Discrete Bids and Empirical Inference in Divisible Good Auctions

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Abstract

I examine a model of a uniform price auction of a perfectly divisible good with private information in which the bidders submit discrete bidpoints, and hence step functions, rather than continuous downward sloping demand functions. I prove equilibrium existence and characterize necessary conditions for equilibrium bidding. The characterization result reveals a close relationship between bidding in multiunit auctions and oligopolistic behavior. I demonstrate that an indirect approach to the revenue comparisons of discriminatory and uniform price auctions is not valid if bid functions have steps. In particular, bidders may bid above their marginal valuation in a uniform price auction.

I also use the necessary conditions for structural estimation. I examine a dataset consisting of individual bids in uniform price treasury auctions of the Czech government. I propose an alternative method for evaluating the performance of the employed mechanism. My results suggest that the uniform price auction performs well, both in terms of efficiency of the allocation and in terms of revenue maximization. I estimate that the employed mechanism failed to extract at most 0.03% (in terms of the annual yield of T-bills) worth of expected surplus while implementing an allocation resulting in almost all of the efficient surplus.

Keywords: multiunit auctions, equilibrium existence, treasury auctions, uniform price auctions, structural estimation, nonparametric identification and estimation

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1 Introduction

There is a consensus among economists that the most effective way to sell government securities is through an auction. There is not a consensus on the best auction mechanism, however. The

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theoretical literature on multiunit auctions does not provide a definitive recommendation whether the ultimate goal is either revenue maximization or efficiency of the allocation. In practice, there is a clear preference between the two most widely employed mechanisms. Bartolini and Cottarelli (1997) report that 39 out of the 42 countries surveyed use the discriminatory auction mechanism ("pay your own bid"), and only 3 countries use a uniform price auction mechanism. In this paper I contribute to the debate on the optimal auction mechanism by providing a method that allows a choice between different auction mechanisms based on data on individual bids, while making as few assumptions as possible. An essential part of my model is that the equilibrium strategies are step functions, which has not been properly modelled in the previous literature.

In both discriminatory and uniform price auctions bidders may submit multiple price-quantity pairs as their bids. These points trace out a bid function. The auctioneer then aggregates these bid functions. The market clearing price is the point at which the aggregate bid function intersects the supply quantity, which is usually preannounced. The securities are then allocated to the bidders for those units for which their bids were higher than the market clearing price. The payments collected from the bidders depend on the auction mechanism used. In the discriminatory auction, also known as a pay-your-bid or multiple-price auction, the bidders pay their full bid for all securities that they are allocated. In the uniform price auction, each bidder pays the market clearing price for every unit won. The auctioneer's revenue in the discriminatory auction is therefore the area under the aggregate bid function up to the supply quantity. Revenue is the product of the market clearing price and the quantity supplied in the uniform price auction. It might be tempting to conclude that the discriminatory auction must therefore lead to a higher revenue, just as a perfectly discriminating monopolist is able to earn more than if she cannot discriminate. This intuition is misleading since the mechanism choice affects bidders' strategic behavior and thus the location and shape of the aggregate bid function. Results from single unit auction settings are also misleading. For example, one might conclude from the similarity between a second price auction and a uniform price auction, or between the first price auction and a discriminatory auction, that the revenue should be the same if the values are private. This intuition is misleading since the revenue equivalence theorem requires that the mechanisms be allocationally equivalent, which is typically not the case in a multi-unit environment.

The strategic considerations are quite different in the two auction formats. In a discriminatory auction a rational bidder would not bid his full marginal valuation for any unit that might be accepted, because he wants to retain some surplus. In a uniform price auction, a bidder may not worry about losing surplus by bidding his marginal valuation, since he pays the market clearing price for all units won. On the other hand, he should shade his bid below his marginal valuation at quantities that might be pivotal and might therefore determine the market clearing price. A lower market clearing price increases his surplus on inframarginal units. Hence, in both auction mechanisms, bidders will not always bid their true marginal valuations. Ausubel and Cramton (2002) show that the comparison of the uniform and discriminatory auction formats, in terms of both efficiency and revenue, is an empirical question. Either format can be better than the other, under either criterion, under some circumstances.

Most of the previous empirical literature that compares these two auction mechanisms focuses on "natural experiments" in which different auction formats have been used in different time periods. Those papers examine the difference between the market clearing auction price and the resale or forward price of the security (Umlauf (1993), Simon (1994), Nyborg and Sundaresan (1996)). A drawback of this approach is that the researcher has to maintain strong assumptions on the information structure across the auctions, especially those involving different auction formats. In particular, observed differences that cannot be explained by observable control variables are attributed solely to the auction format.

My paper instead belongs to a small set of recent papers, discussed in more detail below, that employ structural econometric modelling to compare the alternative auction mechanisms in a divisible good[1] setting. These papers use a bidder's optimality condition to recover structural parameters and, in particular, the distribution of the marginal valuations and private signals, as proposed in Guerre, Perrigne and Vuong (2000) in the single unit setting. This approach avoids problems with comparing realizations of different formats, and it is also amenable to answering counterfactual policy questions. My paper differs from these recent papers in two principle ways. First, most of these papers use parametric assumptions to circumvent the problem of multiple equilibria and for tractability (for example by restricting attention to equilibrium strategies that are linear in private signals). My approach will instead be non-parametric. Second, and more significantly, all of these papers focus on equilibria in strictly downward-sloping continuous bid functions. In the data, however, we instead typically observe step bid functions. This occurs both because bidders are in reality limited in the number of bidpoints they are allowed to submit, and because they choose to submit even fewer bids than the allowed number. For example, in my dataset the bidders are restricted to submit at most 10 bidpoints, yet the average number of submitted bidpoints is less than 3 and the maximum number of submitted bidpoints is 9. My approach takes this feature of the data seriously.

The main contributions of the paper can be classified into two groups. On the theory side, in Section 3 and 4 I introduce a model of a divisible good auction with private information in which the bidders are restricted in number of bids they are allowed to submit and thus submit step bid functions. I prove existence of an equilibrium in distributional strategies in this model and characterize necessary conditions for equilibrium bidding. These necessary conditions differ from those in the differentiabe downward sloping bid functions case.[2] These conditions, which relate the primitives of the model to the observables serve as the basis for the empirical work later in the paper. They also are useful for understanding equilibrium behavior in multiunit auctions of

[1] A divisible good auction is also known as a share auction.

[2] In a companion working paper (Kastl (2005)) I show that these necessary conditions for equilibrium converge to their counterpart in the model with differentiable downward sloping bid functions as the number of submitted bidpoints goes to infinity.

indivisible goods, in which case the observed bid is a discrete vector, since a model with a divisible good can be viewed as the limiting case of such a class of models. My characterization theorem reveals the close relationship between the optimal behavior of a bidder in a uniform price auction and that of an oligopolist facing uncertain demand. I also demonstrate that when bidders are restricted in the number of bids they can submit, they may submit bids higher than their marginal value for some units. This suggests, for example, that recent work comparing uniform and discriminatory auctions by Hortaçsu (2002) may provide an underestimate of the revenue arising for the uniform price auction.

Sections 5 and 6 turn to the empirical side of the paper. In Section 5 I provide conditions under which the primitives of the model can be identified nonparametrically and propose an estimation method using a resampling approach introduced into the literature by Hortaçsu (2002). In Section 6 I describe my data and apply my estimation method to obtain information about bidders' marginal valuations in uniform price treasury auctions of the Czech government. I show that in a nonnegligible share of these auctions, the actual revenue exceeds the revenue that would have been obtained had the bidders bid their true marginal valuation schedules. This result would have been impossible to obtain in the model with continuous downward-sloping demand functions. I propose a new method for evaluating the performance of the auction mechanism using these estimates and find that the uniform price auction performs quite well, both in terms of efficiency and revenue. On average, the employed mechanism implements an allocation that achieves over 99.99% of the efficient surplus. Moreover, the estimated maximum total expected surplus (in terms of the annual yield of T-bills) left to the bidders does not exceed 3 basis points.

In addition, I review the existing literature on structural estimation of divisible good auctions in Section 2, while Section 7 concludes the paper. All proofs are relegated to the appendix.

2 Existing Structural Approaches to Divisible Good Auctions

Fevrier, Preget and Visser (2004) develop an estimation method for a pure common value share auction. They use a two-step estimation procedure. In the first step they estimate the distribution of the bid functions. In the second step, a vector of parameters of the distribution of signals is estimated using generalized method of moments. Finally, in order to evaluate the counterfactual revenue from the uniform price auction they select the equilibrium in which the strategies are linear functions of the private signals. Further, they show that this equilibrium is unique for a class of demand functions under an assumption on parametric distribution of signals. Armantier and Sbaï (2004) also assume pure common values. They parametrize the underlying distribution of signals and choose a functional form for the utility function since they also study the effect of attitudes toward risk. They constrain the set of strategies to finite order simple polynomials and use the method of simulated moments to estimate structural parameters by applying the concept of constrained strategic equilibrium developed in Armantier, Florens and Richard (2002). They can account for risk aversion and bidder asymmetries. In pure common value models, efficiency is not an issue since any allocation is ex post efficient whenever all supply is allocated. These two papers estimate an equilibrium in which the strategies are well-behaved downward sloping functions. In particular, they do not address the issue of discrete bids. The theoretical model they consider requires the implicit assumption that bidders submit downward sloping functions, from which the econometrician observes a few points.

A notable exception is Hortaçsu (2002). He develops a clever nonparametric method to obtain estimates of bidders' private valuations in a discriminatory auction. His approach and in particular his counterfactual revenue comparison can only be applied for data coming from discriminatory auctions environment. Using Turkish treasury data Hortaçsu develops a resampling method that, within an independent private values framework, allows us to nonparametrically estimate the distribution of market clearing prices. Using the estimated distribution he then estimates the marginal valuations for quantities for which a bid was submitted using the optimality equation. He then uses these estimates to calculate the revenue from a hypothetical auction in which each bidder would bid the upper envelope of his possible valuation schedule and each bidder's payment would be determined according to the uniform price auction rule. Within the model that delivers equilibria in strictly downward sloping bid functions, the revenue from such a hypothetical auction constitutes an upper bound on revenue from any equilibrium of the real uniform price auction. The reason is as discussed above; in a uniform price auction each bidder finds it optimal to shade his bid below his marginal valuation for larger quantities. Since he finds that the revenue from this best case scenario would be lower than the observed revenue from the discriminatory auction, he concludes that the discriminatory auction generates higher revenue. If the revenue from this best case scenario turned out to be higher than the observed revenue from the discriminatory auction, we could not make any definitive comparisons between these two auction formats. Moreover, if our data came from the uniform price auction, this best case scenario approach would not be useful; the revenue from an auction in which each bidder bids his true valuation and the payment is according to the discriminatory auction rule would dominate the revenue in any other auction. Moreover, my analysis shows that in a model with equilibria in step functions rather than strictly downward sloping bid functions, the revenue of the hypothetical uniform price auction described above does not constitute an upper bound on the ex post revenue of the uniform price auction. The reason is that bidders might find it optimal to submit bids that are higher than their marginal valuations. In general, the marginal valuation schedule may not be the upper bound on the bid schedule in a uniform price auction, whenever the bidder is not allowed to submit a separate bid for every unit offered for sale.

In a recent paper, Wolak (2004) examines Australian electricity auctions taking into account that the bid functions are step functions. He develops an econometric technique to estimate parameters of parametrically specified cost functions from data on individual bids. His approach is to summarize the price uncertainty each bidder faces due to the behavior of other bidders and unforeseen demand shocks by a single index. He assumes that each bidder knows the joint distribution of these uncertainty indices when deciding on his bid. Finally, he circumvents the non-differentiability problem of the expected profit function by approximating it by a smooth function using a standard normal density. Using the moment conditions implied by each bidder bidding optimally taking into account the uncertainty, he applies GMM to recover the parameters of interest. In his analysis he concludes that there is evidence of the presence of "ramping costs."

3 Model

I will start with the basic uniform price share auction framework of Wilson (1979) with private information, in which both quantity and price are assumed to be continuous. There are N bidders, who are bidding for a share of a perfectly divisible good. Each bidder receives a private real-valued signal, s_i, which is the only private information about the underlying value of the auctioned goods. The joint distribution of the signals will be denoted by $F(\mathbf{s})$.

Assumption 1 *Bidder i's signal s_i is drawn from a common support $[0,1]$ according to an atomless marginal d.f. $F_i(s_i)$ with strictly positive density $f_i(s_i)$.*

Winning q units of the security is valued according to a marginal valuation function $v_i(q, s_i, s_{-i})$. For most of this paper we will deal with the special case of independent private values (IPV). We will discuss the robustness of our estimation method with respect to this assumption later. In the IPV case, the s_i's are distributed independently across bidders, and bidders' valuations do not depend on private information of other bidders. At the estimation stage we will not impose full symmetry, since we will allow for different groups, within which the signal is distributed identically across bidders. We will impose the following assumptions on the marginal valuation function $v(\cdot,\cdot,\cdot)$:

Assumption 2 *$v_i(q, s_i, s_{-i})$ is measurable and bounded, strictly increasing in s_i $\forall (q, s_{-i})$ and weakly decreasing in q $\forall (s_i, s_{-i})$.*

Notice that we do not require any differentiability or continuity assumptions on v. We will denote by $V(q, s_i, s_{-i})$ the gross utility: $V(q, s_i, s_{-i}) = \int_0^q v_i(u, s_i, s_{-i})\,du$.

Bidders' pure strategies are mappings from private signals to bid functions: $\sigma_i : S_i \to \mathcal{Y}$, where the set $\mathcal{Y}$ includes all possible functions $y : \mathbb{R}^+ \to [0,1]$. A bid function for type s_i can thus be summarized by a function, $y_i(\cdot|s_i)$, which specifies for each price p, how big a share $y_i(p|s_i)$ of the securities offered in the auction (type s_i of) bidder i demands. Q will denote the amount of T-bills for sale, i.e., the good to be divided between the bidders. Q might itself be a random variable if it is not announced by the auctioneer ex ante, or if the auctioneer has the right to augment or restrict the supply after he collects the bids. In either case, I will assume that the distribution of Q is common knowledge among the bidders. Furthermore, the number of bidders participating in an auction, denoted by N, is also commonly known. The natural solution concept to apply in this setting is Bayesian Nash Equilibrium. The expected utility of type s_i of bidder i who employs a strategy $y_i(\cdot|s_i)$ in a uniform price auction given that other bidders are using $\{y_j(\cdot|\cdot)\}_{j\neq i}$ can be written as:

$$\begin{aligned} EU_i(s_i) &= E_{Q,s_{-i}|s_i} u(s_i, s_{-i}) \\ &= E_{Q,s_{-i}|s_i}\left[\int_0^{q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s))} v_i(u, s_i, s_{-i})\, du - p^c(Q,\mathbf{s},\mathbf{y}(\cdot|s))\, q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s))\right] \end{aligned}$$

where $q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s))$ is the (market clearing) quantity bidder i obtains if the state (bidders' private information and the supply quantity) is $(\mathbf{s}, Q)$ and bidders bid according to strategies specified in the vector $\mathbf{y}(\cdot|s) = [y_1(\cdot|s_1), ..., y_N(\cdot|s_N)]$, and similarly $p^c(Q,\mathbf{s},\mathbf{y}(\cdot|s))$ is the market clearing price associated with state $(\mathbf{s}, Q)$. A Bayesian Nash Equilibrium in this setting is thus a collection of functions such that almost every type s_i of bidder i is choosing his bid function so as to maximize his expected utility: $y_i(\cdot|s_i) \in \arg\max EU_i(s_i)$ for a.e. s_i and all bidders i.

In most of the previous literature, starting with Wilson (1979), the set $\mathcal{Y}$ of admissible strategies is restricted to continuously differentiable functions so that calculus of variations techniques can be applied. These techniques enable us to show that in an IPV model, and within this restricted class of strategies, a symmetric BNE $y(\cdot|\cdot)$ has to satisfy the following necessary condition for all (p, s_i):

$$v(y(p|s_i), s_i) = p - y(p|s_i)\frac{H_y(p, y(p|s_i))}{H_p(p, y(p|s_i))} \tag{1}$$

where $H(p,x)$ is the probability distribution of the market clearing price when x units are demanded by bidder i and all other bidders $j \neq i$ submit the equilibrium bid functions, i.e., $H(p,x) \equiv \Pr(p^c \leq p|x) = \Pr\left(x \leq Q - \sum_{j\neq i} y(p, s_j)\right)$ (H_p and H_y are the derivatives of $H(\cdot,\cdot)$ with respect to the first and second argument respectively). As Wilson points out, the auction game might have multiple equilibria, some of which lead to low revenue for the auctioneer. Such equilibria, while achieved in a non-cooperative way, are usually called "seemingly collusive" and several authors (e.g., LiCalzi and Pavan (2004)) show how the auctioneer would eliminate at least some of these undesirable equilibria.

Because of the restricted set of strategies, it is an essential feature of a candidate equilibrium that the equilibrium strategies are strictly downward-sloping differentiable functions. One implication of this fact is that the rationing rule does not matter for equilibrium behavior, since rationing does not occur in equilibrium.[3] In other words, we always have $q_i^c\left(Q, \mathbf{s}, \mathbf{y}\left(p|s_i\right)\right) = y_i\left(p^c\left(Q, \mathbf{s}, \mathbf{y}\left(\cdot|s\right)\right)|s_i\right)$. While Wilson's model provides useful insights, and illuminates some of the trade-offs bidders face in share auctions, it cannot account for several features of the data in most actual share auctions.

In the next section, I will introduce a concept of a K-step equilibrium, in which I address directly a central feature of most real-world share auctions: bid functions are step functions, and hence not continuously differentiable. I will argue that accounting for these features has important implications for both the theoretical model and empirical inference in these auctions.

4 K-step equilibrium

Why do bidders submit step functions in these auctions? The first reason is institutional. In the vast majority of actual share auctions, the auctioneer imposes an upper bound on the number of bidpoints that the bidders can submit, which restricts the bidders' strategy space and makes submitting a continuous function impossible. But this is not the whole story. In most auctions the bidders do not attain this institutionally-set upper bound. Moreover, the number of bidpoints bidders submit differs both across bidders within an auction and even for the same bidder across auctions. One way to rationalize this variance is that there is some cost of bid submission that might differ across bidders and/or time and which leads them to submit different number of bids. The presence of such costs would constitute an endogenous, economic restriction on the number of bidpoints. In this section I develop a model that incorporates these features and characterize its equilibrium.

Let us suppose that the cost of submitting K_i steps is private information summarized by a cost function $c\left(K_i, t_i\right)$ where the parameter t_i is private information of bidder i.

Assumption 3 *A bidder submitting K_i bidpoints incurs non-negative cost $c\left(K_i, t_i\right)$ where t_i is private information of bidder i which is drawn from a distribution function $G_i\left(t|s_i\right)$ with the support* $[0, 1]$.

Notice that this formulation nests the original model as a special case in which $c\left(K_i, t_i\right) \equiv 0$ $\forall\left(K_i, t_i\right)$. It also includes the case in which there is an exogenous upper bound $\bar{K}$ on the allowed bidpoints, in which case $c\left(K_i, t_i\right) = \infty$ for $K_i > \bar{K}$ and any t_i. All the results described in the following paragraphs hold true for the model with $c\left(K_i, t_i\right) \equiv 0$ $\forall\left(K_i, t_i\right)$. The important features of the model are (i) bidders can submit only finitely many bidpoints, and (ii) the price and quantity in each bidpoint are continuous choice variables.

