Discrete and Continuous Bodies with Affine Structure (*).

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A DARIO GRAFFI nel suo 70° compleanno

Summary. – *The equations of motion of an affine body are derived (also for the case of variable mass) evidencing in particular the rôle of the generalized moment of momentum of internal forces. Successively, local dynamic and thermodynamic equations of balance for a continuous body with an affine microstructure are proposed, also for the case when such a body is one of the constituents in a mixture.*

1. – Introduction.

In this paper our aim is to motivate the theory of structured media which we have used elsewhere [1] by showing that that theory covers the important special case of bodies which are only capable of homogeneous microdeformations [2], *i.e.*, as we prefer to say, of bodies endowed with an *affine structure.*

It has long been recognized [2] [3] that on the basis of this model it is possible to develop a mechanical theory which is completely equivalent to the theory of an oriented medium in the sense of ERICKSEN and TRUESDELL [4], *i.e.*, a medium at each point of which there are three linearly independent deformable vectors (the directors). A feature of our present theory is that the mass of the unit cell is not necessarily conserved; as a consequence the equation of evolution of microinertia is no longer a kinematical identity; it involves, rather, the assumed mechanism of mass transfer between neighbouring cells. In addition, the global balance equations for momentum, moment of momentum and energy are affected.

We accept a general tensorial form of the balance of moment of momentum; while the skew part of this equation reduces to ordinary statements of balance of moment of momentum for structured continua, the symmetric part requires interpretation. Generalizing an idea put forward by TOUPIN for a hyperelastic continum with directors [3], we show that this equation is the evolution equation for microinertia. We remark also that in some relevant special theories of structured media, such as Ericksen's theory of anisotropic fluids [5] and the directors theory of shells as rendered *e.g.* by NAGHDI [6], equations of balance for the so-called « director momentum » appear which can be shown to correspond exactly to this equation.

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In a companion paper we will show how significant special cases (*e.g.*, Cosserat, Grioli, Mindlin and Tiersten's theories) are obtained from our theory by introducing suitable kinematical constraints on the affine structure [15].

I. - Mechanics of Affine Systems of Mass-Points.

2. – Kinematics.

Let I be an index set. We call a system $\{(P^{(i)}, m^{(i)}) | m^{(i)} > 0, i \in \mathrm{I}\}$ of mass-points *affine* if its motion has the form

$$y^{(i)} = x(t) + G(t)(Y^{(i)} - X) ; \tag{2.1}$$

in (2.1), $x(\cdot)$ and $G(\cdot)$ are defined on a (perhaps infinite) time interval $\mathcal{T}$, and $G(\cdot)$ has values which are invertible tensors so that

$$Y^{(i)} = X + G^{-1}(y^{(i)} - x) . \tag{2.2}$$

We denote an affine system of mass-points by S, and introduce the tensor W defined by

$$W = \dot{G} G^{-1} \ (^1) . \tag{2.3}$$

Using (2.2) and (2.3) we deduce from (2.1) the following identities:

$$\dot{y}^{(i)} = \dot{x} + W(y^{(i)} - x) = (\dot{x} - Wx) + Wy^{(i)} ; \tag{2.4}$$

$$\ddot{y}^{(i)} = \ddot{x} + (\dot{W} + W^2)(y^{(i)} - x) = (\ddot{x} - (\dot{W} + W^2)x) + (\dot{W} + W^2)y^{(i)} . \tag{2.5}$$

We also assume that

$$\sum m^{(i)}(Y^{(i)} - X) = 0 \ (^2) , \tag{2.6}$$

which means that we identify X (x) with the position vector with respect to a fixed origin of the center of mass of S in the reference (current) placement.

Let

$$m = \sum m^{(i)}$$

(1) We mark derivation with respect to time by a dot. In [1] we called « wryness tensor » the transpose of the tensor W defined here. Because this name is already in use for a different object, we suggest here for W the designation « wrench ». Affine motions are considered by ERINGEN [7] in the context of his theory of micromorphic continua.

(2) Here and henceforth we drop the summation limits.

be the *mass* of S, and let

$$m\boldsymbol{I} = \sum m^{(i)}(\boldsymbol{y}^{(i)} - \boldsymbol{x}) \otimes (\boldsymbol{y}^{(i)} - \boldsymbol{x})$$

be the Euler *inertia tensor* of S with respect to the center of mass. We then have for the *momentum* $\boldsymbol{q}$ of S per unit mass

$$m\boldsymbol{q} = \sum m^{(i)} \dot{\boldsymbol{y}}^{(i)},$$

and for the *moment of momentum* $\boldsymbol{s}$ of S per unit mass with respect to the center of mass

$$m\boldsymbol{s} = \sum (\boldsymbol{y}^{(i)} - \boldsymbol{x}) \times m^{(i)} \dot{\boldsymbol{y}}^{(i)} = \sum m^{(i)}(\boldsymbol{y}^{(i)} - \boldsymbol{x}) \times \boldsymbol{W}(\boldsymbol{y}^{(i)} - \boldsymbol{x}).$$

We introduce also a *generalized moment of momentum*, *i.e.* the tensor $\boldsymbol{S}$ defined as follows:

$$m\boldsymbol{S} = \sum m^{(i)}(\boldsymbol{y}^{(i)} - \boldsymbol{x}) \otimes \boldsymbol{W}(\boldsymbol{y}^{(i)} - \boldsymbol{x}),$$

so that $\boldsymbol{s}$ is the axial vector associated with the skew-symmetric part of $\boldsymbol{S}$.

The above definitions have the following easy consequences, which will be relevant for our developments:

$$\boldsymbol{q} = \dot{\boldsymbol{x}}, \qquad \boldsymbol{S} = \boldsymbol{I}\boldsymbol{W}^T, \qquad \dot{\boldsymbol{I}} = 2\,\mathrm{sym}\,\boldsymbol{S}\ (^3). \tag{2.7}$$

We also record here for future use a formula which delivers the generalized moment of momentum with respect to the origin:

$$\boldsymbol{S}_0 = \boldsymbol{x} \otimes \dot{\boldsymbol{x}} + \boldsymbol{S}. \tag{2.8}$$

3. – Dynamics.

Given a *system of forces* $\mathbb{F} = \{(P^{(i)}, \boldsymbol{f}^{(i)}) | i \in \mathbb{I}\}$ for S, we consider the dynamical problem of determining the motion of S under the action of F, *i.e.* the problem of determining the pair $(\boldsymbol{x}(\cdot), \boldsymbol{G}(\cdot))$.

