Discrete Data Assimilation in the Lorenz and 2D Navier–Stokes Equations

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Consider a continuous dynamical system for which partial information about its current state is observed at a sequence of discrete times. Discrete data assimilation inserts these observational measurements of the reference dynamical system into an approximate solution by means of an impulsive forcing. In this way the approximating solution is coupled to the reference solution at a discrete sequence of points in time. This paper studies discrete data assimilation for the Lorenz equations and the incompressible two-dimensional Navier–Stokes equations. In both cases we obtain bounds on the time interval h between subsequent observations which guarantee the convergence of the approximating solution obtained by discrete data assimilation to the reference solution.

1. Introduction

In [11] and [12] Olson and Titi studied the number of determining modes for continuous data assimilation for the incompressible two-dimensional Navier–Stokes equations. As in those papers, the motivating problem for our work is the initialization of weather forecasting models using near continuous in time measurement data obtained, for example, from satellite imaging. In this work, rather than making the idealization that the measurement data is continuous in time, we focus on the case where the measurement data is taken at a sequence of discrete times t_n.

If $t_n = hn$ and h is small then discrete data assimilation can be viewed as near continuous. One expects that if the approximating solution obtained by continuous data assimilation converges to the reference solution then the approximating solution obtained by discrete data assimilation for small h will also converge to the reference solution. Note, however, that near continuous observational data is mathematically quite different from continuous data. If the observations are known continuously in time on some interval then mathematically the n-th time derivatives may be calculated on that same interval for all values of n. It is possible that this derivative information could lead to a reconstruction of the reference solution in cases where near continuous measurement information might

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not. For example, Wingard [18] shows for the Lorenz equations that knowing X and its time derivatives at a single point in time can be used to recover both Y and Z.

In this paper we present a technique to prove a discrete in time determining mode result in the specific context of creating an approximating solution that asymptotically converges to a reference solution. In addition to providing a more realistic framework in which to study near continuous data assimilation, our work can be seen as a discrete in time extension of the theory of determining modes developed by Foias and Prodi [5] and further refined by Jones and Titi in [7] and [8]. Unless otherwise noted, we shall assume the dynamics governing the evolution of the reference solution admit a global attractor and that the reference solution lies on that global attractor. This assumption is made for simplicity, as our analysis actually depends only on the existence of an absorbing ball which contains the reference solution forward in time.

We begin our discussion with the easier case of the Lorenz equations to provide insight and illustrate the methods we will use for the incompressible two-dimensional Navier–Stokes equations. General studies for the synchronization of discrete in time coupled systems for the Lorenz equations were produced by Yang, Yang and Yang [20] and Wu, Lu, Wang and Liu [19]. Note, however, that the matrix corresponding to the observational measurements studied here does not have a suitable spectral radius to apply the theorems of their work.

The method of discrete data assimilation can be described mathematically as follows. Let U be a solution lying on the global attractor of a dissipative continuous dynamical system with initial condition U_0 at time t_0. Let S be the continuous semigroup defined by $U(t) = S(t, t_0, U_0)$. Represent the observational measurements of the reference solution U at time t_n by $PU(t_n)$, where P is a finite-rank orthogonal projection and t_n is an increasing sequence in time. Discrete data assimilation inserts the observational measurements into an approximate solution u as the approximate solution is integrated in time. In particular, let $u_0 = \eta + PU(t_0)$ and $u_{n+1} = QS(t_{n+1}, t_n, u_n) + PU(t_{n+1})$ for $n = 0, 1, \ldots,$ where $P\eta = 0$ and $Q = I - P$. Here η corresponds to an initial guess for the part of the reference solution $QU(t_0)$ that can not be measured. The approximating solution u obtained by discrete data assimilation is defined to be the piecewise continuous in time function

$$u(t) = S(t, t_n, u_n) \quad \text{for} \quad t \in [t_n, t_{n+1}). \tag{1.1}$$

Our goal is to find conditions on P, t_n and η which guarantee that the approximating solution u converges to the reference solution U as $t \to \infty$.

For the Lorenz system the reference solution is a three dimensional vector consisting of the components X, Y and Z whose evolution is governed by the coupled system of three ordinary differential equations

$$\begin{cases} \dot{X} = -\sigma X + \sigma Y \\ \dot{Y} = -\sigma X - Y - XZ \\ \dot{Z} = -bZ + XY - b(r + \sigma) \end{cases} \tag{1.2}$$

where $\sigma = 10$, $b = 8/3$ and $r = 28$. We shall assume that the reference solution lies on the global attractor.

We take the observational measurements of the reference solution to be the values of the variable X at the times t_n. These observations of X are used to create an approximating solution whose components are x, y and z. Note that $x(t_n) = X(t_n)$ where $y(t)$ and $z(t)$ are continuous at $t = t_n$ and x, y and z satisfy

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = -\sigma x - y - xz \\ \dot{z} = -bz + xy - b(r + \sigma) \end{cases} \tag{1.3}$$

on each interval $[t_n, t_{n-1})$ for $n = 0, 1, 2, \ldots$.

For simplicity we take $t_n = hn$ for some fixed $h > 0$. Numerical experiments for the Lorenz system done by Hayden [6] indicate that the approximating solution converges to the reference solution as $t \to \infty$ for values of h as large as 0.175. Figure 1.1 shows the convergence of the approximating solution to the reference solution when $h = 0.1$.

