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Discrete Convex Analysis

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Abstract

A theory of "discrete convex analysis" is developed for integer-valued functions defined on integer lattice points. The theory parallels the ordinary convex analysis, covering discrete analogues of the fundamental concepts such as conjugacy, subgradients, the Fenchel min-max duality, separation theorems and the Lagrange duality framework for convex/nonconvex optimization. The technical development is based on matroid-theoretic concepts, in particular, submodular functions and exchange axioms. Part I extends the conjugacy relationship between submodularity and exchangeability, deepening our understanding of the relationship between convexity and submodularity investigated in the eighties by A. Frank, S. Fujishige, L. Lovász and others. Part II establishes duality theorems for M- and L-convex functions, namely, the Fenchel min-max duality and separation theorems. These are the generalizations of the discrete separation theorem for submodular functions due to A. Frank and the optimality criteria for the submodular flow problem due to M. Iri–N. Tomizawa, S. Fujishige, and A. Frank. A novel Lagrange duality framework is also developed in integer programming. We follow Rockafellar's conjugate duality approach to convex/nonconvex programs in nonlinear optimization, while technically relying on the fundamental theorems of matroid-theoretic nature.

Keywords: convex analysis, combinatorial optimization, discrete separation theorem, integer programming, Lagrange duality, submodular function

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Part I: Conjugacy

1 Introduction

In the field of nonlinear programming (in continuous variables) the generalized Lagrange duality plays a pivotal role both in theory and in practice. The perturbation-based duality scheme, developed primarily by R. T. Rockafellar [43], [44], is based on the concept of conjugate duality for convex functions. The objective of this paper is to develop an analogous theory for discrete optimization (nonlinear integer programming) by combining and generalizing the ideas in convex analysis and the results in matroid theory. This amounts to developing a theory that may be called "discrete convex analysis." To be specific, we give a Lagrange duality scheme for the following optimization problem:

$$\text{Minimize} \quad f(x) \quad \text{subject to} \quad x \in B, \tag{1.1}$$

where $f : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$ and $B \subseteq \mathbf{Z}^n$.

To motivate our framework involving matroid-theoretic concepts, let us first introduce a naive approach (without matroidal structure) and discuss its shortcomings. For $B \subseteq \mathbf{Z}^n$ it seems natural to regard B as a "convex set" (discrete analogue of a convex set) if $B = \mathbf{Z}^n \cap \overline{B}$ holds, where $\overline{B}$ denotes the convex hull of B. Likewise, for $f : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$ it seems natural to define f to be a "convex function" if it can be extended to a convex function in the usual sense, namely, there exists a convex function $\overline{f} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ such that $\overline{f}(x) = f(x)$ for $x \in \mathbf{Z}^n$ and $\mathrm{dom}_{\mathbf{R}}\overline{f} = \overline{\mathrm{dom}_{\mathbf{Z}} f}$, where

$$\mathrm{dom}_{\mathbf{R}}\overline{f} = \{x \in \mathbf{R}^n \mid \overline{f}(x) \text{ is finite}\},$$

$$\mathrm{dom}_{\mathbf{Z}} f = \{x \in \mathbf{Z}^n \mid f(x) \text{ is finite}\}.$$

We can proceed further to imitate the construction in convex analysis [43], [49] by replacing $\mathbf{R}$ to $\mathbf{Z}$ in the defining formulas such as

$$h^*(p) = \sup\{\langle p, x\rangle - h(x) \mid x \in \mathbf{R}^n\} \qquad (p \in \mathbf{R}^n) \tag{1.2}$$

for the (convex) conjugate function (the Fenchel transform) of $h : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ and

$$\partial h(x) = \partial_{\mathbf{R}} h(x) = \{p \in \mathbf{R}^n \mid h(x') - h(x) \geq \langle p, x' - x\rangle \ (\forall x' \in \mathbf{R}^n)\} \tag{1.3}$$

for the subdifferential. To be specific, we can define the "conjugate function" (the "integer Fenchel transform") $f^\bullet : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$ of $f : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$ by

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^n\} \qquad (p \in \mathbf{Z}^n), \tag{1.4}$$

where $\mathrm{dom}_{\mathbf{Z}} f$ is assumed to be nonempty, and the "integer subdifferential" by

$$\partial_{\mathbf{Z}} f(x) = \{p \in \mathbf{Z}^n \mid f(x') - f(x) \geq \langle p, x' - x \rangle \ (\forall x' \in \mathbf{Z}^n)\}. \tag{1.5}$$

Unfortunately, these definitions fail to yield an interesting theory. This is mainly because of the lack of duality theorems (separation theorem, Fenchel's min-max theorem, etc.), which lie at the very heart of the convex analysis. It should be clear in this connection that the fundamental formulas in the convex analysis such as $h^{**} = h$ and $\partial_{\mathbf{R}}(h_1 + h_2) = \partial_{\mathbf{R}} h_1 + \partial_{\mathbf{R}} h_2$ rely on the duality theorems. The following examples demonstrate the failure of such identities in the discrete case and hence limitations of this model for "discrete convex analysis."

Example 1.1 $[f^{\bullet\bullet} \neq f,\ \partial_{\mathbf{Z}} f(0) = \emptyset]$ Let $f : \mathbf{Z}^3 \to \mathbf{Z} \cup \{+\infty\}$ be defined by

$$\mathrm{dom}_{\mathbf{Z}} f = \{(0,0,0), \pm(1,1,0), \pm(0,1,1), \pm(1,0,1)\},$$

$$f(0,0,0) = 0,\ f(1,1,0) = f(0,1,1) = f(1,0,1) = 1,$$
$$f(-1,-1,0) = f(0,-1,-1) = f(-1,0,-1) = -1.$$

This function can be extended to a convex function $\overline{f} : \mathbf{R}^3 \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}_{\mathbf{R}} \overline{f} = \overline{\mathrm{dom}_{\mathbf{Z}} f}$. In fact, $\mathrm{dom}_{\mathbf{Z}} f$ is a "convex set" in the sense that $\mathrm{dom}_{\mathbf{Z}} f = \mathbf{Z}^V \cap \overline{\mathrm{dom}_{\mathbf{Z}} f}$, and $\overline{f}$ is given by

$$\overline{f}(x_1, x_2, x_3) = \begin{cases} (x_1 + x_2 + x_3)/2 & (x \in \overline{\mathrm{dom}_{\mathbf{Z}} f}) \\ +\infty & (x \notin \overline{\mathrm{dom}_{\mathbf{Z}} f}) \end{cases}$$

The "conjugate function" $f^\bullet : \mathbf{Z}^3 \to \mathbf{Z} \cup \{+\infty\}$ of f, as defined in (1.4), is given by

$$\begin{aligned} f^\bullet(p) &= \sup_{x \in \mathbf{Z}^3} (\langle p, x \rangle - f(x)) \\ &= \max(0, |p_1 + p_2 - 1|, |p_2 + p_3 - 1|, |p_3 + p_1 - 1|). \end{aligned}$$

The "biconjugate function", based on (1.4), is given by

$$\begin{aligned} f^{\bullet\bullet}(x) &= \sup_{p \in \mathbf{Z}^3} (\langle p, x \rangle - f^\bullet(p)) \\ &= \sup_{p \in \mathbf{Z}^3} (\langle p, x \rangle - \max(0, |p_1 + p_2 - 1|, |p_2 + p_3 - 1|, |p_3 + p_1 - 1|)). \end{aligned}$$

Hence we have $f^{\bullet\bullet}(0) = -1 \neq 0 = f(0)$. On the other hand,

$$\begin{aligned} \overline{f}^{**}(0) &= \sup_{p \in \mathbf{R}^3} (\langle p, 0 \rangle - f^*(p)) \\ &= \sup_{p \in \mathbf{R}^3} (-\max(0, |p_1 + p_2 - 1|, |p_2 + p_3 - 1|, |p_3 + p_1 - 1|)) \\ &= 0 = \overline{f}(0) \end{aligned}$$

with the supremum attained by $p = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. The subdifferential of $\overline{f}$ at $x = 0$ is given by

$$\partial_{\mathbf{R}}\overline{f}(0) = \{p \in \mathbf{R}^3 \mid \overline{f}(x') - \overline{f}(0) \geq \langle p, x' \rangle \ (\forall x' \in \mathbf{R}^3)\} = \left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\},$$

whereas $\partial_{\mathbf{Z}} f(0) = \emptyset$. The phenomena, $f^{\bullet\bullet} \neq f$ and $\partial_{\mathbf{Z}} f(0) = \emptyset$, demonstrated here are certainly undesirable. $\square$

Example 1.2 $[\partial_{\mathbf{Z}}(f_1 + f_2) \neq \partial_{\mathbf{Z}} f_1 + \partial_{\mathbf{Z}} f_2]$ Consider $f_1, f_2 : \mathbf{Z}^2 \to \mathbf{Z}$ defined by

$$f_1(x_1, x_2) = \max(0, x_1 + x_2), \qquad f_2(x_1, x_2) = -\min(x_1, x_2),$$

which can be extended naturally to $\overline{f}_1, \overline{f}_2 : \mathbf{R}^2 \to \mathbf{R}$ according to their expressions. We have

$$\partial_{\mathbf{Z}} f_1(0) = \{(0,0), (1,1)\}, \qquad \partial_{\mathbf{Z}} f_2(0) = \{(-1,0), (0,-1)\},$$

from which follows

$$\partial_{\mathbf{Z}} f_1(0) + \partial_{\mathbf{Z}} f_2(0) = \{\pm(1,0), \pm(0,1)\}.$$

On the other hand, we have

$$(f_1 + f_2)(x) = \max(|x_1|, |x_2|), \qquad \partial_{\mathbf{Z}}(f_1 + f_2)(0) = \{(0,0), \pm(1,0), \pm(0,1)\}.$$

Hence $\partial_{\mathbf{Z}}(f_1 + f_2)(0) \neq \partial_{\mathbf{Z}} f_1(0) + \partial_{\mathbf{Z}} f_2(0)$, which should be compared with

$$\partial_{\mathbf{R}}(\overline{f}_1 + \overline{f}_2)(0) = \partial_{\mathbf{R}}\overline{f}_1(0) + \partial_{\mathbf{R}}\overline{f}_2(0) = \text{convex hull of } \{\pm(1,0), \pm(0,1)\}.$$

$\square$

Example 1.3 [failure of discrete separation] Let $B = \{(x_1, x_2, x_3) \in \mathbf{Z}^3 \mid x_1 + x_2 + x_3 = 0\}$, and define $f : \mathbf{Z}^3 \to \mathbf{Z} \cup \{+\infty\}$ and $g : \mathbf{Z}^3 \to \mathbf{Z} \cup \{-\infty\}$ by

$$\begin{aligned} f(x_1, x_2, x_3) &= \begin{cases} \max(0, -x_3) & ((x_1, x_2, x_3) \in B) \\ +\infty & ((x_1, x_2, x_3) \notin B) \end{cases} \\ g(x_1, x_2, x_3) &= \begin{cases} \min(x_1, x_2) & ((x_1, x_2, x_3) \in B) \\ -\infty & ((x_1, x_2, x_3) \notin B) \end{cases} \end{aligned}$$

Obviously, f and g can be extended respectively to a convex function $\overline{f} : \mathbf{R}^3 \to \mathbf{R} \cup \{+\infty\}$ and to a concave function $\overline{g} : \mathbf{R}^3 \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom}_{\mathbf{R}}\overline{f} = \mathrm{dom}_{\mathbf{R}}\overline{g} = \overline{B}$. It can be verified that $\overline{f}(x) \geq \overline{g}(x)$ for all $x \in \mathbf{R}^3$ and, by the separation theorem in convex analysis, there exists $\overline{p} \in \mathbf{R}^3$ such that $\overline{f}(x) \geq \langle \overline{p}, x \rangle \geq \overline{g}(x)$ for all $x \in \mathbf{R}^3$. In fact, we can take $\overline{p} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$. However, there exists no integral vector $p \in \mathbf{Z}^3$ such that $f(x) \geq \langle p, x \rangle \geq g(x)$ for all $x \in \mathbf{Z}^3$, although $f(x) \geq \langle \overline{p}, x \rangle \geq g(x)$ for all $x \in \mathbf{Z}^3$ with $\overline{p} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$. This demonstrates the failure of the desired discreteness in the separation theorem. Note also that $\partial_{\mathbf{Z}} f(x) \neq \emptyset$ and $\partial_{\mathbf{Z}}(-g)(x) \neq \emptyset$ for $x \in B$. $\square$

The above examples suggest that we must identify a more restrictive class of well-behaved "convex functions" in order to develop a theory of "discrete convex analysis."

A clue is found in the theory of matroids and submodular functions, which has successfully captured the combinatorial essence underlying the well-solved class of combinatorial optimization problems such as those on graphs and networks (cf., Faigle [11], Fujishige [19], Lawler [30]). The relationship between convex functions and submodular functions was made clear in the eighties through the works of A. Frank, S. Fujishige, and L. Lovász (see Section 3 for details). Fujishige [17], [18] formulated Edmonds' intersection theorem as a Fenchel-type min-max duality theorem, with an integrality assertion in the case of integer-valued functions, and considered further analogies such as subgradients. Frank [14] showed a separation theorem for a pair of submodular/supermodular functions, with an integrality assertion for the separating hyperplane in the case of integer-valued functions. This theorem can also be regarded as being equivalent to Edmonds' intersection theorem. A precise statement, beyond analogy, about the relationship between convex functions and submodular functions was made by Lovász [31]. Namely, a set function is submodular if and only if the so-called Lovász extension of that function is convex. This penetrating remark established a direct link between the duality for convex/concave functions and that for submodular/supermodular functions. The essence of the duality for submodular/supermodular functions is now recognized as the discreteness (integrality) assertion in addition to the duality for convex/concave functions.

In spite of the development in the eighties, our understanding of the relationship between convexity and submodularity seems to be only partial in the sense that it is restricted to the "convexity" of a (discrete) set and its support function.

Let V be a finite set and $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ be a submodular function ($\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$ for all $X, Y \subseteq V$) such that $\rho(\emptyset) = 0$ and $\rho(V)$ is finite[1] . It determines a set $B \subseteq \mathbf{Z}^V$ by

$$B = \{x \in \mathbf{Z}^V \mid \sup_{X \subseteq V} (\langle \chi_X, x \rangle - \rho(X)) = 0; \langle \chi_V, x \rangle - \rho(V) = 0\}, \tag{1.6}$$

where $\chi_X \in \{0,1\}^V$ is the characteristic vector of $X \subseteq V$ (defined by $\chi_X(v) = 1$ if $v \in X$, and $= 0$ otherwise). As is well known, B enjoys the simultaneous exchange property:

(B-EXC) For $x, y \in B$ and for $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$,

[1] Namely, $(\mathrm{dom}\,\rho, \rho|_{\mathrm{dom}\,\rho})$ is a submodular system in the sense of Fujishige [19], where $\mathrm{dom}\,\rho = \{X \subseteq V \mid \rho(X) < +\infty\}$ and $\rho|_{\mathrm{dom}\,\rho}$ is the restriction of ρ to $\mathrm{dom}\,\rho$.

where

$$\mathrm{supp}^+(x-y) = \{u \in V \mid x(u) > y(u)\}, \quad \mathrm{supp}^-(x-y) = \{v \in V \mid x(v) < y(v)\},$$

and $\chi_u = \chi_{\{u\}}$ denotes the characteristic vector of $u \in V$. We say $B \subseteq \mathbf{Z}^V$ is an integral base set if it satisfies (B-EXC). An integral base set B satisfies $B = \mathbf{Z}^V \cap \overline{B}$ (being "convex") and $\overline{B}$ will be called in this paper the integral base polyhedron (which agrees with the base polyhedron of an integral submodular system; see Section 3 for basic facts on submodular functions).

By the submodularity of ρ, its Lovász extension $\hat{\rho} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ agrees with the support function of $\overline{B}$, i.e.,

$$\hat{\rho}(p) = \sup\{\langle p, x\rangle \mid x \in \overline{B}\} = \sup\{\langle p, x\rangle \mid x \in B\},$$

which implies in particular that $\hat{\rho}(p) = \delta_B{}^\bullet(p)$ for $p \in \mathbf{Z}^V$ (i.e., $\hat{\rho}|_{\mathbf{Z}^V} = \delta_B{}^\bullet$), where $\delta_B : \mathbf{Z}^V \to \{0, +\infty\}$ is the indicator function of B (i.e., $\delta_B(x) = 0$ if $x \in B$, and $= +\infty$ otherwise) and $\delta_B{}^\bullet$ is the "conjugate" of δ_B defined in (1.4). It is noted that the equality $\hat{\rho}|_{\mathbf{Z}^V} = \delta_B{}^\bullet$ is essentially a duality (conjugacy) assertion, i.e., $\delta_B{}^{\bullet\bullet} = \delta_B$, since the definition of B in (1.6) says in effect that $\delta_B = (\hat{\rho}|_{\mathbf{Z}^V})^\bullet$.

We also see that Frank's discrete separation theorem asserts that two disjoint integral base sets can be separated by a hyperplane defined in terms of $\{0, \pm 1\}$-vectors (see Theorem 3.6). In this way, the duality results concerning submodular functions are satisfactory enough for us to regard an integral base set as the discrete analogue of a convex set. However, the duality concerning submodular functions does not enjoy full generality from the viewpoint of "discrete convex analysis," in which a "convex function" is involved as the objective function in addition to a "convex set" of feasible solutions.

A discrete analogue of "convex functions" has been identified very recently by the present author (Murota [37], [38], [39]) as a natural extension of the concept of valuated matroid introduced by Dress–Wenzel [7], [8]. In accordance with this, we consider two "convexity" classes that are conjugate through the transformation (1.4). To denote these two classes we coin two words, M-convexity and L-convexity, for functions $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$.

We define $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ to be M-convex if $\mathrm{dom}_{\mathbf{Z}} f \neq \emptyset$ and it satisfies the following quantitative generalization of the simultaneous exchange property:

(M-EXC) For $x, y \in \mathrm{dom}_{\mathbf{Z}} f$ and $u \in \mathrm{supp}^+(x-y)$ there exists $v \in \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

As will be explained later in Theorem 4.11, $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ is M-convex if and only if it can be extended to a convex function $\overline{f} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ such that, for each $p \in \mathbf{R}^V$,

$$\{x \in \mathbf{R}^V \mid \overline{f}(x) - \langle p, x \rangle \leq \overline{f}(x') - \langle p, x' \rangle \ (\forall x' \in \mathbf{R}^V)\}$$

forms an integral base polyhedron. Hence the class of M-convex functions is a subclass of the "convex functions" in the afore-defined sense based on the extendability. Some instances of M-convex functions are listed in Section 2.

Naturally, we call $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ M-concave if $-g$ is M-convex. An M-concave function g with $\mathrm{dom}_{\mathbf{Z}} g \subseteq \{0,1\}^V$ can be identified with a valuated matroid in the sense of Dress–Wenzel [7], [8]. A general M-concave function has been employed by Murota [37] to generalize the cost function in the submodular flow problem.

M-convexity plays the principal role in this paper. The objective function f of the (primal) optimization problem (1.1) is often assumed to be M-convex, though this is not an absolute prerequisite for the development of the Lagrange duality scheme.

The other class, L-convex functions, consists of functions $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ that are submodular on the vector lattice $\mathbf{Z}^V$ and satisfy an additional condition that

$$\exists r \in \mathbf{Z}, \forall p \in \mathbf{Z}^V : \ f(p + \mathbf{1}) = f(p) + r,$$

where $\mathbf{1} = (1, 1, \cdots, 1) \in \mathbf{Z}^V$. It should be clear that submodularity on the vector lattice $\mathbf{Z}^V$ means

$$f(p) + f(q) \geq f(p \vee q) + f(p \wedge q) \qquad (p, q \in \mathbf{Z}^V)$$

with $p \vee q$ and $p \wedge q$ denoting integral vectors defined by $(p \vee q)(v) = \max(p(v), q(v))$ and $(p \wedge q)(v) = \min(p(v), q(v))$. It will be shown that L-convex functions can be extended to convex functions on $\mathbf{R}^V$ and moreover that they are nothing but those functions which are conjugate to M-convex functions. The restriction to $\mathbf{Z}^V$ of the Lovász extension of a submodular function $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ is an L-convex function. The L-convex functions turn out to constitute another class of well-behaved "convex functions" that inherit the nice properties from submodular functions on the boolean lattice 2^V as well as from convex functions on $\mathbf{R}^V$. Naturally, we call $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ L-concave if $-g$ is L-convex. The Lagrange dual of the optimization problem (1.1), to be defined in Section 6, will have an L-concave objective function to be maximized.

This paper is organized as follows. Sections 2 to 5 establish the fundamental theorems in "discrete convex analysis." Section 2 provides a number of natural classes of M-convex functions. Section 3 reviews relevant results on submodular functions. Section 4 shows fundamental properties of M- and L-convex functions, while Section 5 concentrates on duality theorems. Finally, Section 6 employs the established theorems to develop a discrete analogue of the Lagrange duality scheme for the optimization problem (1.1) by adapting Rockafellar's conjugate duality approach in nonlinear optimization. The "convex programs" correspond to the case of (1.1) when f is an M-convex function and B is an integral base set, whereas the duality scheme is capable of dealing with "nonconvex programs" using a discrete analogue of the augmented Lagrangian.

2 Examples of M-convex Functions

We show a number of natural classes of M-convex functions.

Example 2.1 (Affine function) Let $B \subseteq \mathbf{Z}^V$ be an integral base set that satisfies, by definition, the exchange property (B-EXC). For $\eta : V \to \mathbf{Z}$ and $\alpha \in \mathbf{Z}$, the function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ defined by

$$f(x) = \alpha + \langle \eta, x \rangle \qquad (x \in B), \qquad \mathrm{dom}_{\mathbf{Z}} f = B,$$

is M-convex, with equality in (M-EXC). This is an immediate consequence of (B-EXC). □

Example 2.2 (Separable convex function) Let $B \subseteq \mathbf{Z}^V$ be an integral base set. We call $f_0 : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ convex if its piecewise linear extension $\overline{f_0} : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ is a convex function. For a family of convex functions $f_v : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ indexed by $v \in V$, the (separable convex) function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ defined by

$$f(x) = \sum \{f_v(x(v)) \mid v \in V\} \qquad (x \in B), \qquad \mathrm{dom}_{\mathbf{Z}} f = B,$$

is M-convex. To see this, take $x, y \in B$ and $u \in \mathrm{supp}^+(x - y)$. By (B-EXC) there exists $v \in \mathrm{supp}^-(x - y)$ such that $x' \equiv x - \chi_u + \chi_v \in B$, $y' \equiv y + \chi_u - \chi_v \in B$. Then we have

$$\begin{aligned} f(x') + f(y') &= [f_u(x(u) - 1) + f_u(y(u) + 1)] + [f_v(x(v) + 1) + f_v(y(v) - 1)] \\ &\quad + \sum_{w \neq u,v} [f_w(x(w)) + f_w(y(w))]. \end{aligned}$$

By the convexity of f_u and the relations $y(u) \leq x(u) - 1 \leq x(u)$ and $y(u) \leq y(u) + 1 \leq x(u)$, we see that

$$f_u(x(u) - 1) + f_u(y(u) + 1) \leq f_u(x(u)) + f_u(y(u)).$$

Similarly we have

$$f_v(x(v) + 1) + f_v(y(v) - 1) \leq f_v(x(v)) + f_v(y(v)).$$

Hence we obtain $f(x') + f(y') \leq f(x) + f(y)$. The minimization problem of a separable convex function on an integral base set has been considered by Fujishige [19, Section 8]. □

Example 2.3 (Min-cost flow) M-convexity is inherent in the integer minimum-cost flow problem with a separable convex objective function, which has been investigated by Minoux [32], Rockafellar [45] and others.

Let $G = (V, A)$ be a graph with a vertex set V and an arc set A. Assume further that we are given an upper capacity function $\overline{c} : A \to \mathbf{Z} \cup \{+\infty\}$ and a lower capacity function $\underline{c} : A \to \mathbf{Z} \cup \{-\infty\}$. A feasible (integral) flow φ is a function $\varphi : A \to \mathbf{Z}$ such that $\underline{c}(a) \leq \varphi(a) \leq \overline{c}(a)$ for each $a \in A$. Its boundary $\partial\varphi : V \to \mathbf{Z}$ is defined by

$$\partial\varphi(v) = \sum\{\varphi(a) \mid a \in \delta^+ v\} - \sum\{\varphi(a) \mid a \in \delta^- v\}, \tag{2.1}$$

where $\delta^+ v$ and $\delta^- v$ denote the sets of the out-going and in-coming arcs incident to v, respectively. Then $B = \{\partial\varphi \mid \varphi : \text{ feasible flow }\}$ is known (see, e.g., [19]) to satisfy (B-EXC).

Suppose further that we are given a family of convex functions $f_a : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ indexed by $a \in A$ (with the convexity in the sense of Example 2.2). Define $\Gamma(\varphi) = \sum\{f_a(\varphi(a)) \mid a \in A\}$. Then the function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ defined by

$$f(x) = \min\{\Gamma(\varphi) \mid \varphi : \text{ feasible flow with } \partial\varphi = x\} \qquad (x \in B), \qquad \mathrm{dom}_{\mathbf{Z}} f = B,$$

is M-convex under the assumption that the minimum of $\Gamma(\varphi)$ over φ with $\partial\varphi = x$ exists as a finite value for all $x \in B$. To show (M-EXC), consider $x, y \in B$ and $u \in \mathrm{supp}^+(x - y)$ and take feasible φ_x, φ_y such that $f(x) = \Gamma(\varphi_x)$, $f(y) = \Gamma(\varphi_y)$, $\partial\varphi_x = x$, $\partial\varphi_y = y$. By a standard augmenting-path argument we see that there is $\pi : A \to \{0, \pm 1\}$ such that $\mathrm{supp}^+(\pi) \subseteq \mathrm{supp}^+(\varphi_x - \varphi_y)$, $\mathrm{supp}^-(\pi) \subseteq \mathrm{supp}^-(\varphi_x - \varphi_y)$ and that $\partial\pi = \chi_u - \chi_v$ for some $v \in \mathrm{supp}^-(x - y)$. Then we have $\partial(\varphi_x - \pi) = x - \chi_u + \chi_v$ and $\partial(\varphi_y + \pi) = y + \chi_u - \chi_v$. Since

$$\begin{array}{rcll} f_a(\varphi_x(a) - 1) + f_a(\varphi_y(a) + 1) & \leq & f_a(\varphi_x(a)) + f_a(\varphi_y(a)) & \text{if } \ \varphi_x(a) > \varphi_y(a), \\ f_a(\varphi_x(a) + 1) + f_a(\varphi_y(a) - 1) & \leq & f_a(\varphi_x(a)) + f_a(\varphi_y(a)) & \text{if } \ \varphi_x(a) < \varphi_y(a), \end{array}$$

we have

$$\begin{aligned} & f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v) \\ & \quad \leq \Gamma(\varphi_x - \pi) + \Gamma(\varphi_y + \pi) \\ & \quad = \sum_{a:\pi(a)=1} [f_a(\varphi_x(a) - 1) + f_a(\varphi_y(a) + 1)] \\ & \qquad + \sum_{a:\pi(a)=-1} [f_a(\varphi_x(a) + 1) + f_a(\varphi_y(a) - 1)] \\ & \qquad + \sum_{a:\pi(a)=0} [f_a(\varphi_x(a)) + f_a(\varphi_y(a))] \\ & \quad \leq \Gamma(\varphi_x) + \Gamma(\varphi_y) \\ & \quad = f(x) + f(y). \end{aligned}$$

□

Example 2.4 (Determinant [7], [8]) Let $A(t)$ be an $m \times n$ matrix of rank m with each entry being a polynomial in a variable t, and let $\mathbf{M} = (V, \mathcal{B})$ denote the (linear) matroid defined on the column set V of $A(t)$ by the linear independence of the column vectors, where $J \subseteq V$ belongs to $\mathcal{B}$ if and only if $|J| = m$ and the column vectors with indices in J are linearly independent. Let B be the set of the incidence vectors χ_J of the bases J (the members of $\mathcal{B}$), i.e., $B = \{\chi_J \mid J \in \mathcal{B}\}$. Then $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ defined by

$$f(\chi_J) = -\deg_t \det A[J] \qquad (J \in \mathcal{B}), \qquad \mathrm{dom}_{\mathbf{Z}} f = B,$$

is M-convex, where $A[J]$ denotes the $m \times m$ submatrix with column indices in J. In fact the Grassmann-Plücker identity implies the exchange property of f and this example was the motivation for introduction of the concept of valuated matroid in [7], [8].