[3] Because individual bid functions are strictly downward sloping, residual supply is always strictly upward sloping and thus the market always clears exactly.

The expected utility of a bidder of a type (s_i, t_i) in a uniform price auction now becomes:

$$\begin{aligned} EU(s_i, t_i) &= E_{Q,s_{-i},t_{-i}|s_i,t_i} u(s_i, s_{-i}, t_i, t_{-i}) \\ &= E_{Q,s_{-i},t_{-i}|s_i,t_i} \left[\begin{array}{c} \int_0^{q_i^c(Q,\mathbf{s},\mathbf{t},\mathbf{y}(\cdot|s,t))} v_i(u, s_i, s_{-i})\, du \\ -p^c(Q, \mathbf{s}, \mathbf{t}, \mathbf{y}(\cdot|s,t))\, q_i^c(Q, \mathbf{s}, \mathbf{t}, \mathbf{y}(\cdot|s,t)) - c(K_i, t_i) \end{array} \right] \end{aligned}$$

where K_i is the number of steps of $y_i(\cdot|s_i, t_i)$.

Definition 1 *A K-step equilibrium is a collection of functions such that for each bidder i and almost every type (s_i, t_i), $y_i(\cdot|s_i, t_i)$ solves*

$$y_i(\cdot|s_i, t_i) \in \arg\max EU(s_i, t_i)$$

While most of the previous literature restricts bid functions to be continuously differentiable, we allow them to be step functions. With this possibility, we cannot apply the calculus of variations approach to characterize equilibrium strategies. Since for a finite $\bar{K}$ bidders submit left-continuous step functions, we can summarize bidder i's action as a K_i-dimensional vector of bidpoints (b_i, q_i), where the k^{th} point denotes the price (the height of current step) and quantity (strictly speaking, the share of total quantity) at which this step ends (its length). I impose the additional assumption that there is a (price) bid l, at which a bidder is sure to lose and which is isolated from the remaining bids. This is done in order to ensure that all bids are individually rational in the existence proof. All bids other than l will be called *"serious" bids*. I will also assume that there is an upper bound on the maximal bid, which for example in the case of treasury bills could be the face value. In general, any bid above the value of the first infinitessimal unit is weakly dominated by bidding this value, and thus this upper bound is $v(0, \bar{s})$ where $\bar{s}$ is the highest possible signal. To summarize:

Assumption 4 *Each player $i = 1, ..., N$ has an action set:*

$$A_i = \left\{ \begin{array}{c} \left(\vec{b}, \vec{q}, K\right) : \dim\left(\vec{b}\right) = \dim(\vec{q}) = K \in \{0, ..., \bar{K}\}, \\ b_{ik} \in B = \{l\} \cup [0, \bar{p}],\ q_{ik} \in Q = [0, 1],\ b_{ik} \geq b_{ik+1}, q_{ik} \leq q_{ik+1} \end{array} \right\}$$

In what follows when more convenient I use the shorthand vector notation (b_i, q_i) to describe the step function $y(\cdot|s_i, t_i)$ of type (s_i, t_i) of bidder i. Assumptions 1-4 are assumed throughout the analysis.

It is also apparent that because each bidder's bid function is a step function, the residual supply will be a step function, and therefore but for knife-edge cases any equilibrium will involve rationing with probability one. Rationing occurs whenever there is excess demand at the market clearing price, while at all higher prices there is excess supply. On such occasions the auctioneer will determine a *rationing coefficient*, by which demand is adjusted to equal supply. While the theoretical literature has considered a few alternative rationing rules, in our analysis we will consider only the rationing rule that is employed in all uniform price auctions in practice, rationing pro-rata on-the-margin.

Definition 2 *Under rationing* ***pro-rata on-the-margin****, the rationing coefficient satisfies*

$$R(p^c) = \frac{Q - TD_+(p^c)}{TD(p^c) - TD_+(p^c)}$$

where $TD(p^c)$ denote total demand at price p^c, and $TD_+(p^c) = \lim_{p \downarrow p^c} TD(p)$. Only the bids exactly at the market clearing price are adjusted.

Under this rule all bids above the market clearing price are given priority, and only after all such bids are satisfied, the remaining marginal demands at exactly price p^c are reduced proportionally by the rationing coefficient so that their sum exactly equals the remaining supply. An alternative rationing rule would, for example, not give bids at higher prices priority. Kremer and Nyborg (2004) show that, in a complete information framework, this alternative rationing rule encourages competition and may thus be preferred. Notice, however, that this alternative rationing rule may have an adverse effect on allocative efficiency.

Equilibrium existence

Before we turn to the characterization of equilibria of a uniform price auction involving step functions, we need to address the question of existence. Several recent papers (e.g. Reny (1999), Athey (2001), McAdams (2003, 2004), Reny and Zamir (2004), Jackson and Swinkels (2005)) provide nice existence results for games of incomplete information that in many cases guarantee the existence of a pure strategy equilibrium that is non-decreasing in private information. Unfortunately, those results cannot be readily applied in our setting. For our purposes, however, it is enough to prove existence of a K-step equilibrium in distributional strategies.

Proposition 1 *(Existence) With private values, rationing pro-rata on-the-margin and restriction of bid functions to at most $\bar{K}$ steps, there exists a K-step equilibrium in distributional strategies of a uniform price auction.*

The existence result guaranteed by Proposition 1 holds for any specification of the cost $c(K_i, t_i)$ and any distribution $G(t_i|s_i)$. Therefore the model without costly bidding is also covered. Kastl (2005) proves existence of an isotone pure strategy equilibrium for the case of no restriction on the number of steps and no bidding costs.

Characterization of equilibrium

Even though the current problem involves many difficulties due to the lack of differentiability, we can provide the equivalent of a first-order necessary condition by working directly with limit arguments. Before stating the main characterization result, let us first define a tie, and state a lemma, which ensures that a tie is a zero probability event in equilibrium.

Definition 3 *A **tie** occurs whenever there are at least two marginal bidders at the market clearing price, i.e., for some types* (s_i, t_i) *and* (s_j, t_j) *of bidders* i *and* j*, and some steps* k *and* l *in their bid functions, and some state* $(\mathbf{s}, \mathbf{t}, Q)$ *we have* $b_{ik}(s_i, t_i) = b_{jl}(s_j, t_j) = p^c(Q, \mathbf{s}, \mathbf{t}, \boldsymbol{\sigma})$*, where* $\boldsymbol{\sigma}$ *is the vector of employed strategies.*

Lemma 1 *If values are private and rationing is pro-rata on-the-margin then, for a.e. type* (s_i, t_i) *of any bidder* i*, ties at the market clearing price have zero probability in equilibrium.*

The intuitive argument behind Lemma 1 goes as follows. Suppose that for some type of bidder i at a certain step, say $\hat{k}$, there is a positive probability of tying with another bidder. Then submitting a bid $b'_{\hat{k}} = b_{\hat{k}} + \varepsilon$ for quantity $q'_{\hat{k}} = q_{\hat{k}}$, where ε is sufficiently small, will yield a strict increase in expected payoff. The incremental value of an increase in allocation on gross surplus by avoiding the tie is strictly positive, otherwise the bidder would not request $q_{\hat{k}}$ to begin with. The increase in expected payment is arbitrarily small by picking a small enough ε. Hence bidding so that tying another bidder at the market clearing price has a positive probability is not a best response. The crucial assumption delivering this important result is that the space of serious price bids is continuous.

Notice that the argument behind the last lemma uses both the rationing rule and private values. With a common value component, the presence of the winner's curse could make such a deviation upwards unprofitable. This would be the case, for example, if bidder i ties only with bidder j, each requesting 51% of the quantity, and all other bids much lower. In this case, it is likely that the common value lies below the market clearing price set by the tying bids, and hence the above described deviation would no longer be strictly profitable, i.e., in this case, being in a tie is "bad news".

The next proposition characterizes a necessary condition for a K-step equilibrium in a private values model. This result can also be viewed as a characterization of an equilibrium of a limit of a multiunit auction as the units become arbitrarily small, and it reveals the close relationship between the behavior of a bidder in a uniform price auction and that of an oligopolist facing uncertain demand.

Proposition 2 *(Characterization) Suppose values are private and rationing is pro-rata on-the-margin. Then in any Bayesian Nash Equilibrium, for almost every (s_i, t_i), every step k in the K_i-step function $y_i(\cdot|s_i, t_i)$ in the support of i's equilibrium strategy has to satisfy*

$$\Pr(b_k > p > b_{k+1})\left[v(q_k, s_i) - E(p|b_k > p > b_{k+1})\right] = q_k \frac{\partial E(p; b_k \geq p \geq b_{k+1})}{\partial q_k} \tag{2}$$

$$\frac{\partial E\left[V\left(q_i^c\left(Q, \mathbf{s}, \mathbf{t}, \mathbf{y}\left(\cdot|s,t\right)\right), s_i\right); b_{k-1} > p > b_{k+1}\right]}{\partial b_k} = \tag{3}$$

$$= E\left(q_i^c\left(Q, \mathbf{s}, \mathbf{t}, \mathbf{y}\left(\cdot|s,t\right)\right); p = b_k\right) + b_k \frac{\partial E\left(q_i^c\left(Q, \mathbf{s}, \mathbf{t}, \mathbf{y}\left(\cdot|s,t\right)\right); p = b_k\right)}{\partial b_k}$$

$$+ q_k \frac{\partial E(p; b_k > p > b_{k+1})}{\partial b_k} + q_{k-1} \frac{\partial E(p; b_{k-1} > p > b_k)}{\partial b_k}$$

The intuition for the result is the following. Consider the first condition, which rules out profitable local perturbations of q_k. It is this equation that reveals the parallel between the behavior of a bidder in a multiunit uniform-price auction and an oligopolist facing uncertain demand. Since Lemma 1 ensures that a tie (multiple bids at the market clearing price) occurs with probability zero, the only states at which the bidder can affect his payoff by varying the quantity demanded, q_k, are those in which the residual supply cuts the vertical piece of his bid function, i.e., between his adjacent bids $b_k > p > b_{k+1}$. In all states such that the market clearing price is between the two steps of bidder i, he obtains his full quantity request, and the expected marginal cost of quantity shading captured on the LHS is thus the difference between his marginal utility and the expected price. Since in all states that he is rationed he is the only marginal bidder with probability one, there is no cost of quantity shading in those states. On the other hand, the marginal benefit of quantity shading is saving money on the inframarginal units, and this is captured on the RHS. Therefore, the bidder facing random residual supply acts in the same way as a monopolist facing random demand. Notice that (2) can be rewritten as

$$v(q_k, s_i) = E(p|b_k > p > b_{k+1}) + \frac{q_k}{\Pr(b_k > p > b_{k+1})} \frac{\partial E(p; b_k \geq p \geq b_{k+1})}{\partial q_k} \tag{4}$$

which can be interpreted as $MC = MR$, i.e., like an oligopolist's optimality condition.

The second condition makes sure that a local perturbation of b_k is not optimal. Bidder i has to balance the change in the expected prices in the steps above and below the k^{th} one. He also needs to take into account the payoff effect of the perturbation if he is rationed at b_k, which includes the indirect effect on the expected quantity received after rationing. It is this condition that we would regularly obtain in a multiunit auction with discrete units. Equation (2) would become a system of inequalities in that setting.

Equation (2) immediately gives us the following important corollary.

Corollary 3 *Under the hypotheses of Proposition 2, when bidders are restricted to submit step functions, they may optimally bid **above** their marginal valuation schedules in a uniform price auction.*

To see why this corollary holds, it is sufficient to consider one very small bidder, so that she is a "price taker," and let $K_i = 1$.[4] In this case, the RHS of (2) vanishes, and the bidder thus optimally asks for a quantity such that her marginal valuation at that quantity is equal to the expected price conditional on this price being lower than her bid, $v(q_k, s_i) = E_{s_{-i}}(p|b_k > p)$. Therefore, whenever there is a positive probability of a market clearing price below her bid, her bid will be higher than her marginal valuation for that quantity. This important result indicates that the ex post revenue in a uniform price auction is not necessarily bounded by the revenue of the "best case" Vickrey auction, in which each bidder submits his marginal valuation schedule as his bid without getting any transfer from the auctioneer. I should note that this "best case" upper bound is valid for revenues from equilibria in continuously differentiable bid functions since in that setting a bidder never submits a bid above his marginal value such that this bid is in the support of the distribution of the market clearing price. This result is important for empirical work, since calculating counterfactual equilibria and the associated revenues under alternative auction regimes is often an intractable task. The researcher is thus forced to report estimated revenue losses from the realized auction relative to this "best case" Vickrey auction (also sometimes called the "truthful bidding" auction). Corollary 3 reveals, however, that even a uniform price auction can lead to a higher ex post revenue than the "best case" Vickrey auction. As we will see later in the empirical section, this point is not purely theoretical, since in a nonnegligible share of auctions in my dataset the realized ex post revenue is higher than the revenue in an auction in which the bidders submit bids equal to the estimated upper bound of their marginal valuation schedules. This result also suggests that using the model with continuously differentiable bid functions might not be a good approximation.

Another question is the size of set of equilibria (in terms of the range of possible market clearing prices) and whether a proper choice of $\bar{K}$ might restrict this set. Wilson showed that the model with continuously differentiable bid functions has many equilibria, and under some circumstances even a continuum, some of which might yield very low revenue. Back and Zender (1993) describe a class of equilibria that are independent of private information and that they call "collusive," since the resulting revenue is low. LiCalzi and Pavan (2004) show that with pure common values and increasing supply, all possible equilibrium market clearing prices can be achieved by bid functions with just two steps. In both cases, these equilibria depend on the rationing rule (pro-rata on-the-margin), and either on differentiable bid functions (Back and Zender) or on increasing supply

[4] A bidder is a price taker if no small change in her bid has any effect on the distribution of the market clearing price.

(LiCalzi and Pavan). The common feature of these equilibria is that bidders tie at the market clearing price and get rationed. Any deviation to a higher bid results in a large increase in the market clearing price, due to high inframarginal bids by all bidders, rendering such a deviation unprofitable. In the present model neither of those two conditions is satisfied. As Lemma 1 shows, ties at the market clearing price cannot occur in equilibrium with positive probability. Moreover, such equilibria could be easily detected by the auctioneer, since they have to involve ties at the market clearing price together with very high inframarginal bids.[5]

5 Econometric Model and Identification

Suppose we have data on all bids from T auctions. I will impose the following assumption on the data generation process.

Assumption 5 *Bidders have private values and can be split into G groups within which the marginal valuation function is symmetric. Private information is identically distributed within groups and independent across bidders and auctions. The data $\left\{\{b_{it}, q_{it}\}_{i=1}^{N_t}\right\}_{t=1}^{T}$ is generated by K-step equilibrium behavior, where N_t is the number of bidders in auction t.*

The estimation and identification procedure follows the first-order condition approach proposed in Laffont and Vuong (1996) and Guerre, Perrigne and Vuong (2000). In particular, the price-quantity pair submitted as the k^{th} out of K_i total bidpoints has to satisfy condition (4). Thus, if we are able to estimate $E\left(p|b_k > p > b_{k+1}\right)$ and the derivative $\frac{\partial E(p;b_k \geq p \geq b_{k+1})}{\partial q_k}$ we can use (4) to obtain an estimate of the marginal valuation at this particular quantity for a fixed, but unknown, realization of s_i. The general problem of multiunit auctions is that the full marginal valuation function might still not be identified because there may be many functions that are nonincreasing in q and strictly increasing in s that go through all the point estimates obtained from the data. Usually we circumvent this problem by imposing some parametric structure, which ensures unique identification. I will argue below that in theory there are situations under which the marginal valuation functions can be identified nonparametrically from the data. Before stating the main identification results, let us discuss first the method for obtaining the point estimates of marginal valuations at the submitted quantity-bids.

Estimating $E\left(p|b_k > p > b_{k+1}\right)$

To estimate $E\left(p|b_k > p > b_{k+1}\right)$ we can use the resampling strategy proposed by Hortaçsu (2002), which closely follows the usual bootstrapping approach. In particular, under the assumption of independent private values, independence of cost types across bidders, and within-group symmetry of the marginal valuation function and distribution functions of signals and cost types, we can perform the following procedure:

[5] For further discussion of the issue of "collusive seeming" equilibria see Kastl (2005).

1) Fix bidder i from group $g \in G$ among N_{tg} bidders in auction t who belong to group g.

2) From the sample of N_{tg} bid vectors in the data set, draw a random sample of $N_{tg} - 1$ and from all groups h other than g draw N_{th} for $h \in G \backslash \{g\}$ with replacement, giving equal probability of $\frac{1}{N_{tg}}$ (or $\frac{1}{N_{th}}$ respectively) to each bid vector in the original sample.

3) Construct the residual supply function generated by these resampled bid vectors.

4) Intersect this residual supply curve with bidder i's bid function to find the market clearing price.

5) Repeat steps 1-4 B (a large number) times for each bidder and for all bidders in the data set.

This procedure generates B market clearing prices conditional on the bid vector $(\mathbf{b}_i, \mathbf{q}_i)$ and one can estimate $E(p|b_k > p > b_{k+1})$ by looking at the conditional distribution of the market clearing prices which fall in the required interval.

For this method to perform reliably we would like to have a large number of bidders in each group in every auction, so that we observe bid vectors reflecting a large number of signal realizations from the group distribution function of signals. If that is not the case, but we are willing to assume that several auctions are repetitions of the same experiment, we can pool the bid vectors from different auctions. In either case, if we call the estimator obtained by the above procedure the resampling estimator $\hat{E}^R(p|b_k > p > b_{k+1})$, it can be shown (Hortaçsu (2002)) that it is consistent for $E(p|b_k > p > b_{k+1})$ (it converges almost surely) as the number of auctions (regarded as the repetitions of the same experiment) go to infinity, $T \to \infty$.[6]

Estimating $\frac{\partial E(p;b_k \geq p \geq b_{k+1})}{\partial q_k}$

To obtain this piece of equation (4), we can use the same resampling approach described earlier when estimating $E(p|b_k > p > b_{k+1})$ to estimate $E(p|b_k \geq p \geq b_{k+1})$, which together with an estimate of $\Pr(b_k \geq p \geq b_{k+1})$ and Bayes' rule yields an estimate of $E(p; b_k \geq p \geq b_{k+1})$. Call this estimate $E^R(p; b_k \geq p \geq b_{k+1})$. Notice that while obtaining this estimate, we condition on the submitted vector of bidpoints. The natural way to estimate the derivative of this expectation with respect to quantity bid at step k is to perturb q_k in the submitted bid vector to some $q_k - \varepsilon_n$ and obtain an estimate of $E^R(p; b_k \geq p \geq b_{k+1})$ conditional on the perturbed bid vector. We can then construct the estimator of the derivative:

$$\frac{\partial E^R(p; b_k \geq p \geq b_{k+1})}{\partial q_k} = \frac{E^R(p; b_k \geq p \geq b_{k+1}, q_k) - E^R(p; b_k \geq p \geq b_{k+1}, q_k - \varepsilon_n)}{\varepsilon_n}$$

where ε_n is a sequence converging in probability to zero. One difficulty when estimating the slope of this expectation w.r.t. q_k is choosing the appropriate neighborhood ε_n so that the numerical derivative is a consistent estimate. Loosely speaking, this neighborhood should shrink to zero as

[6] It may also be consistent under other conditions - see Hortaçsu (2002).

the sample size increases. Pakes and Pollard (1989) establish that with a regularity condition (on uniformity), such an estimator is consistent whenever $n^{\frac{-1}{2}}\varepsilon^{-1} = O_p(1)$, i.e., whenever ε does not decrease too fast as the sample size increases.