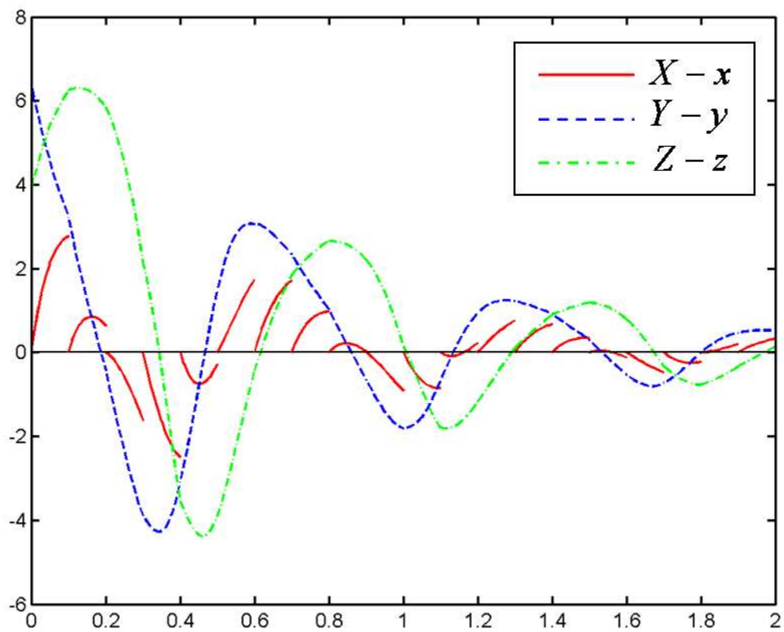

Figure 1.1. Convergence of the approximating solution to the reference solution for the Lorenz system when $h = 0.1$.

Note again that the variables y and z in the approximating solution are continuous in time, in particular at $t = t_n$, whereas x is discontinuous. The discontinuities in x are the result of the assimilation of the observations of X at each time t_n.

Our main result for the Lorenz equations is an analytic proof of Theorem 2.5 which shows there exists $t^* > 0$ depending on σ, β and r such that for any $h \in (0, t^*]$ the approximating solution obtained by discrete data assimilation of measurements of the X variable at times $t_n = hn$, as described in (1.3) above, converges to the reference solution as $t \to \infty$.

The second part of this paper focuses on the incompressible two-dimensional Navier–Stokes equations with L-periodic boundary conditions

$$\begin{cases} \dfrac{\partial U}{\partial t} - \nu\Delta U + (U\cdot\nabla)U + \nabla P = f \\ \hfill \nabla\cdot U = 0 \end{cases} \tag{1.4}$$

where ν is the kinematic viscosity and f is a time independent body forcing. In this case U can be expressed in terms of the Fourier series

$$U = \sum_{k\in\mathcal{J}} U_k e^{ik\cdot x} \quad \text{where} \quad \mathcal{J} = \left\{ \frac{2\pi}{L}(n_1,n_2) : n_i \in \mathbf{Z} \text{ and } (n_1,n_2)\neq(0,0) \right\}$$

where $U_k\cdot k = 0$ for all $k\in\mathcal{J}$ and $U_k = U^*_{-k}$. Again, we assume that the reference solution lies on the global attractor.

Let P_λ be the orthogonal projection defined by

$$P_\lambda U = \sum_{k^2\le\lambda} U_k e^{ik\cdot x}. \tag{1.5}$$

We take the observational measurements of the reference solution to be the values of $P_\lambda U$ at the times t_n. Note that $\lambda^{-1/2}$ represents the smallest length scale of the fluid which can be observed—the resolution of the presumed measuring equipment or distance between measuring stations.

In the context of continuous data assimilation and determining modes, the rank of the smallest projection P_λ such that the approximating solution converges to the reference solution is called the number of determining modes. For discrete and near continuous data assimilation, the parameter λ also depends on the interval of time between the observational measurements. Our main results for the incompressible two-dimensional Navier–Stokes equations are Corollary 3.10 and Corollary 3.11.

Corollary 3.10 shows for any time interval $h>0$ there exists λ large enough so that the approximating solution obtained by discrete data assimilation of the measurements $P_\lambda U(t_n)$ where $t_n = hn$ will converge to the reference solution. This means that increasing the resolution of the measurements can compensate for a large time interval between subsequent observations. Corollary 3.11 shows there is a dimensionless constant C such that if

$$\lambda > \frac{C}{\lambda_1^{5/3}}\left(\frac{\|f\|_{L^2}}{\nu^2}\right)^{8/3}$$

then there exists $h>0$ depending only on $|f|$, ν, Ω and λ small enough such that the approximating solution obtained by discrete data assimilation with initial guess $\eta = 0$ converges to the reference solution. This means that decreasing the time interval between subsequent observations can compensate for a low resolution provided the resolution meets a minimum standard.

2. Lorenz Equations

Following Foias, Jolly, Kukavica and Titi [4] we write the Lorenz system (1.2) as

$$\frac{dU}{dt} + AU + B(U,U) = f \tag{2.1}$$

where

$$U = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad A = \begin{bmatrix} \sigma & -\sigma & 0 \\ \sigma & 1 & 0 \\ 0 & 0 & b \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ 0 \\ -b(r+\sigma) \end{bmatrix},$$

$$B\left(\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \begin{bmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ (X\tilde{Z} + Z\tilde{X})/2 \\ -(X\tilde{Y} + Y\tilde{X})/2 \end{bmatrix}$$

and again $\sigma = 10$, $b = 8/3$ and $r = 28$. One reason for writing the Lorenz equations in this way is to make the similarities and differences in the proofs from this section and following section on the Navier–Stokes equations more transparent. Despite the notational similarities there should be no trouble distinguishing the results on the Lorenz equation that apply only to the Lorenz equations from the results on the Navier–Stokes equations which apply only to the Navier–Stokes equations. We start with some definitions and facts that are easy to verify.