As a concrete instance consider

$$A(t) = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \boxed{\begin{array}{cccc} t+1 & t & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}} \end{array}$$

where $m = 2$ and $V = \{v_1, v_2, v_3, v_4\}$. We have $\mathcal{B} = \binom{V}{2}$ (the family of all the subsets of size two) and $B = B_0 \cup B_1$ with $B_0 = \{(1,1,0,0), (0,0,1,1)\}$, $B_1 = \{(1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1)\}$. The function f is given by

$$f(x) = \begin{cases} 0 & (x \in B_0) \\ -1 & (x \in B_1) \\ +\infty & (\text{otherwise}) \end{cases}$$

$\square$

3 Preliminaries on Submodular Systems

This section is devoted to a summary of the relevant results on submodular functions. See [1], [9], [14], [19], [31], [46], [52], [53] for more accounts.

Let V be a finite nonempty set, $\mathbf{Z}$ be the set of integers, and $\mathbf{R}$ be the set of real numbers. For $u \in V$ we denote by χ_u its characteristic vector, i.e., $\chi_u = (\chi_u(v) \mid v \in V) \in \{0,1\}^V$ such that $\chi_u(v) = 1$ if $v = u$ and $\chi_u(v) = 0$ otherwise. For $x = (x(v) \mid v \in V) \in \mathbf{R}^V$ and $y = (y(v) \mid v \in V) \in \mathbf{R}^V$ we define

$$\mathrm{supp}^+(x) = \{v \in V \mid x(v) > 0\}, \qquad \mathrm{supp}^-(x) = \{v \in V \mid x(v) < 0\},$$

$$x(X) = \sum\{x(v) \mid v \in X\} \qquad (X \subseteq V),$$

$$||x|| = \sum\{|x(v)| \mid v \in V\}, \qquad \langle x, y\rangle = \sum\{x(v)y(v) \mid v \in V\}.$$

For a nonempty set $B \subseteq \mathbf{Z}^V$ we consider the simultaneous exchange property:

(B-EXC) For $x, y \in B$ and for $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$.

Let us say that B is an integral base set if it is nonempty and satisfies (B-EXC). Note that (B-EXC) implies $x(V) = y(V)$ for $x, y \in B$. It can be shown (see also Theorem 3.1 below) that $B = \mathbf{Z}^V \cap \overline{B}$, where $\overline{B}$ denotes the convex hull of B.

A function $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ is said to be submodular if

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y) \qquad (X, Y \subseteq V), \tag{3.1}$$

where it is understood that the inequality is satisfied if $\rho(X)$ or $\rho(Y)$ is equal to $+\infty$. A function $\mu : 2^V \to \mathbf{Z} \cup \{-\infty\}$ is called supermodular if $-\mu$ is submodular.

We define

$$\mathrm{dom}\,\rho = \{X \subseteq V \mid \rho(X) < +\infty\},$$

which is a (distributive) sublattice of the Boolean lattice 2^V if ρ is submodular. Throughout this paper we consider submodular functions ρ that meet an extra condition: $\{\emptyset, V\} \subseteq \mathrm{dom}\,\rho$ and $\rho(\emptyset) = 0$. Namely, we consider ρ such that $(\mathrm{dom}\,\rho, \rho|_{\mathrm{dom}\,\rho})$ is a submodular system in the sense of Fujishige [19], where $\rho|_{\mathrm{dom}\,\rho}$ is the restriction of ρ to $\mathrm{dom}\,\rho$. The base polyhedron of such a submodular function ρ is defined by

$$\mathbf{B}(\rho) = \{x \in \mathbf{R}^V \mid x(X) \leq \rho(X)\ (\forall X \subset V), x(V) = \rho(V)\}. \tag{3.2}$$

In this paper we always consider integer-valued submodular functions, so that the associated base polyhedron is necessarily integral. To emphasize the integrality we refer to $\mathbf{B}(\rho)$ as the integral base polyhedron.

The following fundamental theorem is known as a folklore (according to private communications from W. Cunningham and S. Fujishige; see also [2], [50], [53] in this connection).

Theorem 3.1 *A nonempty set $B \subseteq \mathbf{Z}^V$ satisfies* (B-EXC) *if and only if there exists a submodular function $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ with $\rho(\emptyset) = 0$ and $\rho(V) < +\infty$ such that $\overline{B} = \mathbf{B}(\rho)$ and $B = \mathbf{Z}^V \cap \mathbf{B}(\rho)$. The function ρ associated with B is uniquely determined by*

$$\rho(X) = \sup\{x(X) \mid x \in B\}. \tag{3.3}$$

□

For a function $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ with $\rho(\emptyset) = 0$ and $\rho(V) < +\infty$, the Lovász extension of ρ is a function $\hat{\rho} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\hat{\rho}(p) = \sum_{j=1}^{n} (p_j - p_{j+1})\rho(V_j), \tag{3.4}$$

where, for each $p \in \mathbf{R}^V$, the elements of V are indexed as $\{v_1, v_2, \cdots, v_n\}$ (with $n = |V|$) in such a way that

$$p(v_1) \geq p(v_2) \geq \cdots \geq p(v_n);$$

$p_j = p(v_j)$, $V_j = \{v_1, v_2, \cdots, v_j\}$ for $j = 1, \cdots, n$, and $p_{n+1} = 0$. It is understood that the right-hand side of (3.4) is equal to $+\infty$ if and only if $p_j - p_{j+1} > 0$ and $\rho(V_j) = +\infty$ for some j with $1 \leq j \leq n-1$; no ambiguity arises here since $p_j - p_{j+1} \geq 0$ $(1 \leq j \leq n-1)$ and $\rho(V_n) = \rho(V) < +\infty$.

For any ρ, the Lovász extension $\hat{\rho}$ is nonnegatively homogeneous, i.e., $\hat{\rho}(\lambda p) = \lambda \hat{\rho}(p)$ for $\lambda \geq 0$ and $p \in \mathbf{R}^V$ (with the convention that $0 \times \infty = 0$). If ρ is submodular, the greedy algorithm yields

$$\hat{\rho}(p) = \sup\{\langle p, x\rangle \mid x \in \mathbf{B}(\rho)\} \qquad (p \in \mathbf{R}^V), \tag{3.5}$$

which shows that $\hat{\rho}$ is a convex function. The converse is also true by the following result of Lovász [31] (see also [19, Theorem 6.13]).

Lemma 3.2 ([31]) *ρ is submodular if and only if $\hat{\rho}$ is convex.* □

We introduce the notation $\mathcal{L}_0$ for (the restriction to $\mathbf{Z}^V$ of) the Lovász extensions of submodular functions. Namely,

$$\mathcal{L}_0 = \{g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\} \mid g \text{ satisfies (L1) and (L2) }\}, \tag{3.6}$$

where

(L1) [submodularity] $\quad \rho(X) = g(\chi_X)$ $(X \subseteq V)$ is a submodular function with $\rho(\emptyset) = 0$ and $\rho(V) < +\infty$,

(L2) [greediness] $\quad g(p) = \sum_{j=1}^{n} (p_j - p_{j+1}) g(\chi_{V_j})$,

where, for each $p \in \mathbf{Z}^V$, the elements of V are indexed as $\{v_1, v_2, \cdots, v_n\}$ in such a way that $p(v_1) \geq p(v_2) \geq \cdots \geq p(v_n)$; $p_j = p(v_j)$, $V_j = \{v_1, v_2, \cdots, v_j\}$ for $j = 1, \cdots, n$, and $p_{n+1} = 0$. Also see the convention in (3.4).

Note that (L2) implies the nonnegative homogeneity: $g(\lambda p) = \lambda g(p)$ for $p \in \mathbf{Z}^V$ and $0 \leq \lambda \in \mathbf{Z}$ (in particular $g(0) = 0$), and

$$g(p + \mathbf{1}) = g(p) + r \qquad (p \in \mathbf{Z}^V) \tag{3.7}$$

with $r = \rho(V)$ and $\mathbf{1} = \chi_V = (1, 1, \cdots, 1)$.

The relationship between the exchangeability (B-EXC) of B and the submodularity of ρ, stated in Theorem 3.1, can be expressed as a conjugacy relationship between $\mathcal{L}_0$ and

$$\mathcal{M}_0 = \{\delta_B \mid \emptyset \neq B \subseteq \mathbf{Z}^V, B \text{ satisfies (B-EXC) }\}, \tag{3.8}$$

where $\delta_B : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ is the indicator function of B defined by

$$\delta_B(x) = \begin{cases} 0 & (x \in B) \\ +\infty & (x \notin B) \end{cases} \tag{3.9}$$

This viewpoint is the basis of our subsequent development.

Theorem 3.3 *$\mathcal{M}_0$ and $\mathcal{L}_0$ are in one-to-one correspondence under the integral Fenchel transformation $^\bullet$ of (1.4).*

(Proof) For $g \in \mathcal{L}_0$ there uniquely exists a submodular ρ such that $g = \hat{\rho}|_{\mathbf{Z}^V}$ (the restriction of the Lovász extension $\hat{\rho}$ to $\mathbf{Z}^V$). Then $g^\bullet = \delta_B$ for $B = \mathbf{B}(\rho)$. Conversely, for $\delta_B \in \mathcal{M}_0$ there uniquely exists ρ such that $B = \mathbf{B}(\rho)$. Then ${\delta_B}^\bullet = \hat{\rho}|_{\mathbf{Z}^V}$. □

As a motivation to our subsequent extension of the class $\mathcal{L}_0$ to $\mathcal{L}$ in Section 4.3, we note here the submodularity possessed by a function of $\mathcal{L}_0$ on the vector lattice $\mathbf{Z}^V$, in which, for $p, q \in \mathbf{Z}^V$,

$$\begin{aligned} p \vee q &= (\max(p(v), q(v)) \mid v \in V) \in \mathbf{Z}^V, \\ p \wedge q &= (\min(p(v), q(v)) \mid v \in V) \in \mathbf{Z}^V. \end{aligned}$$

Lemma 3.4 *For $g \in \mathcal{L}_0$ we have*

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \qquad (p, q \in \mathbf{Z}^V), \tag{3.10}$$

where it is understood that the inequality is satisfied if $g(p)$ or $g(q)$ is equal to $+\infty$.

(Proof) Let ρ denote the submodular function associated with g. First we claim

$$p, q \in \mathrm{dom}_{\mathbf{Z}} g \implies p \vee q, p \wedge q, p \pm \mathbf{1} \in \mathrm{dom}_{\mathbf{Z}} g. \tag{3.11}$$

Putting $V(p, \alpha) = \{v \in V \mid p(v) \geq \alpha\}$, we see that $p \in \mathrm{dom}_{\mathbf{Z}} g$ if and only if $V(p, \alpha) \in \mathrm{dom}\, \rho$ for all $\alpha \in \mathbf{Z}$. If $V(p, \alpha), V(q, \alpha) \in \mathrm{dom}\, \rho$, we have $V(p \vee q, \alpha) = V(p, \alpha) \cap V(q, \alpha) \in \mathrm{dom}\, \rho$ and $V(p \wedge q, \alpha) = V(p, \alpha) \cup V(q, \alpha) \in \mathrm{dom}\, \rho$ by the submodularity of ρ. Therefore, $p \vee q, p \wedge q \in \mathrm{dom}_{\mathbf{Z}} g$. The last condition, $p \pm \mathbf{1} \in \mathrm{dom}_{\mathbf{Z}} g$, follows from (3.7).

By (3.11) it suffices (cf. [20]) to prove (3.10) for $p, q \in \mathbf{Z}^V$ with $|p(v) - q(v)| \leq 1$ $(v \in V)$. Then the inequality (3.10) is equivalent to the submodularity of $\rho' : 2^V \to \mathbf{Z} \cup \{+\infty\}$ defined by $\rho'(X) = g((p \wedge q) + \chi_X)$ $(X \subseteq V)$. We will derive the submodularity of ρ' from the expression in (L2) by elementary calculations, though this can be understood also from Theorem 3.1 combined with the fact that the optimal bases of $\mathbf{Z}^V \cap \mathbf{B}(\rho)$ with respect to weight $p \wedge q$ form an integral base set.

Let $\alpha_1 > \alpha_2 > \cdots > \alpha_m$ denote all the distinct values of $(p \wedge q)(v)$ $(v \in V)$, and put $W_k = \{v \in V \mid (p \wedge q)(v) \geq \alpha_k\}$ for $k = 1, 2, \cdots, m$ and $W_0 = \emptyset$. By (L2) we have

$$\rho'(\emptyset) = g(p \wedge q) = \sum_{k=1}^{m} (\alpha_k - \alpha_{k+1}) \rho(W_k),$$

where $\alpha_{m+1} = 0$ by convention. Similarly we have

$$\rho'(X) = \sum_{k=1}^{m} \rho(W_{k-1} \cup (X \cap W_k)) + \sum_{k=1}^{m-1} (\alpha_k - \alpha_{k+1} - 1) \rho(W_k) + \alpha_m \rho(V)$$

also by (L2) since $p \wedge q \in \mathbf{Z}^V$. This expression shows the submodularity of ρ'. □

Finally, we come to the duality. Frank's discrete separation theorem for a pair of sub/supermodular functions reads as follows.

Theorem 3.5 (Discrete Separation Theorem [14]) *Let $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ and $\mu : 2^V \to \mathbf{Z} \cup \{-\infty\}$ be submodular and supermodular functions, respectively, with $\rho(\emptyset) = \mu(\emptyset) = 0$, $\rho(V) < +\infty$, $\mu(V) > -\infty$. If $\mu(X) \leq \rho(X)$ $(X \subseteq V)$, there exists $x^* \in \mathbf{Z}^V$ such that*

$$\mu(X) \leq x^*(X) \leq \rho(X) \qquad (X \subseteq V). \tag{3.12}$$

□

Remark 3.1 The original statement of the discrete separation theorem covers the more general class of sub/supermodular functions on crossing-families. It is also noted that Frank's discrete separation theorem, Edmonds' intersection theorem [9], [10], and Fujishige's Fenchel-type min-max theorem [17] are regarded as equivalent theorems (see [19, §6.1(b)]). See Kindler [26] for a related result. □

The following is a conjugate reformulation of the discrete separation theorem based on the conjugacy between the exchangeability (B-EXC) of an integral base set and the submodularity in its support function, as expounded in Theorem 3.3 (or Theorem 3.1).

Theorem 3.6 *Let B_1 and B_2 be integral base sets ($\subseteq \mathbf{Z}^V$). If they are disjoint ($B_1 \cap B_2 = \emptyset$), there exists $p^* \in \{-1, 0, 1\}^V$ such that*

$$\inf\{\langle p^*, x\rangle \mid x \in B_1\} - \sup\{\langle p^*, x\rangle \mid x \in B_2\} \geq 1. \tag{3.13}$$

In particular, $B_1 \cap B_2 = \emptyset$ is equivalent to $\overline{B_1} \cap \overline{B_2} = \emptyset$.

(Proof) By Theorem 3.1 we have $B_i = \mathbf{Z}^V \cap \mathbf{B}(\rho_i)$ for some submodular function $\rho_i : 2^V \to \mathbf{Z} \cup \{+\infty\}$ with $\rho_i(\emptyset) = 0$ and $\rho_i(V) < +\infty$ $(i = 1, 2)$. If $\rho_1(V) \neq \rho_2(V)$, take $p^* = \chi_V$ or $-\chi_V$. Otherwise, apply Theorem 3.5 with $\rho = \rho_1$ and $\mu = {\rho_2}^\#$ (where ${\rho_2}^\#(X) = \rho_2(V) - \rho_2(V - X)$) to obtain some $X \subseteq V$ with $\rho(X) < \mu(X)$. The choice of $p^* = -\chi_X$ is valid since $\rho(X) = \sup\{\langle \chi_X, x\rangle \mid x \in B_1\}$ and $\mu(X) = \inf\{\langle \chi_X, x\rangle \mid x \in B_2\}$. $\Box$

Remark 3.2 It should be clear that the essence of the discrete separation theorem (Theorem 3.5) lies in the separation for functions in the class $\mathcal{L}_0$, whereas Theorem 3.6 is for $\mathcal{M}_0$. This conjugate pair of separation theorems will be extended in Section 5 to a pair of separation theorems, one for M-convex/concave functions (Theorem 5.3, called the M-separation theorem) and the other for L-convex/concave functions (Theorem 5.4, called the L-separation theorem). $\Box$

4 M-convex and L-convex Functions

4.1 General concepts

For $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{\pm\infty\}$ and $h : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ the effective domains are defined by

$$\begin{aligned} \mathrm{dom}_{\mathbf{Z}} f &= \{x \in \mathbf{Z}^V \mid -\infty < f(x) < +\infty\}, && (4.1) \\ \mathrm{dom}_{\mathbf{R}} h &= \{x \in \mathbf{R}^V \mid -\infty < h(x) < +\infty\}. && (4.2) \end{aligned}$$

We use the notation

$$\begin{aligned} \mathrm{argmin}_{\mathbf{Z}}(f) &= \{x \in \mathbf{Z}^V \mid f(x) \leq f(y)\ (\forall y \in \mathbf{Z}^V)\}, && (4.3) \\ \mathrm{argmin}_{\mathbf{R}}(h) &= \{x \in \mathbf{R}^V \mid h(x) \leq h(y)\ (\forall y \in \mathbf{R}^V)\}. && (4.4) \end{aligned}$$

For $p \in \mathbf{R}^V$ we define $f[p] : \mathbf{Z}^V \to \mathbf{R} \cup \{\pm\infty\}$ and $h[p] : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ by

$$f[p](x) = f(x) + \langle p, x \rangle, \qquad h[p](x) = h(x) + \langle p, x \rangle. \tag{4.5}$$

Obviously, $f[p] : \mathbf{Z}^V \to \mathbf{Z} \cup \{\pm\infty\}$ if $p \in \mathbf{Z}^V$. We say that h is an extension of f if $h(x) = f(x)$ for $x \in \mathbf{Z}^V$.

The (integer, convex) conjugate function $f^\bullet : \mathbf{Z}^V \to \mathbf{Z} \cup \{\pm\infty\}$ is defined by

$$f^\bullet(p) = \sup\{\langle p, x \rangle - f(x) \mid x \in \mathbf{Z}^V\} \qquad (p \in \mathbf{Z}^V). \tag{4.6}$$

The mapping $f \mapsto f^\bullet$ will be called the integer convex Fenchel transformation, or simply the Fenchel transformation. Similarly, the integer concave conjugate function of f is defined to be $f^\circ : \mathbf{Z}^V \to \mathbf{Z} \cup \{\pm\infty\}$ given by

$$f^\circ(p) = \inf\{\langle p, x \rangle - f(x) \mid x \in \mathbf{Z}^V\} \qquad (p \in \mathbf{Z}^V). \tag{4.7}$$

The mapping $f \mapsto f^\circ$ will be called the integer concave Fenchel transformation, or simply the Fenchel transformation if no ambiguity arises.

The (integer, convex) biconjugate function $f^{\bullet\bullet} : \mathbf{Z}^V \to \mathbf{Z} \cup \{\pm\infty\}$ is defined to be $(f^\bullet)^\bullet$, i.e.,

$$f^{\bullet\bullet}(x) = \sup\{\langle p, x \rangle - f^\bullet(p) \mid p \in \mathbf{Z}^V\} \qquad (x \in \mathbf{Z}^V). \tag{4.8}$$

Similarly, $f^{\circ\circ} = (f^\circ)^\circ$. See Examples 1.1 and 1.2 for concrete instances of $f^\bullet$ and $f^{\bullet\bullet}$. The biconjugate function is closely related to the convex closure $\overline{f} : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ of f, which is defined by

$$\overline{f}(x) = \sup_{p \in \mathbf{R}^V} \left[\langle p, x \rangle - \sup_{y \in \mathbf{Z}^V} (\langle p, y \rangle - f(y)) \right] \qquad (x \in \mathbf{R}^V). \tag{4.9}$$

Obviously we have $f^{\bullet\bullet}(x) \leq \overline{f}(x)$ for $x \in \mathbf{Z}^V$.

The subdifferential of f at $x \in \mathrm{dom}_{\mathbf{Z}} f$, denoted $\partial_{\mathbf{R}} f(x)$, is defined by

$$\partial_{\mathbf{R}} f(x) = \{p \in \mathbf{R}^V \mid f(y) - f(x) \geq \langle p, y - x \rangle \ (\forall y \in \mathbf{Z}^V)\}, \tag{4.10}$$

where a vector $p \in \partial_{\mathbf{R}} f(x)$ is called, as usual, a subgradient of f at x. We are particularly interested in subgradients that are integer vectors, and define the integer subdifferential of f at $x \in \mathrm{dom}_{\mathbf{Z}} f$ by $\partial_{\mathbf{Z}} f(x) = \mathbf{Z}^V \cap \partial_{\mathbf{R}} f(x)$, i.e.,

$$\partial_{\mathbf{Z}} f(x) = \{p \in \mathbf{Z}^V \mid f(y) - f(x) \geq \langle p, y - x \rangle \ (\forall y \in \mathbf{Z}^V)\}, \tag{4.11}$$

where $p \in \partial_{\mathbf{Z}} f(x)$ is called an integer subgradient of f at x. See Examples 1.1 and 1.2 for concrete instances of $\partial_{\mathbf{Z}} f$.

Lemma 4.1 *(1) For $x \in \mathrm{dom}_{\mathbf{Z}} f$ and $p \in \mathbf{Z}^V$ we have*

$$p \in \partial_{\mathbf{Z}} f(x) \iff x \in \mathrm{argmin}_{\mathbf{Z}}(f[-p]) \iff f(x) + f^{\bullet}(p) = \langle p, x \rangle. \tag{4.12}$$

(2) For $x \in \mathrm{dom}_{\mathbf{Z}} f$ we have $f^{\bullet\bullet}(x) = f(x) \iff \partial_{\mathbf{Z}} f(x) \neq \emptyset$.

(Proof) (1) This is obvious from the definition.

(2) First note the obvious inequality: $f^{\bullet\bullet}(x) \leq f(x)$. If $\exists\, p \in \partial_{\mathbf{Z}} f(x)$, we see from (4.12) that $f(x) + f^{\bullet}(p) = \langle p, x \rangle$. From this and the definition of $f^{\bullet\bullet}(x)$ we obtain $f^{\bullet\bullet}(x) \geq \langle p, x \rangle - f^{\bullet}(p) = f(x)$. Conversely, if $f^{\bullet\bullet}(x) = f(x)$, there exists $p \in \mathbf{Z}^V$ such that $\langle p, x \rangle - f^{\bullet}(p) = f(x)$, which means $p \in \partial_{\mathbf{Z}} f(x)$. □

For $f_i : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ $(i = 1, 2)$, the integer infimum convolution (or simply the convolution) $f_1 \Box_{\mathbf{Z}} f_2 : \mathbf{Z}^V \to \mathbf{Z} \cup \{\pm\infty\}$ is defined by

$$(f_1 \Box_{\mathbf{Z}} f_2)(x) = \inf\{f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x, x_1 \in \mathbf{Z}^V, x_2 \in \mathbf{Z}^V\}. \tag{4.13}$$

It is easy to verify that, if $f_1 \Box_{\mathbf{Z}} f_2 : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$, the following identities hold true:

$$\mathrm{dom}_{\mathbf{Z}}(f_1 \Box_{\mathbf{Z}} f_2) = \mathrm{dom}_{\mathbf{Z}} f_1 + \mathrm{dom}_{\mathbf{Z}} f_2, \tag{4.14}$$

$$(f_1 \Box_{\mathbf{Z}} f_2)^{\bullet} = {f_1}^{\bullet} + {f_2}^{\bullet}, \tag{4.15}$$

where we use the notation

$$\mathrm{dom}_{\mathbf{Z}} f_1 + \mathrm{dom}_{\mathbf{Z}} f_2 = \{x_1 + x_2 \mid x_1 \in \mathrm{dom}_{\mathbf{Z}} f_1, x_2 \in \mathrm{dom}_{\mathbf{Z}} f_2\} \tag{4.16}$$

on the right-hand side of (4.14).

With the above notations we now introduce two classes of "convex functions", $\mathcal{F}_{\mathrm{E}}$ and $\mathcal{F}_{\mathrm{G}}$. The first class $\mathcal{F}_{\mathrm{E}}$ is defined by

$$\begin{aligned} \mathcal{F}_{\mathrm{E}} = \{ & f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\} \mid \\ & \mathrm{dom}_{\mathbf{Z}} f \neq \emptyset,\ \overline{f}(x) = f(x) \ (x \in \mathbf{Z}^V)\} \end{aligned} \tag{4.17}$$

with $\overline{f}$ denoting the convex closure (4.9) of f. Namely, $\mathcal{F}_{\rm E}$ (with E for Extendability) denotes the class of "convex functions" discussed in Introduction. However, this class is too general to be interesting, as we have seen in Examples 1.1 to 1.3. This motivates us to introduce the second class $\mathcal{F}_{\rm G}$ (with G for subGradient) defined by

$$\begin{aligned}\mathcal{F}_{\rm G} \quad = \quad \{ f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\} \mid & \\ & {\rm dom}_{\mathbf{Z}} f = \mathbf{Z}^V \cap \overline{{\rm dom}_{\mathbf{Z}} f} \neq \emptyset, \\ & \overline{{\rm dom}_{\mathbf{Z}} f} \text{ is rationally-polyhedral}, \\ & \partial_{\mathbf{R}} f(x) = \overline{\partial_{\mathbf{Z}} f(x)} \neq \emptyset \ (x \in {\rm dom}_{\mathbf{Z}} f) \}, \end{aligned} \qquad (4.18)$$

where $\overline{{\rm dom}_{\mathbf{Z}} f}$ and $\overline{\partial_{\mathbf{Z}} f(x)}$ denote the closed convex closures of ${\rm dom}_{\mathbf{Z}} f$ and $\partial_{\mathbf{Z}} f(x)$, respectively, and a closed convex set ($\subseteq \mathbf{R}^V$) is said to be rationally-polyhedral if it is described by a system of finitely many inequalities with coefficients of rational numbers. $\mathcal{F}_{\rm G}$ is a well-behaved proper subclass of $\mathcal{F}_{\rm E}$, as is claimed below, and the function f of Example 1.1 is an instance that belongs to $\mathcal{F}_{\rm E} \setminus \mathcal{F}_{\rm G}$.

