time to merge one by one all these vertices with the first path. This very greedy step makes explode the computational cost when n increases for these graphs.

Detailed results in Appendix B display the average execution time for one execution of the heuristic MaCH for the other collections. These results show that the heuristic MaCH is well-adapted for real-world graphs from hundreds to thousands of vertices. The computational cost prevents however the use of the heuristic MaCH when the graph has millions of vertices. It is nevertheless, in view of our computational results, the fastest solution to minimize the cyclic bandwidth sum on graphs.

6. Applications on complex networks

We focused in this section on the assessment of the performances of the heuristic MaCH on complex networks. Contrary to the previous section, the experiments are guided by the motivation to discover the structure of the network. We then propose to use the value of CBS only to assess the robustness of the heuristic, as performed previously. Then a visual validation of the performance of the heuristic is done to check the match between topology and labeling.

6.1. Properties of complex networks

Three well-known properties of real-world complex networks are often encountered: presence of communities, scale-free property and small-worldness.

Graph with communities (COM) One definition of a community is a group of vertices such that the number of edges between the vertices of a community is significantly higher than between vertices belonging to different communities. This property is for instance well-known in social networks, where people tends to belong to groups of people. We considered the following stochastic block model to build a graph with three communities: each vertex is randomly assigned to one of the three communities. For each pair of vertices, an edge exists with a probability p_{intra} if the two vertices belong to the same community, and with a probability p_{inter} otherwise, with $p_{\text{inter}} < p_{\text{intra}}$. In our experiments, we set $p_{\text{inter}} = 0.01$ and $p_{\text{intra}} = 0.9$.

Scale-free network (SF) Scale-free property means that the distribution of degrees follows a power law. This property has been highlighted in many real-world networks, such that social networks. It exists several methods to generate a scale-free network, among them we used the Barabási-Albert model [1]: from an initial connected graph, vertices are sequentially added, and attached to one existing vertex, chosen according to a probability which depends on the degree of the vertex. The higher the degree is, the higher the probability. The new vertices tend then to connect to vertices with high degree.

Small-world networks (SW) The small-worldness is a property of networks whose the average shortest path length is small with regard of the number of vertices. The generative model used is the Watts-Strogatz model [32]: starting from a regular ring lattice of degree k, an edge between two unlinked vertices is drawn with a probability p. We choose $k = 4$ and $p = 0.1$.

6.2. Robustness of the heuristic

As previously performed in Section 5.4, the robustness of the heuristic is tested on complex networks with 100 vertices. Tab. 5 shows the results of the tests. For each table, the first two columns give the name and the number of graphs in the collection, while the next three columns give the relative distance averaged over all graphs for the different values of k.

Collection		**Relative distance**		
Name	*# graphs*	$k = 10$	$k = 20$	$k = 50$
SF	10	0.02	0.02	0.01
SW	10	0.00	0.00	0.00
COM	10	0.00	0.00	0.00

Table 5: Summary of the study of the robustness of the heuristic MaCH on complex networks. The first two columns indicate the name of the collection and the number of graphs in the collection. The next three columns report the relative distance between the median value of the minimal value of CBS over k repetitions and the minimal value obtained over all repetitions, for three values of k. Detailed results are given in Tab. 16.

The results confirm the good robustness of the heuristic: for complex networks with three different properties, the value of CBS slightly improved for the scale-free network when the number of repetitions increases, and is at a standstill for the other networks.

6.3. Visualization of networks before and after labeling

A visual validation of the consistence between labeling and topology is succinctly performed in this section. For one instance of each type of networks, the graph is displayed using a layout consistent with the topology. The label of each vertex is displayed using a shade of gray defined such that two close colors denotes a short distance within the meaning of Eq.4. Figs. 5, 6 and 7 show for the three type of networks two representations of the graphs, before and after relabeling.

These illustrations show that the three types of networks highlight peculiar structures: the scale-free network has a stringy structure, the small-world has regular structures with random links and the structure in communities present three dense parts. The labels before relabeling do not follow in any sense this structure. The heuristic corrects this by relabeling the vertices in accordance with the topology.