Definition 2.1. $|U| = \sqrt{(U,U)} = \sqrt{X^2 + Y^2 + Z^2}$.

The facts below may be deduced from the preceding definitions and are listed here for reference. First, we state an estimate involving the linear term AU that plays the same role in our treatment of the Lorenz system that Theorem 3.2 plays in our later treatment of the incompressible two-dimensional Navier–Stokes equations:

$$(AU, U) = \sigma X^2 + Y^2 + bZ^2 \geq |U|^2. \tag{2.2}$$

Next we state some algebraic identities analogous to the orthogonality relations (3.3) and (3.4) for the nonlinear term in the two-dimensional Navier–Stokes equations:

$$(B(U,U), U) = 0 \qquad \text{and} \qquad B(U, \tilde{U}) = B(\tilde{U}, U). \tag{2.3}$$

We will also use an estimate on B which is similar to Theorem 3.5:

$$|B(U, \tilde{U})| \leq 2^{-1}|U||\tilde{U}|. \tag{2.4}$$

The following bound on the global attractor was shown in Temam [16] on page 33, see also Foias, Constantin and Temam [3].

Theorem 2.2. Let U be a trajectory that lies on the global attractor of (2.1). Then $|U(t)|^2 \leq K$ for all $t \in \mathbf{R}$ where

$$K = \frac{b^2(r+\sigma)^2}{4(b-1)}. \tag{2.5}$$

Before proceeding, we state Young's inequality which will be used throughout the remainder of this work.

Theorem 2.3. Let $1/p + 1/q = 1$ then $|xy| \le |x|^p/p + |y|^q/q$.

We are now ready to begin our study of discrete data assimilation for the Lorenz equations. Our measurements will consist of the X variable at the times t_n. Therefore, we define the orthogonal projection P as

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad Q = I - P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let U be a solution of the Lorenz equations lying on the global attractor and u the approximating solution given by (1.1) where S is the semigroup generated by (2.1). Since U and u both satisfy (2.1) on the time interval $[t_n, t_{n+1})$ setting $\delta = U - u$ we obtain

$$\frac{d\delta}{dt} + \nu A\delta + B(U,U) - B(u,u) = 0 \quad \text{for} \quad t \in [t_n, t_{n+1})$$

or after some algebra that

$$\frac{d\delta}{dt} + A\delta + B(U,\delta) + B(\delta,U) - B(\delta,\delta) = 0 \quad \text{for} \quad t \in [t_n, t_{n+1}). \tag{2.6}$$

Lemma 2.4. There exists $\beta > 0$ given by (2.8) below, depending on σ, b and r, such that $|\delta(t)|^2 \le |\delta(t_n)|^2 e^{\beta(t-t_n)}$ for $t \in [t_n, t_{n+1})$.

Proof: Take the inner product of δ with (2.6) and apply (2.2) and (2.3) to obtain

$$\frac{1}{2}\frac{d|\delta|^2}{dt} + |\delta|^2 + 2(B(U,\delta),\delta) \le 0. \tag{2.7}$$

Estimating using (2.4) and Theorem 2.2 gives

$$2|(B(U,\delta),\delta)| \le |U||\delta|^2 \le K^{1/2}|\delta|^2.$$

Therefore

$$\frac{d|\delta|^2}{dt} \le \beta|\delta|^2$$

where

$$\beta = 2(K^{1/2} - 1) \tag{2.8}$$

and K is defined in (2.5). Integrating from t_n to t yields

$$|\delta(t)|^2 \le |\delta(t_n)|^2 e^{\beta(t-t_n)},$$

which finishes the proof. ☐

Theorem 2.5. Let U be a solution of the the Lorenz equations (2.1) lying on the global attractor. Then, there exists $t^* > 0$ depending only on σ, b and r such that for any $h \in (0, t^*]$ the approximating solution u given by (1.1), see also (1.3), with $t_n = hn$ converges to U as $t \to \infty$.

Proof: Take the inner product of $P\delta$ with (2.6) to obtain

$$\frac{1}{2}\frac{d|P\delta|^2}{dt} + (A\delta, P\delta) = 0. \tag{2.9}$$

Let

$$w = (PA - AP)\delta = \begin{bmatrix} 0 & -\sigma & 0 \\ -\sigma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta = -\sigma \begin{bmatrix} \delta_Y \\ \delta_X \\ 0 \end{bmatrix} \quad \text{where} \quad \delta = \begin{bmatrix} \delta_X \\ \delta_Y \\ \delta_Z \end{bmatrix}.$$

Then $|P\delta|^2 = \delta_X^2$ and

$$(A\delta, P\delta) = (PA\delta, P\delta) = (AP\delta, P\delta) + (w, P\delta) = \sigma\delta_X^2 - \sigma\delta_X\delta_Y.$$

Therefore (2.9) becomes

$$\frac{1}{2}\frac{d\delta_X^2}{dt} + \sigma\delta_X^2 = \sigma\delta_X\delta_Y \le \frac{\sigma}{2}\delta_X^2 + \frac{\sigma}{2}\delta_Y^2 \le \frac{\sigma}{2}\delta_X^2 + \frac{\sigma}{2}|\delta|^2$$

where we have applied Theorem 2.3 with $p = q = 2$. Using Lemma 2.4 we obtain

$$\frac{d|P\delta|^2}{dt} + \sigma|P\delta|^2 \le \sigma\delta^2 \le \sigma|\delta(t_n)|^2 e^{\beta(t-t_n)}.$$

Multiplying by the integrating factor $e^{\sigma(t-t_n)}$ and integrating from t_n to t gives

$$|P\delta(t)|^2 \le \frac{\sigma|\delta(t_n)|^2}{\beta+\sigma}\big(e^{\beta(t-t_n)} - e^{-\sigma(t-t_n)}\big) \tag{2.10}$$

where we have used the fact that $P\delta(t_n) = 0$.