Lemma 4.2 *For $f \in \mathcal{F}_{\rm G}$ we have the following.*

(1) ${\rm dom}_{\mathbf{Z}} f^\bullet = \bigcup \{ \partial_{\mathbf{Z}} f(x) \mid x \in {\rm dom}_{\mathbf{Z}} f \} \neq \emptyset$.

(2) ${\rm dom}_{\mathbf{Z}} f^{\bullet\bullet} = {\rm dom}_{\mathbf{Z}} f$.

(3) $f^{\bullet\bullet}(x) = f(x) \qquad (x \in \mathbf{Z}^V)$.

(4) $p \in \partial_{\mathbf{Z}} f(x) \iff x \in \partial_{\mathbf{Z}} f^\bullet(p)$, *where* $x \in {\rm dom}_{\mathbf{Z}} f$, $p \in {\rm dom}_{\mathbf{Z}} f^\bullet$.

(5) $\partial_{\mathbf{Z}} f^\bullet(p) \neq \emptyset \qquad (p \in {\rm dom}_{\mathbf{Z}} f^\bullet)$.

(6) $\overline{f} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$.

(7) ${\rm dom}_{\mathbf{R}} \overline{f} = \overline{{\rm dom}_{\mathbf{Z}} f}$.

(8) $\overline{f}(x) = f(x) \qquad (x \in \mathbf{Z}^V)$.

(Proof) First of all, there exists $x_0 \in {\rm dom}_{\mathbf{Z}} f$ and we have $f^\bullet(p) \geq \langle p, x_0 \rangle - f(x_0) > -\infty$, which shows $f^\bullet : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$. Note also

$$f^{\bullet\bullet}(x) \leq \overline{f}(x) \leq f(x) \qquad (x \in \mathbf{Z}^V). \qquad (4.19)$$

(1) For $x \in {\rm dom}_{\mathbf{Z}} f$ and $p \in \partial_{\mathbf{Z}} f(x)$ we have $f^\bullet(p) = \langle p, x \rangle - f(x) < +\infty$ from (4.12). Conversely, if $f^\bullet(p) < +\infty$, there exists $x \in {\rm dom}_{\mathbf{Z}} f$ such that $f^\bullet(p) = \langle p, x \rangle - f(x)$, which implies again by (4.12) that $p \in \partial_{\mathbf{Z}} f(x)$.

(2) Note first that (1) implies $f^{\bullet\bullet} : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$, since $\sup_{y \in \mathbf{Z}^V} (\langle p, y \rangle - f(y)) < +\infty$ for $p \in {\rm dom}_{\mathbf{Z}} f^\bullet$. It follows from (4.19) that ${\rm dom}_{\mathbf{Z}} f^{\bullet\bullet} \supseteq {\rm dom}_{\mathbf{Z}} f$. To show the reverse inclusion, take $x \in \mathbf{Z}^V \setminus {\rm dom}_{\mathbf{Z}} f$. Then $x \notin \overline{{\rm dom}_{\mathbf{Z}} f}$ by the

assumption that $\mathrm{dom}_{\mathbf{Z}} f = \mathbf{Z}^V \cap \overline{\mathrm{dom}_{\mathbf{Z}} f}$. Since $\overline{\mathrm{dom}_{\mathbf{Z}} f}$ is rationally-polyhedral by the assumption, there exists $p^* \in \mathbf{Z}^V$ such that

$$\langle p^*, x\rangle - \sup_{y\in \mathrm{dom}_{\mathbf{Z}} f} \langle p^*, y\rangle \geq 1. \tag{4.20}$$

On the other hand, by (1), there exists $p_0 \in \mathbf{Z}^V$ such that

$$f^\bullet(p_0) = \sup_{y\in\mathbf{Z}^V} [\langle p_0, x\rangle - f(x)] < +\infty.$$

Hence we obtain

$$f^{\bullet\bullet}(x) \geq [\langle p_0, x\rangle - f^\bullet(p_0)] + [\langle p - p_0, x\rangle - \sup_{y\in \mathrm{dom}_{\mathbf{Z}} f} \langle p - p_0, y\rangle]$$

for any $p \in \mathbf{Z}^V$. The right-hand side with $p = p_0 + \alpha p^*$ with $\alpha \in \mathbf{Z}_+$ is at least as large as $[\langle p_0, x\rangle - f^\bullet(p_0)] + \alpha$, which is not bounded for $\alpha \in \mathbf{Z}_+$. Hence $x \notin \mathrm{dom}_{\mathbf{Z}} f^{\bullet\bullet}$.

(3) By Lemma 4.1 and the definition of $\mathcal{F}_{\mathrm{G}}$.

(4) This is immediate from (4.12) combined with (3).

(5) For $p \in \mathrm{dom}_{\mathbf{Z}} f^\bullet$ we can find, by (1), $x \in \mathrm{dom}_{\mathbf{Z}} f$ such that $p \in \partial_{\mathbf{Z}} f(x)$, which is equivalent to $x \in \partial_{\mathbf{Z}} f^\bullet(p)$ by (4.12). Hence $\partial_{\mathbf{Z}} f^\bullet(p) \neq \emptyset$.

(6) In (4.9) we have $\sup_{y\in\mathbf{Z}^V}(\langle p, y\rangle - f(y)) < +\infty$ for $p \in \partial_{\mathbf{Z}} f(x)$ with $x \in \mathrm{dom}_{\mathbf{Z}} f$, just as in the proof of (1).

(7) (4.19) implies $\mathrm{dom}_{\mathbf{R}} \overline{f} \supseteq \mathrm{dom}_{\mathbf{Z}} f$, from which follows $\mathrm{dom}_{\mathbf{R}} \overline{f} \supseteq \overline{\mathrm{dom}_{\mathbf{Z}} f}$ since $\mathrm{dom}_{\mathbf{R}} \overline{f}$ is a closed convex set. The reverse inclusion can be shown as in the proof of (2).

(8) This follows from (3) and (4.19). □

Remark 4.1 A comment is in order on the assumption imposed on $\overline{\mathrm{dom}_{\mathbf{Z}} f}$ in the definition of $\mathcal{F}_{\mathrm{G}}$. First, if $\mathrm{dom}_{\mathbf{Z}} f$ is bounded (i.e., finite), this condition is always fulfilled. Next, this technical condition cannot be omitted in general. We provide here an example of f satisfying the other two conditions but with $\mathrm{dom}_{\mathbf{Z}} f^{\bullet\bullet} \neq \mathrm{dom}_{\mathbf{Z}} f$. Let $V = \{1, 2\}$, $B = \{(x_1, x_2) \in \mathbf{Z}^2 \mid x_2 \geq \sqrt{2}x_1 - 1/2\}$, and $f : \mathbf{Z}^2 \to \mathbf{Z} \cup \{+\infty\}$ be defined by

$$f(x_1, x_2) = \begin{cases} 0 & ((x_1, x_2) \in B) \\ +\infty & (\text{otherwise}) \end{cases}$$

Then $\partial_{\mathbf{R}} f(x) = \partial_{\mathbf{Z}} f(x) = \{(0, 0)\} \neq \emptyset$ for all $x \in B$, but $f^{\bullet\bullet}(x) = 0$ for all $x \in \mathbf{Z}^2$, which shows $\mathrm{dom}_{\mathbf{Z}} f^{\bullet\bullet} = \mathbf{Z}^2 \neq \mathrm{dom}_{\mathbf{Z}} f$. To verify this, note that $f^\bullet(0, 0) = 0$ and $f^\bullet(p) = +\infty$ for $p \in \mathbf{Z}^2 \setminus \{(0, 0)\}$. □

Remark 4.2 Certainly $\mathcal{F}_{\mathrm{G}}$ identifies a class of well-behaved "convex functions," in which $f^{\bullet\bullet} = f$, $\partial_{\mathbf{Z}} f(x) \neq \emptyset$, and $\partial_{\mathbf{Z}} f^\bullet(p) \neq \emptyset$ hold true, but it does not allow us to derive the discrete separation theorem. In fact, Example 1.3 shows the failure of discrete separation for functions f and g with $f, -g \in \mathcal{F}_{\mathrm{G}}$. □

4.2 M-convex functions

We say that f is M-convex if $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$, $\mathrm{dom}_{\mathbf{Z}} f \neq \emptyset$ and the following variant of the simultaneous exchange axiom holds true:

(M-EXC) For $x, y \in \mathrm{dom}_{\mathbf{Z}} f$ and $u \in \mathrm{supp}^+(x - y)$ there exists $v \in \mathrm{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v). \tag{4.21}$$

We use the notation

$$\mathcal{M} = \{f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\} \mid f \text{ is M-convex}\}. \tag{4.22}$$

We say that g is M-concave if $-g$ is M-convex. The name of M-convexity/concavity, which is intended to mean convexity/concavity related to Matroid (or matroidal exchangeability, to be more precise), will be justified by the Extension Theorem (Theorem 4.11) below. Recall Examples 2.1 to 2.4 for instances of M-convex functions.

Obviously, the indicator function $\delta_B : \mathbf{Z}^V \to \{0, +\infty\}$ of a set $B \subseteq \mathbf{Z}^V$ is M-convex if and only if B is an integral base set. Thus the concept of M-convex function is a quantitative generalization of that of integral base set, and accordingly an integral base set could be referred to as an "M-convex set."

We will show fundamental properties of M-convex functions with the analogy to ordinary convex functions in mind. It is remarked that the effective domain is unbounded in general[2].

First we note the following implication of (M-EXC) for the effective domain.

Theorem 4.3 *If f is M-convex, then* $\mathrm{dom}_{\mathbf{Z}} f$ *is an integral base set.*

(Proof) This is immediate from (M-EXC), since in (4.21) we have $x - \chi_u + \chi_v, y + \chi_u - \chi_v \in \mathrm{dom}_{\mathbf{Z}} f$ if $x, y \in \mathrm{dom}_{\mathbf{Z}} f$. □

M-convexity is invariant under the addition/subtraction of linear functions as well as the translation and the negation of the argument, as follows. Recall the notation (4.5).

Theorem 4.4 *Let f be an M-convex function.*

(1) $f[-p](x)$ is M-convex in x for $p \in \mathbf{Z}^V$.

(2) $f(a - x)$ and $f(a + x)$ are M-convex in x for $a \in \mathbf{Z}^V$.

[2]Some of the materials presented in this subsection are immediate consequences and slight modifications of the results of [36], [37], [38], [39], in which M-concave functions are treated under the notation ω. In [38], [39] the effective domain is assumed to be bounded.

(Proof) The proofs are immediate from (M-EXC). $\Box$

For $(a, b) \in \mathbf{Z}^V \times \mathbf{Z}^V$ with $a \leq b$ we define the interval $[a, b]$ by

$$[a, b] = \{x \in \mathbf{R}^V \mid a(v) \leq x(v) \leq b(v) \ (v \in V)\}.$$

We denote the restriction of f to $[a, b]$ by f_a^b, namely,

$$f_a^b(x) = \begin{cases} f(x) & (x \in [a, b]) \\ +\infty & (\text{otherwise}) \end{cases} \tag{4.23}$$

M-convexity is preserved under the restriction operation.

Theorem 4.5 *If f is M-convex, then f_a^b is M-convex for $(a, b) \in \mathbf{Z}^V \times \mathbf{Z}^V$ such that* $\mathrm{dom}_{\mathbf{Z}} f \cap [a, b] \neq \emptyset$. $\Box$

The global optimality is guaranteed by the local optimality expressed in terms of the discrete analogue of the directional derivative defined by

$$f(x, u, v) = f(x - \chi_u + \chi_v) - f(x) \qquad (x \in \mathrm{dom}_{\mathbf{Z}} f; u, v \in V). \tag{4.24}$$

The following theorem is a straightforward extension of a similar result of [7], [8] for a matroid valuation.

Theorem 4.6 ([37]) *Let f be M-convex and $x \in \mathrm{dom}_{\mathbf{Z}} f$. Then $f(x) \leq f(y)$ $(\forall\, y \in \mathbf{Z}^V)$ if and only if*

$$f(x, u, v) \geq 0 \qquad (\forall\, u, v \in V). \tag{4.25}$$

(Proof) The necessity is obvious. We prove the sufficiency by induction on $||x-y||$. The assumption implies $f(x) \leq f(y)$ for $y \in B$ with $||x - y|| = 2$. Suppose $||x-y|| \geq 4$. By (M-EXC) there exists $x', y' \in B$ such that $||x-x'|| = ||y-y'|| = 2$, $||x-y'|| = ||x-y||-2$, and $f(x)+f(y) \geq f(x')+f(y')$. Here we have $f(x') \geq f(x)$ by the assumption and $f(y') \geq f(x)$ by the induction hypothesis. Hence we obtain $f(y) \geq f(x)$. $\Box$

Integer subgradients exist for an M-convex function, namely $\mathcal{M} \subseteq \mathcal{F}_{\mathrm{G}}$ using the notations (4.18) and (4.22). This fact is nontrivial in view of Example 1.1.

Theorem 4.7 *For an M-convex function f and $x \in \mathrm{dom}_{\mathbf{Z}} f$,*

$$\partial_{\mathbf{Z}} f(x) = \{p \in \mathbf{Z}^V \mid f(x, u, v) \geq p(v) - p(u) \ (u, v \in V)\} \neq \emptyset. \tag{4.26}$$

Moreover, $\partial_{\mathbf{R}} f(x) = \overline{\partial_{\mathbf{Z}} f(x)}$, and hence $\mathcal{M} \subseteq \mathcal{F}_{\mathrm{G}}$.

(Proof) Similarly to (4.12), $p \in \partial_{\mathbf{R}} f(x)$ if and only if $x \in \mathrm{argmin}_{\mathbf{R}}(f[-p])$. Then Theorem 4.6 applied to $f[-p]$ implies

$$\partial_{\mathbf{R}} f(x) = \{p \in \mathbf{R}^V \mid f(x, u, v) \geq p(v) - p(u) \ (u, v \in V)\}.$$

(To be precise, $f[-p]$ is not integer-valued, but the extension of Theorem 4.6 to this case is obvious from its proof.)

Consider a directed graph $G = (V, A)$ with vertex set V and arc set $A = \{(u, v) \mid x - \chi_u + \chi_v \in \mathrm{dom}_{\mathbf{Z}} f\}$. We assume that $\gamma(a) = f(x, u, v) \in \mathbf{Z}$ is attached to $a = (u, v) \in A$ as the arc length. Then we can express $\partial_{\mathbf{R}} f(x)$ as

$$\partial_{\mathbf{R}} f(x) = \{p \in \mathbf{R}^V \mid \gamma(a) + p(\partial^+ a) - p(\partial^- a) \geq 0 \ (a \in A)\}, \tag{4.27}$$

where $\partial^+ a$ and $\partial^- a$ denote the initial vertex and the terminal vertex of $a \in A$, respectively. This expression shows that a subgradient is nothing but a valid "potential function" on the vertex set. According to standard results in the network theory (cf., e.g., [30]), we see that (i) $\partial_{\mathbf{R}} f(x) = \overline{\partial_{\mathbf{Z}} f(x)}$ and (ii) $\partial_{\mathbf{R}} f(x) \neq \emptyset$ if there is no negative cycle with respect to γ. To prove the nonexistence of a negative cycle we observe that, for $v_1, v_2, v_3 \in V$ such that $v_1 \neq v_2 \neq v_3$ (with the possibility of $v_1 = v_3$), the exchange axiom (M-EXC) implies the triangle inequality

$$\gamma(v_1, v_2) + \gamma(v_2, v_3) \geq \gamma(v_1, v_3) \quad (> -\infty), \tag{4.28}$$

since (M-EXC) applied to $x' = x - \chi_{v_2} + \chi_{v_3}$ and $y' = x - \chi_{v_1} + \chi_{v_2}$ with $u = v_1 \in \mathrm{supp}^+(x' - y')$ yields $v = v_2 \in \mathrm{supp}^-(x' - y')$. The triangle inequality (4.28) implies in particular that $(v_1, v_3) \in A$. Note that $\gamma(v_1, v_3) = 0$ if $v_1 = v_3$. Furthermore we see that any simple cycle, say consisting of $v_1, v_2, \cdots, v_k$, has a nonnegative length by a repeated application of this relation to

$$\gamma(v_1, v_2) + \gamma(v_2, v_3) + \cdots + \gamma(v_{k-1}, v_k) + \gamma(v_k, v_1).$$

Finally, the effective domain of f is an integral base set by Theorem 4.3, and therefore $\mathrm{dom}_{\mathbf{Z}} f = \mathbf{Z}^V \cap \overline{\mathrm{dom}_{\mathbf{Z}} f} \neq \emptyset$ and $\overline{\mathrm{dom}_{\mathbf{Z}} f}$ is rationally-polyhedral. This establishes $f \in \mathcal{F}_{\mathrm{G}}$. $\square$

The existence of integer subgradients (or the inclusion $\mathcal{M} \subseteq \mathcal{F}_{\mathrm{G}}$ to be more precise) allows us to apply the general lemma (Lemma 4.2) to M-convex functions. In particular we obtain the following statements.

Theorem 4.8 *For an M-convex function f we have the following.*

(1) $f^\bullet : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$.

(2) $\mathrm{dom}_{\mathbf{Z}} f^\bullet = \bigcup \{\partial_{\mathbf{Z}} f(x) \mid x \in \mathrm{dom}_{\mathbf{Z}} f\} \neq \emptyset$.

(3) $f^{\bullet\bullet} = f$. $\square$

The following theorem connects the concepts of M-convex functions ("convex functions") and integral base sets ("convex sets"). It is understood that $\mathrm{argmin}_{\mathbf{Z}}(f[-p])$ is defined by (4.3) with f replaced by $f[-p]$ though $f[-p]$ is not integer-valued.

Theorem 4.9 ([38, Theorem 4.4]) *Let $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be such that $\mathrm{dom}_{\mathbf{Z}} f$ is a bounded integral base set. Then f is M-convex if and only if $\mathrm{argmin}_{\mathbf{Z}}(f[-p])$ is an integral base set for each $p \in \mathbf{R}^V$.*

This theorem can be modified as follows to cover the general case of unbounded effective domain.

Theorem 4.10 *Let $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be such that $\mathrm{dom}_{\mathbf{Z}} f$ is an integral base set.*

(1) If f is M-convex, then $\mathrm{argmin}_{\mathbf{Z}}(f[-p])$ is either an integral base set or an empty set for each $p \in \mathbf{R}^V$.

(2) Let f_a^b denote the restriction of f to $[a, b]$, as defined in (4.23). Then f is M-convex if and only if $\mathrm{argmin}_{\mathbf{Z}}(f_a^b[-p])$ is an integral base set for each $p \in \mathbf{R}^V$ and $(a, b) \in \mathbf{Z}^V \times \mathbf{Z}^V$ such that $\mathrm{dom}_{\mathbf{Z}} f \cap [a, b] \neq \emptyset$.

(Proof) (1) This follows easily from (M-EXC), since in (4.21) we have $x - \chi_u + \chi_v, y + \chi_u - \chi_v \in \mathrm{argmin}_{\mathbf{Z}}(f[-p])$ if $x, y \in \mathrm{argmin}_{\mathbf{Z}}(f[-p])$.

(2) It is easy to see from the definition that f is M-convex if and only if f_a^b is M-convex for each $(a, b) \in \mathbf{Z}^V \times \mathbf{Z}^V$ such that $\mathrm{dom}_{\mathbf{Z}} f_a^b \neq \emptyset$. Then the assertion follows from Theorem 4.9. $\square$

Remark 4.3 The introduction of restrictions in the statement of Theorem 4.10(2) is cumbersome, but seems inevitable. As an example, let $V = \{1, 2\}$ and $f : \mathbf{Z}^2 \to \mathbf{Z} \cup \{+\infty\}$ be defined by

$$f(x_1, x_2) = \begin{cases} 0 & ((x_1, x_2) = (0, 0)) \\ 1 & ((x_1, x_2) \neq (0, 0), x_1 + x_2 = 0) \\ +\infty & (\text{otherwise}) \end{cases}$$

Then $\mathrm{dom}_{\mathbf{Z}} f = \{(x_1, x_2) \in \mathbf{Z}^2 \mid x_1 + x_2 = 0\}$ is an integral base set, and $\mathrm{argmin}_{\mathbf{Z}}(f[-p])$ is $\{(0, 0)\}$ (an integral base set) if $p_1 = p_2$, and empty otherwise. But f is not M-convex. On the other hand, we do not have such a pathological phenomenon for $f \in \mathcal{F}_{\mathrm{E}}$; see Theorem 4.11 below. $\square$

The following theorem justifies the name of M-convexity with reference to our discussion about "convex functions" in Introduction. Note here that $\mathcal{M} \subseteq \mathcal{F}_{\mathrm{G}}$, established in Theorem 4.7, already implies by Lemma 4.2 that M-convex functions can be extended to convex functions. It is mentioned that this theorem has been proven in [38] under the assumption of the boundedness of $\mathrm{dom}_{\mathbf{Z}} f$.

Theorem 4.11 (Extension Theorem) $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ *is M-convex if and only if* $\overline{f} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *defined by (4.9) satisfies* (a), (b) *and* (c) *below:*

(a) $\overline{f}(x) = f(x) \quad (x \in \mathbf{Z}^V)$,

(b) $\mathrm{dom}_{\mathbf{R}} \overline{f}$ *is an integral base polyhedron,*

(c) *For each* $p \in \mathbf{R}^V$,

$$\mathrm{argmin}_{\mathbf{R}}(\overline{f}[-p]) = \{x \in \mathbf{R}^V \mid \overline{f}(x) - \langle p, x\rangle \leq \overline{f}(x') - \langle p, x'\rangle \ (\forall x' \in \mathbf{R}^V)\}$$

is an integral base polyhedron or an empty set.

(Proof) First we prove the "only if" part. We have $f \in \mathcal{F}_{\mathrm{G}}$ by Theorem 4.7. (a) is nothing but Lemma 4.2(8). For (b) we apply Lemma 4.2(7) to obtain $\mathrm{dom}_{\mathbf{R}} \overline{f} = \overline{\mathrm{dom}_{\mathbf{Z}} f}$, which is an integral base polyhedron by Theorem 4.3. For (c) we note $\mathrm{argmin}_{\mathbf{R}} \overline{f}[-p] = \overline{\mathrm{argmin}_{\mathbf{Z}} f[-p]}$, the right-hand side of which is an integral base polyhedron by Theorem 4.10.

Next we prove the "if" part on the basis of Theorem 4.10. It is not difficult to prove $\mathrm{dom}_{\mathbf{R}} \overline{f} = \overline{\mathrm{dom}_{\mathbf{Z}} f}$ in a similar manner as in Lemma 4.2. Then condition (b) means $\overline{\mathrm{dom}_{\mathbf{Z}} f}$ is an integral base polyhedron, whereas (a) shows $\mathrm{dom}_{\mathbf{Z}} f = \mathbf{Z}^V \cap \mathrm{dom}_{\mathbf{R}} \overline{f} = \mathbf{Z}^V \cap \overline{\mathrm{dom}_{\mathbf{Z}} f}$. Therefore $\mathrm{dom}_{\mathbf{Z}} f$ is an integral base set. Let $(a, b) \in \mathbf{Z}^V \times \mathbf{Z}^V$ be such that $\mathrm{dom}_{\mathbf{Z}} f_a^b \neq \emptyset$. It is easy to see that the restriction of $\overline{f}$ to $[a, b]$ defined by

$$\overline{f}_a^b(x) = \begin{cases} \overline{f}(x) & (x \in [a, b]) \\ +\infty & (\text{otherwise}) \end{cases}$$

has the property that $\mathrm{argmin}_{\mathbf{R}}(\overline{f}_a^b[-p])$ is an integral base polyhedron for each $p \in \mathbf{R}^V$. Since $\mathrm{argmin}_{\mathbf{R}}(\overline{f}_a^b[-p])$ contains an integral point and $\overline{f}_a^b[-p](x) = f_a^b[-p](x)$ for $x \in \mathbf{Z}^V$, we have

$$\min\{\overline{f}_a^b[-p](x) \mid x \in \mathbf{R}^V\} = \min\{f_a^b[-p](x) \mid x \in \mathbf{Z}^V\}$$

and

$$\mathbf{Z}^V \cap \mathrm{argmin}_{\mathbf{R}}(\overline{f}_a^b[-p]) = \mathrm{argmin}_{\mathbf{Z}}(f_a^b[-p]).$$

This means that $\mathrm{argmin}_{\mathbf{Z}}(f_a^b[-p])$ is an integral base set. It then follows from Theorem 4.10(2) that f is M-convex. □

Remark 4.4 The condition (b) in Theorem 4.11 is implied by the condition (c), since $\mathrm{dom}_{\mathbf{R}} \overline{f}$ is convex, $\mathrm{dom}_{\mathbf{R}} \overline{f} = \bigcup_p \mathrm{argmin}_{\mathbf{R}}(\overline{f}[-p])$, and the following lemma holds true. □

Lemma 4.12 *Let* $D = \bigcup_i D_i \subseteq \mathbf{R}^V$ *be a convex set with* D_i*'s being integral base polyhedra. Then* D *is an integral base polyhedron.*

(Proof) For $x, y \in D$ the line segment connecting x and y is contained in D. By considering the restrictions of D and D_i's to the interval $[x \wedge y, x \vee y]$ we may assume that there are only finitely many distinct D_i's. Therefore, there exists i such that $x \in D_i$ and $z \equiv x + t(y - x) \in D_i$ for some $t > 0$. For $u \in \mathrm{supp}^+(x - y) = \mathrm{supp}^+(x - z)$, the exchange property of a base polyhedron D_i guarantees the existence of $v \in \mathrm{supp}^-(x - z) = \mathrm{supp}^-(x - y)$ and $\alpha > 0$ such that $x - \alpha(\chi_u - \chi_v) \in D_i \subseteq D$. Hence D is a base polyhedron. The integrality of D is easy to see. □

Next we are interested in a dual view on the subgradient inequality. By the definition (4.10) and Theorem 4.7 (or (4.27) to be more specific) we have, for $x \in \mathrm{dom}_{\mathbf{Z}} f$ and $y \in \mathbf{Z}^V$,

$$\begin{aligned} & f(y) - f(x) \\ & \geq \max\{\langle p, y - x\rangle \mid p \in \partial_{\mathbf{R}} f(x)\} \\ & = \max\{\langle p, y - x\rangle \mid f(x, u, v) \geq p(v) - p(u) \ ((u, v) \in A)\}, \end{aligned} \tag{4.29}$$

where

$$A = \{(u, v) \mid x - \chi_u + \chi_v \in \mathrm{dom}_{\mathbf{Z}} f\}.$$

The last expression of (4.29) can be recognized as a linear programming problem in variable $p \in \mathbf{R}^V$. Here we introduce another linear programming problem, which is closely related to (but not exactly the same as) the dual of this program (see [3], [47] for the basic facts on linear programming). The variable of the new linear program is $\varphi(u, v)$ indexed by $(u, v) \in \hat{A}$, where

$$\hat{A} = \{(u, v) \mid u \in V^+, v \in V^-, x - \chi_u + \chi_v \in \mathrm{dom}_{\mathbf{Z}} f\}$$

with $V^+ = \mathrm{supp}^+(x - y)$ and $V^- = \mathrm{supp}^-(x - y)$ and the objective function is defined in terms of $f(x, u, v)$. The linear program reads

$$\text{Minimize} \quad \sum_{(u,v) \in \hat{A}} f(x, u, v)\varphi(u, v) \tag{4.30}$$

$$\text{subject to} \quad \varphi(u, v) \geq 0 \qquad ((u, v) \in \hat{A}), \tag{4.31}$$

$$\sum_{v:(u,v) \in \hat{A}} \varphi(u, v) = x(u) - y(u) \qquad (u \in V^+), \tag{4.32}$$

$$\sum_{u:(u,v) \in \hat{A}} \varphi(u, v) = y(v) - x(v) \qquad (v \in V^-). \tag{4.33}$$

The optimal value of the objective function of this linear program is denoted by $\widehat{f}(x, y)$, which is equal to $+\infty$, by convention, if the linear program is infeasible. It may be worth mentioning that this linear program represents a transportation problem (see, e.g., [3], [30], [41], [42]) defined on a bipartite graph $G(x, y) =$

$(V^+, V^-; \hat{A})$ (where (V^+, V^-) is the vertex bipartition and $\hat{A}$ the arc set) with arc cost $f(x, u, v)$. Note that the minimum of the objective function can be attained by an integral φ.