We interpret $\boldsymbol{f}^{(i)}$ as the *total* force acting on the mass-point $(P^{(i)}, m^{(i)})$. By taking the zeroth and the first moment with respect to the origin of the set of Newtonian

(3) We write sym $\boldsymbol{A}$ (skw $\boldsymbol{A}$) for the symmetric (skew-symmetric) part of the second-order tensor $\boldsymbol{A}$. We also write tr for the trace functional, and dev $\boldsymbol{A}$ for the deviatoric part of $\boldsymbol{A}$: $\mathrm{dev}\,\boldsymbol{A} = \boldsymbol{A} - (\frac{1}{3}\,\mathrm{tr}\,\boldsymbol{A})\,\mathbf{1}$.

equations

$$m^{(i)}\ddot{\boldsymbol{y}}^{(i)} = \boldsymbol{f}^{(i)}, \qquad i \in \mathbb{I}, \tag{3.1}$$

that rules the motion of $\mathbb{S}$, we deduce two ([4]) global consequences:

A first one, Euler's first law, involving

$$\boldsymbol{r} = \sum \boldsymbol{f}^{(i)},$$

the *resultant force* of $\mathbb{F}$:

$$\sum m^{(i)}\ddot{\boldsymbol{y}}^{(i)} = \sum \boldsymbol{f}^{(i)} \Leftrightarrow m\dot{\boldsymbol{q}} = \boldsymbol{r}\,. \tag{3.2}$$

A second one, involving

$$\boldsymbol{M}_0 = \sum \boldsymbol{y}^{(i)} \otimes \boldsymbol{f}^{(i)} = \sum (\boldsymbol{x} + (\boldsymbol{y}^{(i)} - \boldsymbol{x})) \otimes \boldsymbol{f}^{(i)} = \boldsymbol{x} \otimes \boldsymbol{r} + \boldsymbol{M}, \tag{3.3}$$

the *generalized resultant torque* of $\mathbb{F}$ about the origin:

$$\sum \boldsymbol{y}^{(i)} \otimes m^{(i)}\ddot{\boldsymbol{y}}^{(i)} = \sum \boldsymbol{y}^{(i)} \otimes \boldsymbol{f}^{(i)} \Leftrightarrow m(\dot{\boldsymbol{S}}_0 - \boldsymbol{W}\boldsymbol{I}\boldsymbol{W}^T - \dot{\boldsymbol{x}} \otimes \dot{\boldsymbol{x}}) = \boldsymbol{M}_0\,. \tag{3.4}$$

In view of $(2.7)_2$ and (2.8), we have

$$\dot{\boldsymbol{S}}_0 - \boldsymbol{W}\boldsymbol{I}\boldsymbol{W}^T - \dot{\boldsymbol{x}} \otimes \dot{\boldsymbol{x}} = \boldsymbol{x} \otimes \ddot{\boldsymbol{x}} + \dot{\boldsymbol{S}} - \boldsymbol{W}\boldsymbol{S},$$

so that, taking into account also $(3.3)_3$, equation $(3.4)_2$ can be given the form

$$m(\dot{\boldsymbol{S}} - \boldsymbol{W}\boldsymbol{S}) = \boldsymbol{M}\,. \tag{3.5}$$

As to this last equation, some comments are in order. We put

$$\boldsymbol{M} = \overset{e}{\boldsymbol{M}} + \overset{m}{\boldsymbol{M}},$$

([4]) Consideration of higher order moments of momenta and forces is out of the scope of the present paper, as we delimited it in the Introduction. Had we aimed at theories of more general structured continua than those suffering only affine microdeformations, then consideration of higher moments would have been in order. It would be easy to construct a theory in which the motion of $\mathbb{S}$ is described by a countable set of tensors $\{\boldsymbol{G}_{(n)}\}$ of increasing order, and is governed by a system of differential equations in which successive moments of $\mathbb{F}$ appear. Such a theory would then serve as a model for theories of continuous bodies endowed with a microstructure of a certain grade, *e.g.* the theory sketched by TRUESDELL - TOUPIN in Sections 166, 205 and 232 of [8], or Green-Rivlin's multipolar mechanics (*vid.* [9], and a sequence of related papers). There is also a methodological reason for refraining from this easy greater generality (such as is pursued in some papers by RIVLIN [10], [11]), namely that a model should be simpler than the theory it is devised to enlighten, even at the expense of giving up some subtle correspondences.

where $\overset{e}{\boldsymbol{M}}$ ($\overset{m}{\boldsymbol{M}}$) is the generalized resultant torque of *external* (*mutual*) forces about the center of mass, and we assume

$$\operatorname{skw}\overset{m}{\boldsymbol{M}} = \boldsymbol{0}\ (^5)\,. \tag{3.6}$$

Then, the skew-symmetric part of (3.5) reduces to Euler's second law

$$m\dot{\boldsymbol{s}} = \boldsymbol{m}\,, \tag{3.7}$$

prescribing the balance of moment of momentum with respect to the center of mass (in (3.7) $\boldsymbol{m}$ is the vector associated with $\operatorname{skw}\overset{e}{\boldsymbol{M}}$).

The symmetric part of (3.5) reads

$$m(\operatorname{sym}\dot{\boldsymbol{S}} - \boldsymbol{W}\boldsymbol{S}) = \operatorname{sym}\overset{e}{\boldsymbol{M}} + \overset{m}{\boldsymbol{M}}\,, \tag{3.8}$$

or rather, in view of the kinematical identity $(2.7)_3$,

$$m(\ddot{\boldsymbol{I}} - 2\boldsymbol{W}\boldsymbol{I}\boldsymbol{W}^T) = 2(\operatorname{sym}\overset{e}{\boldsymbol{M}} + \overset{m}{\boldsymbol{M}})\,. \tag{3.9}$$

Thus, (3.9) can be interpreted as the *evolution equation for the Euler tensor* of S (6).

