Discrete Approximations for Multidimensional Singular Integral Operators

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Abstract. For discrete operator generated by singular kernel of Calderon–Zygmund one introduces a finite dimensional approximation which is a cyclic convolution. Using properties of a discrete Fourier transform and a finite discrete Fourier transform we prove a solvability for approximating equation in corresponding discrete space. For comparison discrete and finite discrete solution we obtain an estimate for a speed of convergence for a certain right-hand side of considered equation.

Keywords: Discrete Calderon–Zygmund operator · Discrete fourier transform · Cyclic convolution · Approximation rate

1 Introduction

A basic object of this paper is a multidimensional singular integral

$$v.p.\int\limits_D K(x,x-y)u(y)dy, \quad x\in D,$$

which generates the Calderon–Zygmund operator with variable kernel [1,2], where D is a domain in $\mathbb{R}^m$ unbounded as a rule.

Taking into account forthcoming studies of such a general operator using a local principle here we consider a case of constant coefficients, i.e. when the kernel does not depend on a first variable and the Calderon–Zygmund operator looks as follows

$$(Ku)(x) \equiv v.p.\int\limits_D K(x-y)u(y)dy \equiv \tag{1}$$

$$\lim_{\varepsilon\to 0}\int\limits_{D\cap\{|x-y|>\varepsilon\}} K(x-y)u(y)dy, \ x\in D.$$

We assume here that the kernel $K(x)$ of the integral (1) satisfy the following conditions: (1) $K(x)$ is a homogeneous function of order $-m, K(tx) - t^{-m}K(x), \forall t>0$; (2) $K(x)$ is a differentiable function on a unit sphere S^{m-1}; (3) the function $K(x)$ has zero mean value on the S^{m-1} [1,2].

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1.1 Canonical Domains

We will consider different types of domains D because the theory is essentially depended on this type. So for example cases $D = \mathbb{R}^m$ and $D = \mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x_1, \cdots, x_m, x_m > 0\}$ are essentially distinct. Invertibility conditions for the operator K do not coincide for these cases.

1.2 Infinite Discrete Case–Series

To obtain a good approximation for the integral (1) we will use the following reduction. First instead of the integral (1) we introduce the series

$$\sum_{\tilde{y} \in D \cap h\mathbb{Z}^m} K(\tilde{x} - \tilde{y}) u_d(\tilde{y}) h^m, \tag{2}$$

which generates a discrete operator K_d defined on functions u_d of discrete variable $\tilde{x} \in D \cap h\mathbb{Z}^m$. Since the Calderon–Zygmund kernel has a strong singularity at the origin we mean $K(0) = 0$. Convergence for the series (2) means that the following limit

$$\lim_{N \to +\infty} \sum_{\tilde{y} \in h\mathbb{Z}^m \cap Q_N \cap D} K(\tilde{x} - \tilde{y}) u_d(\tilde{y}) h^m$$

exists, where $Q_N = \{x \in \mathbb{R}^m : \max_{1 \le k \le m} |x_k| < N\}$. It was shown earlier that a norm of the operator $K_d : L_2(h\mathbb{Z}^m \cap D) \to L_2(h\mathbb{Z}^m \cap D)$ does not depend on h [9]. But although the operator is a discrete object it is an infinite one, and to solve equations with a such operator one needs to replace it by a certain finite object.

1.3 Finite Discrete Case–Systems of Linear Algebraic Equations

It is natural to consider a system of linear algebraic equation instead of an infinite system generated by the series (2). This method is called projection method [3]. It proposes following actions. Let $P_N : L_2(D \cap \mathbb{Z}^m) \to L_2(D \cap Q_N \cap \mathbb{Z}^m)$ be a projector on a finite dimensional space. One needs to solve the equation

$$P_N K_d P_N u_d = P_N v_d \tag{3}$$

in the space $L_2(D \cap Q_N \cap \mathbb{Z}^m)$ instead of the infinite system of linear algebraic equations

$$K_d u_d = v_d \tag{4}$$

in the space $L_2(D \cap \mathbb{Z}^m)$.

The following result plays a key role in the theory of projection methods, and it was proved for some special domains D and integrable kernel $K(x)$ [3].