Theorem 4.13 *For an M-convex function f, $x \in \mathrm{dom}_{\mathbf{Z}} f$ and $y \in \mathbf{Z}^V$ we have*

$$\widehat{f}(x, y) = \max\{\langle p, y - x\rangle \mid p \in \partial_{\mathbf{R}} f(x)\} \tag{4.34}$$

and hence

$$f(y) - f(x) \geq \widehat{f}(x, y). \tag{4.35}$$

(Proof) The dual of the linear program in the last expression of (4.29) is given by

$$\begin{aligned} &\text{Minimize} && \sum_{(u,v)\in A} f(x, u, v)\varphi(u, v) && (4.36)\\ &\text{subject to} && \varphi(u, v) \geq 0 \qquad ((u, v) \in A), && (4.37)\\ &&& \partial\varphi(v) = x(v) - y(v) \qquad (v \in V), && (4.38) \end{aligned}$$

where

$$\partial\varphi(v) = \sum_{w:(v,w)\in A} \varphi(v, w) - \sum_{u:(u,v)\in A} \varphi(u, v) \qquad (v \in V).$$

This represents a transshipment problem (see, e.g., [3], [30], [41], [42]) in the network $(G = (V, A), \gamma)$ used in the proof of Theorem 4.7, where $\gamma(a) = f(x, u, v)$ is now interpreted as the cost of arc $a = (u, v) \in A$, $x(v) - y(v)$ is the supply at vertex $v \in V$ and $\varphi : A \to \mathbf{R}$ stands for a flow. Let $\tilde{f}(x, y)$ denote the optimal value of the linear program above. Since γ satisfies the triangle inequality (4.28), we may assume that $\varphi(u, v) > 0$ only if $u \in \mathrm{supp}^+(x - y)$ and $v \in \mathrm{supp}^-(x - y)$. This means that $\tilde{f}(x, y)$ agrees with $\widehat{f}(x, y)$. On the other hand, $\tilde{f}(x, y)$ is equal to $\max\{\langle p, y - x\rangle \mid p \in \partial_{\mathbf{R}} f(x)\}$ by the linear programming duality. Hence follows (4.34), which in turn implies (4.35) when combined with the obvious inequality (4.29). □

Remark 4.5 Theorem 4.13 is a quantitative extension of the well-known fact (cf. [19, Theorem 3.28]) for a submodular system that for $x, y \in \mathbf{B}(\rho)$ there exists $\varphi \in \mathbf{R}^{\hat{A}}$ that satisfy (4.31) –(4.33) (see (3.2) for the notation $\mathbf{B}(\rho)$). This fact is expressed in our notation that $\widehat{f}(x, y) \neq +\infty$ in (4.35), i.e., that the linear program (4.30) – (4.33) has a feasible solution. It is also mentioned that Theorem 4.13 is an extension of the "upper-bound lemma" [34, Lemma 3.4] for a valuated matroid. □

Remark 4.6 It is natural to ask when the inequality $f(y) - f(x) \geq \widehat{f}(x, y)$ of (4.35) is satisfied with equality. A sufficient condition, named the unique-min condition, will be given later in Lemma A.2. This amounts to an extension of the well-known "no-shortcut lemma" for matroids (cf. Kung [29] for this name; also see Bixby-Cunningham [1, Lemma 3.7], Frank [13, Lemma 2], Iri–Tomizawa [25, Lemma 2], Krogdahl [28], Lawler [30, Lemma 3.1 of Chap. 8], Tomizawa–Iri [51, Lemma 3])

□

Remark 4.7 In Theorem 4.13 we have proved the inequality (4.35) by establishing the identity (4.34) by means of the linear programming duality. Here is an alternative (direct) proof of (4.35) on the basis of the exchange property (M-EXC), which remains valid in a more general setting (cf. [37]) but is not suitable to relate $\widehat{f}(x, y)$ to subgradients. For any $u_1 \in \mathrm{supp}^+(x - y)$ there exists $v_1 \in \mathrm{supp}^-(x - y)$ with

$$f(x) + f(y) \geq f(x - \chi_{u_1} + \chi_{v_1}) + f(y + \chi_{u_1} - \chi_{v_1}),$$

which can be rewritten as

$$f(y) \geq f(x, u_1, v_1) + f(y_2)$$

with $y_2 = y + \chi_{u_1} - \chi_{v_1}$. By the same argument applied to (x, y_2) we obtain

$$f(y_2) \geq f(x, u_2, v_2) + f(y_3)$$

for some $u_2 \in \mathrm{supp}^+(x - y_2)$ and $v_2 \in \mathrm{supp}^-(x - y_2)$, where $y_3 = y_2 + \chi_{u_2} - \chi_{v_2} = y + \chi_{u_1} + \chi_{u_2} - \chi_{v_1} - \chi_{v_2}$. Hence

$$f(y) \geq f(y_3) + \sum_{i=1}^{2} f(x, u_i, v_i).$$

Repeating this process we arrive at

$$f(y) \geq f(x) + \sum_{i=1}^{m} f(x, u_i, v_i) \geq f(x) + \widehat{f}(x, y),$$

where $m = ||x - y||/2$, $y = x - \sum_{i=1}^{m} (\chi_{u_i} - \chi_{v_i})$. □

An M-convex function can be transformed to another M-convex function through a network, just as a (poly)matroid is induced by a network [19]. Let $G = (V, A; V^+, V^-)$ be a directed graph with a vertex set V, an arc set A, a set V^+ of entrances and a set V^- of exits such that $V^+, V^- \subseteq V$ and $V^+ \cap V^- = \emptyset$. Also let $\overline{c} : A \to \mathbf{Z} \cup \{+\infty\}$ be an upper capacity function, $\underline{c} : A \to \mathbf{Z} \cup \{-\infty\}$ a lower capacity function, and $\gamma : A \to \mathbf{Z}$ a cost function. Suppose further that we are given a function $f : \mathbf{Z}^{V^+} \to \mathbf{Z} \cup \{+\infty\}$ and put $B = \mathrm{dom}_{\mathbf{Z}} f \subseteq \mathbf{Z}^{V^+}$. A flow is a function $\varphi : A \to \mathbf{Z}$ and its boundary $\partial\varphi : V \to \mathbf{Z}$ is defined by (2.1). We denote by $(\partial\varphi)^+$ (resp. $(\partial\varphi)^-$) the restriction of $\partial\varphi$ to V^+ (resp. V^-). A flow φ is called feasible if

$$\underline{c}(a) \le \varphi(a) \le \overline{c}(a) \qquad (a \in A), \tag{4.39}$$

$$\partial\varphi(v) = 0 \qquad (v \in V - (V^+ \cup V^-)), \tag{4.40}$$

$$(\partial\varphi)^+ \in B. \tag{4.41}$$

Define $\tilde{B} \subseteq \mathbf{Z}^{V^-}$ and $\tilde{f} : \mathbf{Z}^{V^-} \to \mathbf{Z} \cup \{\pm\infty\}$ by

$$\tilde{B} = \{(\partial\varphi)^- \mid \varphi : \text{ feasible flow}\}, \tag{4.42}$$

$$\begin{aligned}\tilde{f}(x) = \inf\{&\langle \gamma, \varphi\rangle_A + f((\partial\varphi)^+) \mid \\ &\varphi : \text{ feasible flow with } (\partial\varphi)^- = x\} \quad (x \in \mathbf{Z}^{V^-}),\end{aligned} \tag{4.43}$$

where $\langle \gamma, \varphi\rangle_A = \sum_{a\in A} \gamma(a)\varphi(a)$ and $\tilde{f}(x) = +\infty$ if there is no feasible φ with $(\partial\varphi)^- = x$. We assume that $\tilde{f} : \mathbf{Z}^{V^-} \to \mathbf{Z} \cup \{+\infty\}$ and $\mathrm{dom}_{\mathbf{Z}}\tilde{f} \neq \emptyset$ (namely, that a feasible flow exists and the minimum is finite for each $x \in \tilde{B}$).

Then we have the following statement, an extension of the well-known fact that, if B is an integral base set, $\tilde{B}$ is also an integral base set.

Theorem 4.14 *If f is M-convex, then $\tilde{f}$ is also M-convex, provided $\tilde{f} : \mathbf{Z}^{V^-} \to \mathbf{Z} \cup \{+\infty\}$ and $\mathrm{dom}_{\mathbf{Z}}\tilde{f} \neq \emptyset$.*

(Proof) This statement has been established in [38], [39] when B is bounded. The general case with unbounded B is an immediate corollary as follows. For $x, y \in \tilde{B}$, take the corresponding flows, say φ_x and φ_y, that attain the minimum on the right-hand side of (4.43). Consider the restriction f_a^b of f to $[a, b]$ with $a = (\partial\varphi_x)^+ \wedge (\partial\varphi_y)^+$ and $b = (\partial\varphi_x)^+ \vee (\partial\varphi_y)^+$. Since $\mathrm{dom}_{\mathbf{Z}} f_a^b$ is bounded, the previous result applies. Let $\widetilde{f_a^b}$ denote the M-convex function induced from f_a^b. Then for any $u \in \mathrm{supp}^+(x - y)$ there exists $v \in \mathrm{supp}^-(x - y)$ such that

$$\begin{aligned}\tilde{f}(x) + \tilde{f}(y) &= \widetilde{f_a^b}(x) + \widetilde{f_a^b}(y) \\ &\ge \widetilde{f_a^b}(x - \chi_u + \chi_v) + \widetilde{f_a^b}(y + \chi_u - \chi_v) \\ &\ge \tilde{f}(x - \chi_u + \chi_v) + \tilde{f}(y + \chi_u - \chi_v).\end{aligned}$$

This shows (M-EXC) for $\tilde{f}$. □

Remark 4.8 The class of M-convex functions is (essentially) closed under the convolution operation (4.13). A precise statement is given later in Theorem 5.8 since the proof depends on coming theorems. □

4.3 L-convex functions

We introduce the other class of "convex functions," called L-convex functions, as a generalization of the class $\mathcal{L}_0$ that consists of the Lovász extensions of submodular functions. The L-convex functions will turn out to be the conjugate of M-convex functions.

As we have seen in Section 3, the class $\mathcal{L}_0$ consists of nonnegatively homogeneous functions g which satisfy (cf. Lemma 3.4 and (3.7))

$$\mathrm{dom}_{\mathbf{Z}} g \neq \emptyset, \tag{4.44}$$

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \qquad (p, q \in \mathbf{Z}^V), \tag{4.45}$$

$$\exists r \in \mathbf{Z}, \forall p \in \mathbf{Z}^V : \; g(p + \mathbf{1}) = g(p) + r. \tag{4.46}$$

We extend the class $\mathcal{L}_0$ by throwing away the homogeneity condition while retaining (4.44), (4.45) and (4.46). Namely we define the class $\mathcal{L}$ of L-convex functions by

$$\mathcal{L} = \{ g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\} \mid g \text{ satisfies (4.44), (4.45), (4.46) } \}. \tag{4.47}$$

Note that (4.46) implies

$$g(p - \chi_X) = g(p + \chi_{V-X}) - r \qquad (p \in \mathbf{Z}^V, X \subseteq V). \tag{4.48}$$

Remark 4.9 L-convexity is intended to be a convexity in a certain sense. In this respect it is worth mentioning that the submodularity (4.45) alone does not imply anything. For example, any function $g : \mathbf{Z}^2 \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}_{\mathbf{Z}} g = \{(p_1, p_2) \in \mathbf{Z}^2 \mid p_1 = p_2\}$ is submodular, whatever values it may take on $\mathrm{dom}_{\mathbf{Z}} g$. Another example is the vector rank function of a submodular system (V, ρ), which is given by $\hat{r}(x) = \min\{\rho(X) + x(V - X) \mid X \subseteq V\}$ $(x \in \mathbf{R}^V)$ and known to be submodular and concave (see [9], [19]). The combination of (4.45) with the other naive-looking condition (4.46) does imply the extendability of g to a convex function, as will be shown in Theorem 4.18(3). □

L-convexity is invariant under the addition/subtraction of linear functions as well as the translation and the negation of the argument, as follows. Recall the notation (4.5).

Theorem 4.15 *Let g be an L-convex function.*

(1) $g[-x](p)$ is L-convex in p for $x \in \mathbf{Z}^V$.

(2) $g(a - p)$ and $g(a + p)$ are L-convex in p for $a \in \mathbf{Z}^V$.

(Proof) The proofs are immediate from the definition. □

As the set version of L-convexity, we define $D \subseteq \mathbf{Z}^V$ to be an L-convex set, if

$$D \neq \emptyset, \tag{4.49}$$
$$p, q \in D \Longrightarrow p \vee q, p \wedge q \in D, \tag{4.50}$$
$$p \in D \Longrightarrow p \pm \mathbf{1} \in D. \tag{4.51}$$

Obviously, $D \subseteq \mathbf{Z}^V$ is an L-convex set if and only if its indicator function δ_D : $\mathbf{Z}^V \to \{0, +\infty\}$ is an L-convex function. We introduce the following notation:

$${}_0\mathcal{L} = \{\delta_D : \mathbf{Z}^V \to \{0, +\infty\} \mid D \text{ is an L-convex set }\}. \tag{4.52}$$

The following two statements hold true, as expected. In the latter theorem it is understood that $\mathrm{argmin}_{\mathbf{Z}}(g[-x])$ is defined by (4.3) with f replaced by $g[-x]$ though $g[-x]$ is not integer-valued.

Theorem 4.16 *Let g be an L-convex function. Then* $\mathrm{dom}_{\mathbf{Z}} g$ *is an L-convex set.*

(Proof) The proof is immediate from (4.45) and (4.46). □

Theorem 4.17 *Let g be an L-convex function. Then* $\mathrm{argmin}_{\mathbf{Z}}(g[-x])$ *is either an L-convex set or an empty set for each $x \in \mathbf{R}^V$.*

(Proof) Note first that $\mathrm{argmin}_{\mathbf{Z}}(g[-x])$ is nonempty only if $g[-x](p+\mathbf{1}) = g[-x](p)$ ($\forall p \in \mathbf{Z}^V$), which implies (4.51). The condition (4.50) follows from (4.45). □

Next we show that the convex closure $\overline{g}$ of $g \in \mathcal{L}$, as defined in (4.9), can be expressed as a localized version of the Lovász extension of the submodular function $\rho_p(X) = g(p + \chi_X) - g(p)$ which represents the local behaviour of g in the neighborhood of p.

Theorem 4.18 *Let g be an L-convex function.*

(1) For $p \in \mathrm{dom}_{\mathbf{Z}} g$ and $q \in [0, 1]^V$ we have

$$\overline{g}(p + q) = g(p) + \sum_{j=1}^{n} (q_j - q_{j+1})(g(p + \chi_{V_j}) - g(p)) \tag{4.53}$$

under the convention $0 \times \infty = 0$ and the indexing convention $q_1 \geq q_2 \geq \cdots \geq q_n$ as (L2) *in Section 3 that, for each q, the elements of V are indexed as $\{v_1, v_2, \cdots, v_n\}$ in such a way that $q(v_1) \geq q(v_2) \geq \cdots \geq q(v_n)$; $q_j = q(v_j)$, $V_j = \{v_1, v_2, \cdots, v_j\}$ for $j = 1, \cdots, n$, and $q_{n+1} = 0$.*

(2) (4.53) can be rewritten as

$$\overline{g}(p + q) = (1 - q_1)g(p) + \sum_{j=1}^{n} (q_j - q_{j+1}) g(p + \chi_{V_j}), \tag{4.54}$$

which is a valid expression for $p \in \mathbf{Z}^V$ and $q \in [0,1]^V$ (being free from $\infty - \infty$ even for $p \notin \mathrm{dom}_{\mathbf{Z}} g$).

(3) $\overline{g}(p) = g(p) \qquad (p \in \mathbf{Z}^V)$.

(4) $\overline{g}(q + \alpha \mathbf{1}) = \overline{g}(q) + \alpha r \qquad (q \in \mathbf{R}^V, \alpha \in \mathbf{R})$.

(5) The expression (4.53) with $p \in \mathrm{dom}_{\mathbf{Z}} g$ remains valid for $q \in N_0$, where

$$N_0 = \{q \in \mathbf{R}^V \mid \max_{v \in V} q(v) - \min_{v \in V} q(v) \leq 1\}. \tag{4.55}$$

(Proof) (1) & (2) First, the rewriting of (4.53) to (4.54) is easy. Denote by $h_p(q)$ the right-hand side of (4.54), which, for each $p \in \mathbf{Z}^V$, represents a piecewise-linear function in $q \in [0,1]^V$. Let $h : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be the piecewise-linear function well-defined as $h(p+q) = h_p(q)$ by the collection $\{h_p \mid p \in \mathbf{Z}^V\}$, where the well-definedness means the consistency between "neighboring pieces," namely, that $h_p(q) = h_{p-\chi_v}(q + \chi_v)$ for $q \in [0,1]^V$ with $q(v) = 0$. By Lemma 3.2 each h_p is convex in the unit hypercube $[0,1]^V$. In other words, h is convex in the hypercube $\{q \in \mathbf{R}^V \mid p \leq q \leq p + \mathbf{1}\}$ for each $p \in \mathbf{Z}^V$. This implies further that h is a convex function over $\mathbf{R}^V$, since we have

$$h(q - \alpha \mathbf{1}) = h(q) - \alpha r \qquad (q \in \mathbf{R}^V, \alpha \in \mathbf{R}) \tag{4.56}$$

from (4.46) and for each $q \in \mathbf{R}^V$ we can find $\alpha \in \mathbf{R}$ such that $q - \alpha\mathbf{1}$ is contained in the interior of some unit hypercube $\{q' \in \mathbf{R}^V \mid p \leq q' \leq p + \mathbf{1}\}$. Therefore h is a convex function such that $h(p) = g(p)$ for $p \in \mathbf{Z}^V$, and obviously h is the maximum of such functions. On the other hand, $\overline{g}$ is characterized as the maximum convex function that satisfies $\overline{g}(p) \leq g(p)$ for $p \in \mathbf{Z}^V$. Hence follows $h = \overline{g}$.

(3) This is immediate from (4.54) with $q = 0$.

(4) This is nothing but (4.56).

(5) For $q \in N_0$ we have $q - \alpha \mathbf{1} \in [0,1]^V$ with $\alpha = \min_{v \in V} q(v)$, and

$$\overline{g}(p + q) = \overline{g}(p + (q - \alpha \mathbf{1})) + \alpha r$$

by (4). The right-hand side can be expanded according to (4.53) as

$$\begin{aligned}
&\overline{g}(p + (q - \alpha \mathbf{1})) + \alpha r \\
&= g(p) + \sum_{j=1}^{n-1} ((q_j - \alpha) - (q_{j+1} - \alpha))(g(p + \chi_{V_j}) - g(p)) \\
&\quad + (q_n - \alpha)(g(p + \mathbf{1}) - g(p)) + \alpha r \\
&= g(p) + \sum_{j=1}^{n} (q_j - q_{j+1})(g(p + \chi_{V_j}) - g(p)).
\end{aligned}$$

□

Corollary 4.19 *$g \in \mathcal{L}_0$ if and only if $g \in \mathcal{L}$ and g is nonnegatively homogeneous (i.e., $g(\lambda p) = \lambda g(p)$ for $0 \leq \lambda \in \mathbf{Z}$ and $p \in \mathbf{Z}^V$).*

(Proof) The "only if" part has been proven in Lemma 3.4 and (3.7). To show the "if" part, first note that the nonnegative homogeneity of g implies that of $\overline{g}$, i.e., $\overline{g}(\lambda q) = \lambda \overline{g}(q)$ for $0 \leq \lambda \in \mathbf{R}$ and $q \in \mathbf{R}^V$. The expression (4.53) with $p = 0$ yields $\overline{g}(q) = \sum_{j=1}^{n}(q_j - q_{j+1})g(\chi_{V_j})$. This expression is valid for $q \in N_0$ by Theorem 4.18(5), and hence for all $q \in \mathbf{R}^V$ by the nonnegative homogeneity of $\overline{g}$. Since $\overline{g}(q) = g(q)$ for $q \in \mathbf{Z}^V$ by Theorem 4.18(3), we obtain $g(q) = \sum_{j=1}^{n}(q_j - q_{j+1})g(\chi_{V_j})$, the condition (L2) for $\mathcal{L}_0$. The other condition (L1) is obvious from the submodularity (4.45). □

Theorem 4.20 *A nonempty set $D \subseteq \mathbf{Z}^V$ is L-convex if and only if $D = \mathbf{Z}^V \cap \overline{D}$ and*

$$\overline{D} = \{p \in \mathbf{R}^V \mid p(v) - p(u) \leq \gamma(u,v) \ (u, v \in V)\} \tag{4.57}$$

for some $\gamma : V \times V \to \mathbf{Z} \cup \{+\infty\}$ such that $\gamma(v,v) = 0$ $(v \in V)$ and

$$\gamma(v_1, v_2) + \gamma(v_2, v_3) \geq \gamma(v_1, v_3) \qquad (v_1, v_2, v_3 \in V). \tag{4.58}$$

Such γ is uniquely determined by D as

$$\gamma(u,v) = \sup_{p \in D}\{p(v) - p(u)\} \qquad (u, v \in V). \tag{4.59}$$

(Proof) ["only if"] First note that γ defined by (4.59) satisfies (4.58). Denote by $P(\gamma)$ the right-hand side of (4.57). The proof of Theorem 4.18 shows that $D = \mathbf{Z}^V \cap \overline{D}$ and $\overline{D} = P(\gamma)$ for some γ, which is given by (4.59).

["if"] For $p, q \in P(\gamma)$ it is easy to derive $(p \vee q)(v) - (p \vee q)(u) \leq \gamma(u,v)$ and $(p \wedge q)(v) - (p \wedge q)(u) \leq \gamma(u,v)$ for $u, v \in V$, which shows $p \vee q, p \wedge q \in P(\gamma)$. Finally, it is obvious that $p \pm \mathbf{1} \in P(\gamma)$.

[Uniqueness] If $P(\gamma_1) = P(\gamma_2)$, we have $\gamma_2(u,v) \geq \sup_{p \in P(\gamma_1)}\{p(v) - p(u)\}$. The right-hand side here is equal, by the linear programming duality, to the shortest path length from u to v with respect to arc length γ_1 in the graph $G = (V, A)$ with $A = \{(u,v) \mid \gamma_1(u,v) < +\infty\}$, and hence is equal to $\gamma_1(u,v)$ by the triangle inequality. Therefore, $\gamma_2(u,v) \geq \gamma_1(u,v)$, and by symmetry, $\gamma_2(u,v) = \gamma_1(u,v)$. □

Integer subgradients exist for an L-convex function, namely $\mathcal{L} \subseteq \mathcal{F}_{\mathrm{G}}$ using the notations (4.18) and (4.47).

Theorem 4.21 *For an L-convex function g and $p \in \mathrm{dom}_{\mathbf{Z}} g$, the integer subdifferential $\partial_{\mathbf{Z}} g(p)$ is an integral base set. Moreover, $\partial_{\mathbf{R}} g(p) = \overline{\partial_{\mathbf{Z}} g(p)}$, and hence $\mathcal{L} \subseteq \mathcal{F}_{\mathrm{G}}$.*

(Proof) First note $\partial_{\mathbf{R}} g(p) = \partial_{\mathbf{R}} \overline{g}(p)$. Since N_0 of (4.55) contains the origin $q = 0$ in its interior, we have

$$x \in \partial_{\mathbf{R}} \overline{g}(p) \iff \overline{g}(p+q) - \overline{g}(p) \geq \langle q, x \rangle \qquad (q \in N_0).$$

Using the expression (4.53) (cf. Lemma 4.18(5)) we see further that

$$\begin{aligned} & \overline{g}(p+q) - \overline{g}(p) \geq \langle q, x \rangle \qquad (q \in N_0) \\ \iff & g(p+\chi_X) - g(p) \geq x(X) \quad (X \subseteq V);\ g(p+\mathbf{1}) - g(p) = x(V) \\ \iff & x \in \mathbf{B}(\rho_p), \end{aligned}$$

where $\rho_p(X) = g(p+\chi_X) - g(p)$ $(X \subseteq V)$ and $\mathbf{B}(\rho_p)$ denotes the integral base polyhedron associated with ρ_p. Hence $\partial_{\mathbf{R}} g(p) = \overline{\mathbf{B}(\rho_p)}$. It then follows from Theorem 3.1 that $\partial_{\mathbf{R}} g(p) = \overline{\partial_{\mathbf{Z}} g(p)} \neq \emptyset$. Finally, $\overline{\mathrm{dom}_{\mathbf{Z}} g}$ is rationally-polyhedral and $\mathrm{dom}_{\mathbf{Z}} g = \mathbf{Z}^V \cap \overline{\mathrm{dom}_{\mathbf{Z}} g}$ by Theorem 4.16 and Theorem 4.20. □

The existence of integer subgradients (or the inclusion $\mathcal{L} \subseteq \mathcal{F}_{\mathrm{G}}$ to be more precise) allows us to apply the general lemma (Lemma 4.2) to L-convex functions. In particular we obtain the following statements.

Theorem 4.22 *For an L-convex function g we have the following.*
(1) $g^\bullet : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$.
(2) $\mathrm{dom}_{\mathbf{Z}} g^\bullet = \bigcup \{\partial_{\mathbf{Z}} g(p) \mid p \in \mathrm{dom}_{\mathbf{Z}} g\} \neq \emptyset$.
(3) $g^{\bullet\bullet} = g$. □

The class of L-convex functions is essentially closed under addition.

Theorem 4.23 *For L-convex functions g_i $(i = 1, 2)$, the sum $g_1 + g_2$ is also L-convex, provided* $\mathrm{dom}_{\mathbf{Z}} g_1 \cap \mathrm{dom}_{\mathbf{Z}} g_2 \neq \emptyset$.

(Proof) It is easy to verify (4.44), (4.45) and (4.46) for $g_1 + g_2$. □

Remark 4.10 After the submission of this paper the author became aware of a paper of Favati–Tardella [12], in which convexity for discrete functions is also discussed. In particular the concept of submodular integrally convex functions introduced there is closely related to that of L-convex functions. In fact, it has been shown very recently by Fujishige–Murota [20] that these two concepts are equivalent in a sense. □

4.4 Conjugacy

The following theorem states that the L-convex functions and the M-convex functions are both extendable to convex functions, and that they are in one-to-one correspondence under the integer Fenchel transformation $^\bullet$ of (4.6). It is emphasized that this is an extension of the conjugacy relation between $\mathcal{M}_0$ and $\mathcal{L}_0$ stated in Theorem 3.3.