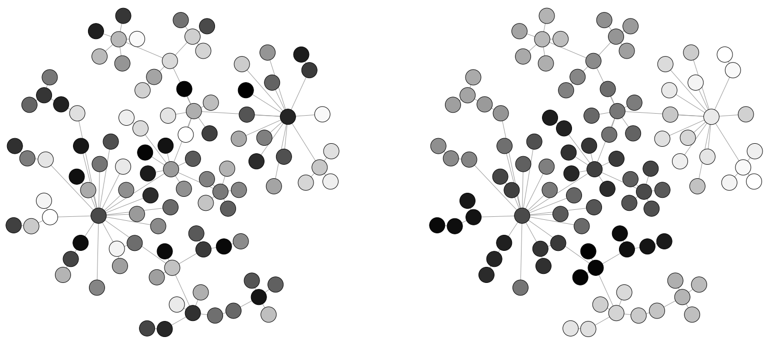


Figure 5: The left and right figures show the vertices of a scale-free network respectively before and after applying the heuristic MaCH to relabel vertices. The color of the vertex depends on the label: two close colors means that the labels are close as well.
CBS value before relabeling: 2734 ; CBS value after relabeling: 375.

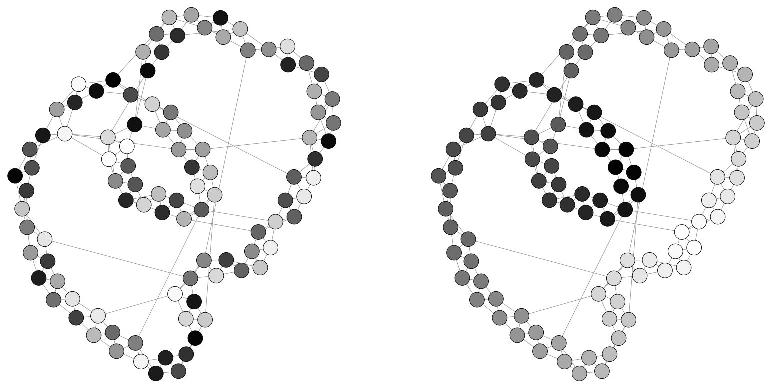


Figure 6: The left and right figures show the vertices of a small-world network respectively before and after applying the heuristic MaCH to relabel vertices. The color of the vertex depends on the label: two close colors means that the labels are close as well.
CBS value before relabeling: 5283 ; CBS value after relabeling: 752.

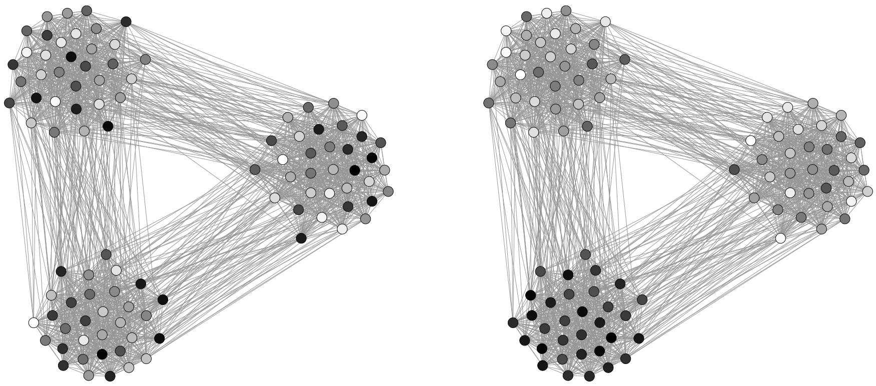


Figure 7: The left and right figures show the vertices of the network with communities respectively before and after applying the heuristic MaCH to relabel vertices. The color of the vertex depends on the label: two close colors means that the labels are close as well.
CBS value before relabeling: 45270 ; CBS value after relabeling: 26589.

7. Extension to weighted graphs

Until now, the analysis was focused on unweighted graphs as up to our knowledge, there is no theoretical study about the cyclic bandwidth sum problem when the graph is considered as weighted. It is hence easier to assess the performance of the heuristic to restrain our study on that category of graphs. It is relevant nevertheless to consider the weight on graphs, as they are very common, especially in real-world graph analysis. The problem of CBSP can be extended to take into account the weight of each edge in the sum of difference of labels. If we note w_{uv} the weight between adjacent vertices u and v, the weighted CBSP is defined by:

$$\min_{\pi} f(\pi) \quad \text{with} \quad f(\pi) = \sum_{\{u,v\}\in E} w_{uv} d_H(\pi(u), \pi(v)) \tag{29}$$

