Discrete Cosine Transforms on Quantum Computers

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Abstract

A classical computer does not allow to calculate a discrete cosine transform on N points in less than linear time. This trivial lower bound is no longer valid for a computer that takes advantage of quantum mechanical superposition, entanglement, and interference principles. In fact, we show that it is possible to realize the discrete cosine transforms and the discrete sine transforms of size $N \times N$ and types I,II,III, and IV with as little as $O(\log^2 N)$ operations on a quantum computer, whereas the known fast algorithms on a classical computer need $O(N \log N)$ operations.

1 Introduction

Feynman proposed in 1982 a computational model that was based on the principles of quantum physics instead of classical physics. The model has been considered a mere curiosity until Peter Shor showed in 1994 that it is possible to factor integers in polynomial time on a quantum computer [1]. Thus, a moderate sized quantum computer is for instance able to break the RSA public key cryptosystem. Quantum computing is an exciting area of emerging signal processing applications. In fact, signal processing methods play a key role in Shor's integer factoring algorithm and in many other quantum algorithms.

A quantum computer is based on the concept of a quantum bit, just as a classical computer is based on the notion of a bit. A single quantum bit represents the state of a two-level quantum system such as a polarized photon or a spin-1/2 system. Unlike a classical computer, adding another quantum bit to the memory of a quantum computer will not increase the dimensionality of the state space by one but will *double* it, allowing for linear combinations of 2^n different base states in the case of n quantum bits.

A program on a quantum computer is composed of a sequence of elementary 'gates', which perform simple unitary transforms (as explained in Section 3). In fact, many algorithms in quantum computing rely on the fast Fourier transforms, the Walsh-Hadamard transforms, or other unitary transforms well-known in signal processing.

The purpose of this paper is to derive (extremely) fast quantum algorithms for the discrete cosine and sine transforms. These algorithms can be implemented on a number of quantum computing technologies based on Raman-coupled low-energy states of trapped ions [2, 3], nuclear spins in silicon [4], electron spins in quantum dots [5], atomic cavity quantum electrodynamics [6], or linear optics [7]. Coherent control of up to four quantum bits has been demonstrated [8], and much progress is expected in the near future.

2 Definitions

Recall the definitions [9] of the *discrete cosine transforms*:

$$C_N^{\mathrm{I}} := \left(\frac{2}{N}\right)^{1/2} \left[k_i \cos\frac{ij\pi}{N}\right]_{i,j=0..N}$$

$$C_N^{\mathrm{II}} := \left(\frac{2}{N}\right)^{1/2} \left[k_i \cos\frac{i(j+1/2)\pi}{N}\right]_{i,j=0..N-1}$$

$$C_N^{\mathrm{III}} := \left(\frac{2}{N}\right)^{1/2} \left[k_i \cos\frac{(i+1/2)j\pi}{N}\right]_{i,j=0..N-1}$$

$$C_N^{\mathrm{IV}} := \left(\frac{2}{N}\right)^{1/2} \left[k_i \cos\frac{(i+1/2)(j+1/2)\pi}{N}\right]_{i,j=0..N-1}$$

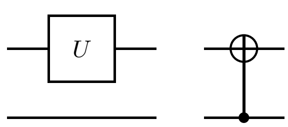
where $k_i := 1$ for $i = 1, \dots, N-1$ and $k_0 := 1/\sqrt{2}$. The numbers k_i ensure that the transforms are orthogonal. The discrete sine transforms S_N^{I}, S_N^{II}, S_N^{III}, and S_N^{IV} are defined accordingly, see [9] for details. Notice that C_N^{III} (resp. S_N^{III}) is the transpose of C_N^{II} (resp. S_N^{II}), hence it suffices to derive circuits for the type II transforms. In the following, we content ourselves to $N = 2^n$, which is justified by the machine model introduced below.