Proposition 1. *If the Eq. (4) is uniquely solvable in the space $L_2(D \cap \mathbb{Z}^m)$ for arbitrary right-hand side $v_d \in L_2(D \cap \mathbb{Z}^m)$ then the Eq. (3) is uniquely solvable in the space $L_2(D \cap Q_N \cap \mathbb{Z}^m)$ for enough large N.*

Here we consider the case $D = \mathbb{R}^m$ and prove this proposition 1 with some novelties because the Eq.(4) is a system of linear algebraic equations and for large N one needs much time to solve it.

Our considerations are based on two steps: continual infinite object (1) $\longrightarrow$ discrete infinite object (2) $\longrightarrow$ discrete finite object (3) with justification and error estimates. Some pieces of this programm were realized in authors' papers [4–9].

2 Discrete Fourier Transform and Symbols

Let us define the **discrete Fourier transform** for functions u_d of a discrete variable $\tilde{x} \in h\mathbb{Z}^m$

$$(F_d u_d)(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}^m} u_d(\tilde{x}) e^{i\tilde{x}\cdot\xi} h^m, \quad \xi \in \hbar\mathbb{T}^m, \hbar = \frac{h^{-1}}{2\pi},$$

where $\mathbb{T}^m$ is m-dimensional cube$[-\pi, \pi]^m$.

Such discrete Fourier transform preserves all basic properties of the classical fourier transform, particularly for a discrete convolution of two discrete functions u_d, v_d

$$(u_d * v_d)(\tilde{x}) \equiv \sum_{\tilde{y} \in h\mathbb{Z}^m} u_d(\tilde{x} - \tilde{y}) v_d(\tilde{y}) h^m$$

we have the well known multiplication property

$$(F_d(u_d * v_d))(\xi) = (F_d u_d)(\xi) \cdot (F_d v_d)(\xi).$$

If we apply this property to the operator K_d we obtain

$$(F_d(K_d u_d))(\xi) = (F_d K_d)(\xi) \cdot (F_d u_d)(\xi).$$

Let us denote $(F_d K_d)(\xi) \equiv \sigma_d(\xi)$ and give the following

Definition 1. *The function $\sigma_d(\xi), \xi \in \hbar\mathbb{T}^m$, is called a periodic symbol of the operator K_d.*

We will assume below that the symbol $\sigma_d(\xi) \in C(\hbar\mathbb{T}^m)$ therefore we have immediately the following

Property 1. The operator K_d is invertible in the space $L_2(h\mathbb{Z}^m)$ iff $\sigma_d(\xi) \neq 0, \forall \xi \in \hbar\mathbb{T}^m$.

Definition 2. *A continuous periodic symbol is called an elliptic symbol if $\sigma_d(\xi) \neq 0, \forall \xi \in \hbar\mathbb{T}^m$.*

So we see that an arbitrary elliptic periodic symbol $\sigma_d(\xi)$ corresponds to an invertible operator K_d in the space $L_2(h\mathbb{Z}^m)$.

Remark 1. It was proved earlier that operators (1) and (2) for cases $D = \mathbb{R}^m, D = \mathbb{R}^m_+$ are invertible or non-invertible in spaces $L_2(\mathbb{R}^m), L_2(\mathbb{R}^m_+)$ and $L_2(h\mathbb{Z}^m), L_2(h\mathbb{Z}^m_+)$ simultaneously [6,8].

3 Periodic Approximation and Cyclic Convolutions

Here we will introduce a special discrete periodic kernel $K_{d,N}(\tilde{x})$ which is defined as follows. We take a restriction of the discrete kernel $K_d(\tilde{x})$ on the set $Q_N \cap \mathbb{Z}^m \equiv Q_N^d$ and periodically continue it to a whole $\mathbb{Z}^m$. Further we consider discrete periodic functions $u_{d,N}$ with discrete cube of periods Q_N^d. We can define a cyclic convolution for a pair of such functions $u_{d,N}, v_{d,N}$ by the formula

$$(u_{d,N} * v_{d,N})(\tilde{x}) = \sum_{\tilde{y} \in Q_N^d} u_{d,N}(\tilde{x} - \tilde{y}) v_{d,N}(\tilde{y}) h^m. \tag{5}$$