Proposition 4 *(Consistency of the resampling estimator)*
(i) If $\Pr(b_k > p > b_{k+1}) > 0$*, then* $\hat{E}^R(p|b_k > p > b_{k+1}) \rightarrow^{a.s.} E(p|b_k > p > b_{k+1})$ *as* $\{\min_h N_{th}\} T \rightarrow \infty$
(ii) If $\Pr(b_k \geq p \geq b_{k+1}) > 0,$ *then* $\frac{\partial \hat{E}^R(p;b_k \geq p \geq b_{k+1})}{\partial q_k} \rightarrow^{a.s.} \frac{\partial E(p;b_k \geq p \geq b_{k+1})}{\partial q_k}$ *as* $\{\min_h N_{th}\} T \rightarrow \infty$

Given consistent estimates of all the pieces of the right hand side of (4), we can obtain the point estimates of the marginal valuations at the submitted bids, conditional on the fixed unobserved private signal. As mentioned above, having these point estimates does not guarantee identification of the entire marginal valuation function. In particular, there could be many functions $v(\cdot,\cdot)$ that: (i) go through the estimated points , (ii) are everywhere nonincreasing in the first argument, (iii) are everywhere strictly increasing in the second. We can also use our second necessary condition, equation (3), to narrow down the set of candidate marginal valuation functions, but this might still not achieve unique identification. The second equation puts restrictions on the area below the marginal valuation function between each two bidpoints, but there is still not enough information to pin down the curvature.

Using bidders with multiple bidpoints

One can exploit information available from bidders submitting more than one bidpoint within an auction. In particular, all bids by one bidder within an auction are generated by the equilibrium strategy for the same type $\hat{s}$. Therefore all point estimates for one bidder from a fixed auction come from the same realization of his signal $\hat{s}$, and hence have to lie on the same level curve, $v(\cdot, \hat{s})$. Since $v(\cdot,\cdot)$ is strictly increasing in the second argument (signal s) by assumption, an inverse function exists, which maps the quantity requested and the marginal valuation back into the signals, $\sigma : [0,1] \times \mathbb{R}^+ \rightarrow [0,1]$. Suppose now we observe two bidpoints (q_1, p_1, q_2, p_2). Using the method described earlier we can recover the marginal valuations at q_1 and q_2. Let us call those v_1 and v_2. Using the inverse function σ we can write:

$$\begin{aligned} v_1 &= v(q_1, \hat{s}) \\ &= v(q_1, \sigma(q_2, v_2)) \\ &= g(q_1, q_2, v_2) \end{aligned}$$

The function g is an estimate of $v\left(\cdot, v^{-1}(\cdot,\cdot)\right)$. This object allows us to construct $v(\cdot,\cdot)$. Keeping the second and third argument of g fixed and varying the first on its domain $[0,1]$ traces out an estimate of a level curve of $v(\cdot, s)$ at a particular s. If we normalize v at some point, for example so that $v(0, s) = s$, we can also trace out all remaining level curves, and thus the whole function. We are free to perform such a normalization (with the appropriate renormalization of the support of signals if needed) since the signal plays a purely informational role. Since the "data" (v_1, v_2, q_1, q_2) is iid across bidders and auctions, we can estimate the function $g(\cdot, \cdot, \cdot)$ by a nonparametric regression.[7] This approach is demanding on the richness of the data, since otherwise the estimates would be extremely imprecise. Moreover, as we will discuss below, we might not be able to obtain point identification of $v(\cdot, \cdot)$ on its full domain unless there is a positive probability of obtaining a datapoint in any neighborhood of any point of its domain at least as the number of bidders and/or auctions goes to infinity.

Proposition 5 *(Identification of the marginal valuation function)*
$\hat{g}_h(q_1, q_2, v_2) \rightarrow^{a.s.} v_h\left(q_1, v_h^{-1}(q_2, v_2)\right)$ as $N_{\geq 2h}T \rightarrow \infty$ for all q_1 such that q_1 is in the support of the equilibrium strategy of type $s = v_h^{-1}(q_2, v_2)$, where $N_{\geq 2h}$ is the number of bidders from group h with at least two bidpoints.

The identification of $v_h(\cdot, \cdot)$ relies on observing bidders with more than one bidpoint. In order to obtain identification of this function on its full domain, we would need that any type s_i that submits at least two bidpoints in an auction submits a bid for any quantity with positive probability as the number of bidders and/or auctions goes to infinity. The reason for this requirement is that in order for the nonparametric regression to be a consistent estimate for v_h on its full domain $[0, 1] \times [0, 1]$, we would need that there is a nonzero probability of observing a bidpoint in any neighborhood of any point in the domain (any type-quantity pair) as the number of datapoints goes to infinity. If this property is not satisfied, we cannot obtain full identification of $v_h(\cdot, \cdot)$ on its domain, since we can identify the function $v_h(\cdot, \cdot)$ only at the points of the support of the equilibrium strategy for players with at least two bidpoints. For example suppose that only one bidder bids, and always submits 2 bidpoints, and follows an equilibrium strategy $[q_1(s), p_1(s), q_2(s), p_2(s)]$. If we fix any signal $\bar{s} > 0$ with $q_1(\bar{s}) > 0$, then we can always find an $\varepsilon > 0$ such that even if the number of auctions goes to infinity we will never observe any bidpoint in a ball around $(0, \bar{s})$ and therefore there would always be a hole around $v_h(0, \bar{s})$. A simple corollary to Proposition 5 would establish that in the model with differentiable downward sloping demand functions from which the researcher observes just a random selection of points we can obtain nonparametric identification of the marginal valuation function on a large part of its domain by performing the nonparametric regression outlined above. How large this part would be depends on the relationship between the private valuantions and the support of market clearing prices. If, for example, equilibrium bids for the first infinitessimal

[7] There is a caveat that the explanatory variables involve $\hat{v}_2$ which itself is an estimate, and therefore we have a problem of a regression with measurement error. In a nonlinear setting such as ours, this problem is difficult to solve and involves complicated econometric techniques (Fan and Truong (1993)).

unit and last unit, $y^{-1}(0|s_i)$ and $y^{-1}(1|s_i)$, lie within the support of the market clearing price for all types, then all types should in equilibrium submit a demand function for all units in $[0,1]$, and therefore the nonparametric regression would achieve point identification on the full domain $[0,1]\times[0,1]$.

In practice it may be hard to show which points of the domain of $v(\cdot,\cdot)$ are actually in the support of the equilibrium strategy without having closed form solution for the equilibrium strategies that generated the data. If we do not want to go into such an argument to obtain point identification, we may approach the identification problem via set identification instead. We may be able to use both necessary conditions for bidding (2) and (3) and inequalities implied by assumptions on the primitives and by the data to obtain the set of all possible marginal valuation functions that would rationalize the data, in a similar way to Haile and Tamer (2003). The difficulty with this approach, however, is that both the necessary condition (3) and inequalities implied by choosing q_k rather than $q_k - \Delta$ involve the gross utility $V(\cdot, s_i)$, which is an integral of the object of interest, $v(\cdot, s_i)$. This research direction is currently left for the future.

Estimating $F_h(s)$

If we impose the normalization that $v_h(0, s_i) = s_i$, for all groups h and for all s_i, then one can get $F_h(\cdot)$ using a consistent estimate of $v_h(0,\cdot)$, such as $\hat{g}_h(0,\cdot,\cdot)$. For all bidders for whom we observe at least one bidpoint in a given auction, we can obtain the point estimate of their signal by evaluating the estimated function $\hat{g}_h$ at 0 and the submitted quantity-bid and the associated estimate of marginal valuation. Using the point estimates of private signals, we can estimate their density by kernel methods. As the number of bidders in a group or auctions grows large (and thus also the number of estimated signals), the kernel estimates of the density converge to the truth.

Proposition 6 *(Identification of the distribution of signals)*
Suppose $v_h(0, s_i) = s_i$ and a consistent estimate of $v_h(0,\cdot)$ is available. Then $\hat{f}_h(s) \to^{a.s.} f_h(s)$ as $N_{th}T \to \infty$, where $\hat{f}_h(s)$ is the kernel estimate of the density of the signals for group h.

While we are able to estimate the distribution of private information whenever we have a consistent estimate of the marginal valuation function, the latter requirement might not be satisfied in many applications. As mentioned above, whenever there is a possibility of not obtaining a datapoint in some part of the domain of v even as the number of auctions and/or bidders increases, which will in general be the case if the bidders submit step functions, the nonparametric regression might not yield a consistent estimate of the function v in that region. One possibility would be to normalize the function v_h to equal the signal at some other point $\bar{q}_h \neq 0$ so that $v_h(\bar{q}_h, s_i) = s_i$, and in particular we would choose this point so as our estimate of v_h would be consistent at that level curve. In this way the distribution of the private signals can again be estimated consistently via kernel methods. If we do not believe that we can obtain a consistent estimate of a level curve of

v_h at which we could estimate the signal, we would have to abandon our hopes for nonparametric point identification. Instead, we might be able to identify a set in which the underlying marginal valuation belongs as discussed above. Alternatively, we could parametrize v_h so that we can obtain point estimates of signals for the chosen functional form.

6 Data and Results

6.1 Description of the Data

My dataset consists of 28 auctions of Treasury bills of the Czech government. The sample period is 11/25/1999 until 12/14/2000. The auctions were conducted by the Czech National Bank. The payment by each bidder whose order was accepted was determined according to the uniform price rule; each bidder paid the market clearing price for all units for which his bid was at least the market clearing price. These auctions of T-bills were conducted weekly, with the auction plan being published quarterly. The T-bills that were sold in different auctions differed in maturities. I will consider only auctions of 3-month T-bills, since they were auctioned most often - usually at least bi-weekly. In the quarterly published auction plan the Bank announces the intentions of the Ministry of Treasury as to how many securities will be sold on a given week and of which maturity. The main purpose of the T-bills is to smooth out the difference between tax revenue and expenditures by the government.

The bidders who wished to participate in an auction of T-bills had to be preregistered by the Czech National Bank. The only requirement for the registration was that the bidder possesses either a banking license or a broker license in the Czech Republic or other EU member country. The list of registered bidders was publicly available. Furthermore, there were limits with which each registered bidder had to comply. Each bidder was obliged to buy at least 3% of the securities offered within a calendar year, and his demand in a given auction could not exceed 50% of the securities offered for sale. The first restriction was usually met by each bidder early in the calendar year. Moreover, since bidders were not given any information about the identities of the winners after any auction, we can safely ignore this restriction in our model, since it is not likely to affect the strategic behavior. The main motive for the bidders to purchase the treasury bills was for their investment portfolios, since T-bills do not carry any risk premium and thus do not have to be outweighed by any cash (or other no-risk) reserves. Moreover, many of the banks involved in these auctions are subject to investment risk regulation for various reasons, and T-bills are one of the few ways to profit from their cash reserves. It is for these reasons that the secondary market for T-bills in the Czech Republic is virtually nonexistent. The absence of active trading on the secondary market suggests that we may not have to worry about an unknown common value component in the auctions. On the other hand, how much bank i valued q units of T-bills depended on its available cash and investment decisions, which were likely to be private information. These two features together lead me to believe that the private values model might be appropriate for this setting.

Table 1 describes the summary statistics of the important data components.

Table 1: Data Summary

	Mean	Min	Max	StDev
Active Bidders in an Auction[a]	13	10	16	1.4
Number of Submitted Bidpoints	2.3	1	9	1.55
Price Bids (in CZK)[b]	986,789	985,919	987,544	252.9
Annual yields corresponding to price bids	5.30	4.99[c]	5.65	0.10
Quantity Bids[d]	0.059	0.0005	0.5	0.082
Noncompetitive Bid[e]	0.36	0	0.75	0.28
Market Clearing Price	986,745	986,190	986,972	192.1
Annual yields corresponding to mkt. cl. price	5.32	5.22	5.54	0.08
Reference interest rate	5.39	5.32	5.74	0.10
Auction Revenue (in mil USD)	423.1	144.1	598.1	170

[a] Active bidder is any bidder actually submitting a serious bid.
[b] 1USD is approximately 38CZK over the sample
[c] Lowest yield corresponds to highest bid
[d] As a share of total quantity offered for sale, across all steps
[e] As a share of total quantity offered for sale

The face value of all T-bills is 1,000,000 Czech Korunas (approximately $26,300). The range of bids in annual yield is 66 basis points, while the range of the market clearing yield is 32 basis points. Bidders submitted bids for as little as 0.05% of total quantity supplied and for as much as 50% which is the maximal amount they can demand in an auction. Bidders are allowed to submit up to 10 bidpoints (price-quantity pairs) in any given auction. Yet the average number of bidpoints submitted by a bidder in an auction is less than 3 and the maximal number of submitted bidpoints is 9. For each auction I observe all individual bids (including the noncompetitive ones placed on behalf of the government, which will be described below), the preannounced supply quantity, and the market clearing price. I also observe the final allocation. My dataset includes 16 unique bidder identities. 7 of these bidders can be classified as belonging to the "small bidder" group, since they request less than 5% of the total quantity in any given auction and also submit fewer bidpoints on average than their larger opponents. The remaining 9 bidders will be treated as belonging to the "large bidder" group. Table 2 offers a split of summary statistics between these groups.

An important feature of many treasury auctions of government securities is the possibility of "noncompetitive bids". These bids specify a quantity which the bidder would like to obtain at the market clearing price no matter what this price will be. Therefore, in terms of modelling, these bids simply decrease the available supply of T-bills in a given auction. While the rules of the auction allow for such bids to be submitted by regular bidders, they rarely use this possibility. In my dataset, none of the bidders submits a noncompetitive bid in any auction. On the other hand, the

Table 2: Data Summary - Large vs Small Bidders

	Large	Small
Active Bidders in an Auction	7.5	5.5
	(0.82)	(0.90)
Number of Submitted Bidpoints	2.45	1.11
	(1.67)	(0.68)
Price Bids[a] (in CZK)	986,792	986,781
	(253)	(251)
Quantity Bids[a,b]	0.075	0.02
	(0.09)	(0.01)

[a] Average taken across all bidpoints.
[b] As a share of total quantity offered for sale.
[c] Standard deviations in parentheses

auctioneer himself, as instructed by the Ministry, can submit such a bid even after observing the bids of regular bidders. In fact, in each announcement about an upcoming auction, which includes the details such as the number of T-bills to be auctioned off, there is a disclaimer that, "The issuer of the security reserves the right to include part or all of the emission in his own portfolio." This possibility then serves as an insurance device against low market clearing prices. Table 1 shows that the auctioneer withdrew as much as 75% of the supply. Further notice that the reference interest rate that the banks use for transactions among themselves has all descriptive statistics only slightly higher than the corresponding statistics of the market clearing yield of T-bills, which suggests that the it might be a factor in the auctioneer's decision how much supply to withdraw. In terms of empirical implementation I treat the noncompetitive bids of the government as a separate bidder group and thus resample from these in the same way as I resampled from the other two groups. This approach may be problematic if bidders' signals were affiliated. For example if bidder i getting a lower signal implied an increase in the probability that the signals of her rivals' signals were also low, then the conditional distributions of the noncompetitive bid would differ depending on the signal received. I will test for signal affiliation in the section discussing the robustness checks.

6.2 Results

Estimating marginal valuations

I first illustrate the resampling procedure, described in Section 5, that I use to estimate the distribution of the market clearing price, and thus the conditional expectation and its derivative. Consider a particular auction labeled as Auction 52 in my data. There are 13 bidders (8 large and 5 small) who actually submitted a bid. For the purposes of resampling, this is not a large number and I therefore pool 4 neighboring auctions, in which T-bills of the same maturity were offered, and consider these auctions to be independent repetitions of the same experiment. Therefore, I split my sample of 28 auctions in 7 groups with 4 auctions in each. In all auctions I assume that the number of potential bidders is the same with one exception. In particular, I assume that there is 7 potential small bidders and 8 potential large bidders[8]. The reason for assuming there is 8 potential large bidders even though there is 9 bidder identities that I classify as large is that one large bidder starts bidding first in auctions later in the sample and another large bidder at that point stops bidding aand never submits a bid again during the sample period. Bids of those two bidders overlap only in two auctions, and therefore for the group of four auctions in which these two particular auctions belong I assume that there are 9 potential large bidders rather than 8. I assume that any bidder for whom I do not observe a bid in a given auction submitted a losing bid l (or a bid of zero for any quantity) and I include such a bid function in the sample from which I resample.

The assumption of the four auctions grouped together being the "same experiment" might be problematic, since as I argued above the private information driving the marginal valuation of each bidder is assumed to come from the current state of its cash reserves and alternative investment opportunities, both of which could be affected by the outcome of previous auctions, or be correlated across auctions. Therefore I will later provide a robustness check against this assumption by testing whether winning larger quantities in earlier auctions results in lower levels of private signals for the later auctions. I decided to pool 4 neighboring auctions for two reasons. For resampling I want to include bid functions from auctions from as short a time-span as possible in order to be more confident about treating the auctions as repetitions of the same experiment, i.e., without the outside environment changing. On the other hand I need a larger number of bid functions so that resampling generates enough variation. Given that there are 15 potential bidders in an auction pooling 4 auctions together yields 60 bid functions for the purposes of resampling which should be enough to generate enough variation. Each four neighboring auctions I pool together were conducted in a time frame of two months, and the macroeconomic variables such as the consumer price index or the interest rate were stable across this period.

In the first three auctions, there are 5 active small bidders and 8 active large bidders. In the fourth auction, there are 6 active small bidders and 8 active large bidders. Under the assumptions of full symmetry and constant number of potential bidders (7 small and 8 large bidders), pooling these four auctions results in 60 ex ante symmetric bidders, who differ ex post because of their private information. Alternatively, with two groups, this results in 28 ex ante symmetric small bidders and 32 large bidders. Let us fix bidder 1's bid function and generate the different residual supply curves he might face by the above described resampling procedure. Figure 1 shows the procedure with 15 different realizations of the residual supply curves.

This process generates a distribution of market clearing prices. The distribution generated by 5000 residual supply draws is depicted in Figure 2.

[8] I also estimated the model assuming that the number of potential bidders differs across the groups of auctions and is equal to the largest number of active bidders within an auction in that group. The results were similar.