The weighted version of the heuristic is very similar to the one in the unweighted case. Two minor modifications have to be considered: the computation of the Jaccard similarity index in Step 1, as the neighborhood is influenced by the weights, and the incremental computation of CBS in Step 2. The first problem can be addressed by defining a weighted similarity index between two vertices u and v as the following:

$$J_w(u,v) = \frac{N(u,v)}{D(u,v)} \tag{30}$$

where $N(u,v)$ represents the weight of neighbors shared by the two vertices and is defined by:

$$N(u,v) = 2w_{u,v} + \sum_{\substack{x\in V \\ x\sim u \\ x\sim v}} \min(w_{ux}, w_{vx}) \tag{31}$$

and $D(u,v)$ represents the total weight of neighborhood of u and v, and is defined by:

$$D(u,v) = 2w_{u,v} + \sum_{\substack{x \in V \\ x \sim u \\ x \sim v}} \frac{w_{ux} + w_{vw}}{2} + \sum_{\substack{x \in V \\ x \sim u \\ x \nsim v}} w_{ux} + \sum_{\substack{x \in V \\ x \nsim u \\ x \sim v}} w_{vx} \tag{32}$$

We can note that if all weights are set to 1 i.e. the graph is unweighted, we have $N(u,v) = 2 + \sharp\{\text{Common neighbors of } u \text{ and } v\}$ and $D = 2 + \sharp\{\text{All neighbors of } u \text{ and } v\}$ which corresponds to the similarity index defined previously.

Adaptation of the incremental CBS is, for its part, trivial, since it is only necessary to multiply each term we add and remove by the weight of the considered edge.

There is no theoretical study about the weighted cyclic bandwidth sum problem to test the validity of the heuristic in the weighted case. Besides, it could be quite tricky to clearly characterize the structure of a weighted graph. We propose here only to illustrate the good behavior of the algorithm, by applying the heuristic on a weighted real-world complex network. The network is collected from data sets of the SocioPatterns projects [15]: it represents a face-to-face networks between students and teachers in a primary, where the people are the vertices of the graph and the weight between the vertices measures the cumulative duration of contact for one day.

Fig. 8 shows the network after relabeling using the same color code as in the previous section. The layout is given by the data.

The relation between the labeling and the topology is less obvious than previously. It stays nevertheless consistent with the positions of the vertices: the colors in the networks looks homogeneous. Each pile of vertices corresponds to a class in the school, and the vertices inside each class have close colors. Even if guarantees about the good behavior of the heuristic are not available, this example shows that the heuristic produces a reasonable result, in line with the ground truth, when it deals with weighted graphs.

8. Conclusion

The topology of a network is a crucial element in the analysis of processes lying over it. If the structure of networks can be described by models, it remains in many cases delicate to easily obtain an ordering of the vertices. We proposed in this paper a heuristic to discover the topology of the network without any assumptions on its structure. This heuristic has been of great interest to find an approximate solution of a known labeling problem called cyclic bandwidth sum problem, which has been used as a criterion to check the consistence between the topology and the labeling. Many extensions of this algorithm can be considered: we considered briefly the weighted graphs in the end of the paper. One could also deal with the case of directed graph, using the same idea of taking into account the local and global structure of the network.

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Figure 8: Visualization of the face-to-face network in a primary school from the SocioPatterns project [15]. Each vertex represents either a student or a teacher. The color of the vertex depends on the label: two close colors means that the labels are close as well. The position of the vertices is from the data and reflects the topology of the network. The thickness of the edge denotes its weights.

A. Incremental CBS

This appendix presents the results of the changes of the CBS value for the incremental update of CBS introduced in Section 2.2. The proofs of the following models follow the same reasoning as the proof of Theorem 2.1.

Theorem A.1. *Edges between k and the vertices of* O_2
Let $u \in O_2$ *and* $\Delta = \pi_i[u] - \pi_i[k]$*. We have:*

1. *if* $\Delta \leq \frac{n}{2} - p$ *then* $CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u) + p$*.*
2. *if* $\Delta \geq \frac{n}{2}$ *then* $CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u) - p$*.*
3. *if* $\frac{n}{2} - p < \Delta < \frac{n}{2}$ *then* $CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u) - 2\Delta + (n-p)$

Theorem A.2. *Edges between k and the vertices of P*
Let $u \in P$ *and* $\Delta = \pi_i[k] - \pi_i[u]$*. We have:*

1. *if* $(p+1) - \frac{n}{2} \leq \Delta \leq \frac{n}{2}$ *then* $CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u) - 2\Delta + (p+1)$*.*
2. *if* $\Delta > \frac{n}{2}$ *then* $CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u) - n + (p+1)$*.*
3. *if* $\Delta < (p+1) - \frac{n}{2}$ *then* $CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u) + n - (p+1)$