(We would like to remind that such convolutions are used in digital signal processing [10]). Further we introduce **finite discrete Fourier transform** by the formula

$$(F_{d,N} u_{d,N})(\tilde{\xi}) = \sum_{\tilde{x} \in Q_N^d} u_{d,N}(\tilde{x}) e^{i\tilde{x}\cdot\tilde{\xi}} h^m, \quad \tilde{\xi} \in R_N^d,$$

where $R_N^d = \hbar\mathbb{T}^m \cap \hbar\mathbb{Z}^m$. Let us note that here $\tilde{\xi}$ is a discrete variable.

According to the formula (5) one can introduce the operator

$$K_{d,N} u_{d,N}(\tilde{x}) = \sum_{\tilde{y} \in Q_N^d} K_{d,N}(\tilde{x} - \tilde{y}) u_{d,N}(\tilde{y}) h^m$$

on periodic discrete functions $u_{d,N}$ and a finite discrete Fourier transform for its kernel

$$\sigma_{d,N}(\tilde{\xi}) = \sum_{\tilde{x} \in Q_N^d} K_{d,N}(\tilde{x}) e^{i\tilde{x}\cdot\tilde{\xi}} h^m, \quad \tilde{\xi} \in R_N^d.$$

Definition 3. *A function $\sigma_{d,N}(\tilde{\xi}), \tilde{\xi} \in R_N^d$, is called s symbol of the operator $K_{d,N}$. This symbol is called an elliptic symbol if $\sigma_{d,N}(\tilde{\xi}) \neq 0, \forall \tilde{\xi} \in R_N^d$.*

Theorem 1. *Let $\sigma_d(\xi)$ be an elliptic symbol. Then for enough large N the symbol $\sigma_{d,N}(\tilde{\xi})$ is elliptic symbol also.*

Proof. The function

$$\sum_{\tilde{x} \in Q_N^d} K_{d,N}(\tilde{x}) e^{i\tilde{x}\cdot\xi} h^m, \quad \xi \in \hbar\mathbb{T}^m,$$

is a segment of the Fourier series

$$\sum_{\tilde{x} \in h\mathbb{Z}^m} K_d(\tilde{x}) e^{i\tilde{x}\cdot\xi} h^m, \quad \xi \in \hbar\mathbb{T}^m,$$

and according our assumptions this is continuous function on $\hbar\mathbb{T}^m$. Therefore values of the partial sum coincide with values of $\sigma_{d,N}$ in points $\tilde{\xi} \in R_N^d$. Besides these partial sums are continuous functions on $\hbar\mathbb{T}^m$. □

As before an elliptic symbol $\sigma_{d,N}(\tilde{\xi})$ corresponds to the invertible operator $K_{d,N}$ in the space $L_2(Q_N^d)$.

4 Approximation Rate

Let $A : B \to B$ be a linear bounded operator acting in a Banach space B, $B_N \subset B$ be its finite dimensional subspace, $P_N : B \to B_N$ be a projector, $A_N : B_N \to B_N$ linear finite-dimensional operator [5].

Definition 4. *Approximation rate for operators A and A_N is called the following operator norm*

$$||P_N A - A_N P_N||_{B \to B_N}$$

We will obtain a "weak estimate" for approximation rate but enough for our purposes. We assume additionally that a function u_d is a restriction on $h\mathbb{Z}^m$ of continuous function with certain estimates [4,5]. Let's define the discrete space $C_h(\alpha,\beta)$ as a functional space of discrete variable $\tilde{x} \in h\mathbb{Z}^m$ with finite norm

$$||u_d||_{C_h(\alpha,\beta)} = ||u_d||_{C_h} + \sup_{\tilde{x},\tilde{y} \in h\mathbb{Z}^m} \frac{|\tilde{x}-\tilde{y}|^\alpha}{(\max\{1+|\tilde{x}|,\,1+|\tilde{y}|\})^\beta}.$$