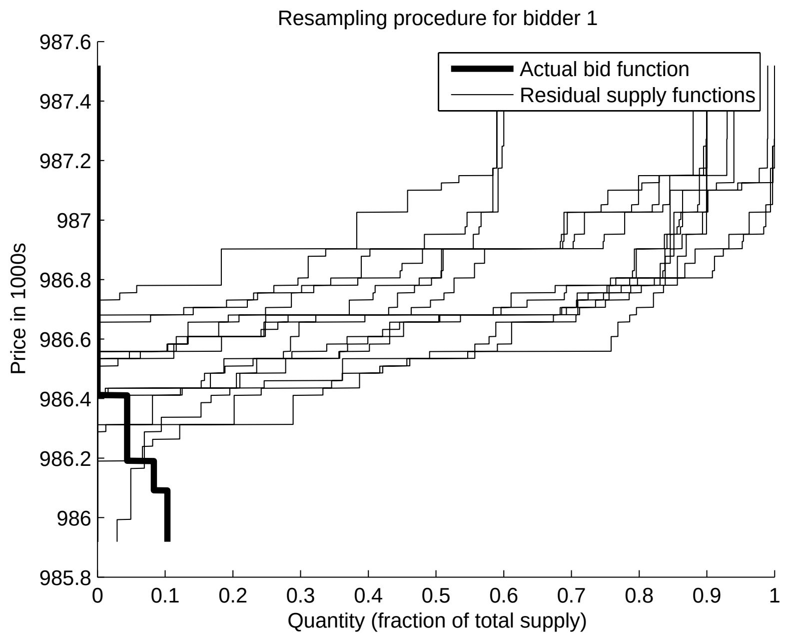

Figure 1: Resampling residual supplies

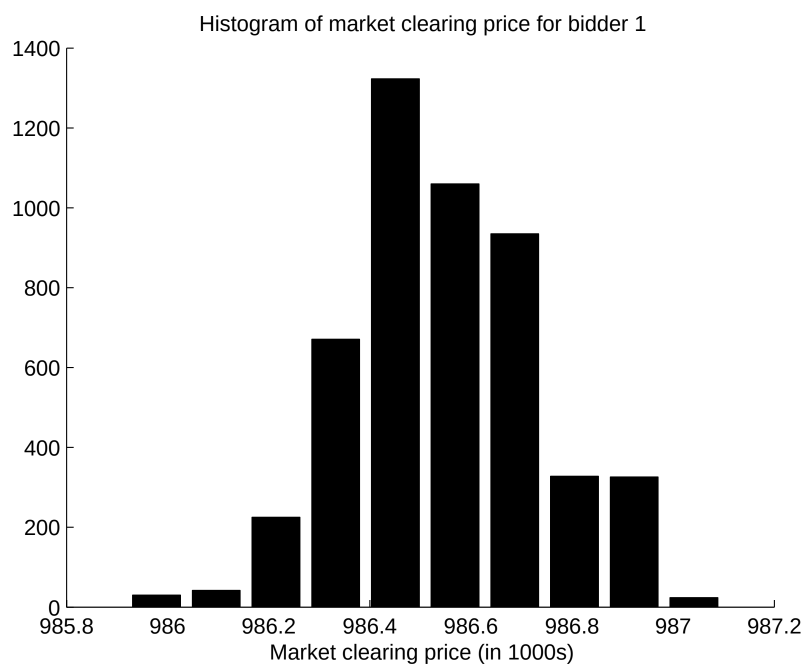

Figure 2: Distribution of market clearing price

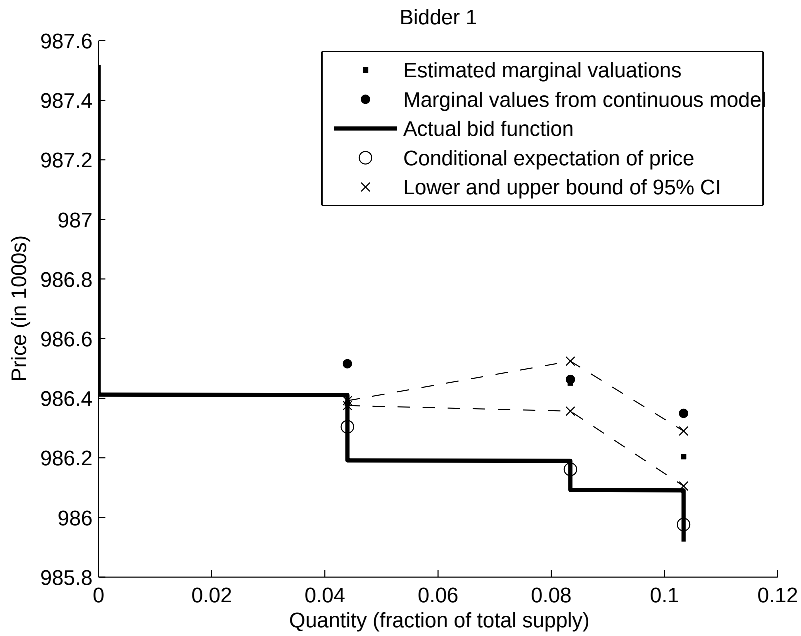

Figure 3: Marginal valuation estimation - bidder 1

With the distribution of the market clearing price, we can recover the marginal valuations for the bidder by using our optimality equation. Figure 3 shows using squares point estimates of marginal valuation of bidder 1 at quantities for which he submitted a bid. Open circles depict the conditional expectation of the market clearing price $\mathbb{E}\left[p^c|b_k > p^c > b_{k+1}\right]$. The distance between these two points is the amount of shading that the bidder executes, which is a direct measure of bidder's market power. Notice that, as suggested in Corollary 3, the actual bid is above the true marginal valuation for the first bidpoint. The fact that it occurs at the first bidpoint is not a coincidence, since the incentives to shade increase in the quantity demanded. Thus, it is more likely that for smaller quantities the marginal valuation will be closer (given market power) to the conditional expectation of the market clearing price and thus below the actual bid.

Similarly, Figure 4 shows the results of the estimation for bidder 4. At smaller quantities, the bid again exceeds the estimated marginal value.

Repeating the same procedure for each bidder in the auction, we obtain point estimates of the marginal valuation function $v(q,s)$ at the different (observed) quantities that the bidders request and at the different (unobserved) signal levels s. As described above, we could use information from bidders who submit at least two bidpoints to estimate $v(q,s)$ nonparametrically, as long as in the limit, as the number of datapoints increases, the whole domain of $v(.,.)$ would be covered. Even if the latter condition were satisfied, however, this exercise would not be useful for empirical estimation with little data, since it involves a three dimensional kernel regression.

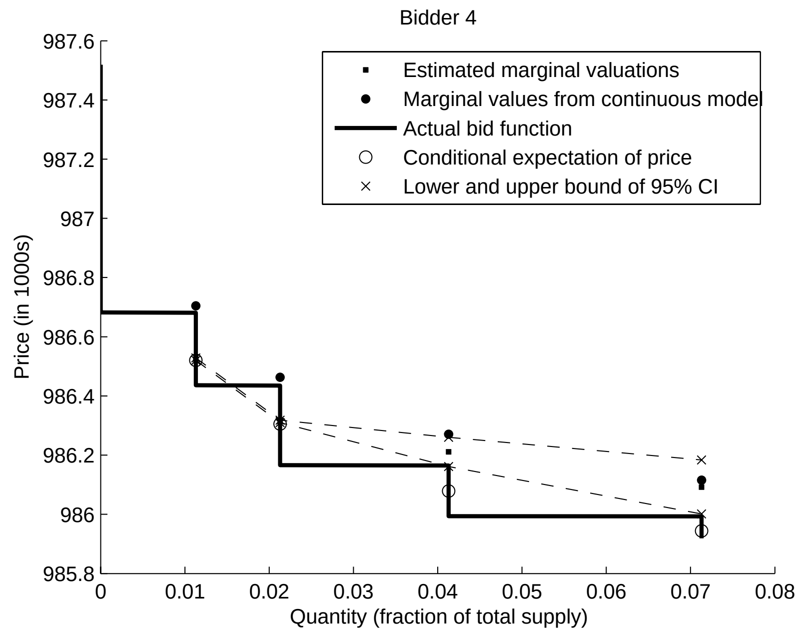

Figure 4: Marginal valuation estimation - bidder 4

Standard Errors

The asymptotic variance of the estimated marginal valuations is difficult to obtain, since our marginal valuation estimator is a nonlinear function of the distribution of the market clearing price. But since this distribution is also estimated, taking the usual delta method approach would be very cumbersome. For this reason I employ bootstrap methods to compute the standard errors of my estimates. The reported standard errors are from the sample of 100 estimates generated by repetitions of the estimation procedure with a new bootstrap sample of bid functions at each round.

Step functions versus continuous downward sloping bids

One might wonder what difference it makes to assume that bidders submit step functions strategically, rather than treating the observed bidpoints as some selection from a downward sloping continuous function, as assumed in previous work. Equation (1) reveals that in the continuous bid functions setting the observed bids should equal marginal valuations less a markup associated with that bidder's market power. In other words, it is necessarily the case that within such a model the marginal valuations are strictly above the observed bids (as long as these bids are within the support of the distribution of the market clearing price), unless the bidder is a pricetaker, in which case the two values coincide. It is not immediately obvious, however, how to estimate such a model using data consisting of few points, rather than full downward sloping continuous functions. First of all, why do we not observe the whole downward sloping functions, i.e., how were the data generated? It seems to me that the researcher would have to assume that he observes some random selection of points from the true bid function. This assumption would allow for use of optimality equation (1), which has to hold at every point of the bid function, and thus also at the observed ones. Do the derivatives of the distribution of the market clearing price exist? The derivative with respect to p should not cause trouble, since that is simply the density of p. The other derivative, however, is not so straightforward. It turns out that this derivative is closely related to the derivative of the expectation of the market clearing price.[9] The problem, however, is that it is no longer an expectation conditional on an interval, but rather on a particular value of the market clearing price. Therefore to estimate this derivative w.r.t. q_k consistently we would have to shrink the neighborhood around both q_k and b_k. After this procedure, we would obtain a similar term related to the expected local market power of a bidder at a particular bidpoint, and the estimate of the corresponding marginal value would thus be just the bid plus this term as seen from Euler equation (1). Therefore to illustrate the difference between using the optimality conditions for the model with step functions from the one with continuously differentiable bids, we can think of the estimates of marginal values in the latter model as adding the estimated shading factor[10] from the model with step functions to the observed bid rather than to the conditional expectation of price. Clearly, if the model with step functions is the one from which the data is generated, then using the necessary conditions from the model with continuously differentiable bid functions would overestimate marginal valuations. If these biased estimates are used for counterfactual exercises, such as the computation of revenues from a discriminatory auction, we would expect the results to be biased towards the discriminatory auction. Figures 3 and 4 show that the model used for estimation can matter, especially when estimating marginal valuations at low quantities, where bidders do not have a lot of market power. Furthermore, I will now show that in a nonnegligible number of the auctions, the actual ex post revenue exceeded the revenue that would have been realized had all bidders bid the upper bound of their estimated marginal valuation functions.

Counterfactual: Truthful Bidding

In my first counterfactual analysis, I compare the actual revenue to the revenue from a best case Vickrey auction, in other words a uniform price auction in which bidders truthfully bid their marginal valuation schedules without actually receiving any payments. To perform this experiment exactly, we need to know the full functional form of $v(q,s)$. Instead, I construct an upper and lower envelope of marginal valuations by using step functions that have steps at the estimated marginal valuations. Unfortunately, we do not have enough information to construct the upper bound on the marginal valuation to the left of the first step. Similarly we can only bound the marginal valuation to the right of the last step from below by zero and from above by the last estimated marginal value. I therefore assume that the estimated first marginal valuation is also equal to the highest possible marginal valuation. This assumption should not be too influential, since for the important

[9] See Kastl (2005) for details.

[10] Recall that the shading factor is the difference between the conditional expectation of price and the estimated marginal value.

(large) bidders whose demands are essential for market clearing, the market usually clears at one of their "interior" steps, and we use the appropriate bounds for those. Nevertheless, to test the robustness of the results with respect to this assumption, I also tried using the first step plus a mark-up as the maximum marginal valuation for smaller quantities, and obtained qualitatively similar results. While the upper bound on the marginal valuation for larger quantities than the last observed bidpoint is the marginal value estimated at this bidpoint, I cannot use such a bound in my analysis. The reason being that there can be a small bidder who demands just a negligible share of the total supply with a high marginal value at his last step, and by bidding such an upper bound for all larger quantities she might win the full supply. I will therefore assume that the marginal value for larger quantities than the one demanded at the last bidpoint is zero. Using these upper and lower envelopes of marginal valuations, I obtain the market clearing price given the same ex post realization of noncompetitive bids as in the actual auction. Tables 3 and 4 report the results in terms of the market clearing price. The first column reports the actual realized market clearing price and the second and third column the market clearing price under bidding truthfully the lower or upper envelope respectively.

These tables reveal that the actual market clearing prices are not far from those that would be obtained under truthful bidding. This suggests that bidders do not have enough (local) market power around the expected market clearing price to adversely affect auction's revenue. In order to offer a better idea about the magnitudes of the differences in revenue, Table 5 reports the same results in terms of annual percentage yield of the T-bills.

In 7 of the 28 auctions, which are highlighted by an asterisk in the table, the actual ex post revenue exceeds the revenue from bidding the upper bound of the marginal valuation schedules, which suggests that the point raised in Corollary 3 is not purely theoretical. These results may cast some doubt on the conclusions that Hortaçsu (2002) reaches in his empirical study of Turkish treasury auctions, which have a discriminatory format. In particular, he concludes that since the revenue generated in a uniform price auction in which bidders submit the upper bound[11] of the estimated marginal valuations as their bids is lower than the actual revenue, the discriminatory auction performs better ex post. (From the ex ante perspective, when he draws the bid functions randomly before the auction, he cannot reject the revenue equivalence hypothesis.) My results suggest that using a model with continuously differentiable bid functions as an approximation to the true model of discrete bidding to conduct any counterfactual exercises will most likely lead to results that are biased towards the discriminatory auction.

Effectiveness of value extraction

How effective a mechanism are these uniform price auctions? Could the Czech government do better by using a discriminatory auction? One way to get a handle on these questions is to compare the performance of the employed mechanism to the ideal mechanism, which would implement

[11] Hortaçsu constructs this upper bound in the same way.

Table 3: Comparison with truthful bidding - part 1

Auction	Actual p	TruthBidMin p[a]	TruthBidMax p[b]
52	986,190	986,347	986,476
		(39.95)	(60.66)
55	986,509	986,475	986,545
		(16.99)	(19.24)
56	986,460	986,509	986,610
		(42.79)	(42.49)
60	986,805	986,755	986,829
		(22.03)	(23.3)
61	986,928	986,879	986,947
		(21.11)	(18.56)
64	986,829	986,831	986,847
		(2.64)	(2.04)
65	986,805	986,832	986,852
		(5.95)	(6.02)
67*	986,903	986,855	986,868
		(6.22)	(12.96)
69	986,834	986,805	986,875
		(10.01)	(9.13)
72	986,854	986,846	986,903
		(7.37)	(1.23)
73	986,903	986,879	986,909
		(10.16)	(5.42)
75*	986,903	986,864	986,892
		(7.48)	(14.20)
76	986,903	986,885	986,917
		(10.19)	(2.41)
81	986,854	986,820	986,861
		(5.63)	(10.65)
82	986,805	986,812	986,834
		(4.82)	(8.48)
85	986,854	986,854	986,878
		(2.58)	(6.60)
Mean (52-108)	986,745	986,729	986,768

* Ex post revenue higher than under truthful bidding

[a] Market clearing price when bidding the lower envelope of marginal valuations

[b] Market clearing price when bidding the upper envelope of marginal valuations

[c] Bootstrap std. errors in parentheses

Table 4: Comparison with truthful bidding - part 2

Auction	Actual p	TruthBidMin p[a]	TruthBidMax p[b]
86*	986,903	986,873	986,887
		(6.98)	(13.60)
87	986,829	986,832	986,832
		(2.10)	(3.37)
91*	986,972	986,887	986,952
		(2.76)	(8.59)
92	986,805	986,805	986,855
		(0.00)	(5.47)
94*	986,878	986,744	986,785
		(41.79)	(46.45)
95*	986,829	986,685	986,688
		(14.87)	(18.42)
99	986,632	986,632	986,659
		(2.71)	(4.28)
100	986,608	986,608	986,636
		(0.00)	(3.21)
103	986,487	986,487	986,540
		(4.89)	(4.28)
104*	986,534	986,500	986,500
		(4.97)	(4.99)
107	986,509	986,522	986,542
		(8.58)	(5.16)
108	986,534	986,544	986,594
		(10.1)	(20.77)
Mean (52-108)	986,745	986,729	986,768

[*] Ex post revenue higher than under truthful bidding
[a] Market clearing price when bidding the lower envelope of marginal valuations
[b] Market clearing price when bidding the upper envelope of marginal valuations
[c] Bootstrap std. errors in parentheses

Table 5: Comparison with truthful bidding - market clearing yield

Auction	Actual yield	Highest yield[a]	Lowest yield[b]
52	5.54	5.48	5.42
55	5.41	5.42	5.40
56	5.43	5.41	5.37
60	5.29	5.31	5.28
61	5.24	5.26	5.23
64	5.28	5.28	5.27
65	5.29	5.28	5.27
67*	5.25	5.27	5.26
69	5.28	5.29	5.26
72	5.27	5.27	5.25
73	5.25	5.26	5.25
75*	5.25	5.27	5.26
76	5.25	5.26	5.24
81	5.27	5.28	5.27
82	5.29	5.29	5.28
85	5.27	5.27	5.26
86*	5.25	5.26	5.26
87	5.28	5.28	5.28
91*	5.22	5.26	5.23
92	5.29	5.29	5.27
94*	5.26	5.31	5.30
95*	5.28	5.34	5.34
99	5.36	5.36	5.35
100	5.37	5.37	5.36
103	5.42	5.42	5.40
104*	5.40	5.41	5.41
107	5.41	5.40	5.40
108	5.40	5.40	5.38
Mean	5.31	5.32	5.30

* Ex post revenue higher than under truthful bidding
[a] Achieved by bidding the lower envelope of marginal valuations
[b] Achieved by bidding the upper envelope of marginal valuations

an efficient allocation and extract full surplus. We can use the upper envelope of the estimated marginal valuations together with the estimated distribution of the market clearing price to obtain estimates of (upper bound of) bidders' expected (interim) utility per T-bill sold in the auction. If this expected utility is close to zero for every bidder, and the allocation is efficient, then the auction mechanism would perform well even from an ex ante perspective. Under the equilibrium hypothesis, the observed bid function of each bidder should be a best response of his type to the equilibrium strategies of other bidders. Using the estimated distribution of the market clearing price conditional on bidder i's bid and setting $c(K_i, t_i) \equiv 0$, I can evaluate i's expected utility given the submitted bid function, i.e., conditional on his type. In equilibrium, this submitted bid function should deliver the highest utility this bidder can obtain (given his type). Therefore this exercise indeed delivers an estimate of the maximal interim utility of each bidder. The results are reported in Tables 6 and 7.

