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DISCRETE CHOICE WITH AN ODDBALL ALTERNATIVE

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Abstract—A discrete choice model is presented that explicitly recognizes differences in the error structure associated with a single "oddball" alternative within the choice set that has properties not common to the other alternatives. The model purportedly resolves questions related to the use of alternative-specific variables in transportation choice models to capture the effects of attributes unique to a single travel alternative, such as "schedule frequency" in the case of modal choice between personal auto and public transit. The model, which shares the general error structure of multinomial logit, is shown to be a modification of a multinomial logit subchoice by terms involving the exponential integral. The model is shown to yield different results from those produced by an equivalent multinomial logit specification. Comparisons to multinomial probit and nested logit formulations are also made.

1. INTRODUCTION

Models of individual choice behavior in transportation are almost universally specified in terms of a common set of abstract attributes that define the utility or value that each available alternative holds in the particular choice situation. In random utility formulations, the specification typically further entails the assumption that the utilities of the respective alternatives are comprised of a deterministic component and an unobserved, random component that is independently, identically distributed (IID). For cases in which the choice alternatives are of such a similar nature that they indeed may be specified and distinguished by gradations among a common set of attributes (e.g., different brands of toothpaste) the IID assumption is warranted, because the random term presumedly is the resultant of unbiased errors in measurement on a common scale. (It is recognized that another common interpretation of the random term is rooted in "differences in personal tastes;" this interpretatoin is not adopted here.) However, it is often the case in transportation applications that one or more of the alternatives in the choice set differ significantly from the others relative to the attributes that define their intrinsic worth. Examples include: modal choice between personal auto and public transit, the latter having schedule, route and fare payments that have meaning for the former at most only in the abstract; route choice between freeways/expressways and surface streets for which the number of traffic signals and turning maneuvers are likely determinants of choice; modal choice among bus, auto, and commuter rail, the last alternative often involving an ancillary access mode not common to the former two. In applications such as these, the approach typically has been to include a set of alternative-specific variables or constants, in cases where particular attributes have little or no meaning for the other alternatives, or to specify the model as a multidimensional choice (e.g., nested logit), in cases for which one or more of the alternatives involves a subordinate choice. Each of these approaches poses problems for the validity of the IID assumption.

In formulations involving assignment of alternative-specific variables or constants, it is reasonable to expect that the measurement of such terms contributes to the random utility component of the respective alternative. However, no such measurement error (or unobserved contribution) can accrue to the utility of alternatives for which the attribute is not present; the value rather is not only totally deterministic but also identically zero.

Formulations that rely on nested choice structures generally must employ restrictive assumptions regarding the distributional properties of the error terms associated with the respective dimensions of the choice situation. In nested logit, for example, it must be assumed that subsets of the error terms associated with the various choice dimensions are sufficiently small so as to be ignored compared to the error terms of dimensions higher in the nesting structure (c.f., Ben Akiva & Lerman, 1979).

An alternative approach is to formulate the choice situation as a multinomial probit (MNP) model in which the deviation from the IID assumption noted above can be captured with the specification of the variance–covariance structure of the error terms. To the extent that the assumption of Gumbel-distributed error terms is merely a mathematical convenience, in approximation to a "true" MNP structure, this approach adequately addresses the theoretical consistency concerns raised; albeit with the introduction of well-known (but increasingly less intimidating) additional complexities in estimation and interpretation associated with MNP formulations. To the extent that the assumption of Gumbel-distributed error terms is rooted in the speculation that it is the extrema of the unobserved components that are likely to affect a stable choice situation, MNP can be viewed as an approximation to a "true" MNL structure.

This article presents a Gumbel-based model form that is not bound by the restrictive assumptions noted. The formulation shares the general error structure that underlies multinomial logit analysis, whereas explicitly addressing the additional complexity of the error arising from a single "oddball" alternative in the choice set that possesses characteristics not common to the other choice alternatives.

2. MODEL FORMULATION

Consider a set of alternatives

$$A = \{1,2, \ldots , n\}$$

with a common set of attributes

$$X = \{x_1,x_2, \ldots , x_q\}$$

Let

$$X_k = \{x_{k1},x_{k2}, \ldots , x_{kq}\} \qquad \forall k \in A$$

define the levels of these attributes for any particular alternative, k.