Theorem A.3. *Edges between the vertices of P and the vertices of* O_1
Let $u \in P$*,* $v \in O_1$ *and* $\Delta = \pi_i[u] - \pi_i[v]$*. We have:*

1. *if* $\Delta \leq \frac{n}{2} - 1$ *then* $CBS^{(i+1)}(u,v) = CBS^{(i)}(u,v) + 1$*.*
2. *if* $\Delta \geq \frac{n}{2}$ *then* $CBS^{(i+1)}(u,v) = CBS^{(i)}(u,v) - 1$*.*
3. *if* $\frac{n}{2} - 1 < \Delta < \frac{n}{2}$ *then* $CBS^{(i+1)}(u,v) = CBS^{(i)}(u,v)$

Theorem A.4. *Edges between the vertices of P and the vertices of* O_2
Let $u \in P$*,* $v \in O_2$ *and* $\Delta = \pi_i[v] - \pi_i[u]$*. We have:*

1. *if* $\Delta \leq \frac{n}{2}$ *then* $CBS^{(i+1)}(u,v) = CBS^{(i)}(u,v) - 1$*.*
2. *if* $\Delta \geq \frac{n}{2} + 1$ *then* $CBS^{(i+1)}(u,v) = CBS^{(i)}(u,v) + 1$*.*
3. *if* $\frac{n}{2} < \Delta < \frac{n}{2} + 1$ *then* $CBS^{(i+1)}(u,v) = CBS^{(i)}(u,v)$

B. Detailed results of computational experiments

This section presents the results for the Cartesian products of graphs described in Section 5, for different values of m and n.

B.1. Performance

For the following tables, the first two columns give the name of the graphs used in the Cartesian products, the third column displays the median values of the distribution of the CBS over 30 repetitions, the fourth column the median absolute deviation of the distribution of the CBS value over 30 repetition and the fifth column the theoretical upper bound. Finally the sixth column displays the mean execution time in seconds for one repetition of the heuristic.

G	H	*median CBS*	*mad CBS*	*ub*	*rd*	*time (s)*
P_5	K_5	200	0	395	-0.49	0.01
P_5	K_{10}	1225	0	3165	-0.61	0.01
P_5	K_{15}	3700	0	10560	-0.65	0.03
P_5	K_{20}	8250	0	25080	-0.67	0.09
P_{10}	K_5	425	0	1545	-0.72	0.01
P_{10}	K_{10}	2550	0	12590	-0.80	0.05
P_{10}	K_{15}	7625	0	42135	-0.82	0.08
P_{10}	K_{20}	16900	0	100180	-0.83	0.15
P_{15}	K_5	650	0	3445	-0.81	0.01
P_{15}	K_{10}	3875	0	28265	-0.86	0.05
P_{15}	K_{15}	11550	0	94710	-0.88	0.13
P_{15}	K_{20}	25550	0	225280	-0.89	0.27
P_{20}	K_5	875	0	6095	-0.86	0.03
P_{20}	K_{10}	5200	0	50190	-0.90	0.08
P_{20}	K_{15}	15475	0	168285	-0.91	0.13
P_{20}	K_{20}	34200	0	400380	-0.91	0.31

Table 6: Results for the Cartesian products of a path and a complete graph

G	H	*median CBS*	*mad CBS*	*ub*	*rd*	*time (s)*
C_5	K_5	225	0	415	-0.46	< 0.01
C_5	K_{10}	1325	0	3205	-0.59	0.04
C_5	K_{15}	3925	0	10620	-0.63	0.05
C_5	K_{20}	8650	0	25160	-0.66	0.07
C_{10}	K_5	450	0	1590	-0.72	0.02
C_{10}	K_{10}	2650	0	12680	-0.79	0.02
C_{10}	K_{15}	7850	0	42270	-0.81	0.07
C_{10}	K_{20}	17300	0	100360	-0.83	0.15
C_{15}	K_5	675	0	3515	-0.81	0.01
C_{15}	K_{10}	3975	0	28405	-0.86	0.07
C_{15}	K_{15}	11775	0	94920	-0.88	0.16
C_{15}	K_{20}	25950	0	225560	-0.88	0.25
C_{20}	K_5	900	0	6190	-0.85	0.03
C_{20}	K_{10}	5300	0	50380	-0.89	0.1
C_{20}	K_{15}	15700	0	168570	-0.91	0.19
C_{20}	K_{20}	34600	0	400760	-0.91	0.24