Equations (3.2), (3.7) and (3.9) rule the motion of S. We now derive an equivalent form of these equations which is perhaps more convenient for certain applications. Let

$$m\boldsymbol{J} = \sum m^{(i)}(\boldsymbol{Y}^{(i)} - \boldsymbol{X}) \otimes (\boldsymbol{Y}^{(i)} - \boldsymbol{X})\,,$$

so that, by (2.1) and the definition of $\boldsymbol{I}$,

$$\boldsymbol{I} = \boldsymbol{G}\boldsymbol{J}\boldsymbol{G}^T\,. \tag{3.10}$$

Once m, $\boldsymbol{J}$ are assigned (with $m > 0$, $\boldsymbol{J}$ a symmetric, positive semidefinite tensor), and $\boldsymbol{r}$, $\overset{e}{\boldsymbol{M}}$, $\overset{m}{\boldsymbol{M}}$ (with $\overset{m}{\boldsymbol{M}}$ a symmetric tensor) are given as functions of $(\boldsymbol{x}, \boldsymbol{G}; \dot{\boldsymbol{x}}, \dot{\boldsymbol{G}}; t)$, a *motion* of S is a solution $(\boldsymbol{x}(\cdot), \boldsymbol{G}(\cdot))$ (with $\boldsymbol{G}$ an invertible tensor) of the differential system

$$\begin{cases} m\ddot{\boldsymbol{x}} = \boldsymbol{r}\,, \\ m\operatorname{skw}(\boldsymbol{G}\boldsymbol{J}\ddot{\boldsymbol{G}}^T) = \operatorname{skw}\overset{e}{\boldsymbol{M}}\,, \\ m\operatorname{sym}(\boldsymbol{G}\boldsymbol{J}\ddot{\boldsymbol{G}}^T) = \operatorname{sym}\overset{e}{\boldsymbol{M}} + \overset{m}{\boldsymbol{M}}\,, \end{cases} \tag{3.11}$$

corresponding to a given set of initial values for $\boldsymbol{x}$, $\boldsymbol{G}$, $\dot{\boldsymbol{x}}$ and $\dot{\boldsymbol{G}}$.

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(5) The statement (3.6) could be derived from (and anyway is suggested by) an appropriate interpretation of Newton's Lex Tertia and the following two Corollaria.

(6) Cf. TOUPIN [3], who put forward this interpretation in the context of his theory of polar hyperelastic material. *Vid.* also [8], Sect. 219.

The kinetic energy theorem for $\mathbb{S}$ can be obtained either directly from equations (3.1) or from the global equations $(3.2)_2$ and (3.5). Let T denote the *kinetic energy* of $\mathbb{S}$ (per unit mass), so that

$$2mT = \sum m^{(i)}\dot{\boldsymbol{y}}^{(i)}\cdot\dot{\boldsymbol{y}}^{(i)},$$

and

$$2T \quad = \dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} + \boldsymbol{I}\boldsymbol{W}^T\cdot\boldsymbol{W}^T. \tag{3.12}$$

We have the identities

$$(\boldsymbol{I}\boldsymbol{W}^T\cdot\boldsymbol{W}^T)^{\cdot} = 2\boldsymbol{I}(\dot{\boldsymbol{W}} + \boldsymbol{W}^2)^T\cdot\boldsymbol{W}^T,$$

$$\boldsymbol{I}(\dot{\boldsymbol{W}} + \boldsymbol{W}^2)^T = \dot{\boldsymbol{S}} - \boldsymbol{W}\boldsymbol{S}.$$

Thus we conclude that

$$m\dot{T} = \boldsymbol{r}\cdot\dot{\boldsymbol{x}} + \boldsymbol{M}\cdot\boldsymbol{W}^T. \tag{3.13}$$

4. – Affine systems with variable mass.

We tacitly assumed in the former sections that the number of mass points of $\mathbb{S}$ were constant in time. With certain applications in mind for our model (7), we now proceed to derive the governing equations for those affine systems which may lose, or gain, mass-points as time progresses. As the imbalance of mass is essentially abrupt in character for these systems, we are naturally driven to use the framework of classical impulsive mechanics (8).

Let ψ denote any tensorial field defined on the current placement of $\mathbb{S}$, so that in particular $\psi^{(i)} = \psi(\boldsymbol{y}^{(i)}, t)$. Let further

$$\{\psi\} = \psi(t+0) - \psi(t-0) = \psi_+ - \psi_- \tag{4.1}$$

denote a jump discontinuity (at the instant t) of ψ as a function of time. If $\{\psi\} \neq 0$ we say that ψ suffers an *impulsive change* at t. We state the general *impulsive balance law* for a field Ψ defined by

$$m\Psi = \sum m^{(i)}\psi^{(i)} \tag{4.2}$$

in the following way:

$$\{m\Psi\} = \hat{s}_\Psi, \tag{4.3}$$

where $\hat{s}_\Psi$ is the *impulsive supply* of Ψ. Taking $\psi^{(i)} = 1$, and using (4.2) and (4.3),

(7) We typically think of mixtures of constituents endowed with an affine microstructure.
(8) A fairly more inclusive treatment of this and related matters will be presented elsewhere.

yields the impulsive balance of mass for S:

$$\{m\} = \hat{s}_m . \tag{4.4}$$

In view of the identity

$$\{m\Psi\} = m_-\{\Psi\} + \{m\}\Psi_+ , \tag{4.5}$$

equations (4.3) and (4.4) combine as to give

$$m_-\{\Psi\} = \hat{s}_\Psi - \hat{s}_m \Psi_+ , \tag{4.6}$$

a statement of the impulsive balance law which is sufficiently general to serve as a basis for the impulsive mechanics of any system of mass-points with variable total mass, be it affine or not.

Specialization of (4.6) yields the following impulsive counterparts of the balance of momentum, inertia, and moment of momentum:

$$m_-\{\boldsymbol{q}\} = \hat{\boldsymbol{s}}_{\boldsymbol{q}} - \hat{s}_m \boldsymbol{q}_+ \quad \left(\text{for } \psi^{(i)} = \dot{\boldsymbol{y}}^{(i)};\ \text{cf. } (3.2)_2\right); \tag{4.7}$$

$$m_-\{\boldsymbol{I}\} = \hat{\boldsymbol{s}}_{\boldsymbol{I}} - \hat{s}_m \boldsymbol{I}_+ \quad \left(\text{for } \psi^{(i)} = (\boldsymbol{y}^{(i)} - \boldsymbol{x}) \otimes (\boldsymbol{y}^{(i)} - \boldsymbol{x});\ \text{cf. } (2.7)_3\right); \tag{4.8}$$

$$m_-\{\boldsymbol{S}\} = \hat{\boldsymbol{s}}_S - \hat{s}_m \boldsymbol{S}_+ \quad \left(\text{for } \psi^{(i)} = (\boldsymbol{y}^{(i)} - \boldsymbol{x}) \otimes \boldsymbol{W}(\boldsymbol{y}^{(i)} - \boldsymbol{x});\ \text{cf. } (3.5)\right). \tag{4.9}$$