A finer analysis of the nonlinear term appearing in (2.7) gives

$$2(B(U,\delta),\delta) = \delta_X(Z\delta_Y - Y\delta_Z).$$

Therefore

$$|2(B(U,\delta),\delta)| \le |U||P\delta||\delta| \le K^{1/2}|P\delta||\delta| \le \frac{1}{2}K|P\delta|^2 + \frac{1}{2}|\delta|^2. \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.7) yields

$$\frac{d|\delta|^2}{dt} + |\delta|^2 \le K\frac{\sigma|\delta(t_n)|^2}{\beta+\sigma}\big(e^{\beta(t-t_n)} - e^{-\sigma(t-t_n)}\big). \tag{2.12}$$

Multiply (2.12) by $e^{(t-t_n)}$ and integrate from t_n to t to obtain

$$|\delta(t)|^2 \le M(t-t_n)|\delta(t_n)|^2$$

where

$$M(\tau) = e^{-\tau}\Big(1 + \frac{\sigma K}{\beta+\sigma}\int_0^\tau \big(e^{(\beta+1)s} - e^{-(\sigma-1)s}\big)ds\Big) \tag{2.13}$$

is a function that doesn't depend on t_n. Note that $M(0) = 1$. Differentiating yields

$$M'(\tau) = -M(\tau) + \frac{\sigma K}{\beta+\sigma}\big(e^{\beta\tau} - e^{-\sigma\tau}\big).$$

Therefore $M'(0) = -1$. It follows that there is $t^* > 0$ such that

$$M(h) = 1 + \int_0^h M'(s)ds < 1$$

for all $h \in (0, t^*]$.

Next fix $h \in (0, t^*]$ and let $\gamma = M(h) < 1$. Then

$$\begin{aligned} |\delta(t_{n+1})|^2 = |Q\delta(t_{n+1})|^2 &= \lim_{t\nearrow t_{n+1}} |Q\delta(t)|^2 \le \lim_{t\nearrow t_{n+1}} |\delta(t)|^2 \\ &\le \lim_{t\nearrow t_{n+1}} M(t-t_n)|\delta(t_n)|^2 = M(h)|\delta(t_n)|^2 = \gamma|\delta(t_n)|^2 \end{aligned}$$

implies by induction that

$$|\delta(t_n)|^2 \le \gamma^n|\delta(t_0)|^2 = \gamma^n|QU(t_0) - \eta|^2 \le \gamma^n R$$

where $R = 2(K + |\eta|^2)$. Now let $t > 0$ and choose n so that $t \in [t_n, t_{n+1})$. Then $n \to \infty$ as $t \to \infty$ and therefore

$$|\delta(t)|^2 \le M(t-t_n)|\delta(t_n)|^2 \le |\delta(t_n)|^2 \le \gamma^n R \to 0 \tag{2.14}$$

shows that the approximating solution converges to the reference solution as $t \to \infty$. □

Corollary 2.6. If $\sigma = 10$, $b = 8/3$ and $r = 28$, then $t^* \approx 0.000129$.

Proof: The result follows using the value of K from Theorem 2.2 and choosing t^* slightly less than the value of t such that $M(t) = 1$. □

Before proving the final result in this section, we extend Theorem 2.5 by proving that the interval of time between t_{n+1} and t_n need not be exactly h for every $n \in \mathbf{N}$.

Corollary 2.7. Let t^* be the bound given in Theorem 2.5. Suppose $t_{n+1} - t_n \le t^*$ where $t_n \to \infty$ as $n \to \infty$. Then the approximating solution u given by (1.1) converges to the reference solution U of the Lorenz equations (2.1) as $t \to \infty$.

Proof: Let $h_n = t_n - t_{n-1}$. If there exists $\epsilon > 0$ such that the set $\mathcal{K} = \{\, k : h_k \ge \epsilon \,\}$ is infinite then $M(h_k) \le \max\{\, M(s) : s \in [\epsilon, t^*] \,\} < 1$ for $k \in \mathcal{K}$ implies that

$$|\delta(t_n)|^2 \le \prod_{k=1}^{n} M(h_k) R \to 0 \quad \text{as} \quad n \to \infty. \tag{2.15}$$

Otherwise, $h_n \to 0$ as $n \to \infty$. By Taylor's theorem $|M(h_n) - M(0) - h_n M'(0)| \le C h_n^2$ where $C = \frac{1}{2} \max\{\, |M''(s)| : s \in [0, t^*] \,\}$. Choose N so large that $Ch_k - 1 \le -1/2$ for $k \ge N$. Since $M(0) = 1$, $M'(0) = -1$ and $\log x \le x - 1$ for $x > 0$ it follows that

$$\begin{aligned} \sum_{k=N}^{n} \log M(h_k) &\le \sum_{k=N}^{n} \big(M(h_k) - 1 \big) \le \sum_{k=N}^{n} \big(M(0) + h_k M'(0) + C h_k^2 - 1 \big) \\ &= \sum_{k=N}^{n} h_k (C h_k - 1) \le -\frac{1}{2} \sum_{k=N}^{n} h_k \to -\infty \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus

$$\prod_{k=1}^{n} M(h_k) R = \exp\Big(\log R + \sum_{k=1}^{n} \log M(h_k) \Big) \to 0 \quad \text{as} \quad n \to \infty.$$

The proof now finishes as in Theorem 2.5. □

Our final result on the Lorenz equations shows that the approximating solution is bounded for any updating time interval of h. If $h \le t^*$ then the approximating solution converges to the reference solution, and since the reference solution is bounded, then the approximating solution will also be bounded. In the case where the approximating solution does not converge to the reference solution then the following result shows that the approximating solution is still bounded.