The minimal estimated interim utility is close to zero, which suggests that submitted bid functions are individually rational. It also suggests that using the upper envelope of the marginal valuations may be close to the true valuation functions. This should hold at least for bidders with interim utility very close to zero, since a lower marginal valuation curve would result in negative interim utility, in which case the observed bid function would not be individually rational. Allocations in all auctions appear to be efficient, since the ratio of the realized surplus to the efficient surplus exceeds 0.9999 on average. Moreover, the sum of expected surpluses across all bidders (strictly speaking, it is the sum across their actual realized types) reported in the columns labelled "Total" of Tables 6 and 7 is close to zero. I conclude that the uniform price auction mechanism performed well, in terms of both efficiency and value extraction. The columns labelled "Total" reveal that in 11 auctions we cannot reject the hypothesis that full expected surplus has been extracted. In 22 out of the 28 auctions we cannot reject the hypothesis that the mechanism failed to extract less than 1 basis point worth of bidders' surplus. In 25 out of the 28 auctions studied we cannot reject that the mechanism failed to extract less than 2 basis points worth of bidders' surplus. On average the mechanism failed to extract less than 3 basis points worth of bidders' surplus[12]. Because the estimated average total expected bidders' surplus is a consistent estimate of the part of the surplus that the mechanism fails to extract ex ante, and because the allocation is nearly efficient, I conclude that the uniform price auction exhibits excellent performance. Value extraction might be even better, since I considered the upper bound on the marginal valuation functions of each realized type when performing the computations. Because the uniform price auction mechanism performed well in terms of both value extraction and allocative efficiency, switching to an alternative auction mechanism is unlikely to result in economically significant improvements in either aspect.

My computations appear to be the best way to assess the performance of an auction mechanism, without having to obtain counterfactual strategies. They are computationally easy to implement, and they can be implemented for data from both uniform price and discriminatory auction mech-

[12]Notice that if we eliminated the insignificant outlier in auction 61, it would be 2 basis points.

Table 6: Interim profit of bidders per T-bill for sale - part 1

Auction | Expected Surplus[b]	Average	Maximal	Minimal	Total	Allocative Efficiency[c]
52	6.71	64.52	0	87.29	0.9995
	(6.34)	(79.11)	(0.003)	(82.41)	(0.003)
55	2.98	13.67	-0.20	38.74	0.99991
	(0.46)	(4.1)	(0.15)	(5.92)	($1*10^{-6}$)
56	9.25	32.44	-0.34	120.29	0.99992
	(2.39)	(17.88)	(0.49)	(31.02)	($4*10^{-6}$)
60	20.29	117.79	-0.21	283.98	0.99977
	(5.33)	(73.12)	(0.004)	(74.67)	($7*10^{-6}$)
61	32.66	411.29	-0.004	457.18	0.99942
	(29.00)	(401.74)	(0.08)	(406.05)	(0.0011)
64	1.65	7.67	-0.20	19.77	0.99996
	(1.15)	(9.38)	(0.11)	(13.80)	($4*10^{-7}$)
65	2.93	14.80	-0.33	35.13	0.99994
	(6.44)	(74.28)	(0.08)	(77.28)	($8*10^{-7}$)
67*	2.27	14.74	-0.22	31.78	0.99994
	(0.31)	(2.57)	(0.11)	(4.32)	($2*10^{-6}$)
69	5.62	60.72	-0.18	73.02	0.99993
	(21.9)	(285.2)	(0.01)	(284.7)	($1*10^{-6}$)
72	1.14	7.37	-0.05	18.27	0.99996
	(0.26)	(3.30)	(0.02)	(4.19)	($5*10^{-7}$)
73	0.66	3.43	-0.09	10.63	0.99996
	(0.20)	(1.33)	(0.05)	(3.18)	($2*10^{-7}$)
75*	1.20	3.67	-0.26	16.90	0.99997
	(0.31)	(2.08)	(0.96)	(4.27)	($3*10^{-7}$)
76	3.11	14.78	-0.07	40.43	0.99997
	(2.00)	(24.18)	(0.20)	(26.01)	($4*10^{-7}$)
81	0.32	3.63	-0.65	4.54	0.99997
	(0.21)	(2.51)	(0.31)	(3.01)	($3*10^{-6}$)
82	0.45	4.09	-0.60	6.31	0.99998
	(0.31)	(2.84)	(0.34)	(4.37)	($5*10^{-7}$)
85	10.01	122.73	-0.47	130.12	0.99937
	(0.27)	(1.88)	(0.59)	(3.48)	(0.00011)
Mean (Auctions 52-108)	5.07	43.24	-0.20	66.14	0.9999

[*] Ex post revenue was higher than under truthful bidding
[a] Standard errors in parentheses
[b] Using the upper envelope of marginal valuations
[c] Defined as (Actual surplus)/(Surplus from the efficient allocation)

Table 7: Interim profit of bidders per T-bill for sale - part 2

Auction | Expected Surplus[b]	Average	Maximal	Minimal	Total	Allocative Efficiency[c]
86*	0.78	3.67	-0.27	10.14	0.99998
	(0.22)	(1.17)	(1.01)	(2.82)	($6*10^{-7}$)
87	0.75	4.98	-0.08	9.01	0.99999
	(0.34)	(2.70)	(0.40)	(4.11)	($2*10^{-7}$)
91*	3.29	29.43	0	39.53	0.99998
	(0.63)	(4.99)	(0.01)	(7.57)	($2*10^{-6}$)
92	9.87	55.25	-0.27	118.49	0.99993
	(3.30)	(31.06)	(4.00)	(39.54)	($8*10^{-8}$)
94*	2.83	14.86	-0.01	28.27	0.99993
	(1.47)	(14.33)	(0.41)	(14.73)	($6*10^{-7}$)
95*	8.88	70.76	-0.82	88.81	0.99999
	(2.51)	(22.15)	(1.97)	(25.08)	($2*10^{-7}$)
99	3.96	52.51	-0.08	55.43	0.99975
	(12.03)	(168.52)	(0.052)	(168.46)	($4*10^{-5}$)
100	3.74	25.19	-0.16	41.12	0.99994
	(2.30)	(22.20)	(0.04)	(25.32)	($6*10^{-7}$)
103	1.51	16.78	0	19.60	0.99967
	(0.55)	(7.05)	(0)	(7.10)	($1*10^{-7}$)
104*	0.54	3.12	-0.20	6.42	0.99999
	(0.23)	(1.90)	(0.16)	(2.81)	($3*10^{-7}$)
107	0.54	4.27	-0.01	6.98	0.99998
	(0.18)	(1.98)	(0.21)	(2.32)	($2*10^{-6}$)
108	4.13	32.62	-0.21	53.64	0.99995
	(14.22)	(181.4)	(0.02)	(184.83)	($2*10^{-6}$)
Mean (Auctions 52-108)	5.07	43.24	-0.20	66.14	0.9999

[*] Ex post revenue was higher than under truthful bidding
[a] Standard errors in parentheses
[b] Using the upper envelope of marginal valuations
[c] Defined as (Actual surplus)/(Surplus from the efficient allocation)

anisms. It would be useful to examine other datasets and auction mechanisms.

Bidding costs

In equilibrium, the cost $c(K_i+1, t_i)$ must be weakly higher than the expected benefit of submitting one more bidpoint. Similarly, $c(K_i, t_i)$ must be weakly less than the expected benefit of going from $K_i - 1$ to K_i bidpoints. This allows us to compute bounds on the implied cost of bidding. I compute the lower bound for bidders submitting one bidpoint by searching for the optimal bid function with two steps given the distribution of residual supplies obtained by the resampling procedure and assuming that bidder i's marginal valuation is the upper envelope of my point estimates. The estimates suggest that the lower bound on costs of going from 1 to 2 bidpoints can total as little as \$2 and as much as \$147. I use the same procedure for bidders who submitted two bidpoints to obtain an upper bound. I estimate that the upper bound on costs of going from 1 to 2 bidpoints can be as low as \$13 and as high as \$360. These computations suggest that the extra benefit of finetuning the bid function a little more may not be that high. I should note that these bounds are for a given specification of the marginal valuation curve (in the case above, the upper envelope of the estimates). It does not necessarily follow that the upper bound obtained from the experiment performed is the largest upper bound for all permissible marginal valuation functions.

Estimation of the distributions of private signals

Using the bounds approach described above does not lead to estimates of bidders' actual signals, and thus does not permit estimation of the distribution of private signals. Hence I will adopt a parametric assumption for $v(\cdot, \cdot)$ that allows me to estimate the private signals and their distribution.

Parametric Approach

To obtain the estimates of private signals, I first specify a parametric functional form for the marginal valuation function. I can then obtain imputed signals corresponding to submitted bids. For simplicity assume that the marginal valuation function is linear in signal and quantity and separable in its two arguments, $v(q, s) = s + \beta_1 q$. The private information s can thus be interpreted as the marginal utility from the first infinitessimal unit consumed $v(0, s)$, and in econometric terms as a fixed effect for a given bid curve.

This parametric structure allows me to identify β by using bidders who submitted at least two bidpoints. As before we can invert for the unobservable signal to obtain a relationship:

$$v_{i1} - v_{i2} = \beta_1 (q_{i1} - q_{i2})$$

Now we can estimate β by standard regression methods. The estimate of β will be consistent as long as the measurement error contained in $v_{i1} - v_{i2}$ is uncorrelated with $q_{i1} - q_{i2}$. In this regression I used all bidders who submitted a bid function with at least two steps. While for a bidder with

Table 8: Marginal valuation function regression (Auctions 52-60)

	Pooled	Large bidders	Small bidders
β_1	-17461*	-17403*	-27818*
	(2180)	(2475)	(8616)
n	39	30	9
R^2	0.63	0.63	0.56

[*] Significant at 5%
[a] Std. errors in parentheses

more than 2 steps any pair of his bidpoints would be a valid observation, we might be worried that the error term might be correlated across observations in that case. Therefore for each such bidder I used only the first two steps. I first estimated this regression using a pooled sample of all bidders, and later using the subsamples of small and large bidders separately. The estimates for the first group of auctions are reported in Table 8. The results suggest that the marginal valuation of bidders from the small group is declining more steeply in quantity obtained than that of large bidders. An increase in quantity bought by a small bidder of one percentage point results in decrease in marginal valuation of the last unit by 278 CZK (cca $8), which is almost twice the decline for a large bidder. Results in all other groups of auctions were qualitatively similar - the marginal valuation of smaller bidders declines significantly faster than that of large bidders. Using the estimates, we can obtain imputations of signals corresponding to submitted bid functions (i.e., the bid functions' fixed effects) and thus obtain an estimate of the distribution of the private information as depicted in Figure 5 for the first group of auctions. Since I cannot obtain an estimate of the signal for small bidders that do not submit a serious bid in an auction, the shown density is that conditional on submitting a serious bid. The estimate shown in the figure was obtained by using a Gaussian kernel with automatic bandwidth selection. As we can see in the figure, small and large bidders indeed differ substantially in the distributions of their private signals.

Since the parametric approach outlined above uses a subsample of bidders with at least two bidpoints to estimate the parameter β, we may worry about a sample selection problem. Conditional on the same cost of bidding (type t) a bidder with higher signal s is more likely to submit more than one bidpoint and hence is more likely to be in the sample. While it is likely that some sample selection takes place, it does not influence the consistency of the estimate of the slope of the marginal valuation function β as long as this slope does not vary with s. To verify the robustness of this parametric approach, I also estimated private information under an alternative scenario. I imposed a simplifying assumption that the first estimated marginal valuation is equal to the private signal received by that particular bidder. In other words, instead of normalizing the function $v(\cdot,\cdot)$ to equal the private signal at a particular quantity level $\bar{q}$, I imposed that $v(q_{i1}, s_i) = s_i$. Notice

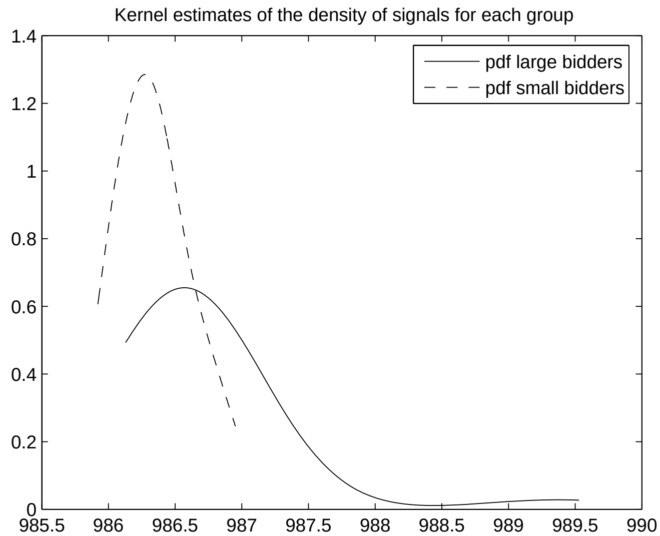

Figure 5: Nonparametric Density Estimation

that this approach does not suffer from using a selected subsample of observed bid functions and it is equivalent to using our bounds on the marginal valuation function constructed above, and evaluating these at $q = 0$, since the marginal valuation was assumed to be constant to the left of the first bidpoint. The results from both approaches were similar.

Robustness checks

I first check whether treating four auctions as repetitions of the same experiment is problematic. A problem might arise if there is some persistent relationship between quantity won in earlier auctions and valuations in the later auctions. For example, if a bidder wins a high quantity in an auction in week 1, his valuation for units offered in the auction in week 2 might decrease. To test this dependence I regress the estimates of signals in auction t on the quantity won in auction $t-1$. The results are reported in Table 9. Under the assumption that the measurement error in the signal estimate from auction t is not correlated with the quantity won in auction $t-1$, the estimates are consistent. The data does not reveal a significant relationship between the signal in auction t and quantity won in auction $t-1$. I therefore conclude that pooling the four auctions together does not constitute a major problem.

Another problem might arise if bidders' signals are affiliated. While affiliation of signals would be a problem on its own for the resampling method, it would also be troublesome because of the presence of the noncompetitive bids by the government. Recall that noncompetitive bids are submitted with the knowledge of the bids submitted by regular bidders. Suppose that the objective of the auctioneer who submits the noncompetitive bid on behalf of the ministry is to maintain a minimal level of the market clearing price, by reducing the supply if necessary. Therefore the supply,

Table 9: Testing dependence of signals and quantities won earlier

Auctions:	$\{52-60\}$	$\{61-67\}$	$\{69-75\}$	$\{76-85\}$	$\{86-92\}$	$\{94-100\}$	$\{103-108\}$	All
Constant	986,477	986,949	986,894	994,878	1,003,891	991,824	986,539	990,818
	(49.93)	(32.3)	(16.7)	(6765)	(8531)	(4015)	(21.2)	(1626)
q_{t-1}	885.6	161.8	193.5	-8302	-92956	-12816	514.6	-13576
	(514.8)	(310.1)	(161)	(55972)	(90899)	(30417)	(340.5)	(15638)
R^2	0.09	0.01	0.04	0.001	0.03	0.01	0.06	0.003
N	33	37	39	39	32	33	37	250

[a] Std. errors in parentheses
[b] Dependent variable: s_t

Table 10: Wilcoxon Rank Sum Test of Equality of Distributions $F_{s_{-1}|s_1}$

Auctions | Sample split[a]	$\{1,2\},\{3,4\}$	$\{1\},\{2,3,4\}$	$\{1\},\{2\}$	$\{3\},\{4\}$
$\{52,55,56,60\}$	0.85	0.14	0.45	0.04
$\{61,64,65,67\}$	0.78	0.99	0.88	0.37
$\{69,72,73,75\}$	0.25	0.37	0.38	0.19
$\{76,81,82,85\}$	0.12	0.95	0.05	0.60
$\{86,87,91,92\}$	0.30	0.02	0.83	0.28
$\{94,95,99,100\}$	0.40	0.09	0.85	0.55
$\{103,104,107,108\}$	0.82	0.79	0.72	0.17

[a] For example splitting the first group of auctions $\{52,55,56,60\}$ according to the split $\{1,2\},\{3,4\}$ means that two samples are created. First sample consisting of auctions $\{52,55\}$ and second sample of auctions $\{56,60\}$
[b] p-values of H_0: Samples are from the same continuous distribution.

even though preannounced, is random from the perspective of the bidders. Therefore, when bidders solve their maximization problem, they have to take an expectation with respect to the distribution of supply. If bidders' signals were affiliated, a lower signal would result in a conditional distribution of supply that is first order stochastically dominated by a conditional distribution obtained after a high signal draw. In this case, we would have to adjust the estimation procedure. To test for signal affiliation, I will employ a nonparametric rank test. I first split the sample of estimated signals from the four auctions from each estimation round into subsamples. I report the results for four particular ways of splitting the sample, but alternative splits led to similar results. I then leave out the signals of bidder 1, and conduct a one-sided Wilcoxon Rank Sum test of equality of distributions $F_{s_{-1}|s_1}$.[13] Under the null hypothesis of no affiliation, the two distributions are equal. Table 10 reports the p-values for which H_0 holds for this test. The results suggest that we cannot reject the null that the signals are not affiliated.

[13]Doing the same exercise for other bidders yields similar results.

7 Conclusion

In this paper I analyze a model of a uniform price auction of a perfectly divisible good with private information. I show that the fact that bidders submit step functions has important implications for equilibrium. I characterize equilibrium strategies in a model in which bidders submit step functions. There is a close relationship between the optimal behavior of an oligopolist facing uncertain demand and that of a bidder in a multiunit auction with private information. My results suggest that it is difficult to make an indirect comparison between a uniform price and discriminatory auction as, for example, is done in Hortaçsu (2002), as in the uniform price auction bidders may submit bids above their marginal valuation schedule when bid functions have finite number of steps. This point is not purely theoretical. In many of the auctions in my empirical analysis, actual revenue exceeds the revenue that would have been achieved had the bidders bid their marginal valuation schedules. Furthermore, the model with continuously differentiable bid functions might not be a good approximation, since the results may be biased towards the discriminatory auction. I propose a new method to evaluate the performance of the employed mechanism, based on estimating the effectiveness of values extraction and the efficiency of the allocation. In the empirical analysis of Czech treasury auctions, I examine the performance of the uniform price auction. I conclude that the uniform price auction performed well. The allocation was nearly efficient, and the mechanism extracted almost all of bidders' values. The method also allows me to obtain an estimate of the implicit bidding costs faced by bidders in these auctions. I find that the bidders may not benefit much from submitting a finer bid function.

For my empirical analysis I used only one of the necessary conditions for equilibrium bidding that allowed me to obtain point estimates of the marginal value at the submitted quantity bids. Using the other necessary condition together with the (infinitely many) inequalities implied by the bid being globally optimal in equilibrium, we may be able to obtain a tighter bound on the marginal valuation function than the upper and lower envelopes of the obtained point estimates used in this paper. Thus improving idenfication of the marginal valuation function is a promissing direction of future research. Apart from this direction it would also be interesting to study the role of noncompetitive bids by the auctioneer. Furthermore, the question of providing methods for computation of counterfactual equilibria is of great interest. Finally, it would be interesting to see whether a strategic choice of the maximal number of bidding steps could eliminate "bad" equilibria that yield low revenue. Does an upper bound on the number of bidpoints increase the lower bound on expected revenue?

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A Appendix

A.1 Proof of Proposition 1

I will follow the strategy of Reny and Zamir (2004) of discretizing the action space to a fine grid and applying existence results from such settings and then taking a sequence of the discretized games as the action spaces become dense in the original ones to argue that the limit of the equilibria from the discretized games constitutes an equilibrium of the original game.