Equation (4.8) shows that the rate of change of inertia is now ruled by a true balance equation in lieu of the kinematical identity $(2.7)_3$. Moreover, the fact that $\boldsymbol{S}$ is affine emerges when one takes care of $(2.7)_2$, and combines (4.8) and (4.9) using also (4.5). It turns out that

$$\hat{\boldsymbol{s}}_S = \hat{\boldsymbol{s}}_I \boldsymbol{W}_+^T + (m\boldsymbol{I})_-\{\boldsymbol{W}^T\} , \tag{4.10}$$

a kinematical identity again which is peculiar of affine systems.

Appendix.

It seems to us that the theory of affine systems of mass-points is a subject of mechanical interest in itself, apart from the rôle of stimulus and support of intuition that we ascribed to it within the limits of the present paper. To substantiate this statement, we collect hereafter (under *A*) some examples of affine systems with invariable mass, as well as (under *B*) further material about affine systems with variable mass which completes the rather formal developments presented in Section 4.

A) *Some examples.*

(i) *Rigidity*. In the important special case when S is rigid, *i.e.* $\boldsymbol{G}$ is orthogonal and $\boldsymbol{W}$ is skew-symmetric, equations $(3.11)_{1,2}$ alone determine the *motion* of S, whereas equation $(3.11)_3$ assigns $\overset{m}{\boldsymbol{M}}$, a global property of the *reaction to the internal constraint* of rigidity.

We observe further that the last term in (3.13) can be written in general

$$\boldsymbol{M}\cdot\boldsymbol{W}^T=\boldsymbol{m}\cdot\boldsymbol{\omega}+(\operatorname{sym}\overset{e}{\boldsymbol{M}}+\overset{m}{\boldsymbol{M}})\cdot\boldsymbol{D}\,, \tag{A.1}$$

where $\boldsymbol{\omega}$ is the vector associated with $\boldsymbol{\Omega}=2\operatorname{skw}\boldsymbol{W}^T$, and $\boldsymbol{D}=\operatorname{sym}\boldsymbol{W}$. In the present instance, $\boldsymbol{\Omega}=-2\boldsymbol{W}$, and $\boldsymbol{D}=\boldsymbol{0}$, so that (A.1) reduces to

$$\boldsymbol{M}\cdot\boldsymbol{W}^T=\boldsymbol{m}\cdot\boldsymbol{\omega}\,. \tag{A.2}$$

(ii) *Pure strain.* In the instance, in a sense complementary to (i), when S is only capable of pure strain, *i.e.* $\boldsymbol{G}$ is a symmetric, invertible tensor, equations $(3.11)_{1,3}$ determine the motion of S, whereas equation $(3.11)_2$ can be viewed either as a *condition of compatibility for the data* or as assigning a global property of the *reaction to the external constraint* that prevents S from rotating.

More particularly, let

$$\boldsymbol{G}=\gamma\mathbf{1}\,,\qquad \gamma(t)>0\qquad \text{for } t\in\mathcal{T}\,,$$

so that, by (3.10),

$$\boldsymbol{I}=\gamma^2\mathbf{1}\,.$$

Then, equations $(3.11)_{2,3}$ reduce to

$$\begin{cases}\boldsymbol{0}=\operatorname{skw}\overset{e}{\boldsymbol{M}}\,,\\ m\gamma\ddot{\gamma}\boldsymbol{J}=\boldsymbol{M}\,.\end{cases} \tag{A.3}$$

If we now take the trace of both sides of equation $(A.3)_2$, and put

$$\mathfrak{m}=\mathfrak{m}(\boldsymbol{x},\gamma;\dot{\boldsymbol{x}},\dot{\gamma};t)=\frac{\operatorname{tr}\boldsymbol{M}}{\operatorname{tr}\boldsymbol{J}}\,,$$

we obtain the *evolution equation* for γ in the form

$$m\gamma\ddot{\gamma}=\mathfrak{m}\,. \tag{A.4}$$

(iii) *Spherical inertia tensor.* There is another interesting case in which $\boldsymbol{G}$ is a scalar multiple of an orthogonal tensor, a case that occurs when one imposes the kinematical constraint

$$\boldsymbol{I}=\iota^2\mathbf{1}\,,\qquad \iota(t)\neq 0\qquad \text{for } t\in\mathcal{T}\,, \tag{A.5}$$

and chooses the reference configuration so that $\boldsymbol{J} = \mathbf{1}$. Under these circumstances it follows from (3.10) that

$$\boldsymbol{G} = \iota \boldsymbol{R}\,, \qquad \text{with } \boldsymbol{R} = \text{an orthogonal tensor}\,.$$

Moreover, by (2.3) we have

$$\boldsymbol{W} = \iota^{-1}\dot{\iota}\mathbf{1} + \dot{\boldsymbol{R}}\boldsymbol{R}^T\,,$$

with

$$\boldsymbol{D} = \iota^{-1}\dot{\iota}\mathbf{1}\,, \qquad \boldsymbol{\Omega} = 2\boldsymbol{R}\dot{\boldsymbol{R}}^T\,;$$

by $(2.7)_2$,

$$\boldsymbol{S} = \iota\dot{\iota}\mathbf{1} + \tfrac{1}{2}\iota^2\boldsymbol{\Omega}\,.$$

Accordingly, equation (3.5) reduces to

$$\text{(A.6)} \qquad m\big(\iota\ddot{\iota}\mathbf{1} + (\tfrac{1}{2}\iota^2\boldsymbol{\Omega})^{\cdot} + \tfrac{1}{4}\iota^2\boldsymbol{\Omega}^2\big) = \boldsymbol{M}\,.$$

Clearly, equation (A.6) has consequences which correspond to equations $(3.11)_{2,3}$. These are, respectively:

$$\text{(A.7)} \qquad m(\tfrac{1}{2}\iota^2\boldsymbol{\omega})^{\cdot} = \boldsymbol{m}\,,$$

which is obtained by taking the skew part of both sides of (A.6); and

$$\text{(A.8)} \qquad \begin{cases} m(\iota\ddot{\iota} - \tfrac{1}{6}\iota^2\omega^2) = \mathfrak{m}\,, \\ \operatorname{dev}(\tfrac{1}{4}\iota^2\boldsymbol{\omega}\otimes\boldsymbol{\omega} - \overset{e}{\boldsymbol{M}}) = \operatorname{dev}\overset{m}{\boldsymbol{M}}\,, \end{cases}$$

which are obtained by taking in turn the trace and the deviator of the symmetric part of (A.6). Equations (A.7) and $(A.8)_1$ form the *system of evolution equations* for $\boldsymbol{\omega}$ and ι (9), whereas a global evaluation of the reaction to the internal constraint (A.5) ensues from equation $(A.8)_2$.