Suppose further that one alternative, say the "rth", has additional attributes, $\{Z_r\}$, with corresponding levels

$$Z_r = \{z_{r1},z_{r2}, \ldots , z_{rs}\}$$

unique to that alternative; and not definable for the other alternatives in A. (Consideration is restricted to a single oddball alternative because of the general intractability of the additional multiple integration required with successive numbers of oddballs. For example, introduction of a second oddball would require integration of functions of an exponential integral.)

Let the utility associated with the common set of attributes be denoted by

$$U_k = V_k + \epsilon_k, \qquad \forall k \in A \tag{1}$$

where

$$\begin{aligned} V_k &= \text{Deterministic component} \\ &= f(x_k) \\ \epsilon_k &= \text{Random component} \end{aligned}$$

Let the utility associated with the unique attributes of alternative r be denoted by

$$\tilde{U}_r = \tilde{V}_r + \nu_r$$

where

$$\tilde{V}_r = \text{Deterministic component}$$
$$= f(z_r)$$
$$\nu_r = \text{Random component}$$

Then, under the standard assumptions of random utility theory, the probabilities of choice are given by

$$P(k|A) = \mathrm{PROB}(U_k > U_\ell,\ \forall \ell \in A,\ \ell \neq k,r;\ U_k > U_r + \tilde{U}_r),\ \forall k \in A,\ k \neq r \tag{3a}$$

$$P(r|A) = 1 - \sum_{\substack{\forall k \in A \\ k \neq r}} P(k|A) \tag{3b}$$

Let $f_k(\cdot)$, $F_k(\cdot)$ denote the density and distribution functions, respectively, for the ϵ_k; $\tilde{f}_r(\cdot)$, $\tilde{F}_r(\cdot)$ the corresponding functions associated with ν_r. Then, Equation (3a) can be rewritten as:

$$P(k|A) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_k(\xi)\tilde{f}_k(\zeta)F_r(V_k - V_r - \tilde{V}_r + \xi - \zeta) \cdot \prod_{\substack{\forall \ell \in R \\ \ell \neq k}} F_\ell(V_k - V_\ell + \xi)d\xi d\zeta,\ \forall k \in A,\ k \neq r \tag{4}$$

Assume that the random disturbances are all Gumbel distributed with the same scale factor. (It is noted that this latter condition on the scale factor is the usual assumption in logit analysis for the ϵ's. Extension to the ν_r term is justified only as a mathematical convenience; closed-form solution to Equation (4) apparently exists only for this case. This restriction limits applicability of the model to situations in which there is justification for assuming that the variances of the common and alternative-specific error terms are approximately equal.). Then, with no loss in generality, assume a scale parameter of unity. It is conventional in logit formulations to also assume a common value of zero for the location (mode) parameters of the Gumbel distributions because its actual value can be taken up by a constant term in the specification of the Vs. In the current formulation, however, this assumption is not made, in order to preserve the proper relationship between the respective contributions of U and $\tilde{U}$ to the overall utility of alternative r. The random disturbances are then specified as follows: the ϵ_ℓ's being Gumbel distributed with parameters $(\eta_\ell,1)$; ν_r Gumbel distributed with parameters $(\tilde{\eta}_r,1)$.

Then, on substitution of these properties, and after some manipulations, Equation (4) can be integrated to the result (Bierens de Haan, 1867):

$$P(k|A) = \frac{e^{(V_k+\eta_k)}}{\sum_{\substack{\forall \ell \in A \\ \ell \neq r}} e^{(V_\ell+\eta_\ell)}}[1 - \phi_r e^{\phi_r} EI(\phi_r)],\ \forall k \in A,\ k \neq r \tag{5}$$

where

$$\phi_r = \frac{e^{(V_r+\tilde{V}_r+\eta_r+\tilde{\eta}_r)}}{\sum_{\substack{\forall \ell \in A \\ \ell \neq k}} e^{(V_\ell+\eta_\ell)}}$$

and where

$$EI(\phi_r) = \int_{\phi_r}^{\infty} \frac{e^{-y}}{y}\, dy$$

is the exponential integral, for which tabulated values are well known.