3 Quantum Gates

A quantum computer consists of a system of n two-level quantum systems each of which represents a quantum bit (qubits). The computational state space is given by $\mathbf{C}^{2^n}$. Denote an orthonormal basis of $\mathbf{C}^{2^n}$ by $|x\rangle$ where x is an n-bit integer. The state of the quantum computer can be manipulated by quantum gates. Two types of operations are considered as elementary: the single qubit operations and the controlled NOT operations [10].

The single qubit operations are given by local unitary operations of the form $\mathbf{1}_{2^{n-t}} \otimes U \otimes \mathbf{1}_{2^{t-1}}$, with $U \in \mathcal{U}(2)$. The controlled NOT gate operates on two qubits. It negates the target qubit if and only if the control qubit is 1. All operations on a quantum computer can be built up from these elementary operations, i.e., they form a universal set of gates. Therefore a basic (and nontrivial) task is to find efficient factorizations for explicitly given transformations into elementary gates

There is a graphical notation for quantum gates that has been introduced by Feynman. Each line denotes a qubit with the most significant bit on top. The circuits are read from left to right like musical scores. The figure on the left shows a single qubit operation $U \otimes \mathbf{1}_2$. The figure on the right shows a controlled NOT operation, that is, the unitary transform $|00\rangle \mapsto |00\rangle$, $|01\rangle \mapsto |11\rangle$, $|10\rangle \mapsto |10\rangle$, $|11\rangle \mapsto |01\rangle$.



We refer the reader to [10] for more information about quantum gates and the graphical notation.

4 DCT and DST of Type I

We derive the circuits for the discrete sine and cosine transforms of type I all at once. Indeed, the $\mathrm{DST_I}$ and $\mathrm{DCT_I}$ can be recovered from the DFT by a base change [11]

$$T_N^\dagger \cdot F_{2N} \cdot T_N = C_N^{\mathrm{I}} \oplus i S_N^{\mathrm{I}}, \tag{1}$$

where

$$T_N = \begin{pmatrix} 1 & & & & & & \\ & \frac{1}{\sqrt{2}} & & & & \frac{i}{\sqrt{2}} & \\ & & \ddots & & & & \ddots \\ & & & \frac{1}{\sqrt{2}} & & & & \frac{i}{\sqrt{2}} \\ & & & & 1 & & & \\ & & & \frac{1}{\sqrt{2}} & & & & -\frac{i}{\sqrt{2}} \\ & & \iddots & & & & \iddots \\ & \frac{1}{\sqrt{2}} & & & & -\frac{i}{\sqrt{2}} & \end{pmatrix}$$

and $F_N = \frac{1}{\sqrt{N}}[\exp(2\pi i\, kl/N)]_{k,l=0..N-1}$ with $i^2 = -1$ denotes the DFT of length N. Since efficient quantum circuits for the DFT are known [1], it remains to find an efficient implementation of the base change matrix T_N.

Denote the basis vectors of $\mathbf{C}^{2^{n+1}}$ by $|bx\rangle$, where b is a single bit and x is an n-bit number. The two's complement of an n-bit unsigned integer x is denoted by x', that is, $x' = 2^n - x$. The action of T_N can be described by

$$\begin{aligned} T_N\,|0\mathbf{0}\rangle &= |0\mathbf{0}\rangle\,, \quad & T_N\,|0x\rangle &= \tfrac{1}{\sqrt{2}}\,|0x\rangle + \tfrac{1}{\sqrt{2}}\,|1x'\rangle\,, \\ T_N\,|1\mathbf{0}\rangle &= |1\mathbf{0}\rangle\,, \quad & T_N\,|1x\rangle &= \tfrac{i}{\sqrt{2}}\,|0x\rangle - \tfrac{i}{\sqrt{2}}\,|1x'\rangle\,, \end{aligned}$$

for all integers x in the range $1 \leq x < 2^n$, where $i^2 = -1$. Ignoring the two's complement in T_N, we can define an operator D by