B) Addendum to Section 4.

With slight abuse of the notation introduced in (4.1), we denote by

$$\{\mathbb{I}\} = (\mathbb{I}_+ \cup \mathbb{I}_-) - (\mathbb{I}_+ \cap \mathbb{I}_-)$$

(9) We note in passing that (non-linear) *oscillations* of S become possible when *e.g.*: (i) external forces vanish; (ii) the resultant torque of mutual forces has negative trace, *i.e.* $\mathfrak{m} < 0$ (a physically reasonable hypothesis); (iii) $|\iota^{-1}\mathfrak{m}|$ is an increasing function of ι.

the impulsive change at t of the index set $\mathbb{I}$ of $\mathbb{S}$ (10). If further we denote by χ the characteristic function of $\mathbb{I}_+ \cup \mathbb{I}_-$, and we put $\chi^{(i)} = \chi(i)$, then clearly

$$\{\chi^{(i)}\} = \begin{cases} 0 & \text{if } i \in \mathbb{I}_+ \cap \mathbb{I}_- ; \\ +1 & \text{if } i \in \mathbb{I}_+ - \mathbb{I}_- ; \\ -1 & \text{if } i \in \mathbb{I}_- - \mathbb{I}_+ ; \end{cases}$$

We then prescribe the behaviour of mass-points in an impulse in the usual way:

(i) individual mass-points conserve their masses and places, *i.e.*

$$\{m^{(i)}\} = 0 \quad \text{and} \quad \{\boldsymbol{y}^{(i)}\} = \boldsymbol{0}, \qquad \forall i \in \mathbb{I}_+ \cup \mathbb{I}_- ;$$

(ii) individual mass-points leave (join) $\mathbb{S}$ according as they lose (gain) instantly the affine velocity, *i.e.* the velocity specified by (2.4), at time $t-0$ $(t+0)$ (11).

It follows from the first part of assumption (i) above that

$$\{m^{(i)}\psi^{(i)}\} = \sum_{\mathbb{I}_- \cap \mathbb{I}_+} m^{(i)}\{\psi^{(i)}\} + \sum_{\mathbb{I}_+ - \mathbb{I}_-} m^{(i)}\psi_+^{(i)} - \sum_{\mathbb{I}_- - \mathbb{I}_+} m^{(i)}\psi_-^{(i)}, \tag{B.1}$$

a formula which clarifies the meaning of the left member of (4.3). Moreover, combining (4.2), (4.5) and (B.1), we have

$$m_-\{\Psi\} + \{m\}\Psi_+ = \sum_{\mathbb{I}_+ \cap \mathbb{I}_-} m^{(i)}\{\psi^{(i)}\} + \sum_{\mathbb{I}_+ - \mathbb{I}_-} m^{(i)}\psi_+^{(i)} - \sum_{\mathbb{I}_- - \mathbb{I}_+} m^{(i)}\psi_-^{(i)}. \tag{B.2}$$

A straightforward consequence of identity (B.2) is obtained by taking $\psi^{(i)} = 1$:

$$\{m\} = \sum_{\{\mathbb{I}\}} m^{(i)}\{\chi^{(i)}\}. \tag{B.3}$$

A less trivial application of (B.2), which is arrived at by taking $\psi^{(i)} = \boldsymbol{y}^{(i)}$ (and, consequently, $\Psi = \boldsymbol{x}$), and using also the second part of assumption (i), is the following statement of the fundamental theorem on the center of mass for systems of mass-points with variable total mass

$$m_-\{\boldsymbol{x}\} + \{m\}\boldsymbol{x}_+ = \sum_{\{\mathbb{I}\}} m^{(i)}\{\chi^{(i)}\}\boldsymbol{y}^{(i)}. \tag{B.4}$$

(10) It should be noticed that $\{\mathbb{I}\}$ is precisely the symmetric difference of the sets $\mathbb{I}_+$ and $\mathbb{I}_-$:

$$\{\mathbb{I}\} = (\mathbb{I}_+ - \mathbb{I}_-) \cup (\mathbb{I}_- - \mathbb{I}_+).$$

(11) These hypotheses imply that $\mathbb{I}_+ - \mathbb{I}_-$ is comprised of mass-points occupying places specified by (2.1), although their velocities do not conform to (2.4), at $t-0$ (whereas $\mathbb{I}_- - \mathbb{I}_+$ is any subset of $\mathbb{I}_-$).

Although the affine structure is unimportant to get the former result, it does lend some special features to the behaviour of S in an impulse. Let $I_+ \cap I_-$ include at least four mass-points placed at the vertices of a non-degenerate tetrahedron. It follows from (2.1) and the second part of assumption (i) that

$$\mathbf{0} = \{\boldsymbol{x} - \boldsymbol{G}\boldsymbol{X}\} + \{\boldsymbol{G}\}\boldsymbol{Y}^{(i)},$$

so that

$$\{\boldsymbol{G}\}(\boldsymbol{Y}^{(i)} - \boldsymbol{Y}^{(j)}) = \mathbf{0}$$

for any pair (i, j) of those mass-points, and hence

$$\{\boldsymbol{G}\} = \mathbf{0} \quad \text{and} \quad \{\boldsymbol{x}\} = \boldsymbol{G}\{\boldsymbol{X}\}. \tag{B.5}$$