For $n = 1, 2, ...$, let G^n denote the modified auction in which bidder i's finite set of possible bids is denoted by B_i^n and common quantity grid Q^n. Further, suppose that $B_i^n \supseteq B_i^{n-1}$ and that $B_i^\infty = \cup_n B_i^n$ is dense in B_i and similarly for Q^n. Let the set of the possible number of steps be the same as in the original game $K = \{0, ..., \bar{K}\}$. Notice that I require that the discretized bid space is individual specific, while the discretized quantity space may be common. The reason for this is that I require that at each discretized game, no two players have a common serious bid, $B_i^n \cap B_j^n = \{l\}$ $\forall i, j$ $\forall n < \infty$, and thus a tie (two bidders with the same bid at the market clearing price) can never occur. Each discretized game G^n is thus a game with finite actions and thus Nash equilibrium in distributional strategies exists by Theorem 1 of Milgrom and Weber (1985). Let $\boldsymbol{\sigma}^n$ denote an equilibrium of G^n. Notice that the set of all probability measures over a product of i's type and all non-increasing left-continuous bid functions with at most $\bar{K}$ steps defined on $B \times Q$ (i.e., the set of his distributional strategies) is a compact metric space with respect to the weak topology and any equilibrium in distributional strategies of the discretized game can be expressed as an element of this set. Therefore as we take the limit of such equilibria there exists a convergent subsequence, so that the limit, $\hat{\sigma}$, is itself a probability measure over $S_i \times T_i \times [\{l\} \cup [0, \bar{p}] \times [0, 1]]^{\bar{K}}$.

Now we need to argue that $\hat{\boldsymbol{\sigma}}$ is an equilibrium of the unrestricted uniform price auction game. The problem in this argument arises because of the rationing pro-rata on-the-margin and the associated possible discontinuity in ex post payoff due to ties. In order to establish existence in the unrestricted game, we need to show the following for a.e. (s_i, t_i):

$$\sup_{b \times q \in (B_i \times Q)^K} EU_i \left(b, q, \hat{\boldsymbol{\sigma}}_{-i} | s_i, t_i\right) \leq \lim_n EU_i \left(\sigma_i^n, \boldsymbol{\sigma}_{-i}^n | s_i, t_i\right) \tag{A-1}$$

and also:

$$\lim_n EU_i \left(\sigma_i^n, \boldsymbol{\sigma}_{-i}^n | s_i, t_i\right) = EU_i \left(\hat{\sigma}_i, \hat{\boldsymbol{\sigma}}_{-i} | s_i, t_i\right) \tag{A-2}$$

$EU_i\left(\sigma_i^n, \boldsymbol{\sigma}_{-i}^n | s_i, t_i\right)$ denotes the conditional expected utility of type (s_i, t_i) obtained using a behavioral strategy[14] of type (s_i, t_i), i.e., a conditional distribution over i's actions, corresponding to his distributional strategy σ_i^n.

Consider an arbitrary bid vector (b, q) (i.e., i's pure strategy). Then we have:

$$EU_i\left(b, q, \hat{\boldsymbol{\sigma}}_{-i} | s_i, t_i\right) \leq \lim_{b' \downarrow b} \lim_{q' \to q} EU_i\left(b', q', \hat{\boldsymbol{\sigma}}_{-i} | s_i, t_i\right) \tag{A-3}$$

where the inequality follows since the only case in which we do not obtain an equality is that some component of b equals somebody else's bid with positive probability and this bid becomes a market clearing price with positive probability. In that case as we take the limit as $b' \downarrow b$ the payoff to i would be strictly higher, since he would be breaking the tie in his favor at any point along the sequence.

As B_i^n becomes dense in B_i, for every $b' \in B_i$, every $\varepsilon > 0$, and a.e. (s_i, t_i) there exists $\bar{n} \geq 1$ and $\bar{b} \in \left(B_i^{\bar{n}}\right)^K, \bar{q} \in \left(Q^{\bar{n}}\right)^K$ such that

$$\begin{aligned}
\lim_{b' \downarrow b} \lim_{q' \to q} EU_i\left(b', q', \hat{\boldsymbol{\sigma}}_{-i} | s_i, t_i\right) &\leq EU_i\left(\bar{b}, \bar{q}, \hat{\boldsymbol{\sigma}}_{-i} | s_i, t_i\right) + \varepsilon \\
&\leq EU_i\left(\bar{b}, \bar{q}, \boldsymbol{\sigma}_{-i}^n | s_i, t_i\right) + 2\varepsilon, \text{ for } n \geq \bar{n} \\
&\leq EU_i\left(\sigma_i^n, \boldsymbol{\sigma}_{-i}^n | s_i, t_i\right) + 2\varepsilon, \text{ for } n \geq \bar{n}
\end{aligned}$$

where the first and second line follow because the bid vector $\bar{b}$ can be chosen so that the probability that any component of $\hat{b}_j \in$supp $\hat{\sigma}_j\left(s_j, t_j\right)$ (there is only finitely price bids, i.e., components of $\hat{b}_j$'s, which may have positive probability) equals any component of $\bar{b}$ is arbitrarily small for any bidder j, and hence the probability of type (s_i, t_i) tying is arbitrarily small. The third line follows since the profile $\left(\bar{b}, \bar{q}\right) \in \left(B_i^{\bar{n}} \times Q^{\bar{n}}\right)^K$ is feasible in G^n for every $n \geq \bar{n}$ and σ_i^n is a best reply to $\boldsymbol{\sigma}_{-i}^n$. Because $\varepsilon > 0$ is arbitrary, and so is (b, q) in (A-3), we obtain the inequality in (A-1) for all i and a.e. (s_i, t_i).

To obtain the equality in (A-2) we need to establish that no two price-bids by two distinct bidders in the support of their limiting strategies can be the same and equal to the market clearing price with positive probability, since the interim payoff would then be continuous at the limiting bid strategy.

Suppose that under $\hat{\boldsymbol{\sigma}}$ there is a price-bid $\hat{b}_{ik}$ that is in the support of the equilibrium distributional strategy of bidder i and that is the same as some price bid by a rival and can at the same time clear the market with positive probability π. Since there can be only finitely many prices that can clear the market with positive probability, there must exist a positive measure subset of types $\hat{S} \times \hat{T} \in [0,1]^{2N}$ such that for some bidder j and some profile of types $(\mathbf{s}, \mathbf{t}) \in \hat{S} \times \hat{T}$ we have $\hat{b}_{ik} = \hat{b}_{jl} = p^c\left(\mathbf{s}, \mathbf{t}, \hat{\boldsymbol{\sigma}}\right)$ with positive probability under $\hat{\boldsymbol{\sigma}}$ for some steps k and l that may depend on

[14] In general, there can be many behavioral strategies that are generated by a distributional strategy - see Milgrom and Weber (1985).

the type. The associated allocation (expected allocation conditional on the tie occuring) is $\hat{q}_i^c$ and $\hat{q}_j^c$ to the players that tie at the market clearing price. Now we will show that this would contradict having an equilibrium in a nearby discretized game by showing that with a sufficiently fine grid, one of these two players would have a profitable deviation from bidding according to σ^n. The idea is that one of these two bidders has to lose in the discretized game, since there can be no ties by construction. With sufficiently fine grid, it will be worthwhile for this player to adjust his bidding strategy up to beat his rival since the discontinuous jump in allocation will outweigh the possible increase in the market clearing price.

Formally, let π_{jn} denote the probability that j beats i in the discretized game G^n but ties in the limit and π_{in} the probability that i beats j in G^n but ties in the limit. Notice that by construction $\lim_{n\to\infty}(\pi_{in}+\pi_{jn}) = \pi$. Suppose without loss that $\lim_{n\to\infty}\pi_{jn} > 0$. This implies that the tie that arises in the limit is broken in j's favor at least in some states of positive measure in the discretized game sufficiently far along the sequence, and thus the (expected) allocation to i conditional on states leading to a tie in the limiting game satisfies $(q_i^c)^n < \hat{q}_i^c$ - in other words i gets allocated even less than in the limit when he ties with j. Let $\bar{b}_j$ denote the supremum of the bid by bidder j that is submitted with positive probability under the strategy σ_j^n and that just beats the market clearing price in states in which these bidders tie in the limit: $\bar{b}_j = \sup\left\{\inf\left\{b_j : b_j > p^c(\mathbf{s},\mathbf{t},\boldsymbol{\sigma}^n),\ \Pr\left(b_j|\sigma_j^n\right) > 0, \text{ and } \ (\mathbf{s},\mathbf{t}) \in \hat{S}\times\hat{T}\right\}\right\}$. Now consider a deviation for all pure strategies in the support of types $(s_i,t_i) \in \hat{S}_i\times\hat{T}_i$ at the relevant step k to $b' > \bar{b}_j$ such that $b' = \inf\left\{b \in B_i^n : b > \bar{b}_j\right\}$. For a sufficiently fine grid (high enough n) this deviation results in an arbitrarily small increase in expected market clearing price, while resulting in a strict increase in expected allocation, thus being strictly profitable. Formally, consider a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty}\varepsilon_n = 0$ and $\varepsilon_n > 0\ \forall n$. Then $\forall\varepsilon_n : \exists\bar{n} : \forall n \geq \bar{n}, \left|b' - \max\left\{\hat{b}_{ik}, b_{ik}^n\right\}\right| < \varepsilon_n$ since in the limit i's action set is dense in $B_i = \{l\}\cup[0,\bar{p}]$. Bidding b' instead of b_{ik}^n results in a change of market clearing price of at most ε_n (since $\left|b' - \max\left\{\hat{b}_{ik}, b_{ik}^n\right\}\right| < \varepsilon_n$), and hence the change in the expected payment is at most ε_n (since the most a bidder can win is $q = 1$). Increasing i's bid also increases the probability of i winning some units. The lowest bound on the change in expected gross utility in game G^n is therefore $\pi_{jn}\left[V\left(q_i^c, s_i\right) - V\left(\left(q_i^c\right)^n, s_i\right)\right]$ where q_i^c is the allocation to i after the deviation (and thus i beats j's bid) and $(q_i^c)^n$ is i's allocation before the deviation (and thus j beats $i's$ bid). π_{jn} must be positive for sufficiently high n since by assumption $\lim_{n\to\infty}\pi_{jn} > 0$. Fix some small $\delta > 0$ such that $\lim_{n\to\infty}\pi_{jn} > \delta$ and let n_1 solve:

$$\forall n \geq n_1 : \left|\pi_{jn} - \lim_{m\to\infty}\pi_{jm}\right| < \delta$$

Let n_2 solve:

$$\forall n \geq n_2 : \varepsilon_n < \left(\lim_{m\to\infty}\pi_{jm} - \delta\right)\left[V\left(q_i^c, s_i\right) - V\left(\left(q_i^c\right)^n, s_i\right)\right] \tag{A-4}$$

The right hand side of (A-4) is strictly positive, since $\lim_{m\to\infty} \pi_{jm} > \delta$, $q_i^c > (q_i^c)^n$, $V(\cdot, s_i)$ is non-decreasing in the first argument since it is defined as $V(q, s_i) = \int_0^q v(u, s_i)\,du$ and $\int_{(q_i^c)^n}^{q_i^c} v(u, s_i)\,du$ must be strictly positive otherwise i would not have demanded $q_{ik} > q_i^c$. Therefore by picking ε_{n^*} where $n^* = \max\{n_1, n_2, \bar{n}\}$, this deviation would be strictly profitable in some discretized game G^{n^*}, a contradiction to $\boldsymbol{\sigma}^{n^*}$ being an equilibrium. Therefore we cannot have ties with positive probability of also setting the market clearing price under the limiting strategies (except possibly for a measure zero subset of i's types).

But this implies that the expected payoff $EU_i(\hat{\sigma}_i, \hat{\boldsymbol{\sigma}}_{-i}|s_i, t_i)$ is continuous in all components of bid vector $\hat{b}_i$ (also at l since l is isolated) at the limit $\hat{\sigma}_i$ for a.e. (s_i, t_i) and all $\hat{b}_i \in \text{supp}\, \hat{\sigma}_i(s_i, t_i)$. Therefore $\lim_n EU_i\left(\sigma_i^n, \boldsymbol{\sigma}_{-i}^n|s_i, t_i\right) = EU_i(\hat{\sigma}_i, \hat{\boldsymbol{\sigma}}_{-i}|s_i, t_i)$ and so (A-1) and (A-2) together imply that $\hat{\boldsymbol{\sigma}}$ is an equilibrium, since they establish that $\hat{\sigma}_i$ is a best reply to $\hat{\boldsymbol{\sigma}}_{-i}$ for a.e. (s_i, t_i). QED

A.2 Proof of Lemma 1

Suppose that there exists an equilibrium, in which for a type (s_i, t_i) of bidder i a tie between at least two bidders can occur with positive probability $\pi > 0$. Since there can be only finitely many prices that can clear the market with positive probability, in order for a tie to be a positive probability event, it has to be the case that there exists a positive measure subset of types $\hat{S}_{-i} \times \hat{T}_{-i} \in [0,1]^{N-1} \times C^{N-1}$ such that for some bidder j, and all profiles of types $(s_{-i}, t_{-i}) \in \hat{S}'_{-i} \times \hat{T}'_{-i} \subseteq \hat{S}_{-i} \times \hat{T}_{-i}$ (another positive measure subset) and some steps k and l we have $b_{ik}(s_i, t_i) = b_{jl}(s_j, t_j) = p^c((s_i, t_i, s_{-i}, t_{-i}), \mathbf{b}, \mathbf{q})$. Without loss suppose that this event occurs at the bid (b_{ik}, q_{ik}), and that the quantity allocated to i after rationing is $q_i^{RAT} < q_{ik}$. Let S_π^R denote the minimal level of the residual supply in the states leading to rationing at b_{ik}.

Consider a deviation to a bidpoint $b'_{ik} = b_{ik} + \varepsilon$ and $q'_{ik} = q_{ik}$ where ε is sufficiently small. This deviation increases the probability of winning $q_{ik} - q_{ik-1}$ units. Most importantly in the states that led to rationing under the original bid, type (s_i, t_i) of bidder i will now obtain $q^* > q_i^{RAT}$, where $q^* \geq \min\left\{q_{ik}, S_\pi^R\right\}$. Notice that $q_{ik-1} = q_i^{RAT}$ is ruled out since the market clearing price has to be the highest price at which aggregate demand weakly exceeds aggregate supply and since $q_{ik-1} = q_i^{RAT}$ would imply that residual supply was vertical at q_i^{RAT}, the market clearing price could not have been b_{ik}. This holds of course also for the other bidder who is being rationed. Therefore, in the states leading to rationing: $\lim_{p\downarrow b_{ik}} S^R > q_i^{RAT} = S^R(b_{ik})$ and hence there is indeed room for a deviation. The probabilities of winning other units remain unchanged. Therefore the lower bound on the increase in s_i's expected gross surplus from such a deviation is $\pi\left(V(q^*, s_i) - V\left(q_i^{RAT}\right)\right) > 0$. The increased bid might also result in an increase in the market clearing price. This increase, however, is bounded by ε, since at prices higher than $b_{ik} + \varepsilon$ bidder i's bid function stays the same. Since the most bidder i wins is $q_i = 1$, the maximum change in the expected payment is ε. Comparing the upper bound on the change in expected payment with the lower bound on the change in expected gross utility, we obtain

$$\varepsilon < \pi \left(V\left(q^*, s_i\right) - V\left(q_i^{RAT}, s_i\right)\right) \tag{A-5}$$

Consider a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $\varepsilon_n > 0\ \forall n$. By definition of a limit there must exist n^* such that for all $n \geq n^*$ we have:

$$\varepsilon_n < \pi \left(V\left(q^*, s_i\right) - V\left(q_i^{RAT}, s_i\right)\right)$$

Therefore setting $\varepsilon = \varepsilon_{n^*}$, the inequality (A-5) will hold, and thus the proposed deviation would indeed be strictly profitable for the type (s_i, t_i). Since there can be only countably many prices at which bidders may tie with positive probability, there can be only countably many types (s_i, t_i) with a profitable deviation (i.e., that can tie with positive probability at one of these prices) such that no bidder j has a profitable deviation and thus for a.e. type (s_i, t_i) ties have zero probability in equilibrium for all bidders i. QED

A.3 Proof of Proposition 2

With a slight abuse of notation, I will summarize a state (Q, s_{-i}, t_{-i}) by s_{-i}. In order to show (local) optimality of a bidpoint (b_k, q_k), we would like to obtain:

$$\lim_{q' \to q_k} \frac{E_{s_{-i}} u\left(s_i | q_k\right) - E_{s_{-i}} u\left(s_i | q'\right)}{q_k - q'}$$

and show that if this limit equals zero, we get our optimality condition, since the bidder does not have a profitable local deviation.

To begin define the following sets given a vector of bidpoints $(\mathbf{p}, \mathbf{q})$:

$$\begin{aligned}
\theta_{1k}\left(q_k\right) &= \left\{s_{-i} : \exists p : b_{k+1} < p < b_k : q_k \in S^R\left(p, s_{-i}\right) \wedge q_k \notin S^R\left(b_k, s_{-i}\right)\right\} \\
\theta_{2k}\left(q_k\right) &= \left\{s_{-i} : \exists q \in S^R\left(b_k, s_{-i}\right) : q_{k-1} < q \leq q_k\right\} \\
\theta_{3k}\left(q_k\right) &= \left\{s_{-i} : \exists q \in S^R\left(b_{k+1}, s_{-i}\right) : q_k < q < q_{k+1} \ \wedge q_k \notin S^R\left(b_k, s_{-i}\right)\right\} \\
\theta_{4k}\left(q_k\right) &= \left\{s_{-i} : S^R\left(b_k, s_{-i}\right) \leq q_{k-1}\right\} \\
\theta_{5k}\left(q_k\right) &= \left\{s_{-i} : S^R\left(b_{k+1}, s_{-i}\right) \geq q_{k+1}\right\}
\end{aligned}$$

The first set includes all vectors s_{-i} such that there is a market clearing price, which is in the interval (b_{k+1}, b_k) and bidder i gets his full demand. The second set includes all vectors s_{-i} such that the market clearing price will be b_k and player i will be rationed (except for the case $q_k \in S^R(b_k, s_{-i})$). The third set includes all s_{-i} such that the market clearing price will be b_{k+1} and player i will be rationed, in which case his payoff might be affected by perturbation of q_k in case of rationing on-the-margin, since his share depends on his marginal demand $q_{k+1} - q_k$. Notice though that $i's$ payoff will be affected only in the case that someone else is being rationed as well, i.e., residual supply is horizontal at b_{k+1}, which is a zero probability event in equilibrium as shown in Lemma 1. The fourth set includes all s_{-i} such that the market clearing price will be strictly above b_k and perturbing q_k does not affect the payoff. The last set includes all s_{-i} such that the market clearing price is weakly less than b_{k+1}, and perturbing q_k will not affect the payoff. Further denote S_{-i} as the set of all possible realizations of the vector of random variables including the signals of all players other than player i.

Notice that $\cup_{j=1}^{5}\theta_{jk}(q_k) = S_{-i}$ and all sets are pairwise disjoint, i.e., any possible vector s_{-i} belongs to exactly one set.