$$\begin{aligned} D\,|0\mathbf{0}\rangle &= |0\mathbf{0}\rangle\,, \quad & D\,|0x\rangle &= \tfrac{1}{\sqrt{2}}\,|0x\rangle + \tfrac{1}{\sqrt{2}}\,|1x\rangle\,, \\ D\,|1\mathbf{0}\rangle &= |1\mathbf{0}\rangle\,, \quad & D\,|1x\rangle &= \tfrac{i}{\sqrt{2}}\,|0x\rangle - \tfrac{i}{\sqrt{2}}\,|1x\rangle\,, \end{aligned}$$

for all integers x in the range $1 \leq x < 2^n$. This operator is essentially block diagonal and easy to implement by a single qubit operation, followed by a correction. Indeed, define the matrix B by $B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$, then Figure 1 gives an implementation of the operator D.

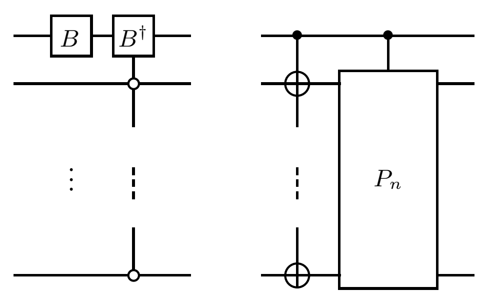


Figure 1. Circuits for the matrix D and the permutation π.

Define π to be the permutation given by a two's complement conditioned on the most significant bit $\pi\,|0x\rangle = |0x\rangle$ and $\pi\,|1x\rangle = |1x'\rangle$ for all n-bit integers x. It is clear that $T_N = \pi D$. The circuit for the permutation π is shown in Figure 1. Here P_n denotes the map $|x\rangle \mapsto |x{+}1 \bmod 2^n\rangle$ on n qubits, see [12] for an implementation.

Theorem 1 *The discrete cosine transform C_N^{I} and the discrete sine transform S_N^{I} can be realized with $O(\log^2 N)$ elementary quantum gates; the quantum circuit for these transforms is shown in Figure 2.*

Proof. Let $N = 2^n$. We note that $O(\log^2 N)$ quantum gates are sufficient to realize the DFT of length $2N$, see [1]. The

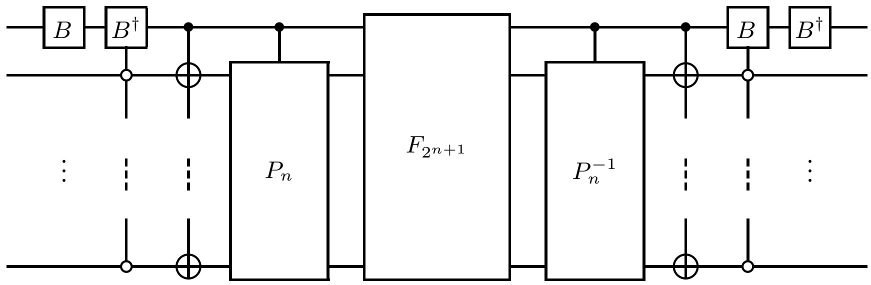


Figure 2. Complete quantum circuit for the $\mathrm{DCT_I}$

permutation π can be implemented with at most $O(\log^2 N)$ elementary gates. At most $O(\log N)$ quantum gates are needed to realize the operator D. This shows that the $\mathrm{DCT_I}$ and the $\mathrm{DST_I}$ can be realized with $O(\log^2 N)$ elementary quantum gates. The preceding discussion shows that Figure 2 realizes the $\mathrm{DCT_I}$ and $\mathrm{DST_I}$. $\square$

5 DCT and DST of Type IV

The trigonometric transforms of type IV are derived from the DFT by

$$e^{\pi i/4N} R_N^t \cdot F_{2N} \cdot R_N = C_N^{\mathrm{IV}} \oplus (-i) S_N^{\mathrm{IV}}. \qquad (2)$$

Here R_N denotes the matrix

$$R_N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & & -i & & & \\ & \omega & & & & -i\omega & & \\ & & \ddots & & & & \ddots & \\ & & & \omega^{N-1} & & & & -i\omega^{N-1} \\ & & & \overline{\omega}^N & & & & 1 \\ & & \iddots & & & & \iddots & \\ & \overline{\omega}^2 & & & & i\overline{\omega}^2 & & \\ \overline{\omega} & & & & i\overline{\omega} & & & \end{pmatrix}$$

with $\omega = \exp(2\pi i/4N)$. Equation (2) is a consequence of Theorem 3.19 in [11] obtained by complex conjugation.