Table 7: Results for the Cartesian products of a cycle and a complete graph

G	H	*median CBS*	*mad CBS*	*ub*	*rd*	*time (s)*
K_5	K_5	547	20	474	0.15	0.01
K_{10}	K_5	2305	2	3324	-0.31	0.02
K_{10}	K_{10}	21349	466	14149	0.51	0.2
K_{15}	K_5	6130	7	10800	-0.43	0.06
K_{15}	K_{10}	33347	18	44475	-0.25	0.17
K_{15}	K_{15}	169974	1766	102900	0.65	1.07
K_{20}	K_5	12571	9	25399	-0.51	0.1
K_{20}	K_{10}	62644	31	103299	-0.39	0.23
K_{20}	K_{15}	187710	13	236199	-0.21	0.47
K_{20}	K_{20}	740458	3582	426599	0.74	3.46

Table 8: Results for the Cartesian products of complete graphs

G	H	*median CBS*	*mad CBS*	*ub*	*rd*	*time (s)*
P_5	C_5	158	7	145	0.09	0.01
P_5	C_{10}	481	28	290	0.66	0.05
P_5	C_{15}	920	50	435	1.11	0.06
P_5	C_{20}	1508	108	580	1.60	0.1
P_{10}	C_5	448	47	545	-0.18	0.02
P_{10}	C_{10}	1472	96	1090	0.35	0.12
P_{10}	C_{15}	3031	218	1635	0.85	0.33
P_{10}	C_{20}	4493	315	2180	1.06	0.65
P_{15}	C_5	739	112	1195	-0.38	0.09
P_{15}	C_{10}	2680	108	2390	0.12	0.35
P_{15}	C_{15}	5464	338	3585	0.52	0.94
P_{15}	C_{20}	8839	410	4780	0.85	1.95
P_{20}	C_5	1000	94	2095	-0.52	0.09
P_{20}	C_{10}	3751	333	4190	-0.10	0.48
P_{20}	C_{15}	8105	519	6285	0.29	1.28
P_{20}	C_{20}	13902	816	8380	0.66	3.74

Table 9: Results for the Cartesian products of a path and a cycle

G	H	*median CBS*	*mad CBS*	*ub*	*rd*	*time (s)*
P_5	P_5	142	13	120	0.18	< 0.01
P_{10}	P_5	426	47	265	0.61	0.03
P_{10}	P_{10}	1397	85	990	0.41	0.08
P_{15}	P_5	759	78	410	0.85	0.05
P_{15}	P_{10}	2671	305	1535	0.74	0.31
P_{15}	P_{15}	5199	621	3360	0.55	0.6
P_{20}	P_5	1164	77	555	1.10	0.12
P_{20}	P_{10}	3882	320	2080	0.87	0.41
P_{20}	P_{15}	7734	791	4555	0.70	1.49
P_{20}	P_{20}	11858	1670	7980	0.49	3.42

Table 10: Results for the Cartesian products of paths

G	*H*	*median CBS*	*mad CBS*	*ub*	*rd*	*time (s)*
C_5	C_5	194	7	165	0.18	0.01
C_{10}	C_5	488	52	330	0.48	0.04
C_{10}	C_{10}	1821	68	1180	0.54	0.18
C_{15}	C_5	853	139	495	0.72	0.11
C_{15}	C_{10}	3376	180	1770	0.91	0.42
C_{15}	C_{15}	6610	357	3795	0.74	0.79
C_{20}	C_5	1086	172	660	0.65	0.15
C_{20}	C_{10}	4606	545	2360	0.95	0.84
C_{20}	C_{15}	9737	728	5060	0.92	2.06
C_{20}	C_{20}	16637	705	8760	0.90	3.65

Table 11: Results for the Cartesian products of cycles

For the following tables, the first three columns give the name of the graph if the graph is from the Harwell-Boeing collection, of the value of p is the graph is a random graph, and the number of vertices ($\#V$) and the number of edges ($\#E$) of the graph. The next three columns give the results for the heuristic MaCH, with the median value of CBS value over 30 repetitions (*median CBS*), the median absolute deviation (*mad CBS*) and the execution time for one execution in seconds. (*time*). The next three column five the results for the heuristic GVNS, with the CBS value obtained without randomizing initial vertex ordering (*CBS w/o r*), the CBS value obtained with randomizing initial vertex ordering (*CBS w/ r*) and the execution time in seconds (*time*). Finally the last column gives the relative distance between *median CBS* and *CBS w/ r*.