Recognizing that the first factor in Equation (5), i.e.,

$$\phi_k = P_{MNL}(k|A - r, k \neq r) = \frac{e^{(V_k+\eta_k)}}{\sum\limits_{\substack{\forall \ell \in A \\ \ell \neq k}} e^{(V_\ell+\eta_\ell)}}$$

is simply the multinomial logit (MNL) choice probability for alternative k given a choice set that excludes alternative r, the choice probability for alternative k may be written as

$$P(k|A) = P_{MNL}(k|A - r, k \neq r)[1 - \phi_r e^{\phi_r} EI(\phi_r)], \ \forall k \in A, k \neq r \tag{6a}$$

and, from Equation (3b)

$$P(r|A) = \phi_r e^{\phi_r} EI(\phi_r) \tag{6b}$$

It is easily verified that the model specified by Equations (6) exhibits the Independence from Irrelevant Alternatives (IIA) property only among the subset of alternatives included in R; changes in the choice probability of the oddball alternative impacts the choice probabilities of all other alternatives proportionately.

Under the usual assumption that the deterministic components of utility are linear in parameters, i.e.,

$$V_k = \beta \cdot X_k^T, \ \forall k \in A \tag{7a}$$

$$\tilde{V}_r = \tilde{\beta} \cdot Z_r^T \tag{7b}$$

where β and $\tilde{\beta}$ are vectors of unspecified utility weights and T is used to denote transpose, the individual choice elasticities of the oddball model are given as:

$$E_j^{kk} = \beta_j X_{kj} \left\{ 1 - P_{MNL}(k|A - r, k \neq r) \left[(1 + \phi_r) - \frac{P(r|A)}{1 - P(r|A)} \right] \right\}, \ \forall k \in A,$$
$$k \neq r \tag{8a}$$

$$E_j^{rr} = \lambda_j \left[(1 + \phi_r) - \frac{\phi_r}{P(r|A)} \right] \tag{8b}$$

where

$$\lambda_j = \begin{cases} \beta_j X_{rj}, & \text{for components } X_{rj} \\ \beta_j Z_{rj}, & \text{for components } Z_{rj} \end{cases}$$

and the cross elasticities by

$$E_j^{k\ell} = -\beta_j X_{\ell j} P_{MNL}(\ell|A - r, \ell \neq r) \left[(1 + \phi_r) - \frac{P(r|A)}{1 - P(r|A)} \right], \ k, \ell \in A,$$
$$k \neq \ell, k, \ell \neq r \tag{9a}$$

$$E_j^{kr} = \lambda_j \left[\phi_r - \frac{P(r|A)}{1 - P(r|A)} \right], \ k \in A, k \neq r \tag{9b}$$

$$E_j^{r\ell} = -\beta_j X_{\ell j} P_{MNL}(\ell | A - r, \ell \neq r)\left[(1 + \phi_r) - \frac{\phi_r}{P(r|A)}\right] \quad \ell \in A, \ell \neq r \quad (9c)$$

where $E_j^{k\ell}$ is defined as the individual choice probability elasticity of alternative k with respect to attribute j of alternative ℓ.

3. LOGICAL CONSISTENCY AND ASYMPTOTIC VALUES

Although the general behavior of the choice probabilities specified by Equations (6) is not immediately apparent from its structure, certain specific properties are easily derived regarding consistency and asymptotic behavior.

First, because (Kreyszig, 1968)

$$\lim_{\phi \to \infty} EI(\phi) = \frac{e^{-\phi}}{\phi} \quad (10a)$$

and (Collins, 1968)

$$\lim_{\phi \to \infty} EI(\phi) = -\gamma - \ell n\phi, \quad (10b)$$

where γ is the Euler's constant, it is easily verified that Equations (6) satisfy the logical consistency conditions

$$0 \leq P(k|A) \leq 1, \quad \sum_{\forall k \in A} P(k|A) = 1, \quad \forall k \in A$$

Moreover, from Equation (10a),

$$\lim_{\tilde{V}_r \to \infty} P(k|A) = 0 \quad (11a)$$

and from Equation (10b),

$$\lim_{\tilde{V}_r \to -\infty} P(k|A) = P_{MNL}(k|A - r, k \neq r) \quad \forall k \in A, k \neq r \quad (11b)$$

$$\lim_{\tilde{V}_r \to -\infty} P(r|A) = 0.$$

4. COMPARISON TO MNL

It is of interest to compare the choice probabilities specified by Equations (6) to those obtained using an equivalent MNL formulation,

$$P_{MNL}(k|A) = \frac{e^{[V_k + \eta_k + \delta_{kr} \cdot (\tilde{V}_r + \tilde{\eta}'_r)]}}{\sum_{\substack{\forall \ell \in A \\ \ell \neq r}} e^{(V_\ell + \eta_\ell)} + e^{(V_r + \tilde{V}_r + \tilde{\eta}'_r)}}, \quad \forall k \in A \quad (12)$$

where

$$\tilde{\eta}'_r = \eta_r + \tilde{\eta}_r,$$

$$\delta_{kr} = \text{Kronecker delta} = \begin{cases} 0, & k \neq r \\ 1, & k = r \end{cases}$$

in which the alternative-specific attributes are merely incorporated in the deterministic component of the utility of alternative r, and the random disturbance assumed to be