Theorem 4.24 *Let $\mathcal{M}$ and $\mathcal{L}$ be the classes of M-convex and L-convex functions, respectively, and $\mathcal{F}_{\rm E}$ and $\mathcal{F}_{\rm G}$ be defined by (4.17) and (4.18).*

(1) $\mathcal{M} \subseteq \mathcal{F}_{\rm G} \subseteq \mathcal{F}_{\rm E}$, $\mathcal{L} \subseteq \mathcal{F}_{\rm G} \subseteq \mathcal{F}_{\rm E}$.

(2) $\mathcal{M}^\bullet = \mathcal{L}$, $\mathcal{L}^\bullet = \mathcal{M}$. More specifically, for $f \in \mathcal{M}$ and $g \in \mathcal{L}$ we have $f^\bullet \in \mathcal{L}$, $g^\bullet \in \mathcal{M}$, $f^{\bullet\bullet} = f$ and $g^{\bullet\bullet} = g$.

(Proof) (1) This has been proven in Theorem 4.7 and Theorem 4.21, as well as in Lemma 4.2.

(2) By Theorem 4.8 and Theorem 4.22 it suffices to show $\mathcal{M}^\bullet \subseteq \mathcal{L}$ and $\mathcal{L}^\bullet \subseteq \mathcal{M}$, which are established in Lemmas 4.25 and 4.26 below. □

Lemma 4.25 $\mathcal{M}^\bullet \subseteq \mathcal{L}$.

(Proof) Let $f \in \mathcal{M}$ and $p \in {\rm dom}_{\mathbf{Z}} f^\bullet$. Firstly, we have $f^\bullet : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ and ${\rm dom}_{\mathbf{Z}} f^\bullet \neq \emptyset$ by Theorem 4.8.

We claim that ${\rm dom}_{\mathbf{Z}} f^\bullet$ is an L-convex set. Since $f \in \mathcal{M} \subseteq \mathcal{F}_{\rm G}$ by Theorem 4.7, we have ${\rm dom}_{\mathbf{Z}} f^\bullet = \mathbf{Z}^V \cap P$ with $P = \bigcup\{\partial_{\mathbf{R}} f(x) \mid x \in {\rm dom}_{\mathbf{Z}} f\}$ by Lemma 4.2(1). For each $x \in {\rm dom}_{\mathbf{Z}} f$, $\partial_{\mathbf{R}} f(x)$ is expressed as (4.27), and therefore P can be represented in the form of (4.57). It then follows from Theorem 4.20 that ${\rm dom}_{\mathbf{Z}} f^\bullet$ is an L-convex set.

For (4.46) we note from Theorem 4.3 that there exists $r \in \mathbf{Z}$ such that $\langle \mathbf{1}, x\rangle = r$ for all $x \in {\rm dom}_{\mathbf{Z}} f$. Therefore

$$f^\bullet(p + \mathbf{1}) = \sup\{\langle p + \mathbf{1}, x\rangle - f(x) \mid x \in {\rm dom}_{\mathbf{Z}} f\} = f^\bullet(p) + r.$$

For the submodularity (4.45) of $f^\bullet$ it suffices to show the submodularity (3.1) of $\rho_p : 2^V \to \mathbf{Z} \cup \{+\infty\}$ defined by $\rho_p(X) = f^\bullet(p + \chi_X) - f^\bullet(p)$ $(X \subseteq V)$, since ${\rm dom}_{\mathbf{Z}} f^\bullet$ is an L-convex set (cf. [20]). Note that $\rho_p(\emptyset) = 0$ and $\rho_p(V) = r$. Put $B_p = {\rm argmin}_{\mathbf{Z}}(f[-p])$, which is nonempty since $p \in {\rm dom}_{\mathbf{Z}} f^\bullet$. Furthermore, B_p is an integral base set by Theorem 4.10. We claim

$$\sup\{\langle p + q, x\rangle - f(x) \mid x \in \mathbf{Z}^V\} = \sup\{\langle p + q, x\rangle - f(x) \mid x \in B_p\} \quad (q \in \{0,1\}^V), \tag{4.60}$$

which is proven later. Since $\langle p, x\rangle - f(x) = f^\bullet(p)$ for $x \in B_p$, we can rewrite (4.60) as

$$f^\bullet(p + q) - f^\bullet(p) = \sup\{\langle q, x\rangle \mid x \in B_p\},$$

which (with $q = \chi_X$) implies $\rho_p(X) = \sup\{x(X) \mid x \in B_p\}$. This expression reveals the submodularity of ρ_p by Theorem 3.1.

It remains to prove (4.60). Obviously, LHS $\geq$ RHS. Assume that RHS is finite, and let $z \in B_p$ be a maximizer, i.e., such that

$$\langle p + q, z\rangle - f(z) = \sup\{\langle p + q, x\rangle - f(x) \mid x \in B_p\}.$$

The local optimality condition in Theorem 4.6 shows

$$f[-p](z,u,v) \geq q(v) - q(u) \qquad \text{if } \ z - \chi_u + \chi_v \in B_p.$$

On the other hand, if $z - \chi_u + \chi_v \notin B_p$, we have

$$f[-p](z - \chi_u + \chi_v) \geq f[-p](z) + 1,$$

which implies

$$f[-p](z,u,v) \geq 1 \geq q(v) - q(u)$$

since $q \in \{0,1\}^V$. Combining these two cases we obtain

$$f[-p](z,u,v) \geq q(v) - q(u) \qquad (u, v \in V),$$

that is,

$$f[-p-q](z,u,v) \geq 0 \qquad (u, v \in V).$$

Then Theorem 4.6 (the reverse direction) shows that z minimizes $f[-p-q]$ over $\mathbf{Z}^V$. Hence follows LHS = RHS in (4.60). □

Lemma 4.26 $\mathcal{L}^\bullet \subseteq \mathcal{M}$.

(Proof) Let $g \in \mathcal{L}$ and put $f = g^\bullet$. We want to apply Theorem 4.10 to derive $f \in \mathcal{M}$.

Firstly we claim that $\mathrm{dom}_{\mathbf{Z}} f$ is an integral base set. Since $g \in \mathcal{L} \subseteq \mathcal{F}_{\mathrm{G}}$ by Theorem 4.21, we have $\mathrm{dom}_{\mathbf{Z}} f = \mathrm{dom}_{\mathbf{Z}} g^\bullet = \mathbf{Z}^V \cap D$ with $D = \bigcup\{\partial_{\mathbf{R}} g(p) \mid p \in \mathrm{dom}_{\mathbf{Z}} g\}$ by Lemma 4.2(1). D is a convex set represented as the union of integral base polyhedra, since $D = \{x \in \mathbf{R}^V \mid \sup\{\langle p, x\rangle - g(p) \mid p \in \mathbf{Z}^V\} < +\infty\}$ and $\partial_{\mathbf{R}} g(p) = \overline{\partial_{\mathbf{Z}} g(p)}$, for each $p \in \mathrm{dom}_{\mathbf{Z}} g$, is an integral base set by Theorem 4.21. It then follows from Lemma 4.12 that D is an integral base polyhedron. Consequently, $\mathrm{dom}_{\mathbf{Z}} f = \mathbf{Z}^V \cap D$ in an integral base set.

Secondly, for $p \in \mathrm{dom}_{\mathbf{Z}} g$ it follows from (4.12) and Theorem 4.22(3) that

$$\mathrm{argmin}_{\mathbf{Z}}(f[-p]) = \mathrm{argmin}_{\mathbf{Z}}(g^\bullet[-p]) = \partial_{\mathbf{Z}} g^{\bullet\bullet}(p) = \partial_{\mathbf{Z}} g(p),$$

which is an integral base set by Theorem 4.21. From this it is easy to see that the restriction f_a^b of f to $[a,b]$ with $(a,b) \in \mathbf{Z}^V \times \mathbf{Z}^V$ has the property that $\mathrm{argmin}_{\mathbf{Z}}(f_a^b[-p])$ is an integral base set for each $p \in \mathbf{R}^V$ (provided $\mathrm{dom}_{\mathbf{Z}} f_a^b \neq \emptyset$). □

Example 4.1 Here is an example of a conjugate pair $f \in \mathcal{M}$ and $g \in \mathcal{L}$ for $V = \{1, 2, 3, 4\}$. Let $f : \mathbf{Z}^4 \to \mathbf{Z} \cup \{+\infty\}$ be defined by

$$f(x) = \begin{cases} 0 & (x = (1,1,0,0), (0,0,1,1)) \\ -1 & (x = (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1)) \\ +\infty & (\text{otherwise}) \end{cases}$$

as in Example 2.4, and $g : \mathbf{Z}^4 \to \mathbf{Z} \cup \{+\infty\}$ be defined by

$$g(p) = \max(p_1 + p_2, p_3 + p_4, p_1 + p_3 + 1, p_1 + p_4 + 1, p_2 + p_3 + 1, p_2 + p_4 + 1).$$

Then $g^\bullet = f \in \mathcal{M}$ and $f^\bullet = g \in \mathcal{L}$. □

Remark 4.11 We can regard an affine function as being M-convex and L-convex at the same time in the following sense. Let $h : \mathbf{Z}^V \to \mathbf{Z}$ be an affine function (or more precisely, the restriction of an affine function over $\mathbf{R}^V$). Consider $\tilde{V} = \{v_0\} \cup V$ by introducing a new element v_0, and define $f : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{+\infty\}$ by

$$f(x_0, x) = \begin{cases} h(x) & (x_0 + \langle \mathbf{1}, x \rangle = 0) \\ +\infty & (\text{otherwise}) \end{cases}$$

and $g : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z}$ by $g(x_0, x) = h(x - x_0\mathbf{1})$, where $x_0 \in \mathbf{Z}$, $x \in \mathbf{Z}^V$ and $(x_0, x) \in \mathbf{Z}^{\tilde{V}}$. Then f is M-convex, g is L-convex and $f(0, x) = h(x) = g(0, x)$ for $x \in \mathbf{Z}^V$. □

In either case of M- and L-convexity we have defined the convexity of a subset of $\mathbf{Z}^V$ through the convexity of its indicator function. Namely, we have introduced

$$\begin{aligned} \mathcal{M}_0 &= \{\delta_B : \mathbf{Z}^V \to \{0, +\infty\} \mid B \text{ is an M-convex set }\}, \\ {}_0\mathcal{L} &= \{\delta_D : \mathbf{Z}^V \to \{0, +\infty\} \mid D \text{ is an L-convex set }\}. \end{aligned}$$

Motivated by the fact that the conjugate of an indicator function is nonnegatively homogeneous, we define

$$\begin{aligned} {}_0\mathcal{M} &= \{f \in \mathcal{M} \mid \ f \text{ is nonnegatively homogeneous }\}, \\ \mathcal{L}_0 &= \{g \in \mathcal{L} \mid \ g \text{ is nonnegatively homogeneous }\}, \end{aligned}$$

where $\mathcal{L}_0$ is consistent with our previous notation for the class of the Lovász extensions of submodular functions (cf. Corollary 4.19). As a consequence of the conjugacy bewteen ${}_0\mathcal{L}$ and ${}_0\mathcal{M}$ and Theorem 4.20, a member of ${}_0\mathcal{M}$ can be identified with a distance function $\gamma : V \times V \to \mathbf{Z} \cup \{+\infty\}$ satisfying the triangle inequality (4.58).

The conjugacy relations are summarized in Table 1.

Table 1: Conjugacy between M- and L-convexity

	M-convexity	L-convexity
Set	$\mathcal{M}_0$ (indicator) integral base set (=M-convex set)	$\mathcal{L}_0$ (homog.) Lovász extension of submodular function
	${}_0\mathcal{M}$ (homog.)	${}_0\mathcal{L}$ (indicator) L-convex set
Function	$\mathcal{M}$ M-convex function	$\mathcal{L}$ L-convex function

Part II: Duality

5 Fenchel-type Duality and Separation Theorems

5.1 Duality theorems

The duality theorems for M- and L-convex functions stated below are derived from Frank's discrete separation theorem for submodular functions (Theorem 3.5) and the following theorem established by the present author in [37].

Theorem 5.1 ([37, Theorem 4.1]) *Assume that* $f_1, f_2 : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ *are M-convex functions and let* $x^* \in \mathrm{dom}_{\mathbf{Z}} f_1 \cap \mathrm{dom}_{\mathbf{Z}} f_2$*. Then*

$$f_1(x^*) + f_2(x^*) \leq f_1(x) + f_2(x) \qquad (\forall\, x \in \mathbf{Z}^V)$$

if and only if there exists $p^* \in \mathbf{Z}^V$ *such that*

$$\begin{aligned} f_1[-p^*](x^*) &\leq f_1[-p^*](x) \qquad && (\forall\, x \in \mathbf{Z}^V), \\ f_2[+p^*](x^*) &\leq f_2[+p^*](x) \qquad && (\forall\, x \in \mathbf{Z}^V). \end{aligned}$$

(Proof) A self-contained proof is provided in Appendix. □

Remark 5.1 When f_1 and f_2 are affine functions (cf. Example 2.1), the above theorem agrees with the optimality criterion (Fujishige's potential characterization [16]) for the weighted intersection problem for a pair of submodular systems (see also [19]). When $\mathrm{dom}_{\mathbf{Z}} f_1, \mathrm{dom}_{\mathbf{Z}} f_2 \subseteq \{0,1\}^V$, on the other hand, which corresponds to a pair of matroids, the above theorem reduces to the optimality criterion

[34, Theorem 4.2] for the valuated matroid intersection problem. If, in addition, f_1 is affine and $f_2 = 0$, this criterion recovers Frank's weight splitting theorem [13] for the weighted matroid intersection problem, which is in turn equivalent to Iri–Tomizawa's potential characterization of the optimality for the independent assignment problem [25]. $\Box$

The following Fenchel-type min-max duality is a discrete analogue of the Fenchel min-max duality in the ordinary convex analysis [43], [49]. This theorem has been proved in Murota [38] under the assumption that both $\mathrm{dom}_{\mathbf{Z}} f$ and $\mathrm{dom}_{\mathbf{Z}} g$ are bounded, in which case we have $\mathrm{dom}_{\mathbf{Z}} f^{\bullet} \cap \mathrm{dom}_{\mathbf{Z}} g^{\circ} = \mathbf{Z}^V$.

Theorem 5.2 (Fenchel-type Duality) *Let* $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ *be an M-convex function and* $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ *be an M-concave function such that* $\mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} g \neq \emptyset$ *or* $\mathrm{dom}_{\mathbf{Z}} f^{\bullet} \cap \mathrm{dom}_{\mathbf{Z}} g^{\circ} \neq \emptyset$*. Then we have*

$$\inf\{f(x) - g(x) \mid x \in \mathbf{Z}^V\} = \sup\{g^{\circ}(p) - f^{\bullet}(p) \mid p \in \mathbf{Z}^V\}. \tag{5.1}$$

If this common value is finite, the infimum is attained by some $x \in \mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} g$ *and the supremum is attained by some* $p \in \mathrm{dom}_{\mathbf{Z}} f^{\bullet} \cap \mathrm{dom}_{\mathbf{Z}} g^{\circ}$*.*

(Proof) In view of the weak duality

$$f(x) - g(x) \geq g^{\circ}(p) - f^{\bullet}(p) \qquad (x \in \mathbf{Z}^V, p \in \mathbf{Z}^V) \tag{5.2}$$

as well as the discreteness of the problem, it suffices to prove the following two claims:

Claim 1: If $\mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} g \neq \emptyset$ and $\inf\{f(x) - g(x) \mid x \in \mathbf{Z}^V\}$ is finite, there exist $x^* \in \mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} g$ and $p^* \in \mathrm{dom}_{\mathbf{Z}} f^{\bullet} \cap \mathrm{dom}_{\mathbf{Z}} g^{\circ}$ such that $f(x^*) - g(x^*) = g^{\circ}(p^*) - f^{\bullet}(p^*)$.

Claim 2: If $\mathrm{dom}_{\mathbf{Z}} f^{\bullet} \cap \mathrm{dom}_{\mathbf{Z}} g^{\circ} \neq \emptyset$ and $\sup\{g^{\circ}(p) - f^{\bullet}(p) \mid p \in \mathbf{Z}^V\}$ is finite, then $\mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} g \neq \emptyset$.

[Proof of Claim 1] Suppose $x^* \in \mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} g$ attains the infimum on the left-hand side of (5.1). It follows from Theorem 5.1 with $f_1 = f$ and $f_2 = -g$ that there exists $p^* \in \mathbf{Z}^V$ such that

$$\begin{array}{rcll} f(x^*) - \langle p^*, x^* \rangle & \leq & f(x) - \langle p^*, x \rangle & \qquad (x \in \mathbf{Z}^V), \\ g(x^*) - \langle p^*, x^* \rangle & \geq & g(x) - \langle p^*, x \rangle & \qquad (x \in \mathbf{Z}^V). \end{array}$$

This means

$$f(x^*) + f^{\bullet}(p^*) = \langle p^*, x^* \rangle, \qquad g(x^*) + g^{\circ}(p^*) = \langle p^*, x^* \rangle,$$

which in turn implies $f(x^*) - g(x^*) = g^\circ(p^*) - f^\bullet(p^*)$.

[Proof of Claim 2] We will show that $\mathrm{dom}_{\mathbf{Z}} f^\bullet \cap \mathrm{dom}_{\mathbf{Z}} g^\circ \neq \emptyset$ and $\mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} g = \emptyset$ imply $\sup\{g^\circ(p) - f^\bullet(p) \mid p \in \mathbf{Z}^V\} = +\infty$. Take $p_0 \in \mathrm{dom}_{\mathbf{Z}} f^\bullet \cap \mathrm{dom}_{\mathbf{Z}} g^\circ$. From the inequalities

$$\begin{aligned} f^\bullet(p) &= \sup_{x \in \mathrm{dom}_{\mathbf{Z}} f} (\langle p - p_0, x\rangle + [\langle p_0, x\rangle - f(x)]) \leq \sup_{x \in \mathrm{dom}_{\mathbf{Z}} f} \langle p - p_0, x\rangle + f^\bullet(p_0), \\ g^\circ(p) &= \inf_{x \in \mathrm{dom}_{\mathbf{Z}} g} (\langle p - p_0, x\rangle + [\langle p_0, x\rangle - g(x)]) \geq \inf_{x \in \mathrm{dom}_{\mathbf{Z}} g} \langle p - p_0, x\rangle + g^\circ(p_0), \end{aligned}$$

we obtain

$$g^\circ(p) - f^\bullet(p) \geq \left[\inf_{x \in \mathrm{dom}_{\mathbf{Z}} g} \langle p - p_0, x\rangle - \sup_{x \in \mathrm{dom}_{\mathbf{Z}} f} \langle p - p_0, x\rangle \right] + g^\circ(p_0) - f^\bullet(p_0). \quad (5.3)$$

On the other hand, since $\mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} g = \emptyset$, Theorem 3.6 (separation theorem for integral base sets) guarantees the existence of $p^* \in \mathbf{Z}^V$ such that

$$\inf\{\langle p^*, x\rangle \mid x \in \mathrm{dom}_{\mathbf{Z}} g\} - \sup\{\langle p^*, x\rangle \mid x \in \mathrm{dom}_{\mathbf{Z}} f\} \geq 1.$$

This means that, for $p = p_0 + \alpha p^*$ with $\alpha \in \mathbf{Z}_+$, the right-hand side of (5.3) is at least as large as $\alpha + g^\circ(p_0) - f^\bullet(p_0)$, which is not bounded for $\alpha \in \mathbf{Z}_+$. □

Remark 5.2 In Theorem 5.2 we cannot get rid of the assumption on the effective domains. In fact, here is an example of M-convex f and M-concave g such that $\mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} g = \emptyset$ and $\mathrm{dom}_{\mathbf{Z}} f^\bullet \cap \mathrm{dom}_{\mathbf{Z}} g^\circ = \emptyset$, for which we have

$$\inf\{f(x) - g(x) \mid x \in \mathbf{Z}^V\} = +\infty, \qquad \sup\{g^\circ(p) - f^\bullet(p) \mid p \in \mathbf{Z}^V\} = -\infty.$$

Let $f : \mathbf{Z}^2 \to \mathbf{Z} \cup \{+\infty\}$ and $g : \mathbf{Z}^2 \to \mathbf{Z} \cup \{-\infty\}$ be defined by

$$f(x_1, x_2) = \begin{cases} x_1 & (x_1 + x_2 = 1) \\ +\infty & (\text{otherwise}) \end{cases} \qquad g(x_1, x_2) = \begin{cases} -x_1 & (x_1 + x_2 = -1) \\ -\infty & (\text{otherwise}) \end{cases}$$

Then

$$f^\bullet(p_1, p_2) = \begin{cases} p_2 & (p_1 - p_2 = 1) \\ +\infty & (\text{otherwise}) \end{cases} \qquad g^\circ(p_1, p_2) = \begin{cases} -p_2 & (p_1 - p_2 = -1) \\ -\infty & (\text{otherwise}) \end{cases}$$

□

From the Fenchel-type duality above we can derive discrete separation theorems for a pair of M-convex/concave functions as well as for a pair of L-convex/concave functions.

Theorem 5.3 (M-Separation Theorem) *Let* $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ *be an M-convex function and* $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ *be an M-concave function such that* $\mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} g \neq \emptyset$ *or* $\mathrm{dom}_{\mathbf{Z}} f^\bullet \cap \mathrm{dom}_{\mathbf{Z}} g^\circ \neq \emptyset$. *If* $f(x) \geq g(x)$ $(x \in \mathbf{Z}^V)$, *there exist* $\alpha^* \in \mathbf{Z}$ *and* $p^* \in \mathbf{Z}^V$ *such that*

$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq g(x) \qquad (x \in \mathbf{Z}^V). \tag{5.4}$$

(Proof) First note that (5.4) is equivalent to

$$f^\bullet(p^*) \leq -\alpha^* \leq g^\circ(p^*). \tag{5.5}$$

In case $\mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} g \neq \emptyset$, we have $0 \leq \inf\{f(x) - g(x) \mid x \in \mathbf{Z}^V\} < +\infty$ and by Theorem 5.2 there exist $x^* \in \mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} g$ and $p^* \in \mathrm{dom}_{\mathbf{Z}} f^\bullet \cap \mathrm{dom}_{\mathbf{Z}} g^\circ$ such that

$$f(x^*) - g(x^*) = g^\circ(p^*) - f^\bullet(p^*).$$

Since the left-hand side is nonnegative by the assumption, there exists $\alpha^* \in \mathbf{Z}$ that satisfies (5.5).

Next we consider the case where $\mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} g = \emptyset$ and $\mathrm{dom}_{\mathbf{Z}} f^\bullet \cap \mathrm{dom}_{\mathbf{Z}} g^\circ \neq \emptyset$. Theorem 5.2 shows $\sup\{g^\circ(p) - f^\bullet(p) \mid p \in \mathbf{Z}^V\} = +\infty$, which implies $f^\bullet(p^*) \leq g^\circ(p^*)$ for some $p^* \in \mathbf{Z}^V$. By choosing $\alpha^* \in \mathbf{Z}$ with (5.5) we obtain (5.4). (An alternative proof can be found in [21].) $\square$

Theorem 5.4 (L-Separation Theorem) *Let* $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ *be an L-convex function and* $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ *be an L-concave function such that* $\mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} g \neq \emptyset$ *or* $\mathrm{dom}_{\mathbf{Z}} f^\bullet \cap \mathrm{dom}_{\mathbf{Z}} g^\circ \neq \emptyset$. *If* $f(p) \geq g(p)$ $(p \in \mathbf{Z}^V)$, *there exist* $\beta^* \in \mathbf{Z}$ *and* $x^* \in \mathbf{Z}^V$ *such that*

$$f(p) \geq \beta^* + \langle p, x^* \rangle \geq g(p) \qquad (p \in \mathbf{Z}^V). \tag{5.6}$$

(Proof) A proof similar to that of Theorem 5.3 works by virtue of the conjugacy between M-convexity and L-convexity in Theorem 4.24. (An alternative proof can be found in [21].) $\square$

It should be clear that M- and L-separation theorems are conjugate to each other, while the Fenchel-type min-max identity is self-conjugate.

Remark 5.3 Let B and B' be integral base sets and δ_B and $\delta_{B'}$ be their indicator functions. The L-separation theorem for $f = {\delta_B}^\bullet$ and $g = (-\delta_{B'})^\circ$ reduces to the discrete separation theorem (Theorem 3.5) for sub/supermodular functions. $\square$

A separation theorem for a pair of L-convex sets is stated below. Compare this with Theorem 3.6, which asserts the discrete separation for M-convex sets (=integral base sets).

Theorem 5.5 *Let D_1 and D_2 be L-convex sets ($\subseteq \mathbf{Z}^V$). If they are disjoint ($D_1 \cap D_2 = \emptyset$), there exists $x^* \in \{-1, 0, 1\}^V$ such that*

$$\inf\{\langle p, x^*\rangle \mid p \in D_1\} - \sup\{\langle p, x^*\rangle \mid p \in D_2\} \geq 1. \tag{5.7}$$

In particular, $D_1 \cap D_2 = \emptyset$ is equivalent to $\overline{D_1} \cap \overline{D_2} = \emptyset$.