Graph			**MaCH**			**GVNS**			
Name	$\#V$	$\#E$	*median CBS*	*mad CBS*	*time (s)*	*CBS w/o r*	*CBS w/ r*	*time (s)*	*rd*
bcspwr01	39	46	106	3	0.01	212	241	104.0	-0.56
bcspwr02	49	59	164	3	0.02	350	388	114.0	-0.58
bcspwr03	118	179	850	64	0.23	2035	3585	270.0	-0.76
bcspwr04	274	669	6280	322	1.5	34135	36627	1060.0	-0.83
bcspwr05	443	590	6032	440	3.78	41194	51486	1590.0	-0.88
bcspwr06	1454	1923	38434	3621	89.0	84704	622141	8620.0	-0.94
dwt59	59	104	322	29	0.03	545	923	189.0	-0.65
dwt72	72	75	204	10	0.08	180	744	127.0	-0.73
dwt87	87	227	1118	86	0.11	2785	3339	392.0	-0.67
dwt162	162	510	3235	669	0.35	5779	15628	349.0	-0.79
dwt193	193	1650	30811	1322	0.77	46150	66408	1590.0	-0.54
dwt221	221	704	6633	642	0.7	15414	30433	972.0	-0.78
dwt419	419	1572	24779	1231	3.8	84803	141523	3510.0	-0.82
dwt592	592	2256	43185	2119	9.47	72302	292871	3700.0	-0.85
dwt992	992	7876	286660	14260	34.1	800566	1827354	23800.0	-0.84
can24	24	68	207	11	0.02	232	229	76.7	-0.10
can61	61	248	1553	0	0.02	2385	2556	291.0	-0.39
can62	62	78	247	15	0.03	389	713	125.0	-0.65
can73	73	152	1003	43	0.12	1413	1838	155.0	-0.45
can96	96	336	2512	145	0.12	3884	5535	237.0	-0.55
can144	144	576	11204	94	0.06	12055	15342	490.0	-0.27
can187	187	652	5363	775	0.57	14112	23658	1070.0	-0.77
can229	229	774	11272	1191	0.83	18362	35230	972.0	-0.68
can268	268	1407	36046	2867	1.03	44034	77680	1510.0	-0.54
can715	715	2975	87773	4651	12.7	149144	473748	4830.0	-0.81
can838	838	4586	373361	16932	15.4	547732	880386	6560.0	-0.58
can1054	1054	5571	335935	30665	33.2	562451	1359525	12800.0	-0.75

Table 12: Results for the graph from the Harwell-Boeing collection.