It means that the function $u_d \in C_h(\alpha,\beta)$ satisfies the following estimates

$$|u_d(\tilde{x}) - u_d(\tilde{y})| \leq c\,\frac{|\tilde{x}-\tilde{y}|^\alpha}{(\max\{1+|\tilde{x}|,\,1+|\tilde{y}|\})^\beta},$$

$$|u_d(\tilde{x})| \leq \frac{c}{(1+|\tilde{x}|)^{\beta-\alpha}}, \qquad \forall \tilde{x},\tilde{y} \in h\mathbb{Z}^m,\ \alpha,\beta > 0,\ 0 < \alpha < 1. \tag{6}$$

Let us note that under required assumptions $C_h(\alpha,\beta) \subset L_2(h\mathbb{Z}^m)$.

Theorem 2. *For operators K_d and $K_{d,N}$ we have the following estimate*

$$||(P_N K_d - K_{d,N} P_N)u_d||_{L_2(Q_N^d)} \leq C N^{m+2(\alpha-\beta)}$$

for arbitrary $u_d \in C_h(\alpha,\beta), \beta > \alpha + m/2$.

Proof. Let us write

$$(P_N K_d - K_{d,N} P_N)u_d = P_N K_d P_N u_d - K_{d,N} P_N u_d + P_N K_d (I - P_N) u_d,$$

where I is an identity operator in $L_2(h\mathbb{Z}^m)$.

First two summands have annihilated, and we need to estimate only the last summand. We have

$$||P_N K_d (I - P_N) u_d|| \leq C||(I-P_N)u_d||$$

because norms of operators K_d are uniformly bounded, and for the last norm taking into account (6) we can write

$$||(I-P_N)u_d||^2 \leq C \sum_{\tilde{x} \in h\mathbb{Z}^m \setminus Q_N} |u_d(\tilde{x})|^2 h^m \leq C \sum_{\tilde{x} \in h\mathbb{Z}^m \setminus Q_N} \frac{h^m}{(1+|\tilde{x}|)^{2(\beta-\alpha)}} \leq$$

and further

$$C \int\limits_{\mathbb{R}^m \setminus Q_N} |x|^{2(\alpha-\beta)} dx$$

The last integral using spherical coordinates gives the estimate $N^{m+2(\alpha-\beta)}$ which tends to 0 under $n \to \infty$ if $\beta > \alpha + m/2$. □

5 Main Theorem on Approximation

Here we consider the equation

$$K_{d,N}u_{d,N} = P_N v_d \tag{7}$$

instead of the Eq. (4) and give a comparison for these two solutions.

Below we assume that operator K_d is invertible in $L_2(h\mathbb{Z}^m)$.

Theorem 3. *If $v_d \in C_h(\alpha,\beta), \beta > \alpha + m/2, u_d$ is a solution of the Eq. (4), $u_{d,N}$ is a solution of (7) then the estimate*

$$||u_d - u_{d,N}||_{L_2(h\mathbb{Z}^m)} \leq CN^{m+2(\alpha-\beta)}$$

is valid, and C is a constant non-depending on N.

Proof. Let us write

$$u_d - u_{d,N} = K_d^{-1}v_d - K_{d,N}^{-1}P_N v_d =$$

$$(I - P_N)K_d^{-1}v_d + P_N K_d^{-1}v_d - K_{d,N}^{-1}P_N v_d$$

For the summand $P_N K_d^{-1}v_d - K_{d,N}^{-1}P_N v_d$ we have a corresponding estimate by the Theorem 2 because the operators K_d^{-1} and $K_{d,N}^{-1}$ are constructed similar initial operators K_d and $K_{d,N}$.

The first summand is estimated like the proof of the Theorem 2 and using the property that operator K_d is uniformly on h is bounded in the space $C_h(\alpha,\beta)$ [9] and the operator K_d^{-1} has a symbol with required properties [8]. □

6 Conclusion

We have introduced such a finite approximation for original integral (1) because there are a lot of algorithms for calculating a finite discrete Fourier transform, these are so called fast Fourier transform algorithms [10]. On the other hand this step-by-step approximation permits to justify mathematically without additional difficulties results on a solvability for a corresponding approximate equation.

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