Theorem 2.8. Let U be a trajectory that lies on the global attractor of (2.1). The approximating solution u given by (1.1) where $t_n = nh$ with $h > 0$ is bounded. Namely, there is a constant M_1 that depends only on η, σ, b and r such that $|u(t)|^2 \le M_1/(1 - e^{-h})$ for all $t \ge 0$ and $h > 0$.

Proof: Taking inner product of (2.1) with u and using (2.2) followed by Young's inequality we obtain

$$\frac{1}{2} \frac{d|u|^2}{dt} + |u|^2 \le (f, u) \le \frac{1}{2} |f|^2 + \frac{1}{2} |u|^2$$

and consequently

$$\frac{d|u|^2}{dt} + |u|^2 \le |f|^2 \tag{2.16}$$

for $t \in [t_n, t_{n+1})$. Grönwall's inequality then implies

$$|u(t)|^2 \le |u_n|^2 e^{-(t - t_n)} + |f|^2 (1 - e^{-(t - t_n)}) \qquad \text{for} \qquad t \in [t_n, t_{n+1}).$$

Defining

$$\bar{u}_0 = \eta \qquad \text{and} \qquad \bar{u}_{n+1} = \lim_{t \nearrow t_{n+1}} u(t) = S(t_{n+1}, t_n, u_n)$$

so that $u_n = Q\bar{u}_n + PU(t_n)$ we obtain

$$\begin{aligned}|\bar{u}_{n+1}|^2 &\le \lim_{t\nearrow t_{n+1}} \big(|u_n|^2 e^{-(t-t_n)} + |f|^2(1-e^{-(t-t_n)})\big)\\ &\le |u_n|^2\gamma + |f|^2(1-\gamma)\end{aligned}$$

where $\gamma = e^{-h}$. Since

$$|u_n|^2 = |Q\bar{u}_n|^2 + |PU(t_n)|^2 \le |\bar{u}_n|^2 + |U(t_n)|^2 \le |\bar{u}_n|^2 + K$$

we obtain

$$|\bar{u}_{n+1}|^2 \le |\bar{u}_n|^2\gamma + C_1 \tag{2.17}$$

where $C_1 = K\gamma + |f|^2(1-\gamma)$.

Induction on (2.17) and summing the series yields

$$|\bar{u}_n|^2 \le |\bar{u}_0|^2\gamma^n + C_1(1+\gamma+\cdots+\gamma^{n-1}) = |\eta|^2\gamma^n + C_1\frac{1-\gamma^n}{1-\gamma}.$$

Given $t > 0$ choose n so that $t \in [t_n, t_{n+1})$. Then $t = t_n + \alpha$ where $\alpha \in [0, h)$ and

$$\begin{aligned}|u(t)|^2 &\le |u_n|^2 e^{-\alpha} + |f|^2(1-e^{-\alpha}) \le |u_n|^2 + |f|^2\\ &\le |\bar{u}_n|^2 + K + |f|^2 \le |\eta|^2\gamma^n + C_1\frac{1-\gamma^n}{1-\gamma} + K + |f|^2 \le \frac{M_1}{1-\gamma}\end{aligned}$$

where $M_1 = |\eta|^2 + C_1 + K + |f|^2$. □

If we take $\eta = 0$ as our initial guess for $QU(t_0)$ when forming the approximating solution u then the constant M_1 depends only on σ, b and r. In either case we obtain the asymptotic bound

$$\limsup_{t\to\infty} |u(t)|^2 \le \frac{M_2}{1-e^{-h}}.$$

where the constant M_2 depends only on σ, b and r.

Theorem 2.8 can be improved using the exact structure of (Au, u) and (f, u). In particular, we obtain using Theorem 2.3 that

$$\begin{aligned}(Au,u) - (f,u) &= \sigma x^2 + y^2 + bz^2 + zb(r+\sigma)\\ &= \sigma x^2 + y^2 + z^2 + (b-1)z^2 + zb(r+\sigma)\\ &\ge \sigma x^2 + y^2 + z^2 - \frac{b^2(r+\sigma)^2}{4(b-1)} \ge |u|^2 - \frac{|f|^2}{4(b-1)}\end{aligned}$$

which improves the bound to $|u(t)|^2 \le M_3/(1-e^{-2h})$. Although an improvement, this bound on u still tends to infinity as h tends to zero. As mentioned earlier, a bound uniform in h can be obtained by combining Theorem 2.5 with Theorem 2.8.

Corollary 2.9. There exists a bound M_4 depending only on η, σ, b and r such that the approximate solution u obtained with $t_n = hn$ is bounded by M_4 for any $h > 0$.

Proof: Let t^* be given as in Theorem 2.5. If $h < t^*$ then (2.14) implies

$$|u(t)| = |U(t) - \delta(t)| \le |U(t)| + |\delta(t)| \le K^{1/2} + R^{1/2}.$$

If $h \ge t^*$ then Theorem 2.8 implies

$$|u(t)| \le M_1^{1/2}/(1 - e^{-h})^{1/2} \le M_1^{1/2}/(1 - e^{-t^*})^{1/2}.$$

Taking

$$M_4 = \max\{K^{1/2} + R^{1/2}, M_1^{1/2}/(1 - e^{-t^*})^{1/2}\}$$

yields a constant that depends only on η, σ, b and r. □

As mentioned before, if we take $\eta = 0$ or consider asymptotic bounds on u as $t \to \infty$ then the dependency on η can be removed from these bounds.