(Proof) Let γ_1 and γ_2 be defined by (4.59) for the polyhedral descriptions of D_1 and D_2, respectively. The expressions (4.57) for $\overline{D_1}$ and $\overline{D_2}$ show that $D_1 \cap D_2 = \emptyset$ implies

$$\overline{D_1} \cap \overline{D_2} = \{p \in \mathbf{R}^V \mid p(v) - p(u) \leq \gamma_{12}(u, v) \ (u, v \in V)\} = \emptyset$$

for $\gamma_{12}(u, v) = \min\{\gamma_1(u, v), \gamma_2(u, v)\}$. This means that the graph $G = (V, A)$ with $A = \{(u, v) \mid \gamma_{12}(u, v) < +\infty\}$ contains a cycle of negative length with respect to arc length γ_{12}. Let $v_0, v_1, v_2, \ldots, v_{k-1}$ be such a cycle with minimum $k \geq 2$. By the triangle inequalities of γ_1 and γ_2, it may be assumed that k is even and $\gamma_1(v_{2i}, v_{2i+1}) \leq \gamma_2(v_{2i}, v_{2i+1})$ and $\gamma_1(v_{2i+1}, v_{2i+2}) \geq \gamma_2(v_{2i+1}, v_{2i+2})$ for $0 \leq i \leq k/2 - 1$, where $v_k = v_0$. Let x^* be such that $x^*(v_{2i}) = 1$, $x^*(v_{2i+1}) = -1$ for $0 \leq i \leq k/2 - 1$, and $x^*(v) = 0$ otherwise. Then

$$\inf_{p \in D_1} \langle p, x^*\rangle = -\sum_{i=0}^{k/2-1} \gamma_1(v_{2i}, v_{2i+1}), \quad \sup_{p \in D_2} \langle p, x^*\rangle = \sum_{i=0}^{k/2-1} \gamma_2(v_{2i+1}, v_{2i+2})$$

by the minimality of k, and therefore

$$\begin{aligned} \sup_{p \in D_2} \langle p, x^*\rangle - \inf_{p \in D_1} \langle p, x^*\rangle &= \sum_{i=0}^{k/2-1} \gamma_1(v_{2i}, v_{2i+1}) + \sum_{i=0}^{k/2-1} \gamma_2(v_{2i+1}, v_{2i+2}) \\ &= \sum_{i=0}^{k-1} \gamma_{12}(v_i, v_{i+1}) \leq -1. \end{aligned}$$

$\square$

Remark 5.4 The Fenchel-type min-max identity, the M- and L-separation theorems can be regarded as equivalent statements, just as the Fenchel min-max identity and the separation theorem are equivalent in the ordinary convex analysis [43], [49]. In fact, the Fenchel-type min-max identity (Theorem 5.2) can be derived from the M-separation theorem as follows. In the proof of Theorem 5.2 Claim 2 has been established on the basis of Theorem 3.6 (separation theorem for integral base sets). For the case of Claim 1, we apply the M-separation theorem to $f(x) - \alpha$ and $g(x)$, where $\alpha = \inf\{f(x) - g(x) \mid x \in \mathbf{Z}^V\}$, to obtain

$$f(x) - \alpha \geq \alpha^* + \langle p^*, x\rangle \geq g(x) \qquad (x \in \mathbf{Z}^V)$$

for some $\alpha^* \in \mathbf{Z}$ and $p^* \in \mathbf{Z}^V$. It then follows that

$$g^\circ(p^*) - f^\bullet(p^*) \geq \alpha = \inf\{f(x) - g(x) \mid x \in \mathbf{Z}^V\},$$

which implies (5.1) by the weak duality (5.2). A similar (conjugate) argument works to derive Theorem 5.2 from the L-separation theorem. It should be clear that Theorem 5.5 above is used in place of Theorem 3.6. $\Box$

Remark 5.5 The M-separation theorem (Theorem 5.3) is a slight extension of the Primal Separation Theorem in [38], [39], in which both $\mathrm{dom}_{\mathbf{Z}} f$ and $\mathrm{dom}_{\mathbf{Z}} g$ are assumed to be bounded. The L-separation theorem (Theorem 5.4), on the other hand, is a reformulation and a slight extension of the Dual Separation Theorem in [38], [39], which is stated for conjugate functions of M-convex/concave functions under the assumption of the boundedness of the effective domains of the M-convex/concave functions. $\Box$

5.2 Consequences of the duality

We go on to deal with subgradients of the sum of two M-convex functions and the convolution of two L-convex functions. These results will have important consequences in the Lagrange duality framework developed later.

The following identity is nontrivial in view of Example 1.2. In fact it is a duality assertion in disguise (and as such the proof relies on the duality theorem). Compare this with a similar identity [43, Theorem 23.8] in the ordinary convex analysis.

Theorem 5.6 *For M-convex functions f_i $(i = 1, 2)$,*

$$\partial_{\mathbf{Z}}(f_1 + f_2)(x) = \partial_{\mathbf{Z}} f_1(x) + \partial_{\mathbf{Z}} f_2(x) \qquad (x \in \mathrm{dom}_{\mathbf{Z}} f_1 \cap \mathrm{dom}_{\mathbf{Z}} f_2).$$

In particular, $\partial_{\mathbf{Z}}(f_1 + f_2)(x) \neq \emptyset$ for $x \in \mathrm{dom}_{\mathbf{Z}} f_1 \cap \mathrm{dom}_{\mathbf{Z}} f_2$.

(Proof) For each $i = 1, 2$, we see $p_i \in \partial_{\mathbf{Z}} f_i(x) \iff x \in \mathrm{argmin}_{\mathbf{Z}}(f_i[-p_i])$, as noted in (4.12). Obviously, the latter conditions for $i = 1, 2$ imply that $x \in \mathrm{argmin}_{\mathbf{Z}}(f_1[-p_1] + f_2[-p_2]) = \mathrm{argmin}_{\mathbf{Z}}((f_1 + f_2)[-p_1 - p_2])$, i.e., $p_1 + p_2 \in \partial_{\mathbf{Z}}(f_1 + f_2)(x)$. Therefore $\partial_{\mathbf{Z}}(f_1 + f_2)(x) \supseteq \partial_{\mathbf{Z}} f_1(x) + \partial_{\mathbf{Z}} f_2(x)$. To show the reverse inclusion we need a duality result. Suppose $p \in \partial_{\mathbf{Z}}(f_1 + f_2)(x)$. Since it is equivalent to $x \in \mathrm{argmin}_{\mathbf{Z}}((f_1 + f_2)[-p]) = \mathrm{argmin}_{\mathbf{Z}}(f_1 + f_2[-p])$, Theorem 5.1 applied to f_1 and $f_2[-p]$ implies the existence of $p_1 \in \mathbf{Z}^V$ such that $x \in \mathrm{argmin}_{\mathbf{Z}}(f_1[-p_1])$ and $x \in \mathrm{argmin}_{\mathbf{Z}}(f_2[-p + p_1])$. Putting $p_2 = p - p_1$ we obtain $p = p_1 + p_2$ with $p_i \in \partial_{\mathbf{Z}} f_i(x)$ for $i = 1, 2$. Therefore $\partial_{\mathbf{Z}}(f_1 + f_2)(x) \subseteq \partial_{\mathbf{Z}} f_1(x) + \partial_{\mathbf{Z}} f_2(x)$. The final claim follows from the established identity since $\partial_{\mathbf{Z}} f_i(x) \neq \emptyset$ for $i = 1, 2$ by Theorem 4.7. $\Box$

Remark 5.6 The special case of the above theorem for matroid valuations has been given in [36, Theorem 4.7]. $\Box$

The following assertion, though looking similar to an obvious relation (4.15), relies also on the duality in the form encoded in Theorem 5.6.

Theorem 5.7 *Let f_i ($i = 1, 2$) be M-convex functions.*

(1) $(f_1 + f_2)^\bullet = f_1{}^\bullet \Box_{\mathbf{Z}} f_2{}^\bullet$, *provided* $\mathrm{dom}_{\mathbf{Z}} f_1 \cap \mathrm{dom}_{\mathbf{Z}} f_2 \neq \emptyset$.

(2) $(f_1 + f_2)^{\bullet\bullet} = f_1 + f_2$.

(Proof) (1) Let $p \in \mathrm{dom}_{\mathbf{Z}}(f_1 + f_2)^\bullet$. If $p = p_1 + p_2$ we have

$$\begin{aligned} (f_1 + f_2)^\bullet(p) &= \sup\{\langle p, x\rangle - f_1(x) - f_2(x) \mid x \in \mathbf{Z}^V\} \\ &\leq \sum_{i=1}^{2} \sup\{\langle p_i, x\rangle - f_i(x) \mid x \in \mathbf{Z}^V\} \\ &= f_1{}^\bullet(p_1) + f_2{}^\bullet(p_2), \end{aligned}$$

which shows $(f_1+f_2)^\bullet(p) \leq (f_1{}^\bullet \Box_{\mathbf{Z}} f_2{}^\bullet)(p)$. For the reverse direction of the inequality, take $x \in \arg\min_{\mathbf{Z}}((f_1 + f_2)[-p])$, which means $p \in \partial_{\mathbf{Z}}(f_1 + f_2)(x)$ by (4.12). Then Theorem 5.6 guarantees the decomposition $p = p_1 + p_2$ with $p_i \in \partial_{\mathbf{Z}} f_i(x)$ ($i = 1, 2$), and therefore

$$\begin{aligned} (f_1 + f_2)^\bullet(p) &= \langle p, x\rangle - f_1(x) - f_2(x) \\ &= \sum_{i=1}^{2}(\langle p_i, x\rangle - f_i(x)) \\ &= f_1{}^\bullet(p_1) + f_2{}^\bullet(p_2) \\ &\geq (f_1{}^\bullet \Box_{\mathbf{Z}} f_2{}^\bullet)(p). \end{aligned}$$

(2) Both $(f_1 + f_2)^{\bullet\bullet}$ and $f_1 + f_2$ are identically equal to $+\infty$ if $\mathrm{dom}_{\mathbf{Z}} f_1 \cap \mathrm{dom}_{\mathbf{Z}} f_2 = \emptyset$. Otherwise, by (1), (4.15) and Theorem 4.8(3) we obtain $(f_1+f_2)^{\bullet\bullet} = (f_1{}^\bullet \Box_{\mathbf{Z}} f_2{}^\bullet)^\bullet = f_1{}^{\bullet\bullet} + f_2{}^{\bullet\bullet} = f_1 + f_2$. $\Box$

Remark 5.7 The second statement of Theorem 5.6 (i.e., $\partial_{\mathbf{Z}}(f_1 + f_2)(x) \neq \emptyset$) is not contained in the latter half of Theorem 4.7. Nor is Theorem 5.7(2) a special case of Theorem 4.8(3). This is because the sum of two M-convex functions is not necessarily M-convex. $\Box$

The class of M-convex functions is essentially closed under the infimum convolution operation (4.13).

Theorem 5.8 *Let f_i ($i = 1, 2$) be M-convex functions and $f_1 \Box_{\mathbf{Z}} f_2 : \mathbf{Z}^V \to \mathbf{Z} \cup \{\pm\infty\}$ be the convolution defined by (4.13).*

(1) $f_1 \Box_{\mathbf{Z}} f_2$ *is M-convex, provided* $f_1 \Box_{\mathbf{Z}} f_2 : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ *(i.e., if it does not take the value* $-\infty$*).*

(2) If $(f_1 \Box_{\mathbf{Z}} f_2)(x_0) = -\infty$ *for some* $x_0 \in \mathbf{Z}^V$*, then*

$$(f_1 \Box_{\mathbf{Z}} f_2)(x) = \begin{cases} -\infty & (x \in \mathrm{dom}_{\mathbf{Z}} f_1 + \mathrm{dom}_{\mathbf{Z}} f_2) \\ +\infty & (\textit{otherwise}) \end{cases}$$

This is the case if and only if $\mathrm{dom}_{\mathbf{Z}} {f_1}^{\bullet} \cap \mathrm{dom}_{\mathbf{Z}} {f_2}^{\bullet} = \emptyset$.

(Proof) (1) First recall the relation (4.15): $(f_1 \Box_{\mathbf{Z}} f_2)^{\bullet} = {f_1}^{\bullet} + {f_2}^{\bullet}$. By the conjugacy between M-convexity and L-convexity (Theorem 4.24) the assertion is equivalent to the fact (Theorem 4.23) that the sum of two L-convex functions is again L-convex, if their effective domains are not disjoint.

(2) The Fenchel-type min-max identity (Theorem 5.2) shows that

$$\inf_{y \in \mathbf{Z}^V} (f_1(y) + f_2(x - y)) = \sup_{p \in \mathbf{Z}^V} (\langle p, x \rangle - {f_1}^{\bullet}(p) - {f_2}^{\bullet}(p)),$$

for each $x \in \mathrm{dom}_{\mathbf{Z}} f_1 + \mathrm{dom}_{\mathbf{Z}} f_2$. The assumption $(f_1 \Box_{\mathbf{Z}} f_2)(x_0) = -\infty$ implies that $x_0 \in \mathrm{dom}_{\mathbf{Z}} f_1 + \mathrm{dom}_{\mathbf{Z}} f_2$ and hence the right-hand side above for $x = x_0$ is equal to $-\infty$, which is equivalent to $\mathrm{dom}_{\mathbf{Z}} {f_1}^{\bullet} \cap \mathrm{dom}_{\mathbf{Z}} {f_2}^{\bullet} = \emptyset$. Hence follows $(f_1 \Box_{\mathbf{Z}} f_2)(x) = -\infty$ for all $x \in \mathrm{dom}_{\mathbf{Z}} f_1 + \mathrm{dom}_{\mathbf{Z}} f_2$. Finally it is obvious that $(f_1 \Box_{\mathbf{Z}} f_2)(x) = +\infty$ for $x \notin \mathrm{dom}_{\mathbf{Z}} f_1 + \mathrm{dom}_{\mathbf{Z}} f_2$. □

Remark 5.8 We have $f_1 \Box_{\mathbf{Z}} f_2 : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ if $\mathrm{dom}\, f_1$ or $\mathrm{dom}\, f_2$ is bounded, and [38, Theorem 6.10] gives (1) of the above theorem in the case where both $\mathrm{dom}\, f_1$ and $\mathrm{dom}\, f_2$ are bounded. □

We now turn to the infimum convolution $g_1 \Box_{\mathbf{Z}} g_2$ of two L-convex functions g_i $(i = 1, 2)$. First we observe the following fact.

Lemma 5.9 *Let* g_i $(i = 1, 2)$ *be L-convex functions and* $g_1 \Box_{\mathbf{Z}} g_2 : \mathbf{Z}^V \to \mathbf{Z} \cup \{\pm\infty\}$ *be the convolution defined by (4.13). If* $(g_1 \Box_{\mathbf{Z}} g_2)(p_0) = -\infty$ *for some* $p_0 \in \mathbf{Z}^V$*, then*

$$(g_1 \Box_{\mathbf{Z}} g_2)(p) = \begin{cases} -\infty & (p \in \mathrm{dom}_{\mathbf{Z}} g_1 + \mathrm{dom}_{\mathbf{Z}} g_2) \\ +\infty & (\textit{otherwise}) \end{cases}$$

This is the case if and only if $\mathrm{dom}_{\mathbf{Z}} {g_1}^{\bullet} \cap \mathrm{dom}_{\mathbf{Z}} {g_2}^{\bullet} = \emptyset$.

(Proof) The proof is similar to that of Theorem 5.8(2). □

The subdifferential of the infimum convolution of two L-convex functions has a nice combinatorial characterization. Namely, it is a nonempty set that can be expressed as the intersection of two integral base sets.

Theorem 5.10 *Let g_i $(i = 1, 2)$ be L-convex functions such that $g_1 \Box_{\mathbf{Z}} g_2 : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$. For $p \in \mathrm{dom}_{\mathbf{Z}}(g_1 \Box_{\mathbf{Z}} g_2)$ there exists $p_i \in \mathrm{dom}_{\mathbf{Z}} g_i$ $(i = 1, 2)$ such that $p = p_1 + p_2$ and*

$$\partial_{\mathbf{Z}}(g_1 \Box_{\mathbf{Z}} g_2)(p) = \partial_{\mathbf{Z}} g_1(p_1) \cap \partial_{\mathbf{Z}} g_2(p_2) \neq \emptyset. \tag{5.8}$$

(Proof) Put $f_i = g_i^{\bullet}$ $(i = 1, 2)$, $g = g_1 \Box_{\mathbf{Z}} g_2$ and $f = f_1 + f_2$. Then $f_i \in \mathcal{M}$ $(i = 1, 2)$ by Theorem 4.24, and $g^{\bullet} = f$ by (4.15) and $f^{\bullet} = g$ by Theorem 5.7(1). Then (4.12) shows

$$x \in \partial_{\mathbf{Z}} g(p) \iff p \in \partial_{\mathbf{Z}} f(x) \iff x \in \mathrm{argmin}_{\mathbf{Z}}(f[-p]).$$

Note here that $\mathrm{argmin}_{\mathbf{Z}}(f[-p]) \neq \emptyset$ for $p \in \mathrm{dom}_{\mathbf{Z}} f^{\bullet} = \mathrm{dom}_{\mathbf{Z}} g$. Also it follows from Theorem 5.6 that $p \in \partial_{\mathbf{Z}} f(x)$ if and only if $p = p_1 + p_2$ for some $p_i \in \partial_{\mathbf{Z}} f_i(x)$ $(i = 1, 2)$, in which $p_i \in \partial_{\mathbf{Z}} f_i(x)$ is equivalent to $x \in \partial_{\mathbf{Z}} g_i(p_i)$. $\Box$

Theorem 5.11 *For L-convex functions g_i $(i = 1, 2)$ such that $g_1 \Box_{\mathbf{Z}} g_2 : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$, we have $(g_1 \Box_{\mathbf{Z}} g_2)^{\bullet\bullet} = g_1 \Box_{\mathbf{Z}} g_2$.*

(Proof) By (4.15) we have $(g_1 \Box_{\mathbf{Z}} g_2)^{\bullet} = g_1{}^{\bullet} \Box_{\mathbf{Z}} g_2{}^{\bullet}$. Since $g_i{}^{\bullet} \in \mathcal{M}$ $(i = 1, 2)$, we have by Theorem 5.7(1) and Theorem 4.22 that

$$(g_1{}^{\bullet} \Box_{\mathbf{Z}} g_2{}^{\bullet})^{\bullet} = g_1{}^{\bullet\bullet} \Box_{\mathbf{Z}} g_2{}^{\bullet\bullet} = g_1 \Box_{\mathbf{Z}} g_2.$$

$\Box$

The above theorems would justify to give names to the class of functions that can be represented as the sum of two M-convex functions as well as to the class of functions that can be represented as the convolution of two L-convex functions. We say $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ is M_2-convex if it can be represented as the sum of two M-convex functions, and $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ is L_2-convex if it can be represented as the convolution of two L-convex functions. The classes of the M_2- and L_2-convex functions are denoted respectively by

$$\begin{aligned} \mathcal{M}_2 &= \{f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\} \mid \\ &\qquad \mathrm{dom}_{\mathbf{Z}} f \neq \emptyset, f = f_1 + f_2 \ (\exists f_1, f_2 \in \mathcal{M})\}, &(5.9) \\ \mathcal{L}_2 &= \{g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\} \mid \\ &\qquad \mathrm{dom}_{\mathbf{Z}} g \neq \emptyset, g = g_1 \Box_{\mathbf{Z}} g_2 \ (\exists g_1, g_2 \in \mathcal{L})\}. &(5.10) \end{aligned}$$

Then the above theorems are rephrased as follows. Recall the notations $\mathcal{F}_{\mathrm{E}}$ and $\mathcal{F}_{\mathrm{G}}$ defined by (4.17) and (4.18).

Theorem 5.12

(1) $\mathcal{M}_2 \subseteq \mathcal{F}_{\mathrm{G}} \subseteq \mathcal{F}_{\mathrm{E}}$. *In particular,* $\partial_{\mathbf{Z}} f(x) \neq \emptyset$ $(x \in \mathrm{dom}_{\mathbf{Z}} f)$ *and* $f^{\bullet\bullet} = f$ *for* $f \in \mathcal{M}_2$.

(2) $\mathcal{L}_2 \subseteq \mathcal{F}_{\mathrm{G}} \subseteq \mathcal{F}_{\mathrm{E}}$. *In particular,* $\partial_{\mathbf{Z}} g(p) \neq \emptyset$ $(p \in \mathrm{dom}_{\mathbf{Z}} g)$ *and* $g^{\bullet\bullet} = g$ *for* $g \in \mathcal{L}_2$.

(3) ${\mathcal{M}_2}^{\bullet} = \mathcal{L}_2$, ${\mathcal{L}_2}^{\bullet} = \mathcal{M}_2$.

(Proof) (1) This follows from Theorems 5.6, 5.7(2) and 4.7.

(2) This follows from Theorems 5.10, 5.11 and 4.21.

(3) This follows from Theorem 5.7(1) and (4.15). $\square$

6 Lagrange Duality for Optimization

6.1 Overview

On the basis of the fundamental theorems established so far, we shall develop in this section a Lagrange duality theory for (nonlinear) integer program:

$$\text{P:} \qquad \text{Minimize} \;\; c(x) \quad \text{subject to} \quad x \in B, \tag{6.1}$$

where $c : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ and $\emptyset \neq B \subseteq \mathbf{Z}^V$. The canonical "convex" case consists of the problems in which

(REG) B is an integral base set,

(OBJ) c is M-convex.

We follow Rockafellar's conjugate duality approach [44] to convex/nonconvex programs in nonlinear optimization. In fact, the whole scenario of the present theory is a straightforward adaptation of it, whereas the technical development leading to a strong duality assertion for "convex" programs relies heavily on the fundamental theorems established in the previous sections. An adaptation of the augmented Lagrangian function in nonlinear programming affords a duality framework that covers "nonconvex" programs.

In the canonical "convex" case, the problem dual to P turns out to be a maximization of an L-concave (or L_2-concave) function and the strong duality holds between the primal/dual pair of problems. This is a consequence of the conjugacy between M- and L-convexity (or between M_2- and L_2-convexity) and the Fenchel-type duality theorem (Theorem 5.2). Thus the classes of M_2-/L_2-convex functions, introduced in Section 5.2 as extensions of M-/L-convex functions, play an important role in this section.

In the literature of integer programming [41], [42], can be found a number of duality frameworks such as the subadditive duality. The present approach is distinguished from those existing ones in that:

1. It is primarily concerned with nonlinear objective functions.

2. The theory parallels the perturbation-based duality formalism in nonlinear programming.

3. In particular, the dual problem is derived from an embedding of the given problem in a family of perturbed problems with a certain convexity in the direction of perturbation.

4. It identifies M-convex programs as the well-behaved core structure to be compared to convex programs in nonlinear programming.

6.2 General duality framework

We describe the general framework, in which neither (REG) nor (OBJ) is assumed. This is a straightforward adaptation of the existing Lagrange duality framework in nonlinear programming, in particular, Rockafellar's conjugate duality approach [44] to convex/nonconvex programs. We follow the notation of [44] to clarify the parallelism.

First we rewrite the problem P as

$$\text{P:} \qquad \text{Minimize} \quad f(x) \quad \text{subject to} \quad x \in \mathbf{Z}^V \tag{6.2}$$

with

$$f(x) = c(x) + \delta_B(x), \tag{6.3}$$

where $\delta_B : \mathbf{Z}^V \to \{0, +\infty\}$ is the indicator function of B (i.e., $\delta_B(x) = 0$ if $x \in B$, and $= +\infty$ otherwise). Let us say that the problem P is feasible if $f(x) < +\infty$ for some $x \in \mathbf{Z}^V$.

Adopting the basic idea of the Lagrange duality theory in nonlinear programming we embed the optimization problem P in a family of perturbed problems. As the perturbation of f we consider $F : \mathbf{Z}^V \times \mathbf{Z}^U \to \mathbf{Z} \cup \{+\infty\}$, with U being a finite set, such that

$$F(x, 0) = f(x), \tag{6.4}$$

$$F(x, \cdot)^{\bullet\bullet} = F(x, \cdot) \quad \text{for each } x \in \mathbf{Z}^V. \tag{6.5}$$

Here the second condition (6.5) means that the integer biconjugate of $F(x, u)$ as a function in u for each fixed x agrees with $F(x, u)$ itself.

Remark 6.1 Lemma 4.2(3) shows that the condition (6.5) is satisfied if, for each x, either $F(x, \cdot) \equiv +\infty$ or $F(x, \cdot) \in \mathcal{F}_{\mathrm{G}}$, where $\mathcal{F}_{\mathrm{G}}$ is defined in (4.18). Recall in this connection that $\mathcal{M} \subseteq \mathcal{M}_2 \subseteq \mathcal{F}_{\mathrm{G}}$ and $\mathcal{L} \subseteq \mathcal{L}_2 \subseteq \mathcal{F}_{\mathrm{G}}$. □

The resulting family of optimization problems, parametrized by $u \in \mathbf{Z}^U$, reads:

$$\text{P}(u): \qquad \text{Minimize} \quad F(x, u) \quad \text{subject to} \quad x \in \mathbf{Z}^V. \tag{6.6}$$

We define the optimal value function $\varphi : \mathbf{Z}^U \to \mathbf{Z} \cup \{\pm\infty\}$ by

$$\varphi(u) = \inf\{F(x, u) \mid x \in \mathbf{Z}^V\} \qquad (u \in \mathbf{Z}^U), \tag{6.7}$$

and the Lagrangian function $K : \mathbf{Z}^V \times \mathbf{Z}^U \to \mathbf{Z} \cup \{\pm\infty\}$ by

$$K(x, y) = \inf\{F(x, u) + \langle u, y \rangle \mid u \in \mathbf{Z}^U\} \qquad (x \in \mathbf{Z}^V, y \in \mathbf{Z}^U). \tag{6.8}$$

For each $x \in \mathbf{Z}^V$, the function $K(x, \cdot) : y \mapsto K(x, y)$ is the integer concave Fenchel conjugate of the function $-F(x, \cdot) : u \mapsto -F(x, u)$.

Our assumptions (6.4) and (6.5) on the perturbation $F(x, u)$ enable us to claim the following.

Lemma 6.1

(1) $F(x,u) = \sup\{K(x,y) - \langle u, y\rangle \mid y \in \mathbf{Z}^U\} \qquad (x \in \mathbf{Z}^V, u \in \mathbf{Z}^U)$.

(2) $f(x) = \sup\{K(x,y) \mid y \in \mathbf{Z}^U\} \qquad (x \in \mathbf{Z}^V)$.

(Proof) (1) Abbreviate $F(x,u)$ and $K(x,y)$ to $F(u)$ and $K(y)$, respectively. We have $F^{\bullet}(y) = -K(-y)$ by the definition (6.8), while $F(u) = F^{\bullet\bullet}(u)$ by the assumption (6.5). Therefore we obtain

$$F(u) = F^{\bullet\bullet}(u) = \sup_y(\langle u, y\rangle + K(-y)) = \sup_y(K(y) - \langle u, y\rangle).$$

(2) This follows from (1) with $u = 0$ and (6.4). □

We define the dual problem to P as

$$\text{D:} \qquad \text{Maximize } \ g(y) \ \text{ subject to } \ y \in \mathbf{Z}^U, \tag{6.9}$$

where the objective function $g : \mathbf{Z}^U \to \mathbf{Z} \cup \{\pm\infty\}$ is defined by

$$g(y) = \inf\{K(x,y) \mid x \in \mathbf{Z}^V\} \qquad (y \in \mathbf{Z}^U). \tag{6.10}$$

We say that the problem D is feasible if $g(y) > -\infty$ for some $y \in \mathbf{Z}^U$. It is remarked that $g(y) < +\infty$ for all $y \in \mathbf{Z}^U$ if the problem P is feasible; for, $g(y) \leq K(x,y) \leq F(x,0) = f(x)$ for all $x \in \mathbf{Z}^V$.

We use the following notations:

$$\begin{aligned}
\inf(\mathrm{P}) &= \inf\{f(x) \mid x \in \mathbf{Z}^V\}, \\
\sup(\mathrm{D}) &= \sup\{g(y) \mid y \in \mathbf{Z}^U\}, \\
\mathrm{opt}(\mathrm{P}) &= \{x \in \mathbf{Z}^V \mid f(x) = \inf(\mathrm{P})\}, \\
\mathrm{opt}(\mathrm{D}) &= \{y \in \mathbf{Z}^U \mid g(y) = \sup(\mathrm{D})\}.
\end{aligned}$$

We write min(P) instead of inf(P) if the problem P is feasible and the infimum is finite, where it is noted that the infimum is attained (i.e., $\mathrm{opt}(\mathrm{P}) \neq \emptyset$) in such a case. Similarly, we write max(D) for sup(D) if the problem D is feasible and the supremum is finite, in which case the supremum is attained (i.e., $\mathrm{opt}(\mathrm{D}) \neq \emptyset$).

Theorem 6.2 (Weak Duality) $\quad \inf(\mathrm{P}) \geq \sup(\mathrm{D})$.