Theorem 2 *The discrete cosine transform C_N^{IV} and the discrete sine transform S_N^{IV} can be realized with $O(\log^2 N)$ elementary quantum gates; the quantum circuit for these transforms is shown in Figure 3.*

Proof. It remains to show that there exists an efficient quantum circuit for the matrix R_N in equation (2). A factorization of R_N can be obtained as follows. Denote by $\overline{x}$ the one's complement of an n-bit integer x. We define a permutation matrix π_1 by $\pi_1 |0x\rangle = |0x\rangle$ and $\pi_1 |1x\rangle = |1\overline{x}\rangle$ for all integers x in the range of $0 \le x < 2^n$. Denote by D_1 the diagonal matrix

$$D_1 = \mathrm{diag}(1, \omega, \ldots, \omega^{N-1}, \overline{\omega}^N, \ldots, \overline{\omega}^2, \overline{\omega}).$$

Then R_N can be factored as $R_N = \pi_1 \cdot D_1 \cdot (\overline{B} \otimes \mathbf{1}_N)$.

Note that $\overline{B} \otimes \mathbf{1}_N$ is a single qubit operation, and π_1 can be realized by controlled not operations. The implementation of the diagonal matrix D_1 is more interesting. Note that

$$\Delta_1 = \mathrm{diag}(1, \omega, \ldots, \omega^{N-1}) = L_n \otimes \cdots \otimes L_2 \otimes L_1$$
$$\Delta_2 = \mathrm{diag}(\overline{\omega}^{N-1}, \ldots, \overline{\omega}, 1) = K_n \otimes \cdots \otimes K_2 \otimes K_1$$

where $L_j = \mathrm{diag}(1, \omega^{2^{j-1}})$ and $K_j = \mathrm{diag}(\overline{\omega}^{\,2^{j-1}}, 1)$. Therefore, it is possible to write D_1 in the form $D_1 = (C \otimes \mathbf{1}_N) \cdot (\Delta_1 \oplus \Delta_2)$ with $C = \mathrm{diag}(1, \overline{\omega})$.

The complete quantum circuit for the $\mathrm{DCT_{IV}}$ is shown in Figure 3. Note that the last three single qubit gates C, $B^\dagger$, and $M = \mathrm{diag}(e^{\pi i/4N}, e^{\pi i/4N})$ can be combined into a single gate $MB^\dagger C$. $\square$

6 DCT and DST of Type II

The implementation of the trigonometric transforms of type II follows a similar pattern. Both transforms can be recovered from the DFT of length 2N after multiplication with certain sparse matrices, cf. Theorem 3.13 in [11]:

$$U_N^\dagger \cdot F_{2N} \cdot V_N = C_N^{\mathrm{II}} \oplus (-i) S_N^{\mathrm{II}}, \qquad (3)$$

where

$$V_N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & 1 \\ & & 1 & & & -1 \\ & \iddots & & & \iddots & \\ 1 & & & -1 & & \end{pmatrix}$$

and

$$U_N = \begin{pmatrix} 1 & & & & 0 & & & \\ & \frac{\overline{\omega}}{\sqrt{2}} & & & -\frac{i\overline{\omega}}{\sqrt{2}} & & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & \frac{\overline{\omega}^{N-1}}{\sqrt{2}} & & & -\frac{i\overline{\omega}^{N-1}}{\sqrt{2}} & 0 \\ & & & 0 & & & & -1 \\ & & & \frac{\omega^{N-1}}{\sqrt{2}} & & & \frac{i\omega^{N-1}}{\sqrt{2}} & \\ & & \iddots & & & \iddots & & \\ 0 & \frac{\omega}{\sqrt{2}} & & & \frac{i\omega}{\sqrt{2}} & & & \end{pmatrix},$$