Gumbel distributed with an identical scale factor of unity. Then, from Equations (6) and (12),

$$\frac{P(k|A)}{P_{MNL}(k|A)} = (1 + \phi_r)[1 - \phi_r e^{\phi_r} EI(\phi_r)], \ \forall k \in A, k \neq r \tag{13a}$$

$$\frac{P(r|A)}{P_{MNL}(r|A)} = (1 + \phi_r) e^{\phi_r} EI(\phi_r) \tag{13b}$$

It is easily verified that the following asymptotic relationships hold:

$$\lim_{\tilde{V}_r \to \pm\infty} \frac{P(k|A)}{P_{MNL}(k|A)} = 1, \ \forall k \in A, k \neq r \tag{14a}$$

$$\lim_{\tilde{V}_r \to -\infty} \frac{P(r|A)}{P_{MNL}(r|A)} = \infty, \ \lim_{\tilde{V}_r \to \infty} \frac{P(r|A)}{P_{MNL}(r|A)} = 1 \tag{14b}$$

These asymptotic results are consistent with the behavioral assumptions that, as the expected utility of the oddball alternative becomes either vanishingly small or infinitely large compared to the utilities of the other alternatives, the choice probabilities of both models approach the appropriate limiting values; although both model formulations yield a zero choice probability for alternative r as $\tilde{V}_r \to -\infty$, the MNL model approaches that limiting value more quickly than does the oddball formulation.

Comparative values of the choice probabilities over the range of ϕ_r are displayed in Fig. 1 and Fig. 2. These results indicate that the conventional MNL model consistently

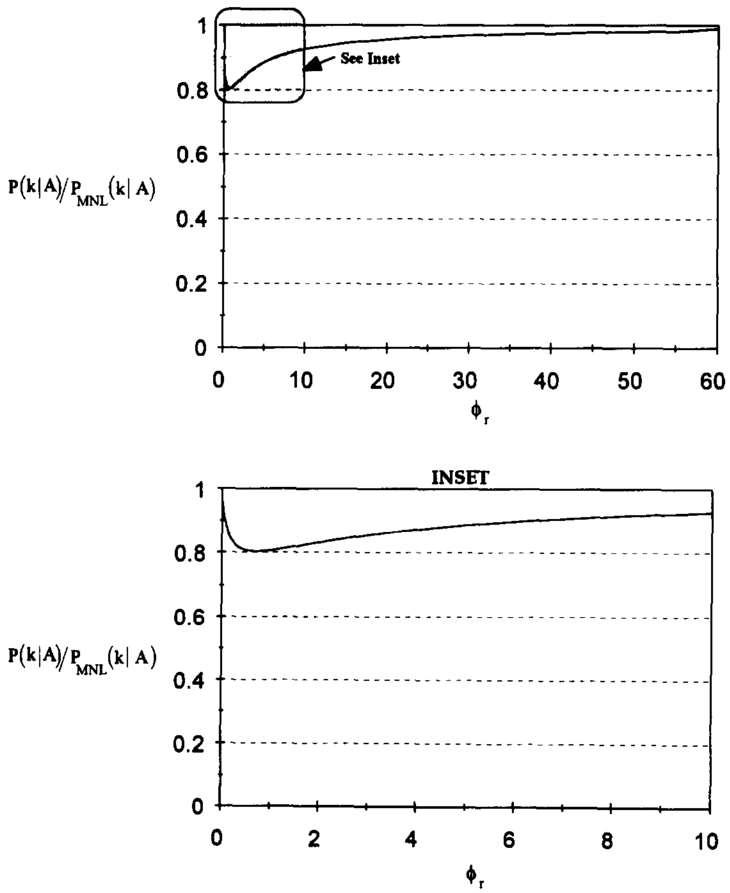

Fig. 1. Ratio of choice probabilities for non-oddball alternatives.