(Proof) We have

$$\begin{aligned}
g(y) &= \inf_x K(x,y) = \inf_x \inf_u (F(x,u) + \langle u, y\rangle) \\
&\leq \inf_x (F(x,0) + \langle 0, y\rangle) = \inf_x f(x) = \inf(\mathrm{P}).
\end{aligned}$$

Hence, $\sup(\mathrm{D}) = \sup_y g(y) \leq \inf(\mathrm{P})$. □

Corollary 6.3 *If* $\inf(\mathrm{P}) = -\infty$ *or* $\sup(\mathrm{D}) = +\infty$*, then* $\inf(\mathrm{P}) = \sup(\mathrm{D})$. □

Our main interest lies, of course, in the strong duality, namely, the case where the equality holds with a finite value in the weak duality above.

Theorem 6.4

(1) $g(y) = -\varphi^\bullet(-y)$.

(2) $\sup(\mathrm{D}) = \varphi^{\bullet\bullet}(0)$.

(3) $\inf(\mathrm{P}) = \varphi(0)$.

(4) $\inf(\mathrm{P}) = \sup(\mathrm{D}) \iff \varphi(0) = \varphi^{\bullet\bullet}(0)$.

(5) *If* $\inf(\mathrm{P})$ *is finite, then* $\min(\mathrm{P}) = \max(\mathrm{D}) \iff \partial_{\mathbf{Z}}\varphi(0) \neq \emptyset$.

(6) *If* $\min(\mathrm{P}) = \max(\mathrm{D})$*, then* $\mathrm{opt}(\mathrm{D}) = -\partial_{\mathbf{Z}}\varphi(0)$.

(Proof) (1) By the definitions we have

$$\begin{aligned} -\varphi^\bullet(-y) &= -\sup_u(\langle u, -y\rangle - \varphi(u)) = \inf_u(\varphi(u) + \langle u, y\rangle) \\ &= \inf_u(\inf_x F(x,u) + \langle u, y\rangle) = \inf_x \inf_u(F(x,u) + \langle u, y\rangle) \\ &= \inf_x K(x,y) = g(y). \end{aligned}$$

(2) By using (1) we have

$$\begin{aligned} \sup(\mathrm{D}) &= \sup_y g(y) = \sup_y(\langle 0, -y\rangle - \varphi^\bullet(-y)) \\ &= \sup_y(\langle 0, y\rangle - \varphi^\bullet(y)) = \varphi^{\bullet\bullet}(0). \end{aligned}$$

(3) This is obvious from (6.4) and (6.7).

(4) The equivalence is due to (2) and (3).

(5)–(6) We have the following chain of equivalence: $\overline{y} \in -\partial_{\mathbf{Z}}\varphi(0) \iff \varphi(u) - \varphi(0) \geq \langle u, -\overline{y}\rangle\ (\forall u \in \mathbf{Z}^U) \iff \inf_u(\varphi(u) + \langle u, \overline{y}\rangle) = \varphi(0) \iff g(\overline{y}) = \varphi(0)$, where $\inf_u(\varphi(u) + \langle u, \overline{y}\rangle) = g(\overline{y})$ is shown in the proof of (1). This implies the claims when combined with the weak duality (Theorem 6.2). □

Theorem 6.5 (Saddle-Point Theorem)
$\min(\mathrm{P}) = \max(\mathrm{D})$ *if and only if there exist* $\overline{x} \in \mathbf{Z}^V, \overline{y} \in \mathbf{Z}^U$ *such that* $K(\overline{x}, \overline{y})$ *is finite and*

$$K(\overline{x}, y) \leq K(\overline{x}, \overline{y}) \leq K(x, \overline{y}) \qquad (x \in \mathbf{Z}^V, y \in \mathbf{Z}^U).$$

If this is the case, we have $\overline{x} \in \mathrm{opt}(\mathrm{P})$ *and* $\overline{y} \in \mathrm{opt}(\mathrm{D})$.

(Proof) By Lemma 6.1(2) we have $f(\overline{x}) = \sup_y K(\overline{x}, y)$, whereas $g(\overline{y}) = \inf_x K(x, \overline{y})$ by the definition (6.10). In view of the weak duality (Theorem 6.2) and the relation:

$$f(\overline{x}) = \sup_y K(\overline{x}, y) \geq K(\overline{x}, \overline{y}) \geq \inf_x K(x, \overline{y}) = g(\overline{y}),$$

we see that

$$\begin{aligned} & \min(\mathrm{P}) = \max(\mathrm{D}) \\ \Longleftrightarrow\ & \exists\, \overline{x}, \exists\, \overline{y}:\ f(\overline{x}) = g(\overline{y}) \\ \Longleftrightarrow\ & \exists\, \overline{x}, \exists\, \overline{y}:\ \sup_y K(\overline{x}, y) = K(\overline{x}, \overline{y}) = \inf_x K(x, \overline{y}). \end{aligned}$$

□

6.3 Augmented Lagrangian based on M-convexity

As the perturbation F, we choose $F = F_r : \mathbf{Z}^V \times \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ defined by

$$F_r(x, u) = c(x) + \delta_B(x + u) + r(u) \qquad (x, u \in \mathbf{Z}^V), \tag{6.11}$$

where $r : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ is an M-convex function with $r(0) = 0$. (We take V as the U in the general framework.) The special case with $r = 0$ is naturally distinguished by subscript 0. Namely,

$$F_0(x, u) = c(x) + \delta_B(x + u) \qquad (x, u \in \mathbf{Z}^V). \tag{6.12}$$

We single out the case of $r = 0$ because the technical development in this special case can be made within the framework of M-/L-convex functions, whereas the general case involves M_2-/L_2-convex functions.

Throughout this section we assume (REG), i.e., that B is an integral base set, which assumption is not substantially restrictive, as is discussed later in Remark 6.2.

We use subscript r to denote the quantities derived from F_r, namely,

$$\begin{aligned} \varphi_r(u) &= \inf\{F_r(x, u) \mid x \in \mathbf{Z}^V\} \qquad (u \in \mathbf{Z}^V), & (6.13) \\ K_r(x, y) &= \inf\{F_r(x, u) + \langle u, y\rangle \mid u \in \mathbf{Z}^V\} \qquad (x, y \in \mathbf{Z}^V), & (6.14) \\ g_r(y) &= \inf\{K_r(x, y) \mid x \in \mathbf{Z}^V\} \qquad (y \in \mathbf{Z}^V), & (6.15) \\ \sup(\mathrm{D}_r) &= \sup\{g_r(y) \mid y \in \mathbf{Z}^V\}. \end{aligned}$$

Our choice of the perturbation (6.11) is legitimate, meeting the requirements (6.4) and (6.5), as follows.

Lemma 6.6 *Assume* (REG).

(1) $F_r(x, 0) = f(x) \qquad (x \in \mathbf{Z}^V)$.

(2) For each $x \in \mathbf{Z}^V$, $F_0(x, u)$ *is M-convex in* u *or* $F_0(x, u) = +\infty$ *for all* u.

(3) For each $x \in \mathbf{Z}^V$, $F_r(x, u)$ *is* M_2*-convex in* u *or* $F_r(x, u) = +\infty$ *for all* u.

(4) $F_r(x, u) = \sup\{K_r(x, y) - \langle u, y\rangle \mid y \in \mathbf{Z}^V\} \qquad (x, u \in \mathbf{Z}^V)$.

Assume (REG) *and* (OBJ).

(5) For each $u \in \mathbf{Z}^V$, $F_r(x, u)$ *is* M_2*-convex in* x *or* $F_r(x, u) = +\infty$ *for all* x.

(Proof) (1) This follows from $r(0) = 0$.

(2) We have $F_0(x, u) = +\infty$ unless $x \in \mathrm{dom}_{\mathbf{Z}} c$. For each x, $\delta_B(x+u) = \delta_{B-x}(u)$ is the indicator function of $B - x$ (translation of B by x), which is again an integral base set. Therefore $\delta_B(x + u)$ is M-convex in u.

(3) We have $F_r(x, u) = +\infty$ unless $x \in \mathrm{dom}_{\mathbf{Z}} c$. Besides $\delta_B(x + u)$, $r(u)$ is also M-convex by the assumption. Hence $F_r(x, \cdot)$ is the sum of two M-convex functions for each $x \in \mathrm{dom}_{\mathbf{Z}} c$. By definition, such a function is either M_2-convex or identically equal to $+\infty$.

(4) It follows from (3) and Theorem 5.12(1) that the condition (6.5) is satisfied. Then Lemma 6.1(1) establishes this.

(5) We have $F_r(x, u) = +\infty$ unless $u \in \mathrm{dom}_{\mathbf{Z}} r$. $\delta_B(x + u) = \delta_{B-u}(x)$ is M-convex in x and $c(x)$ is M-convex by the assumption. Hence $F_r(\cdot, u)$ is the sum of two M-convex functions for each $u \in \mathrm{dom}_{\mathbf{Z}} r$. By definition, such a function is either M_2-convex or identically equal to $+\infty$. $\square$

The Lagrangian function $K_r(x, y)$ enjoys the following properties. It should be clear that $\delta_B{}^\bullet$ is the integer convex Fenchel conjugate of the indicator function of B, and that $\delta_{-B} \Box_{\mathbf{Z}}\, r[y]$ means the infimum convolution of the indicator function of $-B = \{x \mid -x \in B\}$ and $r[y](u) = r(u) + \langle u, y\rangle$.

Lemma 6.7

(1) $K_0(x, y) = \begin{cases} c(x) - \langle x, y\rangle - \delta_B{}^\bullet(-y) & (x \in \mathrm{dom}_{\mathbf{Z}} c, y \in \mathbf{Z}^V) \\ +\infty & (x \notin \mathrm{dom}_{\mathbf{Z}} c, y \in \mathbf{Z}^V) \end{cases}$

(2) $K_r(x, y) = \begin{cases} c(x) + (\delta_{-B} \Box_{\mathbf{Z}}\, r[y])(-x) & (x \in \mathrm{dom}_{\mathbf{Z}} c, y \in \mathbf{Z}^V) \\ +\infty & (x \notin \mathrm{dom}_{\mathbf{Z}} c, y \in \mathbf{Z}^V) \end{cases}$

(Proof) (1) Assume $x \in \mathrm{dom}_{\mathbf{Z}} c$. Substituting (6.12) into (6.14) we obtain

$$\begin{aligned} K_0(x, y) &= \inf\{c(x) + \delta_B(x + u) + \langle u, y\rangle \mid u \in \mathbf{Z}^V\} \\ &= \inf\{c(x) + \delta_B(u) + \langle u - x, y\rangle \mid u \in \mathbf{Z}^V\} \\ &= c(x) - \langle x, y\rangle + \inf\{\langle u, y\rangle \mid u \in B\} \\ &= c(x) - \langle x, y\rangle - \delta_B{}^\bullet(-y). \end{aligned}$$

(2) Assume $x \in \mathrm{dom}_{\mathbf{Z}} c$. Substituting (6.11) into (6.14) we obtain

$$\begin{aligned} K_r(x,y) &= \inf\{c(x) + \delta_B(x+u) + r(u) + \langle u, y\rangle \mid u \in \mathbf{Z}^V\} \\ &= c(x) + \inf\{\delta_{-B}(-x-u) + r[y](u) \mid u \in \mathbf{Z}^V\} \\ &= c(x) + (\delta_{-B} \Box_{\mathbf{Z}}\, r[y])(-x). \end{aligned}$$

$\Box$

Theorem 6.8 *Assume* (REG).

(1) For each $x \in \mathbf{Z}^V$, $K_0(x,y)$ *is L-concave in* y *or* $K_0(x,y) = +\infty$ *for all* y.

(2) For each $x \in \mathbf{Z}^V$, $K_r(x,y)$ *is* L_2*-concave in* y *or* $K_r(x,y) = +\infty$ *for all* y.

Assume (REG) *and* (OBJ).

(3) For each $y \in \mathbf{Z}^V$, $K_0(x,y)$ *is M-convex in* x *or* $K_0(x,y) \in \{+\infty, -\infty\}$ *for all* x.

(4) For each $y \in \mathbf{Z}^V$, $K_r(x,y)$ *is* M_2*-convex in* x *or* $K_r(x,y) \in \{+\infty, -\infty\}$ *for all* x.

(Proof) (1) For each $x \in \mathbf{Z}^V$, $K_0(x,\cdot)$ is the integer concave Fenchel conjugate of the function $-F_0(x,\cdot)$. Since $F_0(x,\cdot) \in \mathcal{M}$ or $\equiv +\infty$ by Lemma 6.6(2) and $\mathcal{M}^\bullet = \mathcal{L}$ by Theorem 4.24, we see that $K_0(x,\cdot)$ is L-concave or $\equiv +\infty$.

(An alternative direct proof can be obtained from the expression of Lemma 6.7(1) as follows: Since δ_B is M-convex by (REG), ${\delta_B}^\bullet$ is L-convex by the conjugacy. It then follows that $\langle x, y\rangle + {\delta_B}^\bullet(-y)$ is also L-convex in y.)

(2) Similarly, for each $x \in \mathbf{Z}^V$, $K_r(x,\cdot)$ is the integer concave Fenchel conjugate of the function $-F_r(x,\cdot)$. Since $F_r(x,\cdot) \in \mathcal{M}_2$ or $\equiv +\infty$ by Lemma 6.6(3) and ${\mathcal{M}_2}^\bullet = \mathcal{L}_2$ by Theorem 5.12(3), we see that $K_r(x,\cdot)$ is L_2-concave or $\equiv +\infty$.

(3) The M-convexity of c implies that of $c(x) - \langle x, y\rangle$ for each y in the expression of K_0 in Lemma 6.7(1). Hence $K_0(\cdot, y)$ is M-convex if ${\delta_B}^\bullet(-y)$ is finite. (4) In the expression of $K_r(x,y)$ in Lemma 6.7(2) we see that the second term $(\delta_{-B} \Box_{\mathbf{Z}}\, r[y])(-x)$ is M-convex or $\in \{+\infty, -\infty\}$ since it is the convolution of two M-convex functions (cf. Theorems 4.4 and 5.8). $\Box$

In the case of M-convex programs the dual objective function g_r and the optimal value function φ_r are well-behaved, as follows.

Theorem 6.9 *Assume* (REG) *and* (OBJ).

(1) g_0 *is L-concave or* $g_0(y) = -\infty$ *for all* y.

(2) g_r *is* L_2*-concave, or* $g_r(y) = -\infty$ *for all* y, *or* $g_r(y) = +\infty$ *for all* y.

(3) φ_0 *is M-convex or* $\varphi_0(u) \in \{+\infty, -\infty\}$ *for all* u.

(4) φ_r *is* M_2*-convex or* $\varphi_r(u) \in \{+\infty, -\infty\}$ *for all* u.

(Proof) We prove (1), (3), (4) and finally (2).

(1) Using Lemma 6.7(1) we obtain

$$g_0(y) = \inf_x K_0(x,y) = \inf_x (c(x) - \langle x, y\rangle) - \delta_B{}^\bullet(-y) = -c^\bullet(y) - \delta_B{}^\bullet(-y). \quad (6.16)$$

This shows g_0 is L-concave or $\equiv -\infty$, since the sum of two L-concave functions is again L-concave, provided the effective domains of the summands are not disjoint (cf. Theorem 4.23).

(3) We have $\varphi_0(u) = \inf_x(c(x) + \delta_B(x+u)) = (c \Box_{\mathbf{Z}}\, \delta_{-B})(-u)$. The assertion follows from Theorem 5.8.

(4) It follows from $F_r(x,u) = F_0(x,u) + r(u)$ that $\varphi_r(u) = +\infty$ unless $u \in \mathrm{dom}_{\mathbf{Z}} r$ and that $\varphi_r(u) = \varphi_0(u) + r(u)$ if $u \in \mathrm{dom}_{\mathbf{Z}} r$. If $\varphi_0 \in \mathcal{M}$, then $\varphi_r \in \mathcal{M}_2$ or $\varphi_r \equiv +\infty$. If $\varphi_0(u) \in \{+\infty, -\infty\}$, then $\varphi_r(u) \in \{+\infty, -\infty\}$.

(2) First recall the relation $g_r(y) = -\varphi_r{}^\bullet(-y)$ (cf. Theorem 6.4(1)). If $\varphi_r \in \mathcal{M}_2$, the conjugacy between $\mathcal{L}_2$ and $\mathcal{M}_2$ (cf. Theorem 5.12) implies the L_2-concavity of g_r. If $\varphi_r \equiv +\infty$, then $g_r \equiv -\infty$. If $\varphi_r(u) = -\infty$ for some u, then $g_r \equiv +\infty$. □

The strong duality holds true for M-convex programs with the Lagrangian function $K_r(x,y)$.

Theorem 6.10 (Strong Duality) *Assume* (REG), (OBJ) *and that the problem* P *is feasible and bounded from below (i.e.,* $\inf(\mathrm{P}) \neq \pm\infty$*).*

(1) $\min(\mathrm{P}) = \varphi_r(0) = \varphi_r{}^{\bullet\bullet}(0) = \max(\mathrm{D}_r)$.

(2) $\mathrm{opt}(\mathrm{D}_r) = -\partial_{\mathbf{Z}}\varphi_r(0)$.

(Proof) Since $\varphi_r(0)$ is finite by the assumption, φ_r is M_2-convex by Theorem 6.9(4) and hence $\partial_{\mathbf{Z}}\varphi_r(0) \neq \emptyset$ by Theorem 5.12(1). Then the assertions follow from Theorem 6.4. □

It should be emphasized that the M-convexity of the objective function c is a sufficient condition and not an absolute prerequisite for the strong duality to hold.

Example 6.1 Let us consider the case where $c(x)$ is a linear function on another integral base set $B' \subseteq \mathbf{Z}^V$. The primal problem with $c(x) = \langle x, w\rangle + \delta_{B'}(x)$ (where $w \in \mathbf{Z}^V$ denotes the weight vector) reads:

$$\mathrm{P}: \qquad \text{Minimize } \langle x, w\rangle \quad \text{subject to} \quad x \in B \cap B'. \quad (6.17)$$

The Lagrangian function K_0 is given by

$$K_0(x,y) = \begin{cases} \langle x, w-y\rangle + \inf_{z\in B}\langle z, y\rangle & (x \in B', y \in \mathbf{Z}^V) \\ +\infty & (x \notin B', y \in \mathbf{Z}^V) \end{cases} \quad (6.18)$$

from which is derived the dual problem:

$$\text{D:} \qquad \text{Maximize} \quad \left(\inf_{x \in B'} \langle x, w - y \rangle + \inf_{z \in B} \langle z, y \rangle \right) \quad \text{subject to} \quad y \in \mathbf{Z}^V. \tag{6.19}$$

The strong duality for this pair of problems is equivalent to Fujishige's potential characterization [16] (see also [19]) for the weighted intersection problem for a pair of submodular systems, which is an extension of Frank's weight splitting theorem [13] for the weighted matroid intersection problem and Iri–Tomizawa's potential characterization of the optimality for the independent assignment problem [25].

For a concrete instance, take $V = \{1, 2\}$, $B = \{(x_1, x_2) \in \mathbf{Z}^2 \mid x_1 \geq 0, x_1 + x_2 = 0\}$, $B' = \{(x_1, x_2) \in \mathbf{Z}^2 \mid x_1 \leq 1, x_1 + x_2 = 0\}$. We have $B \cap B' = \{(0, 0), (1, -1)\}$ and

$$K_0(x, y) = \begin{cases} \langle x, w - y \rangle & ((x_1, x_2) \in B', y_1 \geq y_2) \\ -\infty & ((x_1, x_2) \in B', y_1 < y_2) \\ +\infty & (\text{otherwise}). \end{cases} \tag{6.20}$$

$\Box$

Remark 6.2 In this section we have assumed (REG) throughout. This is not really a restriction, since any problem can be transformed into such a form. Given a problem described by (c, B), consider $\tilde{V} = \{v_0\} \cup V$ by introducing a new element v_0, choose $\tilde{B} \subseteq \mathbf{Z}^{\tilde{V}}$ to be an integral base set such that $\tilde{B} \supseteq \tilde{B}'$, where

$$\tilde{B}' = \{(x_0, x) \in \mathbf{Z}^{\tilde{V}} \mid x \in B, x_0 + \langle \mathbf{1}, x \rangle = 0\},$$

and define $\tilde{c} : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{+\infty\}$ by

$$\tilde{c}(x_0, x) = \begin{cases} c(x) & (x_0 + \langle \mathbf{1}, x \rangle = 0, (x_0, x) \notin \tilde{B} \setminus \tilde{B}') \\ c(x) + \alpha & (x_0 + \langle \mathbf{1}, x \rangle = 0, (x_0, x) \in \tilde{B} \setminus \tilde{B}') \\ +\infty & (x_0 + \langle \mathbf{1}, x \rangle \neq 0) \end{cases}$$

with a big number α. Then the problem described by $(\tilde{c}, \tilde{B})$ meets the condition (REG). For example, we can take $\tilde{B} = \{(x_0, x) \in \mathbf{Z}^{\tilde{V}} \mid x_0 + \langle \mathbf{1}, x \rangle = 0\}$. $\Box$

6.4 Symmetry in duality

In the previous section we have derived the dual problem D from the primal P by means of a perturbation function $F(x, u)$ such that $F(x, 0) = f(x)$ and $F(x, \cdot) \in \mathcal{M}_2 \subseteq \mathcal{F}_{\mathrm{G}}$. Namely,

$$\text{P : min. } f(x) \quad \xrightarrow[\substack{F(x,0) = f(x) \\ F(x,\cdot) \in \mathcal{M}_2 \subseteq \mathcal{F}_{\mathrm{G}}}]{\qquad\qquad} \quad \text{D : max. } g(y)$$

We have seen that g is L_2-concave, i.e., $-g \in \mathcal{L}_2$ in the "convex" case where (REG) and (OBJ) are satisfied.

We are now interested in the reverse process, i.e., how to restore the primal problem P from the dual D in a way consistent with the general duality framework of Section 6.1. We embed the dual problem D in a family of maximization problems defined in terms of another perturbation function $G(y, v)$ such that $G(y, 0) = g(y)$ and $-G(y, \cdot) \in \mathcal{L}_2 \subseteq \mathcal{F}_{\mathrm{G}}$. Namely,

$$\mathrm{P} : \text{min. } f(x) \quad \xleftarrow[\substack{G(y,0)=g(y) \\ -G(y,\cdot) \in \mathcal{L}_2 \subseteq \mathcal{F}_{\mathrm{G}}}]{\qquad\qquad} \quad \mathrm{D} : \text{max. } g(y)$$

With reference to (6.10) we define a perturbation function $G : \mathbf{Z}^U \times \mathbf{Z}^V \to \mathbf{Z} \cup \{\pm\infty\}$ by

$$G(y, v) = \inf\{K(x, y) - \langle x, v\rangle \mid x \in \mathbf{Z}^V\} \qquad (y \in \mathbf{Z}^U, v \in \mathbf{Z}^V). \tag{6.21}$$

By this we intend to consider a family of maximization problems parametrized by $v \in \mathbf{Z}^V$:

$$\text{Maximize } \ G(y, v) \ \text{ subject to } \ y \in \mathbf{Z}^U.$$

The optimal value function $\gamma : \mathbf{Z}^V \to \mathbf{Z} \cup \{\pm\infty\}$ is accordingly defined by

$$\gamma(v) = \sup\{G(y, v) \mid y \in \mathbf{Z}^U\} \qquad (v \in \mathbf{Z}^V). \tag{6.22}$$

It is then natural to introduce the dual Lagrangian function $\tilde{K}(x, y) : \mathbf{Z}^V \times \mathbf{Z}^U \to \mathbf{Z} \cup \{\pm\infty\}$ defined by

$$\tilde{K}(x, y) = \sup\{G(y, v) + \langle x, v\rangle \mid v \in \mathbf{Z}^V\} \qquad (x \in \mathbf{Z}^V, y \in \mathbf{Z}^U). \tag{6.23}$$

The problem dual to the problem D is to minimize

$$\tilde{f}(x) = \sup\{\tilde{K}(x, y) \mid y \in \mathbf{Z}^U\} \qquad (x \in \mathbf{Z}^V). \tag{6.24}$$

As can be imagined from the corresponding constructions in the ordinary convex analysis ([44, Section 4], in particular), the objects $\tilde{K}(x, y)$ and $\tilde{f}(x)$ thus constructed are not guranteed to agree with the original objects $K(x, y)$ and $f(x)$. We show, however, that the dual of the dual comes back to the primal in the canonical case with a bounded integral base set B using the augmented Lagrangian function.

Example 6.2 Continuing Example 6.1 we consider the example with $V = \{1, 2\}$, $B = \{(x_1, x_2) \in \mathbf{Z}^2 \mid x_1 \geq 0, x_1 + x_2 = 0\}$, $B' = \{(x_1, x_2) \in \mathbf{Z}^2 \mid x_1 \leq 1, x_1 + x_2 = 0\}$, and $c(x) = \langle x, w\rangle + \delta_{B'}(x)$. From K_0 of (6.20) we can calculate

$$\tilde{K}_0(x, y) = \begin{cases} \langle x, w - y\rangle & ((x_1, x_2) \in B', y_1 \geq y_2) \\ +\infty & ((x_1, x_2) \notin B', y_1 \geq y_2) \\ -\infty & (y_1 < y_2) \end{cases}$$

We observe that $\tilde{K}_0(x, y) = K_0(x, y)$ where they take finite values. □

In what follows we always assume (REG), (OBJ) and that B is bounded. We consider the augmented Lagrangian K_r.

Lemma 6.11 *Assume* (REG), (OBJ) *and that B is bounded. Then*

$$\tilde{K}_r(x, y) = K_r(x, y) \quad (x, y \in \mathbf{Z}^V), \qquad \tilde{f}(x) = f(x) \quad (x \in \mathbf{Z}^V).$$

(Proof) The definitions (6.21) and (6.23) show $G_r(y, \cdot) = -(K_r(\cdot, y))^\bullet$ and $\tilde{K}_r(\cdot, y) = (-G_r(y, \cdot))^\bullet$ for each y. Since $K_r(\cdot, y) \in \mathcal{M}_2$ or $\equiv +\infty$ by Theorem 6.8(4) when B is bounded, we have $K_r(\cdot, y) = (K_r(\cdot, y))^{\bullet\bullet}$ for each y. Hence follows $\tilde{K}_r = K_r$. Then Lemma 6.1(2) and (6.24) imply $\tilde{f} = f$. □

Lemma 6.12 *Assume* (REG), (OBJ) *and that B is bounded.*

(1) $G_r(y, 0) = g_r(y) \qquad (y \in \mathbf{Z}^V)$.

(2) For each $y \in \mathbf{Z}^V$, $G_r(y, v)$ is L_2-concave in v or $G_r(y, v) = +\infty$ for all v.

(3) For each $v \in \mathbf{Z}^V$, $G_r(y, v)$ is L_2-concave in y, or $G_r(y, v) = -\infty$ for all y, or $G_r(y, v) = +\infty$ for all y.

(Proof) (1) This is obvious from (6.15) and (6.21).

(2) The definition (6.21) shows $G_r(y, \cdot) = -(K_r(\cdot, y))^\bullet$ for each y, while $K_r(\cdot, y) \in \mathcal{M}_2$ or $\equiv +\infty$ by Theorem 6.8(4) when B is bounded. Hence $-G_r(y, \cdot) \in \mathcal{L}_2$ or $\equiv -\infty$ by Theorem 5.12(3).