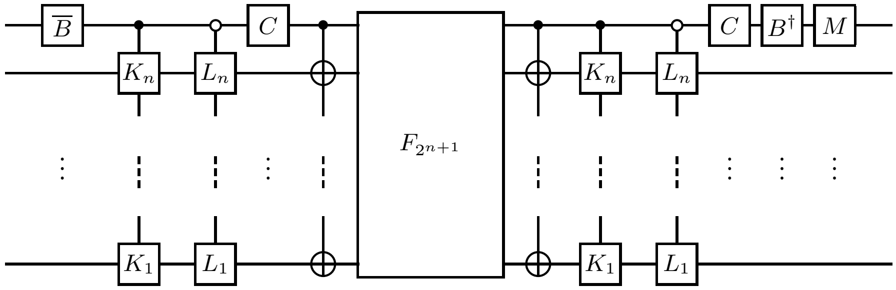


Figure 3. Complete quantum circuit for $\mathrm{DCT_{IV}}$

and $\omega = \exp(2\pi i/4N)$ with $i^2 = -1$.

Theorem 3 *The discrete cosine transform C_N^{II} and the discrete sine transform S_N^{II} can be realized with $O(\log^2 N)$ elementary quantum gates; the quantum circuit for these transforms is shown in Figure 4.*

Proof. We need to derive efficient quantum circuits for the matrices V_N and U_N in equation (3). The matrix V_N has a fairly simple decomposition in terms of quantum circuits.

Lemma 4 $V_N = \pi_1(H \otimes \mathbf{1}_N)$.

Proof. It is clear that the Hadamard transform on the most significant bit $H \otimes \mathbf{1}_N$ is – up to a permutation of rows – equivalent to V_N. The appropriate permutation of rows has been introduced in the previous section, namely $\pi_1 |0x\rangle = |1x\rangle$ and $\pi_1 |1x\rangle = |1\overline{x}\rangle$ for all $0 \leq x < 2^n$. We can conclude that $V_N = \pi_1(H \otimes \mathbf{1}_N)$ as desired. □

The decomposition of U_N is more elaborate. Notice that

$$U_N |0\mathbf{0}\rangle = |0\mathbf{0}\rangle \qquad U_N |1\mathbf{1}\rangle = (-1)\, |1\mathbf{0}\rangle$$
$$U_N |0x\rangle = \frac{\overline{\omega}^{\,x}}{\sqrt{2}} |0x\rangle + \frac{\omega^x}{\sqrt{2}} |1x'\rangle$$
$$U_N |1y\rangle = -\frac{i\overline{\omega}^{\,y+1}}{\sqrt{2}} |0\,(y+1 \bmod 2^n)\rangle + \frac{i\omega^{y+1}}{\sqrt{2}} |1\overline{y}\rangle$$

for all integers x in the range $1 \leq x < 2^n$ and all integers y in $0 \leq y < 2^n - 1$. Here $\mathbf{0}$ and $\mathbf{1}$ denote the n-bit integers 0 and $2^n - 1$ respectively.

Define D_0 by $D_0 |1\mathbf{0}\rangle = i\,|1\mathbf{0}\rangle$ and $D_0 |x\rangle = |x\rangle$ otherwise. We define a permutation π_2 by $\pi_2 |0x\rangle = |0x\rangle$ and $\pi_2 |1x\rangle = |1(x + 1 \bmod 2^n)\rangle$ for all integers x in $0 \leq x < 2^n$.

Lemma 5 $U_N = D_1^\dagger\, \overline{T}_N\, D_0\, \pi_2$.