3. Navier–Stokes Equations

This section contains results for the two-dimensional incompressible Navier–Stokes equations that are similar to the results proved in the previous section for the Lorenz equations. Let $\Omega = [0, L] \times [0, L]$, $\mathcal{V}$ be the space of divergence-free vector-valued L-periodic trigonometric polynomials from Ω into $\mathbf{R}^2$ with $\int_\Omega u = 0$, H be the closure of $\mathcal{V}$ with respect to the L^2 norm and P_H be the L^2-orthogonal projection $P_H\colon L^2(\Omega) \to H$, referred to as the Leray-Helmholtz projector. Following the notations of Constantin and Foias [2], Robinson [15] and Temam [17] we write the incompressible two-dimensional Navier–Stokes equations (1.4) as

$$\frac{du}{dt} + \nu A u + B(u, u) = f \tag{3.1}$$

where

$$Au = -P_H \Delta u \qquad \text{and} \qquad B(u, v) = P_H\big[(u \cdot \nabla)v\big]. \tag{3.2}$$

Note that we have assumed $f \in H$ so that $P_H f = f$. Also notice that $A = -\Delta$ in the periodic case.

Let V be the closure of $\mathcal{V}$ with respect to the H^1 norm and $\mathcal{D}(A) = H \cap H^2(\Omega)$ be the domain of A. We may define norms on H, V and $\mathcal{D}(A)$ which are equivalent to the L^2, H^1 and H^2 norms respectively by

$$|u| = L^2 \sum_{k \in \mathcal{J}} |u_k|^2, \qquad \|u\| = L^2 \sum_{k \in \mathcal{J}} k^2 |u_k|^2 \qquad \text{and} \qquad |Au| = L^2 \sum_{k \in \mathcal{J}} k^4 |u_k|^2.$$

Here u has been expressed in terms of the Fourier series

$$u = \sum_{k \in \mathcal{J}} u_k e^{ik\cdot x} \quad \text{where} \quad \mathcal{J} = \left\{ \frac{2\pi}{L}(n_1, n_2) : n_i \in \mathbf{Z} \text{ and } (n_1, n_2) \ne 0 \right\}.$$

The mathematical theory proving the existence and uniqueness of strong solutions to the two-dimensional incompressible Navier–Stokes equations (3.1) with initial data in V may be found for example in [2], [15] or [17]. Specifically we have

Theorem 3.1. Let $u_0 \in V$ and $f \in H$. Then (3.1) has unique strong solutions that satisfy

$$u \in L^\infty\big((0,T);V\big) \cap L^2\big((0,T);\mathcal{D}(A)\big) \quad \text{and} \quad \frac{du}{dt} \in L^2\big((0,T);H\big)$$

for any $T > 0$. Furthermore, this solution is in $C\big([0,T];V\big)$ and depends continuously on the initial data u_0 in the V norm.

Let $S(t,t_0,u_0)$ for $t \geq t_0$ denote the unique strong solution of (3.1) given by Theorem 3.1 with initial condition u_0 at time t_0. Let $\lambda_1 = (2\pi)^2/L^2$ be the smallest eigenvalue of A on $\mathcal{D}(A)$ and define P_λ as in (1.5). Further define $Q_\lambda = I - P_\lambda$ to be the orthogonal complement of P_λ. The Poincaré inequalities for u, $P_\lambda u$ and $Q_\lambda u$ on Ω can now be summarized as

Theorem 3.2. Given P_λ, Q_λ and λ_1 as defined above then

$$|u| \leq \lambda_1^{-1/2}\|u\|, \qquad |Q_\lambda u| \leq \lambda^{-1/2}\|Q_\lambda u\|, \qquad \|P_\lambda u\| \leq \lambda^{1/2}|P_\lambda u|$$

and

$$\|u\| \leq \lambda_1^{-1/2}|Au|, \qquad \|Q_\lambda u\| \leq \lambda^{-1/2}|AQ_\lambda u|, \qquad |AP_\lambda u\| \leq \lambda^{1/2}\|P_\lambda u\|.$$

provided the norms exist and are finite.

Theorem 3.3. If $u \in \mathcal{D}(A)$ then

$$|Au|^{1/4} \leq 2^{1/8}(\lambda^{1/8}\|u\|^{1/4} + |Q_\lambda Au|^{1/4}).$$

Proof: Since P_λ and Q_λ are orthogonal then

$$|Au|^2 = |P_\lambda Au|^2 + |Q_\lambda Au|^2 \leq \lambda\|P_\lambda u\|^2 + |Q_\lambda Au|^2 \leq 2\max\{\lambda\|P_\lambda u\|^2, |Q_\lambda Au|^2\}.$$

Therefore

$$|Au|^{1/4} \leq 2^{1/8}\max\{\lambda^{1/8}\|P_\lambda u\|^{1/4}, |Q_\lambda Au|^{1/4}\} \leq 2^{1/8}(\lambda^{1/8}\|u\|^{1/4} + |Q_\lambda Au|^{1/4}),$$

which completes the proof. □

Let us now recall some algebraic properties of the non-linear term that can also be found in [2], [15] or [17] that play an important role in our analysis. They are

$$(B(u,v),w) = (B(u,w),v), \tag{3.3}$$

$$(B(u,v),v) = 0 \qquad \text{and} \qquad (B(u,u),Au) = 0. \tag{3.4}$$

We now move on to some inequalities which we shall refer to later.