Graph			MaCH			GVNS			
p	#V	#E	*median CBS*	*mad CBS*	*time (s)*	*CBS w/o r*	*CBS w/ r*	*time (s)*	*rd*
0.1	0	455	11236	0	0.32	9315	9355	413.0	0.20
0.1	1	524	13169	0	0.29	10600	10883	470.0	0.21
0.1	2	497	12682	0	0.31	10379	10423	376.0	0.22
0.1	3	501	13013	0	0.26	10194	10225	383.0	0.27
0.1	4	533	14011	0	0.3	10901	11191	408.0	0.25
0.1	5	488	12295	0	0.28	10105	10317	401.0	0.19
0.1	6	510	12772	0	0.28	10783	10303	520.0	0.24
0.1	7	487	12439	0	0.21	10081	10094	578.0	0.23
0.1	8	465	11654	0	0.14	9529	9401	578.0	0.24
0.1	9	517	13036	0	0.32	10703	10866	660.0	0.20
0.3	0	1500	37587	0	0.52	33331	34209	2110.0	0.10
0.3	1	1449	35784	0	0.24	32354	33373	2420.0	0.07
0.3	2	1494	38841	0	0.36	34174	34139	3120.0	0.14
0.3	3	1537	39000	0	0.16	35006	35281	3080.0	0.11
0.3	4	1541	39524	0	0.4	35246	35424	2660.0	0.12
0.3	5	1475	36670	0	0.35	33872	33975	2450.0	0.08
0.3	6	1463	36893	0	0.24	33488	32930	2390.0	0.12
0.3	7	1511	38280	0	0.25	34399	34551	2720.0	0.11
0.3	8	1482	38131	0	0.32	33813	33962	2770.0	0.12
0.3	9	1506	37388	0	0.41	34492	34374	3000.0	0.09
0.5	0	2406	60937	0	0.4	56308	55603	3880.0	0.10
0.5	1	2482	62238	0	0.32	57704	58516	6050.0	0.06
0.5	2	2446	61217	0	0.26	57293	56651	5290.0	0.08
0.5	3	2437	61659	0	0.42	57678	56448	5650.0	0.09
0.5	4	2471	62437	0	0.4	58010	57770	5200.0	0.08
0.5	5	2503	62970	0	0.7	58613	58754	4480.0	0.07
0.5	6	2500	63012	0	0.47	58612	58699	4800.0	0.07
0.5	7	2453	61796	0	0.58	57868	57506	2820.0	0.07
0.5	8	2468	61736	0	0.45	58727	58030	3590.0	0.06
0.5	9	2425	60957	0	0.42	57344	56866	3730.0	0.07
0.7	0	3468	87649	0	0.66	83380	84035	8580.0	0.04
0.7	1	3470	88077	0	0.46	82910	83215	9900.0	0.06
0.7	2	3432	86837	0	0.61	82959	82992	8300.0	0.05
0.7	3	3437	87006	0	0.66	83165	82268	7250.0	0.06
0.7	4	3450	87801	0	0.46	83799	82603	4770.0	0.06
0.7	5	3436	87372	0	0.48	83288	82933	5360.0	0.05
0.7	6	3473	88352	0	0.65	84383	83628	6700.0	0.06
0.7	7	3483	88322	0	0.46	84468	84277	8700.0	0.05
0.7	8	3443	87339	0	0.78	83285	81972	6640.0	0.07
0.7	9	3518	88014	0	0.64	84904	84211	6210.0	0.05
0.9	0	4428	111117	0	0.67	108761	109564	13200.0	0.01
0.9	1	4480	113297	0	0.67	110700	110148	12500.0	0.03
0.9	2	4459	113026	0	0.87	110413	110317	9540.0	0.02
0.9	3	4468	112714	0	0.92	110782	109953	6480.0	0.03
0.9	4	4469	113236	0	0.68	109824	109934	8890.0	0.03
0.9	5	4460	112747	0	0.67	110390	109940	11100.0	0.03
0.9	6	4433	111787	0	0.68	108919	108971	8780.0	0.03
0.9	7	4471	112802	0	0.88	110596	110817	8070.0	0.02
0.9	8	4457	112511	0	0.5	109174	109762	6560.0	0.03
0.9	9	4451	112342	0	0.66	110106	109668	5300.0	0.02

Table 13: Results for the random graphs.

B.2. Robustness

For the following tables, the first column gives the name of the graph if the graph is from the Harwell-Boeing collection, of the value of p is the graph is a random graph. The next two columns give the results for the heuristic MaCH, with the median value of CBS value over 30 repetitions (*median CBS*) and the median absolute deviation (*mad CBS*) when k repetitions are performed. The next four columns give the results for respectively $k = 20$ and $k = 30$. Finally the last column gives the minimum value of CBS achieved for all repetitions.

Graph	**k = 10**		**k = 20**		**k = 50**		
Name	*median CBS*	*mad CBS*	*median CBS*	*mad CBS*	*median CBS*	*mad CBS*	*min CBS*
bcspwr01	102	1	101	1	100	1	99
bcspwr02	157	1	156	0	155	0	154
bcspwr03	784	17	767	10	756	6	722
bcspwr04	5159	112	5083	200	4918	121	4543
bcspwr05	5290	147	5166	105	5049	106	4469
dwt59	280	4	277	4	274	3	261
dwt72	184	6	183	5	177	4	169
dwt87	1011	12	1000	8	995	5	979
dwt162	2309	127	2228	75	2155	68	2025
dwt193	28335	568	27340	478	27364	523	25693
dwt221	5078	179	5087	147	4773	92	4475
can24	192	2	192	2	190	0	190
can61	1553	0	1553	0	1553	0	1553
can62	226	5	220	5	215	4	203
can73	949	17	940	14	914	16	888
can96	2055	86	2051	91	1929	32	1832
can144	11110	0	11110	0	11110	0	11110
can187	4073	84	4031	135	3920	117	3544
can229	8923	361	8530	260	8435	298	7574
can268	30414	962	29929	969	28289	654	24826