Proof. Since $D_1^\dagger |0x\rangle = \overline{\omega}^x |0x\rangle$ and $D_1^\dagger |1x\rangle = \omega^{x'} |1x\rangle$, we obtain

$$D_1^\dagger \overline{T}_N |0x\rangle = \quad \frac{\overline{\omega}^x}{\sqrt{2}} |0x\rangle + \frac{\omega^x}{\sqrt{2}} |1x'\rangle$$
$$D_1^\dagger \overline{T}_N |1x\rangle = -\frac{i\overline{\omega}^x}{\sqrt{2}} |0x\rangle + \frac{i\omega^x}{\sqrt{2}} |1x'\rangle$$

We have $D_0\pi_2 |0x\rangle = |0x\rangle$ and moreover $D_0\pi_2 |1x\rangle = |1(x + 1 \bmod 2^n)\rangle$ for all integers x in $0 \leq x < 2^n - 1$, and $D_0\pi_2 |1\mathbf{1}\rangle = i\,|1\mathbf{0}\rangle$. We note that $(x+1 \bmod 2^n)' = \overline{x}$, whence combining $D_1\overline{T}_N$ with $D_0\pi_2$ shows the result. □

Recall that $T_N = \pi D$. It follows that

$$U_N^\dagger = \pi_2^{-1}(\overline{D}_0 D^t)\pi^{-1} D_1.$$

The implementation of D_1 has been described in the section on the $\mathrm{DCT_{IV}}$, and the implementation of π (and hence π^{-1}) is contained in the section on the $\mathrm{DCT_I}$. The implementation of π_2^{-1} is also straightforward. It remains to find an implementation of $\overline{D}_0 D^t$. We observe that

$$\overline{D}_0 D^t |0\mathbf{0}\rangle = |0\mathbf{0}\rangle\,, \qquad \overline{D}_0 D^t |0x\rangle = \tfrac{1}{\sqrt{2}} |0x\rangle + \tfrac{i}{\sqrt{2}} |1x\rangle\,,$$
$$\overline{D}_0 D^t |1\mathbf{0}\rangle = -i\,|1\mathbf{0}\rangle\,, \; \overline{D}_0 D^t |1x\rangle = \tfrac{1}{\sqrt{2}} |0x\rangle - \tfrac{i}{\sqrt{2}} |1x\rangle\,.$$

This can be accomplished by a single qubit operation followed by a multiply conditioned gate, where the single qubit operation is given by $B^t \otimes \mathbf{1}_N$ and the conditional gate acts via

$$J = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

The full circuit is shown in Figure 4. The statement about the complexity is clear. □

7 Conclusions

Signal processing methods have proved to be useful in virtually all known quantum algorithms. A basic problem in the design of quantum algorithms is to choose a well-adapted basis to recoup relevant information about the state of the system. The well-known decorrelation properties of DCTs may prove to be useful within this framework. We have shown that the DCT and DST of types I, II, III, and IV can be realized with a polylogarithmic number of elementary operations on a quantum computer. Compared to the classical realization of the DCT, this is a tremendous speed-up, making the DCT attractive in the design of other quantum algorithms.

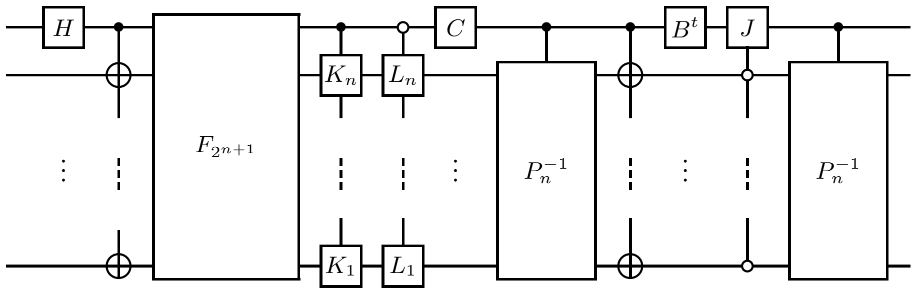


Figure 4. Complete quantum circuit for $\mathrm{DCT_{II}}$

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