Theorem 3.4. There exists a dimensionless constant c such that if $u \in V$ then

$$\|u\|_{L^{8/3}}^2 \leq c|u|^{3/2}\|u\|^{1/2}, \qquad \|u\|_{L^4}^2 \leq c|u|\|u\|, \qquad \|u\|_{L^8}^2 \leq c|u|^{1/2}\|u\|^{3/2}$$

and if $u \in \mathcal{D}(\mathcal{A})$ then

$$\|u\|_{L^\infty} \le c\lambda_1^{-1/8}\|u\|^{3/4}|Au|^{1/4}.$$

The first three inequalities above may be obtained from the Sobolev inequalities followed by interpolation, see for example (6.2) and (6.7) in [2]. The inequality bounding L^4 is sometimes referred to as Ladyzhenskaya's inequality and appears as Lemma 1 on page 8 of Ladyzhenskaya [10]. The last inequality is a form of Agmon's inequality which appears as (2.23) on page 11 in Temam [17]. These inequalities may be used along with Hölder's inequality to estimate the nonlinear term.

Theorem 3.5. Let $u, v, w \in V$ then

$$|(B(u,v),w)| \le c|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2}$$

if further $v \in \mathcal{D}(A)$ then

$$|(B(u,v),w)| \le c\lambda_1^{-1/8}\|u\|\|v\|^{3/4}|Av|^{1/4}|w|$$
$$|(B(v,u),w)| \le c\lambda_1^{-1/8}\|u\|\|v\|^{3/4}|Av|^{1/4}|w|$$

where c is the constant appearing in Theorem 3.4.

We consider a reference solution U to the incompressible two-dimensional Navier–Stokes equations and the approximating solution u given by (1.1) where S is the semigroup generated by (3.1). As with the Lorenz equations, we shall assume that the reference solution lies on the global attractor. As proved in Jones and Titi [7] we have

Theorem 3.6. A solution U that lies on the global attractor of (3.1) satisfies the bound $\|U\|^2 \le K$ where

$$K = \frac{|f|^2}{\lambda_1 \nu^2}. \tag{3.5}$$

A similar result also appears in [2] for establishing estimates on the global attractor. As mentioned in the introduction, any reference solution U which satisfies a bound such as (3.5) forward in time is suitable for our analysis, whether that solution is on the attractor or not. However, for simplicity we continue to assume U lies on the global attractor.

Now setting $\delta = U - u$ and following the same algebra as for the Lorenz equations we arrive at

$$\frac{d\delta}{dt} + \nu A\delta + B(U,\delta) + B(\delta,U) - B(\delta,\delta) = 0, \tag{3.6}$$

an equation that looks like (2.6) where A and B have been given the meanings in (3.2).

Before starting with the proof of our main result we begin with a simpler theorem that finds values of λ large enough for the approximating solution u to start converging to a reference solution U lying on the attractor but does not provide the uniform bound on h necessary to maintain convergence as $t \to \infty$.

Theorem 3.7. If $\lambda > c^2|f|^2/(\lambda_1\nu^4)$ there exists $t > t_n$ such that $|\delta(t)| < |\delta(t_n)|$.

Proof: Multiply (3.6) by δ and integrate over Ω. The first of the orthogonality relationships given in (3.4) yields

$$\frac{1}{2}\frac{d|\delta|^2}{dt} + \nu\|\delta\|^2 = -(B(\delta, U), \delta). \tag{3.7}$$

Now applying Theorem 3.5 followed by Young's and Theorem 3.6 we obtain

$$\frac{1}{2}\frac{d|\delta|^2}{dt} + \nu\|\delta\|^2 \le c|\delta|\|\delta\|\|U\| \le \frac{\nu}{2}\|\delta\|^2 + \frac{c^2K}{2\nu}|\delta|^2$$

or

$$\frac{d|\delta|^2}{dt} \le \frac{c^2K}{\nu}|\delta|^2 - \nu\|\delta\|^2.$$

Integrating from t_n to t where $t \in [t_n, t_{n+1})$ results in

$$|\delta(t)|^2 - |\delta(t_n)|^2 \le \int_{t_n}^{t} \Big(\frac{c^2K}{\nu}|\delta(s)|^2 - \nu\|\delta(s)\|^2\Big) ds.$$

Therefore

$$|\delta(t)|^2 \le M_n(t - t_n)|\delta(t_n)|^2$$

where

$$M_n(\tau) = 1 + \frac{1}{|\delta(t_n)|^2}\int_0^{t-t_n} \Big(\frac{c^2K}{\nu}|\delta(t_n + s)|^2 - \nu\|\delta(t_n + s)\|^2\Big) ds.$$

Note $M_n(\tau)$ is a differentiable function with $M_n(0) = 1$.

Differentiating yields

$$M_n'(\tau) = \frac{c^2K}{\nu}\frac{|\delta(t_n + \tau)|^2}{|\delta(t_n)|^2} - \nu\frac{\|\delta(t_n + \tau)\|^2}{|\delta(t_n)|^2}.$$

By Theorem 3.2 and the hypothesis $\lambda > c^2K/\nu^2$ we obtain

$$M_n'(0) = \frac{c^2K}{\nu} - \nu\frac{\|\delta(t_n)\|^2}{|\delta(t_n)|^2} = \frac{c^2K}{\nu} - \nu\frac{\|Q_\lambda\delta(t_n)\|^2}{|\delta(t_n)|^2} \le \frac{c^2K}{\nu} - \nu\lambda < 0.$$

Therefore there exists $\tau > 0$ such that

$$M_n(\tau) = 1 + \int_0^{\tau} M'(s) ds < 1.$$

It follows that for $t = t_n + \tau$ that $|\delta(t)| < |\delta(t_n)|$. □

It should be pointed out that the value t in the above theorem depends on $M_n(\tau)$ and thus on $\delta(s)$ for $s \ge t_n$. Unfortunately, this provides us with no way of knowing whether the family of functions M_n is equicontinuous or not. This means Theorem 3.7 can not be used to provide a uniform bound t^* on the discrete measurement time interval h that ensures the approximate solution will converge to the reference solution as $t \to \infty$. In

Theorem 3.9 below we circumvent this issue at the expense of a more stringent condition on the minimum size of λ.