Clearly, $(B.5)_1$ implies that

$$\{\boldsymbol{W}\} = \{\dot{\boldsymbol{G}}\}\boldsymbol{G}^{-1}. \tag{B.6}$$

Let further the manner of impulsive imbalance of the total mass of S be *affine*: *i.e.* such that

$$\{\boldsymbol{X}\} = \mathbf{0}, \tag{B.7}$$

or equivalently, due to $(B.5)_2$,

$$\{\boldsymbol{x}\} = \mathbf{0}. \tag{B.8}$$

(B.8) implies the coalescence of the centers of mass, not only of I_+ and I_- but also of any set resulting *via* union, intersection or difference from these two sets. Another consequence of (B.8) and (B.3) is the following reduction of (B.4):

$$\sum_{\{I\}} m^{(i)}\{\chi^{(i)}\}(\boldsymbol{y}^{(i)} - \boldsymbol{x}) = \mathbf{0}. \tag{B.9}$$

Finally, in order to show how to identify the right member of (4.3), let us consider the case $\Psi = \boldsymbol{q}$. We write the impulsive supply of $\boldsymbol{q}$ as follows:

$$\hat{\boldsymbol{s}}_q = \hat{\overset{e}{\boldsymbol{r}}} + \hat{\overset{m}{\boldsymbol{r}}}, \tag{B.10}$$

where

$$\hat{\overset{e}{\boldsymbol{r}}} = \sum_{I_+ \cap I_-} \hat{\overset{e}{\boldsymbol{f}}}{}^{(i)} \tag{B.11}$$

is interpreted as the resultant impulsive force acting on S independently of the imbalance of total mass, whereas $\hat{\overset{m}{\boldsymbol{r}}}$ is interpreted as the resultant impulsive force acting on S as a consequence of the abrupt imbalance of mutual forces going along with the imbalance of total mass.

Because on the one hand

$$\{m\boldsymbol{q}\} = \sum_{I_+} m^{(i)}\dot{\boldsymbol{y}}^{(i)}_+ - \sum_{I_-} m^{(i)}\dot{\boldsymbol{y}}^{(i)}_- = \sum_{I_+\cap I_-} m^{(i)}\{\dot{\boldsymbol{y}}^{(i)}\} + \sum_{I_+-I_-} m^{(i)}\dot{\boldsymbol{y}}^{(i)} - \sum_{I_--I_+} m^{(i)}\dot{\boldsymbol{y}}^{(i)}_-,$$

and on the other hand

$$\sum_{I_+\cap I_-} m^{(i)}\{\dot{\boldsymbol{y}}^{(i)}\} = \sum_{I_+\cap I_-} \hat{\boldsymbol{f}}^{(i)},$$

one has

$$\{m\boldsymbol{q}\} = \sum_{I_+\cap I_-} \hat{\boldsymbol{f}}^{(i)} + \sum_{I_+-I_-} m^{(i)}\dot{\boldsymbol{y}}^{(i)}_+ - \sum_{I_--I_+} m^{(i)}\dot{\boldsymbol{y}}^{(i)}_-.$$

But

$$\sum_{I_+\cap I_-} \hat{\boldsymbol{f}}^{(i)} = \hat{\boldsymbol{r}}^{e} - \sum_{\{I\}} m^{(i)}\{\dot{\boldsymbol{y}}^{(i)}\}$$

so that finally

$$\{m\boldsymbol{q}\} = \hat{\boldsymbol{r}}^{e} - \sum_{I_--I_+} m^{(i)}\dot{\boldsymbol{y}}^{(i)} + \sum_{I_+-I_-} m^{(i)}\dot{\boldsymbol{y}}^{(i)}$$

and

(B.12) $$\hat{\boldsymbol{r}}^{m} = \sum_{I_+-I_-} m^{(i)}\dot{\boldsymbol{y}}^{(i)}_- - \sum_{I_--I_+} m^{(i)}\dot{\boldsymbol{y}}^{(i)}_+ .$$

II. - Continuous Bodies with Affine Structure.

The developments of Chapter I suggest ideas for a definition of a continuous body with affine structure.

We consider first here the case of a *single body* of this type, and lay down the governing balance equations, which, on accepting a well known « metaphysical » principle of TRUESDELL (cf. *e.g.* [12], Lect. 5), remain formally unaltered also for a *mixture* of such bodies. We further list the peculiar balance equations for a generic *constituent* of a reacting mixture, when that constituent is endowed with the affine structure.

5. – Balance equations for a single body.

We begin by sketching the definition of a continuous body endowed with affine structure along lines first made precise by NOLL ([12]).

([12]) As far as possible we follow NOLL [13], [14] for terminology and notations; we refer the reader to Noll's papers, where the notions used here in a hasty way are most precisely introduced and described.

Let $\mathcal{B}$ model a set of material particles X, which is equipped with a class $\mathbb{P}$ of mappings $\varkappa$, called *placements*, from $\mathcal{B}$ into open subsets $B_\varkappa$ of the three-dimensional Euclidean point space $\mathcal{E}$. Let further $\mathbb{P}$ obey a number of axioms sufficient to induce on $\mathcal{B}$ a structure of differentiable manifold, so that a tangent space $\mathfrak{T}_X$ is attached to each particle $X \in \mathcal{B}$. In addition to $\mathbb{P}$, we equip $\mathcal{B}$ with a class $\mathbb{G}$ of mappings Γ, such that, at any $X \in \mathcal{B}$, $\Gamma_X = \Gamma(X)$ is an invertible linear mapping from $\mathfrak{T}_X$ into $\mathcal{V}$, the translation space of $\mathcal{E}$. We say that such a continuous body $\mathcal{B}$ has *affine structure* determined by $\mathbb{G}$, and call the pair $(\varkappa, \Gamma)$ a *complete placement* of $\mathcal{B}$. We single out a particular placement $(\varkappa^*, \Gamma^*)$, and we use also Γ_X^* to induce an inner product on $\mathfrak{T}_X$: then, any other placement can be uniquely specified in terms of two fields

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}) ; \qquad \boldsymbol{G} = \boldsymbol{G}(\boldsymbol{X}) ,$$

where

$$\boldsymbol{X} \in B_{\varkappa^*} = \{\boldsymbol{X} | \boldsymbol{X} = \varkappa^*(X),\ X \in \mathcal{B}\} ;$$
$$\boldsymbol{x} = \varkappa(\varkappa^{*-1}(\boldsymbol{X})) ; \qquad \boldsymbol{G} = \Gamma_X(\Gamma_X^{*-1})) .$$

Actually, we will assume that $\boldsymbol{x}$ and $\boldsymbol{G}$ be given for all t of a time interval $\mathcal{T}$:

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t) ; \qquad \boldsymbol{G} = \boldsymbol{G}(\boldsymbol{X}, t) , \qquad \text{for } (\boldsymbol{X}, t) \in B_{\varkappa^*} \times \mathcal{T} ,$$

and that appropriate regularity conditions prevail. Accordingly, the kinematical state of $\mathcal{B}$ is determined by a *velocity* field

$$\dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}(\boldsymbol{X}, t) ,$$

and a *wrench* field

$$\boldsymbol{W} = \dot{\boldsymbol{G}}(\boldsymbol{X}, t)\boldsymbol{G}^{-1}(\boldsymbol{X}, t) . \tag{5.1}$$

We now pass on to establish the balance equations which rule the thermomechanics of our affine body; in these equations, the definitions of the kinematical entities and the form of their supplies are directly inspired by the corresponding discrete objects in Sections 2 and 3 ([13]).