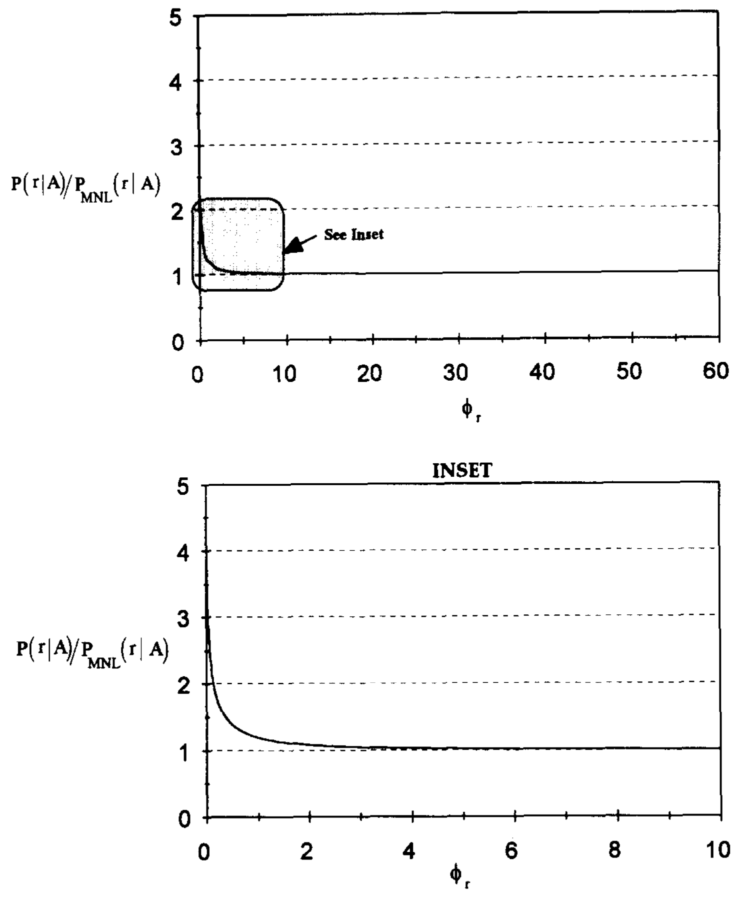

Fig. 2. Ratio of choice probabilities for oddball alternatives.

over predicts the oddball formulation choice probabilities of the nonoddball alternatives, reaching an extrema of $P(k|A)/P_{MNL}(k|A) \approx 0.80$ at a value of $\phi_r^* \approx 0.86$, determined by the solution to

$$P(r|A) = \frac{(1 + \phi_r^*)^2 - 2}{(1 + \phi_r^*)^2 + 1} \tag{15}$$

The results generally support empirical results reported by Horowitz (1982) for the specification error caused by random utility components with unequal variances.

As an example of the implications of this disparity between the two model forms, consider the case in which the shared attributes of each of the n alternatives offer an identical deterministic component of utility, say V, to the individual. Suppose further that the "unique" attributes of the oddball alternative r offer a proportional component, say αV, where α is the corresponding proportionality constant. Under such conditions, the question arises as to how comparable might estimations of the two model forms be. Some information in this regard is provided by Fig. 3, in which the ratios of the estimates of the proportionality constants (α_{MNL} and α, for the MNL and oddball formulations, respectively) required to yield equivalent choice probabilities for the two model forms are plotted as a function of α for various numbers of choice alternatives, n. These results indicate that the respective estimates of utility weights associated with attributes comprising Z_r generally can be expected to coincide as these unique attributes dominate the choice situation. Alternatively, in situations (for this case of constant utilities) in which the utility of the unique attributes of the oddball alternative r is comparable in magnitude to that of the common attributes, there is a significant discrepancy between the expected

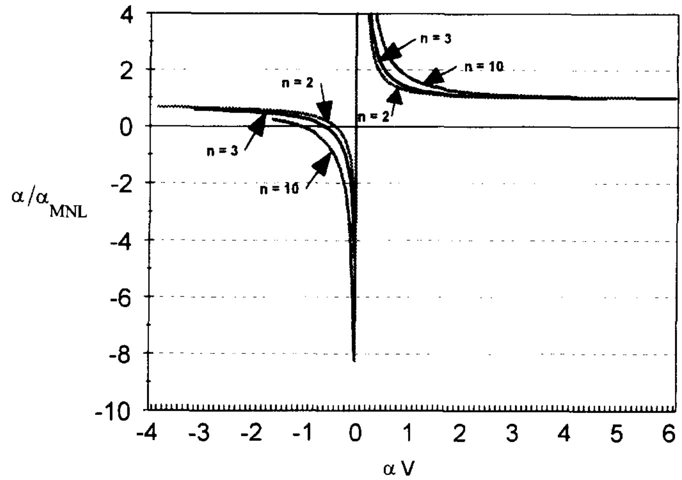

Fig. 3. Ratio of estimated utility proportionality constant.