(3) We have the expression

$$\begin{aligned} G_r(y, v) &= \inf_x \inf_u (F_r(x, u) + \langle u, y\rangle - \langle x, v\rangle) \\ &= \inf_x \inf_u (c[-v](x) + \delta_B(x + u) + r(u) + \langle u, y\rangle), \end{aligned}$$

in which $c[-v]$ is M-convex. On the other hand, Theorem 6.9(2) shows that

$$g_r(y) = \inf_x \inf_u (c(x) + \delta_B(x + u) + r(u) + \langle u, y\rangle)$$

is L_2-concave, or $g_r(y) = -\infty$ for all y, or $g_r(y) = +\infty$ for all y. By replacing c with $c[-v]$ we obtain the claim. □

The optimal value function γ_r, defined by (6.22) with reference to G_r, enjoys the following properties.

Theorem 6.13 *Assume* (REG), (OBJ) *and that B is bounded.*

(1) $f(x) = -\gamma_r{}^\circ(-x)$.

(2) γ_r is L_2-concave or $\gamma_r(v) = +\infty$ for all v.

(Proof) Using Lemma 6.1(2) and Lemma 6.11 we obtain

$$\begin{aligned}
f(x) &= \sup_y K_r(x,y) = \sup_y \sup_v (G_r(y,v) + \langle x, v\rangle) \\
&= \sup_v \left((\sup_y G_r(y,v)) + \langle x, v\rangle \right) \\
&= \sup_v (\gamma_r(v) + \langle x, v\rangle) = -\gamma_r{}^{\circ}(-x).
\end{aligned}$$

(2) Since $f \in \mathcal{M}_2$ or $\equiv +\infty$, the assertion follows from (1) and the conjugacy between $\mathcal{L}_2$ and $\mathcal{M}_2$. □

Theorem 6.14 *Assume* (REG), (OBJ) *and that the problem* P *is feasible and* B *is bounded.*

(1) $\min(\mathrm{P}) = \gamma_r(0) = \gamma_r{}^{\circ\circ}(0) = \max(\mathrm{D}_r)$.

(2) $\mathrm{opt}(\mathrm{P}) = \partial_{\mathbf{Z}}(-\gamma_r)(0) \neq \emptyset$.

(Proof) The proof is essentially the same as that of Theorem 6.4. To be specific, we have the following chain of equivalence: $\overline{x} \in \partial_{\mathbf{Z}}(-\gamma_r)(0) \iff \gamma_r(v) - \gamma_r(0) \leq \langle v, -\overline{x}\rangle$ $(\forall v \in \mathbf{Z}^V) \iff -\inf_v(\langle v, -\overline{x}\rangle - \gamma_r(v)) = \gamma_r(0) \iff f(\overline{x}) = \gamma_r(0)$. This implies the claim when combined with the weak duality (Theorem 6.2). □

7 Concluding Remarks

Remark 7.1 In this paper we have concentrated on the structural aspects of the discrete convex analysis, with emphasis on conjugacy and duality. Algorithmic and computational issues are left for future research. In this respect it is worth while investigating whether the algorithmic results for nonlinear objective functions in integers [12], [22], [23], [24], [32], as well as for matroid valuations and related structures [4], [5], [6], [7], [27], [33], [35] can be extended to M-convex functions. See a recent paper of Shioura [48] for an interesting result in this direction. □

Remark 7.2 In this paper we have been interested in M-/L-convexity of integer-valued functions defined on integer lattice points. Obviously, the exchange property (M-EXC) makes sense for real-valued functions on integer lattice points, as in [38], [39]. The Extension Theorem (Theorem 4.11) suggests a further generalization of the concept of M-convexity to piecewise linear functions $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$. Namely we may define $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ to be M-convex if (a) f is convex, (b) $\mathrm{dom}_{\mathbf{R}} f$ is a base polyhedron, (c) For each $p \in \mathbf{R}^V$, $\mathrm{argmin}_{\mathbf{R}}(f[-p])$ is a base polyhedron or an empty set. Also the defining conditions (4.44)–(4.46) for L-convexity admit an obvious adaptation to $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$. Further study in this direction is left for the future. □

Remark 7.3 The concept of g-polymatroid (generalized polymatroid), introduced by A. Frank (cf. [15]), is closely related to that of base polyhedron. In fact, it is known [19, Theorem 3.58] that a g-polymatroid can be identified as a projection of a base polyhedron. This fundamental fact indicates that the g-polymatroid version of M-convex functions can be obtained by replacing "base polyhedron" with "g-polymatroid" in the characterization given in the Extension Theorem (Theorem 4.11). This idea has been worked out recently by Murota–Shioura [40]. $\square$

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Appendix: Proof of Theorem 5.1

A self-contained proof of Theorem 5.1, originally established in [37] as an extension of the result of [34], is provided here using the present notations.

A.1 Unique-min condition

In Theorem 4.13 we have seen

$$f(y) - f(x) \geq \widehat{f}(x, y) = \max\{\langle p, y - x\rangle \mid p \in \partial_{\mathbf{R}} f(x)\} \tag{A.1}$$

for an M-convex function f, $x \in \mathrm{dom}_{\mathbf{Z}} f$ and $y \in \mathbf{Z}^V$. Here we give a sufficient condition, the unique-min condition below, for this inequality to be an equality.

Recall from Section 4.2 the bipartite graph $G(x, y) = (V^+, V^-; \hat{A})$ with $V^+ = \mathrm{supp}^+(x - y)$, $V^- = \mathrm{supp}^-(x - y)$ and

$$\hat{A} = \{(u, v) \mid u \in V^+, v \in V^-, x - \chi_u + \chi_v \in \mathrm{dom}_{\mathbf{Z}} f\}.$$

In case $|x(v) - y(v)| \leq 1$ $(v \in V)$ holds, the linear program (4.30) – (4.33) describes a minimum-weight perfect matching problem on $G(x, y)$ and $\widehat{f}(x, y)$ is equal to the minimum weight of a perfect matching in $G(x, y)$.

For (x, y) with $x \in \mathrm{dom}_{\mathbf{Z}} f$, $y \in \mathbf{Z}^V$ and $|x(v) - y(v)| \leq 1$ $(v \in V)$, we consider

[Unique-Min Condition]

There exists exactly one minimum-weight perfect matching in $G(x, y)$.

We first note that the unique-min condition can be expressed in terms of "potentials" associated with vertices as follows. The essence of this lemma lies in the duality in the weighted bipartite matching problem, being independent of the M-convexity of f.

Lemma A.1 ([34, Lemma 3.5]) *Let (x, y) be such that $x \in \mathrm{dom}_{\mathbf{Z}} f$, $y \in \mathbf{Z}^V$ and $|x(v) - y(v)| \leq 1$ $(v \in V)$.*

(1) $G(x, y)$ has a perfect matching if and only if there exist $\widehat{p} : V^+ \cup V^- \to \mathbf{Z}$ and indexings of the elements of V^+ and V^-, say $V^+ = \{u_1, \cdots, u_m\}$, $V^- = \{v_1, \cdots, v_m\}$, such that, for any $(u_i, v_j) \in \hat{A}$,

$$f(x, u_i, v_j) + \widehat{p}(u_i) - \widehat{p}(v_j) \begin{cases} = 0 & (1 \leq i = j \leq m) \\ \geq 0 & (1 \leq i, j \leq m) \end{cases} \tag{A.2}$$

Furthermore, the relation (A.2) implies

$$\widehat{f}(x, y) = \sum_{i=1}^{m} \left(\widehat{p}(v_i) - \widehat{p}(u_i)\right). \tag{A.3}$$

(2) The pair (x, y) satisfies the unique-min condition if and only if there exist $\widehat{p} : V^+ \cup V^- \to \mathbf{Z}$ and indexings of the elements of V^+ and V^-, say $V^+ = \{u_1, \cdots, u_m\}$, $V^- = \{v_1, \cdots, v_m\}$, such that, for any $(u_i, v_j) \in \hat{A}$,

$$f(x, u_i, v_j) + \widehat{p}(u_i) - \widehat{p}(v_j) \begin{cases} = 0 & (1 \leq i = j \leq m) \\ \geq 0 & (1 \leq j < i \leq m) \\ > 0 & (1 \leq i < j \leq m) \end{cases} \tag{A.4}$$

The following lemma shows that the unique-min condition guarantees the equality in (A.1). This is a reformulation of the "unique-max lemma" of [34].

Lemma A.2 ([34, Lemma 3.8]) *Let f be M-convex, and let (x, y) be such that $x \in \mathrm{dom}_{\mathbf{Z}} f$, $y \in \mathbf{Z}^V$ and $|x(v) - y(v)| \leq 1$ $(v \in V)$. If (x, y) satisfies the unique-min condition, then $y \in \mathrm{dom}_{\mathbf{Z}} f$ and*

$$f(y) - f(x) = \widehat{f}(x, y). \tag{A.5}$$

(Proof) Consider the restriction $f' = f_{x \wedge y}^{x \vee y}$ of f to $[x \wedge y, x \vee y]$, which is M-convex by Theorem 4.5. Let $\widehat{p} : V^+ \cup V^- \to \mathbf{Z}$, $V^+ = \{u_1, \cdots, u_m\}$ and $V^- = \{v_1, \cdots, v_m\}$ be as in Lemma A.1(2). We regard $\widehat{p}$ as $\widehat{p} : V \to \mathbf{Z}$ by defining $\widehat{p}(v) = 0$ for $v \in V - (V^+ \cup V^-)$. We claim

$$f'[-\widehat{p}](x, u, v) = f'(x, u, v) + \widehat{p}(u) - \widehat{p}(v) \geq 0 \qquad (u, v \in V). \tag{A.6}$$

To show this it suffices to consider (u, v) with $u \in V^+$ and $v \in V^-$, since $f'(x, u, v) = +\infty$ unless $x - \chi_u + \chi_v \in [x \wedge y, x \vee y]$. Then (A.4) implies (A.6).

It follows from the above claim and Theorem 4.6 that $x \in \mathrm{argmin}_{\mathbf{Z}}(f'[-\widehat{p}])$. In other words we have $V^+ \in \mathcal{B}$, where

$$\mathcal{B} = \{X \subseteq V \mid (x \wedge y) + \chi_X \in \mathrm{argmin}_{\mathbf{Z}}(f'[-\widehat{p}])\}.$$

Here $\mathcal{B}$ forms the basis family of a matroid, since $\mathrm{argmin}_{\mathbf{Z}}(f'[-\widehat{p}])$ is an integral base set by Theorem 4.10.

Since $V^+ - u_i + v_i \in \mathcal{B}$ $(1 \leq i \leq m)$ and $V^+ - u_i + v_j \notin \mathcal{B}$ $(1 \leq i < j \leq m)$ by (A.4), we can apply the "no-shortcut lemma" (cf. Kung [29] for this name; also see Bixby-Cunningham [1, Lemma 3.7], Frank [13, Lemma 2], Iri–Tomizawa [25, Lemma 2], Krogdahl [28], Lawler [30, Lemma 3.1 of Chap. 8], Tomizawa–Iri [51, Lemma 3]) to the pair (V^+, V^-) in the matroid $(V, \mathcal{B})$ to conclude $V^- \in \mathcal{B}$. This means that $f[-\widehat{p}](y) = f[-\widehat{p}](x)$, i.e.,

$$f(y) - f(x) = \sum_{i=1}^{m} (\widehat{p}(v_i) - \widehat{p}(u_i)) = \widehat{f}(x, y),$$

where the last equality is due to (A.3). $\square$

A.2 Generalized submodular flow problem

We prove Theorem 5.1 by establishing a more general result on the optimality for an extension of the submodular flow problem as follows [37]. See, e.g., [1], [11], [14], [19], [41] for the ordinary submodular flow problem.

Let $G = (V, A)$ be a graph with a vertex set V and an arc set A. We are given an upper capacity function $\overline{c} : A \to \mathbf{Z} \cup \{+\infty\}$, a lower capacity function $\underline{c} : A \to \mathbf{Z} \cup \{-\infty\}$, a cost function $\gamma : A \to \mathbf{Z}$, and an M-convex function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$. For a flow $\varphi : A \to \mathbf{Z}$ we define its boundary $\partial\varphi : V \to \mathbf{Z}$ by

$$\partial\varphi(v) = \sum\{\varphi(a) \mid a \in \delta^+ v\} - \sum\{\varphi(a) \mid a \in \delta^- v\}, \tag{A.7}$$

where $\delta^+ v$ and $\delta^- v$ denote the sets of the out-going and in-coming arcs incident to v, respectively.

We consider the following generalization of the submodular flow problem, in which the cost of a flow φ involves a term $f(\partial\varphi)$ for the boundary in addition to the usual arc cost $\langle \gamma, \varphi \rangle_A = \sum_{a \in A} \gamma(a)\varphi(a)$.

[Generalized submodular flow problem]

Minimize

$$\Gamma(\varphi) = \langle \gamma, \varphi \rangle_A + f(\partial\varphi)$$

subject to

$$\underline{c}(a) \le \varphi(a) \le \overline{c}(a) \qquad (a \in A), \tag{A.8}$$

$$\partial\varphi \in \mathrm{dom}_{\mathbf{Z}} f. \tag{A.9}$$

The optimality criteria refer to an auxiliary network and a "potential" function. The auxiliary network $(G_\varphi, \gamma_\varphi)$ associated with a feasible flow φ is defined as follows. The underlying graph $G_\varphi = (V, A_\varphi)$ has the vertex set V and the arc set A_φ consisting of three disjoint parts: $A_\varphi = A_\varphi^* \cup B_\varphi^* \cup C_\varphi$, where

$$\begin{aligned} A_\varphi^* &= \{a \mid a \in A, \varphi(a) < \overline{c}(a)\}, \\ B_\varphi^* &= \{\overline{a} \mid a \in A, \underline{c}(a) < \varphi(a)\} \qquad (\overline{a}\text{: reorientation of } a), \\ C_\varphi &= \{(u, v) \mid u, v \in V, u \ne v, \partial\varphi - \chi_u + \chi_v \in \mathrm{dom}_{\mathbf{Z}} f\}. \end{aligned}$$

The length function $\gamma_\varphi : A_\varphi \to \mathbf{Z}$ is defined by

$$\gamma_\varphi(a) = \begin{cases} \gamma(a) & (a \in A_\varphi^*) \\ -\gamma(\overline{a}) & (a = (u, v) \in B_\varphi^*, \overline{a} = (v, u) \in A) \\ f(\partial\varphi, u, v) & (a = (u, v) \in C_\varphi) \end{cases} \tag{A.10}$$

where $f(\partial\varphi, u, v) = f(\partial\varphi - \chi_u + \chi_v) - f(\partial\varphi)$ as in (4.24). We call a directed cycle of negative length a negative cycle.

By a “potential” function we mean a function $p : V \to \mathbf{Z}$ (or $p \in \mathbf{Z}^V$), and we define $\gamma_p : A \to \mathbf{Z}$ by

$$\gamma_p(a) = \gamma(a) + p(\partial^+ a) - p(\partial^- a) \qquad (a \in A) \tag{A.11}$$

(with the optimism that the notations γ_φ and γ_p are distinctive enough).

Theorem A.3 *(1) The following three conditions are equivalent for a feasible flow φ:*

(OPT) *φ is optimal.*

(NNC) *There is no negative cycle in the auxiliary network $(G_\varphi, \gamma_\varphi)$.*

(POT) *There exists a potential $p : V \to \mathbf{Z}$ such that*

(i) for each $a \in A$,

$$\gamma_p(a) > 0 \quad \Longrightarrow \quad \varphi(a) = \underline{c}(a), \tag{A.12}$$

$$\gamma_p(a) < 0 \quad \Longrightarrow \quad \varphi(a) = \overline{c}(a), \tag{A.13}$$

(ii) $\partial\varphi$ minimizes $f[-p]$, that is, $f[-p](\partial\varphi) \leq f[-p](x)$ for any $x \in \mathbf{Z}^V$.

(2) Let p be a potential that satisfies (i)–(ii) above for some (optimal) flow φ. A feasible flow φ' is optimal if and only if it satisfies (i)–(ii) (with φ replaced by φ'). □

We prove (OPT) ⇒ (NNC) ⇒ (POT) ⇒ (OPT) and finally the second part (2).

(OPT) ⇒ (NNC): Suppose $(G_\varphi, \gamma_\varphi)$ has a negative cycle. Let Q ($\subseteq A_\varphi$) be the arc set of a negative cycle having the smallest number of arcs, and let $\overline{\varphi} : A \to \mathbf{Z}$ be defined by

$$\overline{\varphi}(a) = \begin{cases} \varphi(a) + 1 & (a \in Q \cap A_\varphi^*) \\ \varphi(\overline{a}) - 1 & (a = (u,v) \in Q \cap B_\varphi^*, \overline{a} = (v,u) \in A) \\ \varphi(a) & (\text{otherwise}) \end{cases} \tag{A.14}$$

Note that $|\partial\varphi(v) - \partial\overline{\varphi}(v)| \leq 1$ $(v \in V)$.

Lemma A.4 *$(\partial\varphi, \partial\overline{\varphi})$ satisfies the unique-min condition.*

(Proof) We make use of an extension of Fujishige’s proof technique [16], [19, Lemma 5.4]. Consider the bipartite graph $G(\partial\varphi, \partial\overline{\varphi}) = (V^+, V^-; \hat{A})$ with $V^+ = \mathrm{supp}^+(\partial\varphi - \partial\overline{\varphi})$, $V^- = \mathrm{supp}^-(\partial\varphi - \partial\overline{\varphi})$ and

$$\hat{A} = \{(u,v) \mid u \in V^+, v \in V^-, x - \chi_u + \chi_v \in \mathrm{dom}_{\mathbf{Z}} f\},$$

and take a minimum-weight perfect matching $M = \{(u_i, v_i) \mid i = 1, \cdots, m\}$ (where $m = ||\partial\varphi - \partial\overline{\varphi}||/2$) in $G(\partial\varphi, \partial\overline{\varphi})$ with respect to the arc weight $f(\partial\varphi, u, v)$ as well as the potential function $\widehat{p}$ in Lemma A.1(1). Then M is a subset of

$$C^*_\varphi = \{(u, v) \mid u \in V^+, v \in V^-, f(\partial\varphi, u, v) + \widehat{p}(u) - \widehat{p}(v) = 0\}.$$

Put $Q' = (Q - C_\varphi) \cup M$, where M is now regarded as a subset of C_φ. Q' is a disjoint union of cycles in G_φ and

$$\gamma_\varphi(Q') = \gamma_\varphi(Q) + [\gamma_\varphi(M) - \gamma_\varphi(Q \cap C_\varphi)] \leq \gamma_\varphi(Q) < 0, \qquad \text{(A.15)}$$

since $\gamma_\varphi(M)$ is equal to the minimum weight of a perfect matching in $G(\partial\varphi, \partial\overline{\varphi})$ and $Q \cap C_\varphi$ can be identified with a perfect matching in $G(\partial\varphi, \partial\overline{\varphi})$. The minimality of Q (with respect to the number of arcs) implies that Q' itself is a negative cycle having the smallest number of arcs.

Suppose, to the contrary, that $(\partial\varphi, \partial\overline{\varphi})$ does not satisfy the unique-min condition. Since $(u_i, v_i) \in C^*_\varphi$ for $i = 1, \cdots, m$, it follows from Lemma A.1(2) that there are distinct indices i_k $(k = 1, \cdots, q; q \geq 2)$ such that $(u_{i_k}, v_{i_{k+1}}) \in C^*_\varphi$ for $k = 1, \cdots, q$, where $i_{q+1} = i_1$. That is,

$$f(\partial\varphi, u_{i_k}, v_{i_{k+1}}) = -\widehat{p}(u_{i_k}) + \widehat{p}(v_{i_{k+1}}) \qquad (k = 1, \cdots, q). \qquad \text{(A.16)}$$

On the other hand we have

$$f(\partial\varphi, u_{i_k}, v_{i_k}) = -\widehat{p}(u_{i_k}) + \widehat{p}(v_{i_k}) \qquad (k = 1, \cdots, q). \qquad \text{(A.17)}$$

It then follows that

$$\sum_{k=1}^{q} f(\partial\varphi, u_{i_k}, v_{i_{k+1}}) = \sum_{k=1}^{q} f(\partial\varphi, u_{i_k}, v_{i_k}) \quad \left(= \sum_{k=1}^{q} [-\widehat{p}(u_{i_k}) + \widehat{p}(v_{i_k})] \right)$$

i.e.,

$$\sum_{k=1}^{q} \gamma_\varphi(u_{i_k}, v_{i_{k+1}}) = \sum_{k=1}^{q} \gamma_\varphi(u_{i_k}, v_{i_k}). \qquad \text{(A.18)}$$

For $k = 1, \cdots, q$, let $P'(v_{i_{k+1}}, u_{i_k})$ denote the path on Q' from $v_{i_{k+1}}$ to u_{i_k}, and let Q'_k be the directed cycle formed by arc $(u_{i_k}, v_{i_{k+1}})$ and path $P'(v_{i_{k+1}}, u_{i_k})$. Obviously,

$$\gamma_\varphi(Q'_k) = \gamma_\varphi(u_{i_k}, v_{i_{k+1}}) + \gamma_\varphi(P'(v_{i_{k+1}}, u_{i_k})) \qquad (k = 1, \cdots, q). \qquad \text{(A.19)}$$

A simple but crucial observation here is that

$$\left(\bigcup_{k=1}^{q} P'(v_{i_{k+1}}, u_{i_k}) \right) \cup \{(u_{i_k}, v_{i_k}) \mid k = 1, \cdots, q\} = q' \cdot Q'$$

for some q' with $1 \le q' < q$, where the union denotes the multiset union, and this expression means that each element of Q' appears q' times on the left hand side. Hence by adding (A.19) over $k = 1, \cdots, q$ we obtain

$$\begin{aligned}
\sum_{k=1}^{q} \gamma_\varphi(Q'_k) &= \sum_{k=1}^{q} \gamma_\varphi(u_{i_k}, v_{i_{k+1}}) + \sum_{k=1}^{q} \gamma_\varphi(P'(v_{i_{k+1}}, u_{i_k})) \\
&= \left[\sum_{k=1}^{q} \gamma_\varphi(u_{i_k}, v_{i_{k+1}}) - \sum_{k=1}^{q} \gamma_\varphi(u_{i_k}, v_{i_k}) \right] + q' \cdot \gamma_\varphi(Q') \\
&= q' \cdot \gamma_\varphi(Q') \ < 0,
\end{aligned}$$

where the last equality is due to (A.18) and the last inequality is due to (A.15). This implies that $\gamma_\varphi(Q'_k) < 0$ for some k, while Q'_k has a smaller number of arcs than Q'. This contradicts the minimality of Q'. Therefore $(\partial\varphi, \partial\overline{\varphi})$ satisfies the unique-min condition. □

The following lemma shows "(OPT) ⇒ (NNC)".

Lemma A.5 *For a negative cycle Q in G_φ having the smallest number of arcs, $\overline{\varphi}$ defined by (A.14) is a feasible flow with* $\Gamma(\overline{\varphi}) \le \Gamma(\varphi) + \gamma_\varphi(Q)$ $(< \Gamma(\varphi))$.

(Proof) By Lemma A.4 and Lemma A.2 we have

$$f(\partial\overline{\varphi}) \;=\; f(\partial\varphi) + \widehat{f}(\partial\varphi, \partial\overline{\varphi}) \le f(\partial\varphi) + \gamma_\varphi(Q \cap C_\varphi).$$

Also we have

$$\langle \gamma, \overline{\varphi} \rangle_A = \langle \gamma, \varphi \rangle_A + \gamma_\varphi(Q \cap (A^*_\varphi \cup B^*_\varphi)).$$

These expressions, when added, yield $\Gamma(\overline{\varphi}) \le \Gamma(\varphi) + \gamma_\varphi(Q)$. □

(NNC) ⇒ (POT): By the well-known fact in graph theory, (NNC) implies the existence of a function $p : V \to \mathbf{Z}$ such that $\gamma_\varphi(a) + p(\partial^+ a) - p(\partial^- a) \ge 0$ $(a \in A_\varphi)$. This condition for $a \in A^*_\varphi \cup B^*_\varphi$ is equivalent to the condition (i) in Theorem A.3. For $a = (u, v) \in C_\varphi$, on the other hand, it means

$$f[-p](\partial\varphi, u, v) = f[-p](\partial\varphi - \chi_u + \chi_v) - f[-p](\partial\varphi) = f(\partial\varphi, u, v) + p(u) - p(v) \ge 0.$$

This implies the condition (ii) by Theorem 4.6. Thus "(NNC) ⇒ (POT)" has been shown.

(POT) ⇒ (OPT): For any $\varphi : A \to \mathbf{Z}$ and $p : V \to \mathbf{Z}$ we have

$$\begin{aligned}
\Gamma(\varphi) &= \langle \gamma, \varphi \rangle_A + f(\partial\varphi) \\
&= \langle \gamma_p, \varphi \rangle_A + (f(\partial\varphi) - \langle p, \partial\varphi \rangle) \\
&= \langle \gamma_p, \varphi \rangle_A + f[-p](\partial\varphi).
\end{aligned}$$

Suppose φ and p satisfy (i)–(ii) of Theorem A.3, and take an arbitrary feasible flow φ'. Since

$$\begin{aligned} &\langle \gamma_p, \varphi' - \varphi \rangle_A \\ &= \sum_{a:\gamma_p(a)>0} \gamma_p(a)(\varphi'(a) - \underline{c}(a)) + \sum_{a:\gamma_p(a)<0} \gamma_p(a)(\varphi'(a) - \overline{c}(a)) \ \geq 0, \end{aligned} \tag{A.20}$$

we have

$$\Gamma(\varphi') = \langle \gamma_p, \varphi' \rangle_A + f[-p](\partial\varphi') \geq \langle \gamma_p, \varphi \rangle_A + f[-p](\partial\varphi) = \Gamma(\varphi).$$

This shows that φ is optimal, establishing "(POT) $\Rightarrow$ (OPT)".

For the second half of Theorem A.3 we note in the above inequality that $\Gamma(\varphi') = \Gamma(\varphi)$ if and only if $\langle \gamma_p, \varphi' \rangle_A = \langle \gamma_p, \varphi \rangle_A$ and $f[-p](\partial\varphi') = f[-p](\partial\varphi)$. From (A.20) we note further that $\langle \gamma_p, \varphi' \rangle_A = \langle \gamma_p, \varphi \rangle_A$ is equivalent to the condition (i) for φ'. We have completed the proof of Theorem A.3.

Finally, Theorem 5.1 can be formulated as a special case of Theorem A.3, as follows. Given $f_1, f_2 : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$, let V_1 and V_2 be disjoint copies of V, and consider the generalized submodular flow problem on the graph $G = (V_1 \cup V_2, A)$ with $A = \{(v_1, v_2) \mid v \in V\}$, where $v_1 \in V_1$ and $v_2 \in V_2$ denote the copy of $v \in V$. We choose $\overline{c}(a) = +\infty$, $\underline{c}(a) = -\infty$, $\gamma(a) = 0$ for all $a \in A$, and define $f : \mathbf{Z}^{V_1} \times \mathbf{Z}^{V_2} \to \mathbf{Z} \cup \{+\infty\}$ by

$$f(x_1, x_2) = f_1(x_1) + f_2(-x_2) \qquad (x_1 \in \mathbf{Z}^{V_1}, x_2 \in \mathbf{Z}^{V_2}).$$

The optimality criterion (POT) in Theorem A.3 gives Theorem 5.1.

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