To economize on space I will write $\Pr(\theta_{jk}(q_k))$ for $\Pr(s_{-i} \in \theta_{jk}(q_k))$. By the law of total probability, we can rewrite the $E_{s_{-i}}u(s_i)$ as:

$$\begin{aligned} E_{s_{-i}}u(s_i) &= \sum_{j=1}^{5} \Pr(\theta_{jk}(q_k))\, E_{s_{-i}}[u(s_i)\,|\theta_{jk}(q_k)] \\ &= \Pr\left(\cup_{j=1}^{3}\theta_{jk}(q_k)\right) E_{s_{-i}}\left[u(s_i)\,|\cup_{j=1}^{3}\theta_{jk}(q_k)\right] + \\ &\quad + \sum_{j=4}^{5} \Pr(\theta_{jk}(q_k))\, E_{s_{-i}}[u(s_i)\,|\theta_{jk}(q_k)] \end{aligned} \tag{A-6}$$

Notice that $\Pr\left(\cup_{j=1}^{3}\theta_{jk}(q_k)\right)$ is constant for any local perturbation of q_k, since any such perturbation only causes some reshuffling of states s_{-i} between θ_{1k}, θ_{2k}, and θ_{3k}. Since in states in θ_{4k} and θ_{5k} bidder i actually obtains at most q_{k-1} or at least q_{k+1} respectively, perturbing q_k will not result in any change in (conditional) expected utility in these states.

The main point of the following long derivation is to show that the terms obtained by direct differentiation of the expected payment $b_k \frac{\partial E_{s_{-i}}(q(s_{-i});p=b_k)}{\partial q_k}$, $b_{k+1}\frac{\partial E_{s_{-i}}(q(s_{-i});p=b_{k+1})}{\partial q_k}$ and $q_k \frac{\partial E_{s_{-i}}(p^c(s_{-i});b_{k+1}<p<b_k)}{\partial q_k}$ can be combined into one term: $q_k \frac{\partial E_{s_{-i}}(p^c(s_{-i});b_{k+1}\leq p\leq b_k)}{\partial q_k}$ and that this object exists in equilibrium for a.e. type (s_i, t_i).

For easier exposition, consider now a perturbation of q_k down to $q' = q_k - \varepsilon$. Let $\mathbf{q}'$ be the perturbed quantity-bid vector, i.e., $q'_m = q_m \forall m \neq k$ and $q'_k \neq q_k$. Define the following subsets of θ_{2k} and θ_{3k}:

$$\begin{aligned} \omega_{1k}(q') &= \{s_{-i} : s_{-i} \in \theta_{2k}(q_k) \cap \theta_{1k}(q')\} \\ \omega_{2k}(q') &= \{s_{-i} : s_{-i} \in \theta_{2k}(q_k) \cap \theta_{3k}(q')\} \\ \omega_{3k}(q') &= \{s_{-i} : s_{-i} \in \theta_{1k}(q_k) \cap \theta_{3k}(q')\} \end{aligned}$$

These subsets include all states that get transferred from one θ to another one. The set ω_{1k} includes states in which bidder i was rationed at price b_k originally, and after perturbing q_k down to q' he gets his full demand. Set ω_{3k} includes states in which he originally got q_k, but after perturbation the market is going to clear at b_{k+1} and bidder i will thus be rationed and obtains a higher quantity. Finally set ω_{2k} includes states in which he was rationed at b_k and after perturbing his demand q_k, he will be rationed at b_{k+1} instead.

Notice that with these sets we can now express the probabilities of sets $\theta_{jk}(q')$ as follows:

$$\begin{aligned}
\Pr\left(\theta_{1k}\left(q'\right)\right) &= \Pr\left(\theta_{1k}\left(q_k\right)\right) + \Pr\left(\omega_{1k}\left(q'\right)\right) - \Pr\left(\omega_{3k}\left(q'\right)\right) \\
\Pr\left(\theta_{2k}\left(q'\right)\right) &= \Pr\left(\theta_{2k}\left(q_k\right)\right) - \Pr\left(\omega_{1k}\left(q'\right)\right) - \Pr\left(\omega_{2k}\left(q'\right)\right) \\
\Pr\left(\theta_{3k}\left(q'\right)\right) &= \Pr\left(\theta_{3k}\left(q_k\right)\right) + \Pr\left(\omega_{3k}\left(q'\right)\right) + \Pr\left(\omega_{2k}\left(q'\right)\right)
\end{aligned}$$

Now we have all the necessary notation. First some preliminary results and observations. We have already shown in Lemma 1 that ties at the market clearing price are zero probability events in equilibrium. In other words this implies that with probability one only one player has a bid exactly at the market clearing price and thus under rationing pro-rata on-the-margin he is the only one who is rationed if necessary. Now we will show that in equilibrium for a.e. type (s_i, t_i) for every step k (i) $\Pr\left(\theta_{jk}\left(q_k\right)\right)$ is continuous at q_k and (ii) $E_{s_{-i}}\left[p|\theta_{1k}\left(q_k\right)\right]$ is continuous at q_k, and hence $E_{s_{-i}}\left[p; b_k \geq p \geq b_{k+1}\right]$ is continuous at q_k, thus locally differentiable a.e.

First, I begin with a helpful lemma, which guarantees that in equilibrium the probability of residual supply having a common vertical segment with i's bid at any $q_k(s_i, t_i)$ such that $q_k(s_i, t_i)$ is a quantity bid submitted by type (s_i, t_i) of bidder i with positive probability is zero for a.e. type (s_i, t_i).

Lemma 2 *In equilibrium,* $\Pr\left\{s_{-i} : \exists p_L, p_U \text{ such that } p_U \leq b_k\left(s_i, t_i\right) \text{ and } \forall p \in \left[p_L, p_U\right]\ S^R\left(p, s_{-i}\right) = \hat{q}\right\} = 0$ *for all bidpoints* $(b_k, q_k = \hat{q})$ *that are submitted with positive probability by type* (s_i, t_i) *of bidder* i*, for a.e.* (s_i, t_i) *and every step* k*.*

Proof. Suppose for contradiction that in equilibrium residual supply can be vertical at q_k with probability π and let Π be the set of states leading to this event. Let p_L be the lowest price such that $S^R(p, s_{-i}) = q_k$ and let $p^c = E(p|\Pi)$. It follows from Lemma 1 and from residual supply being non-decreasing in price that $p_L < b_k(s_i, t_i)$.

Suppose first that $p_U = b_k(s_i, t_i)$. Consider a deviation of type (s_i, t_i) for all bid functions such that $q_k(s_i, t_i)$ is submitted by this type with positive probability at some step to $b_k(s_i, t_i) - \varepsilon$ with the same quantity bid $q_k(s_i, t_i)$. This deviation decreases the probability of winning units in $\left(q_k(s_i, t_i), q_{k-1}(s_i, t_i)\right)$, but this decrease can be made arbitrarily small by a proper selection of ε (by the same argument as in the proof of Lemma 1). For any ε this deviation leads to saving in expected payment of at least $\pi\varepsilon q_k$. Therefore for ε sufficiently small this deviation is strictly profitable. Hence in equilibrium only zero measure of types of bidder i can have such a profitable deviation.

Now suppose that $p_U < b_k(s_i)$. But this implies that the residual supply is horizontal at p_U with probability π. Therefore there must be some bidder j and positive measure subset of his types for whom the residual supply is vertical at his step $q_m(s_j, t_j)$ and $p_U = b_m(s_j, t_j)$. Hence all such types of bidder j would have a profitable deviation. Therefore in order for this to be an equilibrium it must again be that no positive measure of types of bidder i submit a bid at q_k (otherwise bidder j would have a strictly profitable deviation). ■

For the following lemmas, we will make use of the fact that $\lim_{q' \to q_k} \omega_{jk}(q') = 0 \; \forall j, k$ which is a direct corollary to the last lemma.

Lemma 3 *In equilibrium,* $\Pr(\theta_{jk}(q_k))$ *is continuous at* q_k *(*k^{th} *component of* $\mathbf{q}$*)* $\forall k$*,*j *for a.e. type* (s_i, t_i)*.*

Proof. We will show this for θ_{1k}. Pick $\varepsilon > 0$. Then we need to show that $\exists \delta$ such that $\forall q' \in [q_k - \delta, q_k + \delta]$, $|\Pr(\theta_{1k}(q')) - \Pr(s_{-i} \in \theta_{1k}(q_k))| \leq \varepsilon$, where $q'_m = q_m \; \forall m \neq k$.

Let's first consider $q_k - \delta$. Using our notation, $\Pr(\theta_{1k}(q_k - \delta)) = \Pr(\theta_{1k}(q_k)) + \Pr(\omega_{1k}(q_k - \delta)) - \Pr(\omega_{3k}(q_k - \delta))$. Therefore to prove continuity we need to show that $|\Pr(\omega_{1k}(q_k - \delta)) - \Pr(\omega_{3k}(q_k - \delta))| \leq \varepsilon$, which is implied if $\max\{\Pr(\omega_{1k}(q_k - \delta)), \Pr(\omega_{3k}(q_k - \delta))\} \leq \varepsilon$.

Consider first $\Pr(\omega_{1k}(q_k - \delta))$.

Consider a decreasing sequence $\{\delta_n\}$, such that $\lim \delta_n = 0$. Now we have a decreasing sequence of sets: $\omega_{1k}(q_k - \delta_1) \supset \omega_{1k}(q_k - \delta_2) \ldots \supset \omega_{1k}(q_k)$. By the elementary theorem from probability theory, the limit of the probabilities of the sets along the sequence is equal to probability of the limiting set. The limiting set has zero measure by definition of θ's and by Lemma 2, and hence $\Pr(\omega_{1k}(q_k - \delta_n)) \to 0$. By definition of a limit, we must have: $\exists m : \forall n \geq m : \Pr(\omega_{1k}(q_k - \delta_n)) - 0 \leq \varepsilon$.

Now consider a similar argument for $\Pr(\omega_{3k}(q_k - \delta))$. The set $\omega_{3k}(q')$ includes all states that cut the vertical part of i's bid function under q_k, but cut the horizontal part under $q\prime$. By the same argument as above, this set becomes arbitrarily small as $\delta \to 0$, and therefore we can pick $\delta_{m'}$ such that $\Pr(\omega_{3k}(q_k - \delta_{m'})) \leq \varepsilon$. Choosing $\delta = \min\{\delta_m, \delta_{m'}\}$ concludes the proof since the case $q_k + \delta$ is analogous. A similar argument establishes continuity of $\Pr(s_{-i} \in \theta_{jk} | \mathbf{p}, \mathbf{q})$ for $j \in \{2, 3\}$, and of course since $\Pr(\theta_{jk}(q_k)) = \Pr(\theta_{jk}(q'))$ for $j \in \{4, 5\}$ and any q' continuity is satisfied for these states as well. ■

Lemma 4 *In equilibrium,* $E_{s_{-i}}[p^c(s_{-i}, q_k) | \theta_{1k}(q_k)]$ *is continuous at* q_k $\forall k$ *for a.e. type* (s_i, t_i)*.*

Proof. By Lemma 3, $\Pr(\theta_{1k}(q_k))$ is continuous in q_k. The conditional expectation is:

$$E(p^c | b_k > p^c > b_{k+1}, q_k) = \int_{s_{-i} \in \theta_{1k}(q_k)} p^c(s_{-i}, q_k) \frac{dF(s_{-i})}{\Pr(\theta_{1k}(q_k))}$$

where $p^c(s_{-i}, q_k)$ solves: $\sup_p p$ s.t. $q_k \in 1 - \sum_{j \neq i} q_j(s_j, p)$. Let's fix $\varepsilon > 0$. Now we want to show that there is $\delta > 0$, s.t. $\forall q \in B(q_k, \delta) : |E(p^c | b_k > p^c > b_{k+1}, q_k) - E(p^c | b_k > p^c > b_{k+1}, q)| \leq \varepsilon$. Perturbing q will have two effects on the conditional expectation: a direct effect through changing $p^c(s_{-i}, q_k)$ and an indirect effect through changing the set $\theta_{1k}(q_k)$. We want to pick δ such that neither of these effects is larger than $\frac{\varepsilon}{2}$.

Consider first the direct effect. The change in the market clearing price for a state s_{-i} can happen only in the case that the residual supply corresponding to this state has at least one vertical piece between q' and q_k, call the set of such states $\eta_1(q', q_k)$. But under the BNE hypothesis the probability measure of a set of states s_{-i} that lead to a vertical residual supply exactly at q_k between prices b_k and b_{k+1} and must be zero by Lemma 2. $\eta_1(q', q_k)$ is therefore continuous by the same argument as in Lemma 3 and in a neighborhood sufficiently close to q_k the probability measure of this set is arbitrarily small. Moreover, since the new market clearing price still has to fall between $b_k(s_i, t_i)$ and $b_{k+1}(s_i, t_i)$, the induced direct change is bounded by $|b_k(s_i, t_i) - b_{k+1}(s_i, t_i)|$, and therefore we can pick δ_1 such that:

$$|b_k(s_i, t_i) - b_{k+1}(s_i, t_i)| \max[\Pr(\eta_1(q_k - \delta_1, q_k)), \Pr(\eta_1(q_k + \delta_1, q_k))] \leq \frac{\varepsilon}{2}$$

Now consider the indirect effect. Changing q_k to q' can result in some states s_{-i} that originally led to market clearing price between $b_k(s_i, t_i)$ and $b_{k+1}(s_i, t_i)$ to no longer satisfy this restriction. Call the set of such states $\eta_2(q', q_k)$. On the other hand there might be other states s_{-i} which originally did not lead to prices between $b_k(s_i, t_i)$ and $b_{k+1}(s_i, t_i)$, which now do; call this set $\eta_3(q', q_k)$. Again by the same argument as in Lemma 3, as q' becomes arbitrarily close to q_k the probability measure of either of these sets is arbitrarily close to zero, and it is continuous and limiting to 0 as $\delta \to 0$ on $[q_k - \delta, q_k]$ and on $[q_k + \delta, q_k]$. Since the change in expectation cannot exceed $|b_k(s_i, t_i) - b_{k+1}(s_i, t_i)|$, we can pick δ_2 and δ_3 such that

$$\begin{aligned}
|b_k(s_i, t_i) - b_{k+1}(s_i, t_i)| \max[\Pr(\eta_2(q - \delta_2, q_k)), \eta_2(q + \delta_2, q_k)] &\leq \frac{\varepsilon}{4} \\
|b_k(s_i, t_i) - b_{k+1}(s_i, t_i)| \max[\Pr(\eta_3(q - \delta_3, q_k)), \eta_3(q + \delta_3, q_k)] &\leq \frac{\varepsilon}{4}
\end{aligned}$$

Therefore we can pick $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ concluding the proof. ■

Lemma 5 *In equilibrium, $E_{s_{-i}}[p^c(s_{-i}, q_k); \theta_{1k}, \theta_{2k}, \theta_{3k}] = E_{s_{-i}}[p^c(s_{-i}, q_k); b_k \geq p^c \geq b_{k+1}]$ is continuous at q_k $\forall k$ and thus locally differentiable a.e. for a.e. type.*

Proof. We have:

$$E_{s_{-i}}\left[p^c(s_{-i}, q_k); \cup_{j=1}^3 \theta_{jk}(q_k)\right] = \Pr(\theta_{1k}(q_k)) E_{s_{-i}}[p^c(s_{-i}, q_k) | \theta_{1k}(q_k)] + \Pr(\theta_{2k}(q_k)) b_k + \Pr(\theta_{3k}(q_k)) b_{k+1}$$

By Lemma 3, $\Pr(\theta_{jk}(q_k))$ is continuous in q_k and by Lemma 4 $E_{s_{-i}}[p^c(s_{-i}, q_k) | \theta_{1k}(q_k)]$ is also

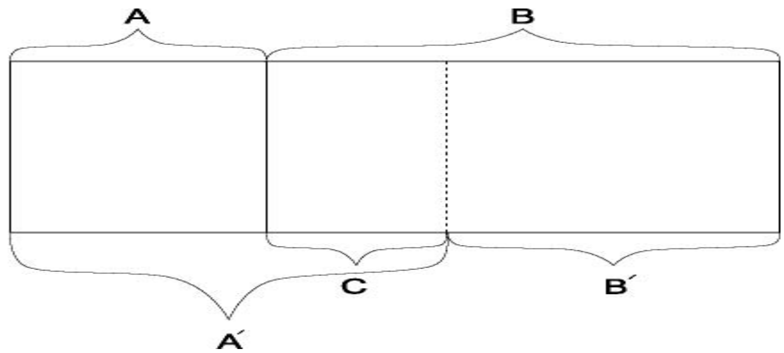

Figure 6: Different Cell Partitions

continuous. Therefore the object of interest is a sum and product of continuous functions, and hence is itself continuous. ■

With the preliminaries in hand, we are now ready for the main derivation.

Let us focus on $\Pr\left(\cup_{j=1}^{3}\theta_{jk}(q_k)\right) E_{s_{-i}}\left[u(s_i) \mid \cup_{j=1}^{3}\theta_{jk}(q_k)\right]$. First $E_{s_{-i}}u(s_i,t_i)$ can further be split into two parts: (i) the expected gross utility $E_{s_{-i}}V(y(s_{-i},q_k),s_i)$ where $y(s_{-i},q_k)$ is either q_k in case of a state in θ_{1k}, the rationed quantity $q^{RAT}(s_{-i},q_k-q_{k-1})$ in case of a state in θ_{2k}, or $q^{RAT}(s_{-i},q_{k+1}-q_k)$ in case of a state in θ_{3k}; and (ii) the expected payment $E_{s_{-i}}[y(s_{-i},q_k)p^c(s_{-i},q_k)]$ where both $y(s_{-i},q_k)$ and $p^c(s_{-i},q_k)$ depend on the state: e.g., $y(q_k)=q_k$ in θ_{1k}, but $p^c(s_{-i},q_k)$ is random, in θ_{2k} on the other hand $p^c(s_{-i},q_k)=b_k$, but $y(s_{-i},q_k)$ is random due to rationing and similarly for θ_{3k}. Recall that

$$\begin{aligned}
\Pr\left(\theta_{1k}\left(q'\right)\right) &= \Pr\left(\theta_{1k}(q_k)\right)+\Pr\left(\omega_{1k}\left(q'\right)\right)-\Pr\left(\omega_{3k}\left(q'\right)\right)\\
\Pr\left(\theta_{2k}\left(q'\right)\right) &= \Pr\left(\theta_{2k}(q_k)\right)-\Pr\left(\omega_{1k}\left(q'\right)\right)-\Pr\left(\omega_{2k}\left(q'\right)\right)\\
\Pr\left(\theta_{3k}\left(q'\right)\right) &= \Pr\left(\theta_{3k}(q_k)\right)+\Pr\left(\omega_{3k}\left(q'\right)\right)+\Pr\left(\omega_{2k}\left(q'\right)\right)
\end{aligned}$$

The difficulty we are facing is that $y(s_{-i},q_k)$ and $p^c(s_{-i},q_k)$ are not continuous over the cells of our partition - in particular they are different functions at each cell, and hence the usual Leibnitz rule fails. To illustrate this, consider Figure 6. $y(s_{-i},q_k)$ and $p^c(s_{-i},q_k)$ are the same functions on A and A' evaluated at q_k and q' respectively (for example if the set A is our θ_{1k}, then $y(\cdot,x)=x$). But in states falling to set C under q', these functions would be different under q_k. We can, however, always "pretend" that the same continuous function f that we are integrating on cell A under q_k is also valid on cell A under q' and add to it the integral of the same function on cell C under q'. Similarly we can pretend that the same function f that we are integrating on B under q_k will hold on B under q' and then subtract the integral of the same function on set C under q'.