contribution of the unique attributes to choice as determined by the respective model forms; this discrepancy is exaggerated as the deterministic component of the utility of the attributes comprising Z_r approaches zero, where it is impossible to achieve equivalent choice-probabilities, owing to the added skewness of the error term associated with $\tilde{U}_r$.

5. COMPARISON TO NESTED LOGIT AND MULTINOMIAL PROBIT

Complications introduced by complexities in the assumed distributional properties of the error terms in discrete choice modeling are, of course, not new. Typically, such complications have been addressed either through generalizations to multinomial logit (e.g., nested logit) or through alternative (and less restrictive) specification of the error terms (e.g., multinomial probit). In that regard, the error structure addressed by the oddball model shares features associated both with nested logit and with multinomial probit.

In the case of nested logit, the multiple error terms in the utility specification arising from shared random components are sequentially treated by defining a nesting structure that takes advantage of the property that the maximum of a series of Gumbel-distributed variables is itself Gumbel-distributed, leading to the well-known "logsum" term. In both the nested logit and the oddball cases the complication introduced is in the form of an additive error term in the utility specification. However, the genesis of the additive error term is completely different; in nested logit it results from shared random components, while in the oddball case it results from unique random components. This difference has a profound impact on any simplification afforded by the nested approach.

The logical nested structure for the oddball case is a two-level model with the oddball alternative (r) in the upper nest and all other alternatives ($\forall k \in A,\ k \neq r$) in the lower nest. Then the "upper-nest" choice probabilities are given as:

$$
\begin{aligned}
P(k|A, k \neq r) &= Pr\left(\underset{\substack{\forall k \in A \\ k \neq r}}{\text{MAX}}\ U_k \geq U_r\right) \\
&= Pr\left(\underset{\substack{\forall k \in A \\ k \neq r}}{\text{MAX}}\ V_k + \epsilon_k \geq V_r + \tilde{V}_r + \epsilon_r + \nu_r\right) \\
&= Pr(V^* + \epsilon^* \geq V_r + \tilde{V}_r + \epsilon_r + \nu_r)
\end{aligned}
\tag{16}
$$

where V^* is the logsum term and ϵ^* is Gumbel distributed. Since the distribution function $G(y)$ for $\epsilon_r + \nu_r$ is given by:

$$G(y) = \int_0^\infty e^{-[\gamma + e^{-y/\gamma}]} d\gamma, \tag{17}$$

Equation (16) integrates to the same result as Equation (5). The nested formulation, while affording no simplification of the error structure associated with the oddball case, nonetheless provides an alternative derivation of the governing choice probabilities developed earlier.

An alternative to the assumption of Gumbel-distributed error terms is that of normal deviates, leading to a multinomial probit formulation. Unlike the case presented by Gumbel-distributed error terms, an assumption of normally-distributed error terms introduces no fundamental obstacles in applying multinomial probit to the oddball case, because the sum of normal deviates is itself normal. In the trinomial case, for example, the probit formulation analogous to the assumptions leading to Equation (5) involves specifying the covariance matrix as:

$$\Sigma = \frac{\pi^2}{6} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparative values of the choice probabilities for the oddball alternative relative to the corresponding multinomial probit choice probabilities for this example are shown in Fig. 4, both for the oddball formulation presented here as well as for an equivalent multinomial logit formulation. Both models are seen to converge to the multinomial probit model for large ϕ_r but to deviate significantly (and in opposite directions) from that model for small ϕ_r.