Table 14: Results for the graph from the Harwell-Boeing collection

Graph	**k = 10**		**k = 20**		**k = 50**		
p	*median CBS*	*mad CBS*	*median CBS*	*mad CBS*	*median CBS*	*mad CBS*	*min CBS*
0.9	11236	0	11236	0	11236	0	11236
0.9	13169	0	13169	0	13169	0	13169
0.9	12682	0	12682	0	12682	0	12682
0.9	13013	0	13013	0	13013	0	13013
0.9	14011	0	14011	0	14011	0	14011
0.9	12295	0	12295	0	12295	0	12295
0.9	12772	0	12772	0	12772	0	12772
0.9	12439	0	12439	0	12439	0	12439
0.9	11654	0	11654	0	11654	0	11654
0.9	13036	0	13036	0	13036	0	13036
0.9	37587	0	37587	0	37587	0	37587
0.9	35784	0	35784	0	35784	0	35784
0.9	38841	0	38841	0	38841	0	38841
0.9	39000	0	39000	0	39000	0	39000
0.9	39524	0	39524	0	39524	0	39524
0.9	36670	0	36670	0	36670	0	36670
0.9	36893	0	36893	0	36893	0	36893
0.9	38280	0	38280	0	38280	0	38280
0.9	38131	0	38131	0	38131	0	38131
0.9	37388	0	37388	0	37388	0	37388
0.9	60937	0	60937	0	60937	0	60937
0.9	62238	0	62238	0	62238	0	62238
0.9	61217	0	61217	0	61217	0	61217
0.9	61659	0	61659	0	61659	0	61659
0.9	62437	0	62437	0	62437	0	62437
0.9	62970	0	62970	0	62970	0	62970
0.9	63012	0	63012	0	63012	0	63012
0.9	61796	0	61796	0	61796	0	61796
0.9	61736	0	61736	0	61736	0	61736
0.9	60957	0	60957	0	60957	0	60957
0.9	87649	0	87649	0	87649	0	87649
0.9	88077	0	88077	0	88077	0	88077
0.9	86837	0	86837	0	86837	0	86837
0.9	87006	0	87006	0	87006	0	87006
0.9	87801	0	87801	0	87801	0	87801
0.9	87372	0	87372	0	87372	0	87372
0.9	88352	0	88352	0	88352	0	88352
0.9	88322	0	88322	0	88322	0	88322
0.9	87339	0	87339	0	87339	0	87339
0.9	88014	0	88014	0	88014	0	88014
0.9	111117	0	111117	0	111117	0	111117
0.9	113297	0	113297	0	113297	0	113297
0.9	113026	0	113026	0	113026	0	113026
0.9	112714	0	112714	0	112714	0	112714
0.9	113236	0	113236	0	113236	0	113236
0.9	112747	0	112747	0	112747	0	112747
0.9	111787	0	111787	0	111787	0	111787
0.9	112802	0	112802	0	112802	0	112802
0.9	112511	0	112511	0	112511	0	112511
0.9	112342	0	112342	0	112342	0	112342

Table 15: Results for the random graphs.

Graph		k = 10		k = 20		k = 50		
Name	*Repetition*	*median CBS*	*mad CBS*	*median CBS*	*mad CBS*	*median CBS*	*mad CBS*	*min CBS*
SF	0	373	7	373	7	373	7	358
SF	1	374	2	374	2	374	2	366
SF	2	381	3	381	3	381	3	366
SF	3	315	3	315	3	315	3	305
SF	4	391	1	391	1	391	1	388
SF	5	306	3	306	3	306	3	296
SF	6	351	1	351	1	351	1	345
SF	7	379	1	379	1	379	1	374
SF	8	401	2	401	2	401	2	394
SF	9	354	3	354	3	354	3	347
SW	0	752	0	752	0	752	0	752
SW	1	937	0	937	0	937	0	937
SW	2	662	0	662	0	662	0	662
SW	3	977	0	977	0	977	0	977
SW	4	571	0	571	0	571	0	571
SW	5	741	0	741	0	741	0	741
SW	6	742	0	742	0	742	0	742
SW	7	891	0	891	0	891	0	891
SW	8	563	0	563	0	563	0	563
SW	9	1035	0	1035	0	1035	0	1035
COM	0	26338	20	26338	20	26338	20	26292
COM	1	27352	37	27352	37	27352	37	27315
COM	2	27880	10	27880	10	27880	10	27831
COM	3	31573	0	31573	0	31573	0	31573
COM	4	31599	49	31599	49	31599	49	31515
COM	5	27067	49	27067	49	27067	49	26822
COM	6	29546	22	29546	22	29546	22	29400
COM	7	28102	136	28102	136	28102	136	27890
COM	8	27571	22	27571	22	27571	22	27549
COM	9	26943	0	26943	0	26943	0	26943

Table 16: Results for the complex networks.

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