Let's consider first the effect that a perturbation in q_k would have on the expected gross utility. Deriving it indirectly using the limit:

$$
\begin{aligned}
&\lim_{q'\to q_k} \frac{E_{s_{-i}} V\left(y\left(s_{-i}, q'\right), s_i\right) - E_{s_{-i}} V\left(y\left(s_{-i}, q_k\right), s_i\right)}{q' - q_k} \\
&= \lim_{q'\to q_k} \frac{\sum_{j=1}^{3} \left[E_{s_{-i}}\left[V\left(y\left(s_{-i}, q'\right), s_i\right); \theta_{jk}\left(q'\right)\right] - E_{s_{-i}}\left[V\left(y\left(s_{-i}, q_k\right), s_i\right); \theta_{jk}\left(q_k\right)\right]\right]}{q' - q_k} \\
&= \lim_{q'\to q_k} \frac{E_{s_{-i}}\left[V\left(q', s_i\right) - V\left(q_k, s_i\right); \theta_{1k}\left(q_k\right)\right] + \left[\Pr\left(\omega_{1k}\right) - \Pr\left(\omega_{3k}\right)\right] V\left(q', s_i\right)}{q' - q_k} \\
&+ \lim_{q'\to q_k} \frac{\left[\begin{array}{c} E_{s_{-i}}\left[V\left(q^{RAT}\left(q' - q_{k-1}, s_{-i}\right), s_i\right) - V\left(q^{RAT}\left(q_k - q_{k-1}, s_{-i}\right), s_i\right); \theta_{2k}\left(q_k\right)\right] - \\ E_{s_{-i}}\left[V\left(q^{RAT}\left(q' - q_{k-1}, s_{-i}\right), s_i\right); \omega_{1k}\right] - E_{s_{-i}}\left[V\left(q^{RAT}\left(q' - q_{k-1}, s_{-i}\right), s_i\right); \omega_{2k}\right] \end{array}\right]}{q' - q_k} \\
&+ \lim_{q'\to q_k} \frac{\left[\begin{array}{c} E_{s_{-i}}\left[V\left(q^{RAT}\left(q_{k+1} - q', s_{-i}\right), s_i\right) - V\left(q^{RAT}\left(q_{k+1} - q_k, s_{-i}\right), s_i\right); \theta_{3k}\left(q_k\right)\right] + \\ E_{s_{-i}}\left[V\left(q^{RAT}\left(q_{k+1} - q', s_{-i}\right), s_i\right); \omega_{3k}\right] + E_{s_{-i}}\left[V\left(q^{RAT}\left(q_{k+1} - q', s_{-i}\right), s_i\right); \omega_{2k}\right] \end{array}\right]}{q' - q_k}
\end{aligned}
$$

where the first equality follows by the law of total probability and the fact that on θ_{4k} and θ_{5k} perturbing k^{th}-step q_k to q' does not alter the gross utility and also not their respective probabilities. The second equality results after plugging in the conditional gross utility before and after the perturbation using the approach described above - extending the continuous functions to the partition cells under q_k and collecting terms.

Now invoking the definition of the derivative and noting that $\lim_{q'\to q_k}\left[q^{RAT}\left(.\right)|\omega_{jk}\right] = q_k$ and $\lim_{q'\to q_k}\Pr\left(\omega_{jk}\left(q', q\right)\right) = 0$ and hence after applying l'Hospital's rule all terms involving ω_{jk} vanish in the limit, we can simplify the last expression above to:

$$
\begin{aligned}
&\Pr\left(\theta_{1k}\left(q_k\right)\right) v\left(q_k, s_i\right) + \\
&+ E_{s_{-i}}\left[v\left(q^{RAT}\left(s_{-i}, q_k - q_{k-1}\right), s_i\right) \frac{\partial q^{RAT}\left(s_{-i}, q_k - q_{k-1}\right)}{\partial q_k}; \theta_{2k}\left(q_k\right)\right] + \\
&+ E_{s_{-i}}\left[v\left(q^{RAT}\left(s_{-i}, q_{k+1} - q_k\right), s_i\right) \frac{\partial q^{RAT}\left(s_{-i}, q_{k+1} - q_k\right)}{\partial q_k}; \theta_{3k}\left(q_k\right)\right] \\
&= \Pr\left(\theta_{1k}\left(q_k\right)\right) v\left(q_k, s_i\right)
\end{aligned}
$$

where the last equality follows by Lemma 1 together with the fact that in the zero probability event that q^{RAT} might be responsive to changes in q_k, this response is bounded (since it is the rationing coefficient which is less than 1).

Now let us move to the key step in the proof - the effect of the perturbation in q_k on the expected payment. Again using the limit derivation:

$$
\begin{aligned}
& \lim_{q' \to q_k} \frac{E_{s_{-i}}\left[y\left(s_{-i}, q'\right) p^c\left(s_{-i}, q'\right); \cup_{j=1}^3 \theta_{jk}\left(q'\right)\right] - E_{s_{-i}}\left[y\left(s_{-i}, q_k\right) p^c\left(s_{-i}, q_k\right); \cup_{j=1}^3 \theta_{jk}\left(q_k\right)\right]}{q' - q_k} \\
= & \lim_{q' \to q_k} \frac{\sum_{j=1}^3 \left[E_{s_{-i}}\left[y\left(s_{-i}, q'\right) p^c\left(s_{-i}, q'\right); \theta_{jk}\left(q'\right)\right] - E_{s_{-i}}\left[y\left(s_{-i}, q_k\right) p^c\left(s_{-i}, q_k\right); \theta_{jk}\left(q_k\right)\right]\right]}{q' - q_k} \\
= & \lim_{q' \to q_k} \frac{\left[\begin{array}{c} E_{s_{-i}}\left[q' p^c\left(s_{-i}, q'\right); \theta_{1k}\left(q_k\right)\right] + E_{s_{-i}}\left[q' p^c\left(s_{-i}, q'\right); \omega_{1k}\right] \\ -E_{s_{-i}}\left[q' p^c\left(s_{-i}, q'\right); \omega_{3k}\right] - E_{s_{-i}}\left[q_k p^c\left(s_{-i}, q_k\right); \theta_{1k}\left(q_k\right)\right] \end{array}\right]}{q' - q_k} + \\
& + \lim_{q' \to q_k} \frac{\left[\begin{array}{c} E_{s_{-i}}\left[q^{RAT}\left(q' - q_{k-1}, s_{-i}\right) b_k; \theta_{2k}\left(q_k\right)\right] - E_{s_{-i}}\left[q^{RAT}\left(q' - q_{k-1}, s_{-i}\right) b_k; \omega_{1k}\right] \\ -E_{s_{-i}}\left[q^{RAT}\left(q' - q_{k-1}, s_{-i}\right) b_k; \omega_{2k}\right] - E_{s_{-i}}\left[q^{RAT}\left(q_k - q_{k-1}, s_{-i}\right) b_k; \theta_{2k}\left(q_k\right)\right] \end{array}\right]}{q' - q_k} + \\
& + \lim_{q' \to q_k} \frac{\left[\begin{array}{c} E_{s_{-i}}\left[q^{RAT}\left(q_{k+1} - q', s_{-i}\right) b_{k+1}; \theta_{1k}\left(q_k\right)\right] + E_{s_{-i}}\left[q^{RAT}\left(q_{k+1} - q', s_{-i}\right) b_{k+1}; \omega_{3k}\right] \\ +E_{s_{-i}}\left[q^{RAT}\left(q_{k+1} - q', s_{-i}\right) b_{k+1}; \omega_{2k}\right] - E_{s_{-i}}\left[q^{RAT}\left(q_{k+1} - q_k, s_{-i}\right) b_{k+1}; \theta_{3k}\left(q_k\right)\right] \end{array}\right]}{q' - q_k}
\end{aligned}
$$

where the second equality follows by the law of total probability after substituting in for the probabilities of the different partition cells after perturbation and extending (or reducing) the functions to the old partition cells as described earlier.

By adding and subtracting $E_{s_{-i}}\left[q_k p^c\left(s_{-i}, q'\right); \theta_{1k}\left(q_k\right)\right]$, collecting terms and using the definition of a derivative, we can rewrite the last expression as:

$$
\begin{aligned}
& \lim_{q'\to q_k} \frac{E_{s_{-i}}\left[q'p^c\left(s_{-i},q'\right);\theta_{1k}\left(q_k\right)\right]-E_{s_{-i}}\left[q_kp^c\left(s_{-i},q'\right);\theta_{1k}\left(q_k\right)\right]}{q'-q_k}+ \\
& +\lim_{q'\to q_k} \frac{\left[\begin{array}{c} E_{s_{-i}}\left[q_kp^c\left(s_{-i},q'\right);\theta_{1k}\left(q_k\right)\right]+E_{s_{-i}}\left[q'p^c\left(s_{-i},q'\right);\omega_{1k}\right] \\ -E_{s_{-i}}\left[q'p^c\left(s_{-i},q'\right);\omega_{3k}\right]-E_{s_{-i}}\left[q_kp^c\left(s_{-i},q_k\right);\theta_{1k}\left(q_k\right)\right] \end{array}\right]}{q'-q_k}+ \\
& +b_kE_{s_{-i}}\left[\frac{\partial q^{RAT}\left(s_{-i},q_k-q_{k-1}\right)}{\partial q_k};\theta_{2k}\left(q_k\right)\right]+b_{k+1}E_{s_{-i}}\left[\frac{\partial q^{RAT}\left(s_{-i},q_{k+1}-q_k\right)}{\partial q_k}b_{k+1};\theta_{3k}\left(q_k\right)\right]+ \\
& +\lim_{q'\to q_k} \frac{b_{k+1}E_{s_{-i}}\left[q^{RAT}\left(q_{k+1}-q',s_{-i}\right);\omega_{3k}\cup\omega_{2k}\right]-b_kE_{s_{-i}}\left[q^{RAT}\left(q'-q_{k-1},s_{-i}\right);\omega_{1k}\cup\omega_{2k}\right]}{q'-q_k} \\
= \; & E_{s_{-i}}\left[p^c\left(s_{-i},q_k\right);\theta_{1k}\left(q_k\right)\right]+ \\
& \lim_{q'\to q_k} \frac{\left[\begin{array}{c} q_k\left[E_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\theta_{1k}\left(q_k\right)\right]+E_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\theta_{2k}\left(q_k\right)\right]+E_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\theta_{3k}\left(q_k\right)\right]\right] \\ -q_k\left[E_{s_{-i}}\left[p^c\left(s_{-i},q_k\right);\theta_{1k}\left(q_k\right)\right]+E_{s_{-i}}\left[p^c\left(s_{-i},q_k\right);\theta_{2k}\left(q_k\right)\right]+E_{s_{-i}}\left[p^c\left(s_{-i},q_k\right);\theta_{3k}\left(q_k\right)\right]\right] \end{array}\right]}{q'-q_k}+ \\
& +\lim_{q'\to q_k} \frac{\begin{array}{c} E_{s_{-i}}\left[q'p^c\left(s_{-i},q'\right);\omega_{1k}\right]-E_{s_{-i}}\left[q'p^c\left(s_{-i},q'\right);\omega_{3k}\right] \\ -q_kE_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\omega_1\right]+q_kE_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\omega_{3k}\right] \end{array}}{q'-q_k} \\
= \; & E_{s_{-i}}\left[p^c\left(s_{-i},q_k\right);\theta_{1k}\left(q_k\right)\right]+q_k\frac{\partial E_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\cup_{j=1}^{k}\theta_{jk}\left(q_k\right)\right]}{\partial q_k}
\end{aligned}
$$

where the first equality is the key step: (i) first term is obtained by simplification; (ii) using Lemma 1 we eliminate the derivatives of the rationed quantity; (iii) we add and subtract terms to complete the function $q_kp^c\left(s_{-i},q'\right)$ to full $\cup_{j=1}^{3}\theta_{jk}$. In doing that we make use of the following facts:

$$
\begin{aligned}
& E_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\theta_{2k}\left(q_k\right)\right]-E_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\theta_{2k}\left(q_k'\right)\right] \\
& -E_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\omega_{1k}\right]-E_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\omega_{2k}\right]=0 \\
& E_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\theta_{3k}\left(q_k\right)\right]-E_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\theta_{3k}\left(q_k'\right)\right] \\
& +E_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\omega_{3k}\right]+E_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\omega_{2k}\right]=0
\end{aligned}
$$

Therefore we can multiply all terms by q_k and add them to our limit. Final expression following the first equality obtains by rearranging terms. Finally the last equality then follows by definition of the derivative and because

$$
\begin{aligned}
&\lim_{q'\to q_k} \frac{E_{s_{-i}}\left[q'p^c\left(s_{-i},q'\right);\omega_{1k}\right]-E_{s_{-i}}\left[q'p^c\left(s_{-i},q'\right);\omega_{3k}\right]-q_kE_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\omega_1\right]+q_kE_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\omega_{3k}\right]}{q'-q_k}\\
&=\lim_{q'\to q_k}\frac{\partial \Pr\left(\omega_{1k}\right)}{\partial q'}\left[q'E_{s_{-i}}\left[p^c\left(s_{-i},q'\right)|\omega_{1k}\right]-q_kE_{s_{-i}}\left[p^c\left(s_{-i},q'\right)|\omega_{1k}\right]\right]\\
&+\lim_{q'\to q_k}\frac{\partial \Pr\left(\omega_{3k}\right)}{\partial q'}\left[q'E_{s_{-i}}\left[p^c\left(s_{-i},q'\right)|\omega_{3k}\right]-q_kE_{s_{-i}}\left[p^c\left(s_{-i},q'\right)|\omega_{3k}\right]\right]\\
&+\lim_{q'\to q_k}\left[\Pr\left(\omega_{1k}\right)K_1+\Pr\left(\omega_{3k}\right)K_2\right]\\
&=0
\end{aligned}
$$

where the first equality follows after first splitting the expectations, which can be done because q' is constant on ω_{jk}.

$$E_{s_{-i}}\left[q'p^c\left(s_{-i},q'\right);\omega_{jk}\right]=q'\Pr\left(\omega_{jk}\right)E_{s_{-i}}\left(p^c\left(s_{-i},q'\right)|\omega_{jk}\right)$$

and applying l'Hospital's rule (note that $\Pr\left(\omega_{jk}\right)$ is a function of q'). Finally as we noted earlier $\lim_{q'\to q_k}E_{s_{-i}}\left[q'|\omega_{jk}\right]=q_k$ and $\lim_{q'\to q_k}\Pr\left(\omega_{jk}\right)=0$, and since both K_1 and K_2 are bounded ($\frac{\partial \Pr\left(\omega_{jk}\right)}{\partial q'}$ is also bounded since roughly speaking this is just an integral of some density of s_{-i} which is bounded by assumption), all terms vanish in the limit.

The last step is to note that the event $\{s_{-i}\in\theta_{1k}\}$ is equivalent to the event $\{b_k>p^c>b_{k+1}\}$ and collecting terms our FOC becomes:

$$\Pr\left[b_k>p^c>b_{k+1}\right]\left[v\left(q_k,s_i\right)-E_{s_{-i}}\left(p^c\left(s_{-i},q_k\right)|b_k>p^c>b_{k+1}\right)\right]=q_k\frac{\partial E_{s_{-i}}\left(p^c\left(s_{-i},q_k\right);b_k\geq p^c\geq b_{k+1}\right)}{\partial q_k}$$

The second part of the proposition, equation (3), follows from a much simpler argument. Notice that expected payment can be written as

$$
\begin{aligned}
&E_{s_{-i}}\left[p^c\left(s_{-i}\right)q\left(s_{-i}\right)\right]\\
=\;&\Pr\left(b_k<p<b_{k-1}\right)q_{k-1}E_{s_{-i}}\left[p^c\left(s_{-i}\right)|b_k<p<b_{k-1}\right]+\\
&\Pr\left(p=b_k\right)b_kE_{s_{-i}}\left[q\left(s_{-i}\right)|p=b_k\right]+\\
&\Pr\left(b_{k+1}<p<b_k\right)q_kE_{s_{-i}}\left[p^c\left(s_{-i}\right)|b_{k+1}<p<b_k\right]+\\
&\Pr\left(p\leq b_{k+1}\cup p\geq b_k\right)E_{s_{-i}}\left[p^c\left(s_{-i}\right)q\left(s_{-i}\right)|p\leq b_{k+1}\cup p\geq b_k\right]\\
=\;&q_{k-1}E_{s_{-i}}\left[p^c\left(s_{-i}\right);b_k<p<b_{k-1}\right]+q_kE_{s_{-i}}\left[p^c\left(s_{-i}\right);b_{k+1}<p<b_k\right]\\
&+b_kE_{s_{-i}}\left[q\left(s_{-i}\right);p=b_k\right]+E_{s_{-i}}\left[p^c\left(s_{-i}\right)q\left(s_{-i}\right);p\leq b_{k+1}\cup p\geq b_{k-1}\right]
\end{aligned}
$$

where the last term does not depend on b_k. Taking the derivative w.r.t. b_k delivers

$$q_{k-1}\frac{\partial E_{s_{-i}}\left[p^c\left(s_{-i}\right);b_k<p<b_{k-1}\right]}{\partial b_k}+\frac{q_k E_{s_{-i}}\left[p^c\left(s_{-i}\right);b_{k+1}<p<b_k\right]}{\partial b_k}+ \tag{A-7}$$
$$+E_{s_{-i}}\left[q\left(s_{-i}\right);p=b_k\right]+b_k\frac{\partial E_{s_{-i}}\left[q\left(s_{-i}\right);p=b_k\right]}{\partial b_k}$$

Notice that doing the same simple exercise w.r.t. q_k would not lead directly to our FOC, since the heart of the argument of the proof above involves combining the terms $b_k\frac{\partial E_{s_{-i}}(q(s_{-i});p=b_k)}{\partial q_k}$, $b_{k+1}\frac{\partial E_{s_{-i}}(q(s_{-i});p=b_{k+1})}{\partial q_k}$ and $q_k\frac{\partial E_{s_{-i}}(p^c(s_{-i});b_{k+1}<p<b_k)}{\partial q_k}$ into one term: $q_k\frac{\partial E_{s_{-i}}(p^c(s_{-i});b_{k+1}\leq p\leq b_k)}{\partial q_k}$. Combining the derivative of the expected payment w.r.t. b_k with the derivative of the gross utility yields (3). Also notice that by similar arguments as in Lemmas 3 and 2 we can establish contiuity and local differentiability a.e. of all expectations involved in (A-7) with respect to the bid b_k. QED

Notice that we can also express the expected gross utility as

$$\begin{aligned}
& E_{s_{-i}}\left[V\left(q\left(s_{-i}\right),s_i\right)\right] \\
= \; & \Pr\left(b_{k+1}<p<b_k\right)V\left(q_k,s_i\right)+\Pr\left(b_{k-1}<p<b_k\right)V\left(q_{k-1},s_i\right)+ \\
& +E_{s_{-i}}\left[V\left(q\left(s_{-i}\right),s_i\right);p=b_k\right]
\end{aligned}$$

But taking the derivative of this expression would not yield any simpler expression.

A.4 Proofs of identification results

For the proof of Proposition 4 see Hortaçsu (2002). Propositions 5 and 6 follow from application of standard results in consistency of a nonparametric regression and kernel density estimators.