6. SPECIFICATION AND ESTIMATION

In the development of the oddball model, it is presumed a priori that certain characteristics associated with the oddball alternative both are determinants to the choice as well as being unique to the oddball alternative. Although in many instances the latter presumption is evident (e.g., the time required for recharging an electric vehicle alternative), in other instances alternative specifications of a particular characteristic may cloud such conviction (e.g., the specification of the "unique" feature that public transit typically involves sharing a vehicle with a number of strangers while personal auto does not, as the number of persons in the vehicle—a characteristic definable for both alternatives). An indication of whether or not specific attributes are more appropriately modeled as being

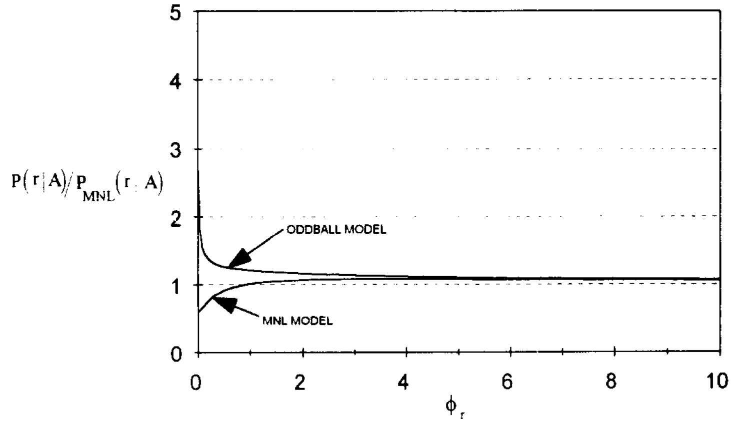

Fig. 4. Ratio of MNL and oddball model choice probabilities for oddball alternative to MNP choice probability.

unique to a particular oddball alternative than as being included among a common set of attributes can be obtained by either the Cox test of separate families of hypotheses or from comparisons of the likelihood ratio index, as proposed by Horowitz (1983), because such alternative specifications involve comparisons of non-nested models; i.e., neither model form can be obtained from the other through manipulation of model coefficients. This same approach can be used to evaluate the appropriateness of the oddball formulation against alternative model structures, such as multinomial probit and logit formulations. Tests of whether or not to include such unique characteristics at all in the model fall under the general category of nested model comparisons, which can be conducted using likelihood ratio and Lagrangian multiplier tests, as also described by Horowitz (1982).

Estimation of the oddball model can be achieved via standard maximum likelihood procedures, as in the multinomial logit case. Subscripting by n, terms associated with the nth observation in a set of N total observations, the appropriate first order conditions for the usual linear-in-parameters specification, are easily verified to be given by:

$$\sum_{n=1}^{N}\left\{\sum_{\substack{\forall k\in A\\ k\neq r}}\Omega_{kn}\cdot\left(x_{kj}^{n}-\sum_{\substack{\forall \ell\in A\\ \ell\neq r}}x_{\ell j}^{n}\phi_{kn}\right)+(1+\phi_{rn})\cdot[1+P_n(r|A)]\right.$$
$$\left.\cdot\left[\frac{\Omega_{rn}}{P_n(r|A)}-\sum_{\substack{\forall k\in A\\ k\neq r}}\frac{\Omega_{kn}}{[1-P_n(r|A)]}\right]\cdot\left(x_{rj}^{n}-\sum_{\substack{\forall \ell\in A\\ \ell\neq r}}x_{\ell j}^{n}\phi_{k}\right)\right\}=0,$$
$$j=1,\ldots q \tag{18}$$
$$\sum_{n=1}^{N}\left\{(1+\phi_{rn})\cdot[1+P_n(r|A)]\cdot\left[\frac{\Omega_{rn}}{P_n(r|A)}-\sum_{\substack{\forall k\in A\\ k\neq r}}\frac{\Omega_{kn}}{[1-P_n(r|A)]}\right]\cdot z_{rj}^{n}\right\}$$
$$=0,\ j=1,\ldots s$$

where

$$\Omega_{kn}=\begin{cases}1 \text{ if observation } n \text{ chose alternative } k\\ 0 \text{ otherwise.}\end{cases}$$

Investigation of the associated second-order conditions is beyond the scope of the current research.

7. CONCLUSIONS

A discrete choice model has been developed for application to choice situations in which one of the alternatives in the choice set has features that are not properly defined for or associated with the other alternatives. Such situations arise commonly in travel demand modeling, particularly in choices between public transit and auto. Whereas more complex than multinomial logit, the new model nonetheless is compactly defined in terms of known functions, as are its standard ancillary properties. The model is shown to yield results that, for a range of comparative utility values, are significantly different from those that would be expected from MNL. The model is also restricted to applications in which the variance of the error terms associated with the unique features of the oddball alternative are of the same order of magnitude as the common attributes.

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