We assume that volume densities exist for mass (ϱ), inertia ($\varrho \boldsymbol{I}$), momentum ($\varrho \dot{\boldsymbol{x}}$), and generalized moment of momentum ($\varrho \boldsymbol{S}_0$) with respect to $\boldsymbol{0}$. Moreover, we take

([13]) The developments of this Chapter are based on a straightforward adaptation to a continuum of the results of Ch. I. It is possible to pursue different courses leading to models of different physical systems; for instance, one can define generalized moment of momentum in such a way that $\boldsymbol{S}$ vanishes when $\boldsymbol{W} = \operatorname{grad} \dot{\boldsymbol{x}}$, etc.

for the supply of momentum:

$$\boldsymbol{r} = \int_B \varrho \boldsymbol{b}\, dB + \int_{\partial B} \boldsymbol{T}\boldsymbol{n}\, ds \text{ [14]}, \tag{5.2}$$

where $\boldsymbol{b}$ is the density of body force, $\boldsymbol{T}$ is the Cauchy stress tensor and $\boldsymbol{n}$ is the unit outward normal to ∂B. To write the correct expression of supply of generalized moment of momentum we must keep in mind the developments of Sect. 3; it appears from the discrete model that such supply does not derive from external agencies alone, but that also internal actions contribute to it (though these do not contribute to the classical moment). Thus we are led to assume that there exists a density $\boldsymbol{Z}$ of generalized moment of momentum of internal forces (which is symmetric for consistency with condition (3.6)) and to accept the following expression of $\boldsymbol{M}_0$:

$$\boldsymbol{M}_0 = \int_B \varrho(\boldsymbol{x} \otimes \boldsymbol{b} + \boldsymbol{L})\, dB + \int_{\partial B} (\boldsymbol{x} \otimes \boldsymbol{T} + \mathrm{H})\boldsymbol{n}\, dS + \int_B \boldsymbol{Z}\, dB\,; \qquad \operatorname{skw} \boldsymbol{Z} = \boldsymbol{0}\,, \tag{5.3}$$

where $\varrho\boldsymbol{L}$ is the density of external body couples, and H is a third-order hyperstress tensor. Then, we write down the following *balance laws*:

(5.4) (mass) $$\Big(\int_B \varrho\, dB\Big)^{\bullet} = 0\,;$$

(5.5) (inertia) $$\Big(\int_B \varrho \boldsymbol{I}\, dB\Big)^{\bullet} = 2\int_B \varrho \operatorname{sym}(\boldsymbol{I}\boldsymbol{W}^T)\, dB \qquad \text{(cf. eq. } (2.7)_{2,3});$$

(5.6) (momentum) $$\Big(\int_B \varrho \dot{\boldsymbol{x}}\, dB\Big)^{\bullet} = \boldsymbol{r}\,;$$

(5.7) (generalized moment of momentum) $$\Big(\int_B \varrho \boldsymbol{S}_0\, dB\Big)^{\bullet} - \int_B \varrho(\dot{\boldsymbol{x}} \otimes \dot{\boldsymbol{x}} + \boldsymbol{W}\boldsymbol{I}\boldsymbol{W}^T)\, dB = \boldsymbol{M}_0 \quad \text{(cf. eq. } (3.4)_2)\,.$$

Under usual assumptions of smoothness, equations (5.4), (5.5) and (5.6) have local consequences. These are:

$$\begin{aligned} \dot{\varrho} &= -\varrho \operatorname{div} \dot{\boldsymbol{x}}\,, \\ \dot{\boldsymbol{I}} &= 2\operatorname{sym}(\boldsymbol{I}\boldsymbol{W}^T)\,, \\ \varrho\ddot{\boldsymbol{x}} &= \varrho\boldsymbol{b} + \operatorname{div}\boldsymbol{T}\,. \end{aligned} \tag{5.8}$$

If we recall $(2.7)_2$ and (2.8), and use (5.8) to perform some obvious cancellations, equation (5.7) can be given a form corresponding to (3.5), which generalizes Euler's

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[14] From now on we write B in place of $B_{\varkappa^*}$.

second law:

(5.9) $$\int_B \varrho(\dot{\boldsymbol{S}} - \boldsymbol{W}\boldsymbol{S})\,dB = \int_B (\varrho\boldsymbol{L} + \boldsymbol{T}^T)\,dB + \int_{\partial B} \mathrm{H}\boldsymbol{n}\,dS + \int_B \boldsymbol{Z}\,dB\,.$$

The local counterpart of equation (5.9) reads

(5.10) $$\varrho(\dot{\boldsymbol{S}} - \boldsymbol{W}\boldsymbol{S}) = \varrho\boldsymbol{L} + \boldsymbol{T}^T + \operatorname{div}\mathrm{H} + \boldsymbol{Z}\,.$$

We then preliminarly state the balance of energy in the form:

(5.11) $$\left(\int_B \varrho(\varepsilon + \tfrac{1}{2}\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} + \tfrac{1}{2}\boldsymbol{I}\boldsymbol{W}^T\cdot\boldsymbol{W}^T)\,dB\right)^{\bullet} = \int_B \varrho(\boldsymbol{b}\cdot\dot{\boldsymbol{x}} + \boldsymbol{L}\cdot\boldsymbol{W}^T + q)\,dB + \int_{\partial B}(\boldsymbol{T}\boldsymbol{n}\cdot\dot{\boldsymbol{x}} + \mathrm{H}\boldsymbol{n}\cdot\boldsymbol{W}^T + \boldsymbol{h}\cdot\boldsymbol{n})\,dS \qquad (\text{cf. eq. } (3.12))\,,$$

where ε is the density of internal energy, ϱq the density of heat source, and $\boldsymbol{h}$ is the heat flux per unit area. Combining (5.11) with (5.8) and (5.10) and using a number of trivial differential identities, yields the following reduced form of the so-called « first law » of thermodynamics:

(5.12) $$\left(\int_B \varrho\varepsilon\,dB\right)^{\bullet} = \int_B (\boldsymbol{T}\cdot(\operatorname{grad}\dot{\boldsymbol{x}} - \boldsymbol{W}) + \mathrm{H}\cdot\operatorname{grad}\boldsymbol{W}^T - \boldsymbol{Z}\cdot\boldsymbol{W} + \varrho q)\,dB + \int_{\partial B}\boldsymbol{h}\cdot\boldsymbol{n}\,dS\,,$$

which admits of the local version

(5.13) $$\varrho\dot{\varepsilon} = \varrho q + \operatorname{div}\boldsymbol{h} + \boldsymbol{T}\cdot(\operatorname{grad}\dot{\boldsymbol{x}} - \boldsymbol{W}) + \mathrm{H}\cdot\operatorname{grad}\boldsymbol{W}^T - \boldsymbol{Z}\cdot\boldsymbol{W}\,.$$

Finally, we introduce a density $\varrho\gamma$ of entropy production, and define the total entropy production as

(5.14) $$\int_B \varrho\gamma\,dB = \left(\int_B \varrho\eta\,dB\right)^{\bullet} - \int_B \varrho\frac{1}{\vartheta}q\,dB - \int_{\partial B}\frac{1}{\vartheta}\boldsymbol{h}\cdot\boldsymbol{n}\,dS\,,$$

where $\varrho\eta$ is the density of entropy, and ϑ is the temperature; as a statement of the entropy imbalance, we lay down the inequality

(5.15) $$\int_B \varrho\gamma\,dB \geqslant 0\,;$$

denoting by

$$\varrho\psi = \varrho(\varepsilon - \vartheta\eta)$$

the density of free energy, and using (5.13) to eliminate q, we manipulate (5.14) and (5.15) to arrive at the following local reduced form of the « second law » of thermodynamics:

$$-\varrho(\dot{\psi}+\vartheta\dot{\eta})+\frac{1}{\vartheta}\boldsymbol{h}\cdot\operatorname{grad}\vartheta+\boldsymbol{T}\cdot(\operatorname{grad}\dot{\boldsymbol{x}}-\boldsymbol{W})+\mathrm{H}\cdot\operatorname{grad}\boldsymbol{W}^{T}-\boldsymbol{Z}\cdot\boldsymbol{W}\geqslant 0\,. \tag{5.16}$$

REMARK. We recall from the Introduction that a primary scope of the present paper is to motivate the basic fields equations used in [1].

As the comparison of $(5.8)_{1,2,3}$, (5.10), (5.13) and (5.16) with equations (3.4), (3.15), (4.4), (4.11), (5.3) and (6.14), respectively, of [1] is disconcerting, we must declare that we were unable to have corrected in [1] many misprints, or even to have inserted missing lines. We remark also that an attempt was made in [1] to start with such equations of balance for inertia and generalized moment of momentum as to cover at the same time the case explicitly studied here and that envisaged in footnote (13). Therefore our present formulae are to a certain extent more special.

6. – Structured constituent in a mixture.

Perusal of Section 4 suggests immediately the equations of balance for a constituent in a mixture; in fact our equation (4.5) is the discrete impulsive equivalent of equation (3.10) of [1]. Corresponding to equations (4.4) and (4.7), we have, adopting standard notation ([15]).

$$\Big(\int_B \underset{\alpha}{\varrho}\, dB\Big)^{\backprime}=\int_B \underset{\alpha}{\varrho}\,\underset{\alpha}{\hat{c}}\, dB\,, \tag{6.1}$$

$$\int_B \underset{\alpha}{\varrho}\,\underset{\alpha}{\grave{\dot{\boldsymbol{x}}}}\, dB-\int_B \underset{\alpha}{\varrho}\,\underset{\alpha}{\boldsymbol{b}}\, dB-\int_{\partial B} \underset{\alpha}{\boldsymbol{T}}\boldsymbol{n}\, dS=\int_B \varrho(\underset{\alpha}{\hat{\boldsymbol{m}}}-\underset{\alpha}{\hat{c}}\,\underset{\alpha}{\dot{\boldsymbol{x}}})\, dB\,; \tag{6.2}$$

corresponding to equations (4.8), (4.9) we have

$$\int_B \underset{\alpha}{\varrho}(\underset{\alpha}{\grave{\boldsymbol{I}}}-2\operatorname{sym}\underset{\alpha}{\boldsymbol{S}})\, dB=\int_B \varrho(\underset{\alpha}{\hat{\boldsymbol{i}}}-\underset{\alpha}{\hat{c}}\,\underset{\alpha}{\boldsymbol{I}})\, dB\,, \tag{6.3}$$

$$\int_B \underset{\alpha}{\varrho}(\underset{\alpha}{\grave{\boldsymbol{S}}}-\underset{\alpha}{\boldsymbol{W}}\,\underset{\alpha}{\boldsymbol{I}}\,\underset{\alpha}{\boldsymbol{W}}^{T})\, dB-\int_B (\underset{\alpha}{\varrho}\,\underset{\alpha}{\boldsymbol{L}}+\underset{\alpha}{\boldsymbol{T}}^{T})\, dB-\int_{\partial B} \underset{\alpha}{\mathrm{H}}\boldsymbol{n}\, dS-\int_B \underset{\alpha}{\boldsymbol{Z}}\, dB=\int_B \varrho(\underset{\alpha}{\hat{\boldsymbol{s}}}-\underset{\alpha}{\hat{c}}\,\underset{\alpha}{\boldsymbol{S}})\, dB\,. \tag{6.4}$$

([15]) The subscript α indicates the α-th constituent, and a backward prime denotes material differentiation following that constituent.

REMARK. – From equations (6.1), (6.2), (6.3), (6.4) local equations follow which should be compared, with caution (*vid.* the final Remark of Sect. 5), with equations (3.1), (4.1), (3.13), (4.8), respectively, of [1].

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