Note that the bound on λ given in Theorem 3.7 above is the same as the bound given by Jones and Titi in [7] on the number of determining modes for the incompressible two-dimensional Navier–Stokes equations. This bound was later improved by Jones and Titi in [8] using estimates on the time averages of the term $\|Au\|^2$. However, in the case of discrete assimilation the same technique did not achieve similar improvements.

We commence with the detailed analysis that allows us to prove our main result for the incompressible two-dimensional Navier–Stokes equations. The first theorem we prove is an analog of Lemma 2.4.

Theorem 3.8. There exists $\beta > 0$ depending on $|f|$, Ω and ν such that the solution δ to (3.6) satisfies $\|\delta(t)\|^2 \le \|\delta(t_n)\|^2 e^{\beta(t-t_n)}$ for $t \in [t_n, t_{n+1})$.

Proof: Multiply (3.6) by $A\delta$ and integrate over Ω to obtain

$$\frac{1}{2}\frac{d\|\delta\|^2}{dt} + \nu|A\delta|^2 + (B(U,\delta), A\delta) + (B(\delta,U), A\delta) = 0.$$

By Theorem 3.5 we have that

$$|(B(\delta,U), A\delta)| \le c\lambda_1^{-1/8}\|\delta\|^{3/4}\|U\||A\delta|^{5/4}$$

and

$$|(B(U,\delta), A\delta)| \le c\lambda_1^{-1/8}\|\delta\|^{3/4}\|U\||A\delta|^{5/4}.$$

Therefore applying Theorem 2.3 with $p = 8/3$ and $q = 8/5$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{d\|\delta\|^2}{dt} + \nu|A\delta|^2 &\le 2c\lambda_1^{-1/8}\|\delta\|^{3/4}\|U\||A\delta|^{5/4} \\ &\le C_1\nu^{-5/3}\lambda_1^{-1/3}\|\delta\|^2\|U\|^{8/3} + \nu|A\delta|^2 \\ &\le C_1\nu^{-5/3}\lambda_1^{-1/3}K^{4/3}\|\delta\|^2 + \nu|A\delta|^2\end{aligned}$$

where C_1 is a dimensionless constant related to c. Hence

$$\frac{d\|\delta\|^2}{dt} \le \beta\|\delta\|^2$$

where $\beta = 2C_1\nu^{-5/3}\lambda_1^{-1/3}K^{4/3}$. Integrating in time from t_n to t and applying Grönwall's inequality yields $\|\delta(t)\|^2 \le \|\delta(t_n)\|^2 e^{\beta(t-t_n)}$ for $t \in [t_n, t_{n+1})$. □

We are now ready to prove the main results of this paper.

Theorem 3.9. If

$$\lambda > \frac{9}{\lambda_1^{1/3}}\Big(\frac{2c|f|/(\lambda^{1/2}\nu) + c\|\delta(t_0)\|}{\nu}\Big)^{8/3} \tag{3.8}$$

then there exists $t^* > 0$ depending only on K, $\|\delta(t_0)\|$, ν, Ω and λ such that for any $h \in (0, t^*]$ the approximating solution u given by (1.1) with $t_n = hn$ converges to the reference solution U of (3.1) as $t \to \infty$.

Proof: The proof is similar to the proof of Theorem 2.5 for the Lorenz equations with the addition that we first use induction to show the bound $R = \|\delta(t_0)\|^2$ on the difference of the initial conditions ensures that $\|\delta(t_n)\|^2 \le R$ holds for each each t_n.

Define

$$g(\tau) = C_2\Big(\frac{\lambda}{\nu^4\lambda_1}\Big)^{1/4} g_1(\tau) + C_3\Big(\frac{1}{\nu^5\lambda_1}\Big)^{1/3} g_2(\tau) \tag{3.9}$$

where

$$C_2 = c^2(2^{5/4}), \qquad C_3 = 3c^{8/3}(5^{5/3})(2^{-10/3}) \tag{3.10}$$

and

$$g_1(\tau) = e^{\beta\tau}\big(R^{1/2}e^{\beta\tau/2} + 2K^{1/2}\big)^2, \qquad g_2(\tau) = e^{\beta\tau}\big(R^{1/2}e^{\beta\tau/2} + 2K^{1/2}\big)^{8/3}.$$

Further define

$$M(\tau) = e^{-\nu\lambda\tau}\Big(1 + \int_0^\tau g(s)e^{\nu\lambda s}ds\Big). \tag{3.11}$$

Note that $M(0) = 1$. Differentiating yields

$$M'(\tau) = -\nu\lambda M(\tau) + g(\tau).$$

Therefore

$$M'(0) = -\nu\lambda + C_2\Big(\frac{\lambda}{\nu^4\lambda_1}\Big)^{1/4}\big(R^{1/2} + 2K^{1/2}\big)^2 + C_3\Big(\frac{1}{\nu^5\lambda_1}\Big)^{1/3}\big(R^{1/2} + 2K^{1/2}\big)^{8/3}.$$

Taking $C_4 = 9c^{8/3} \ge \max\{(2C_2)^{4/3}, 2C_3\} = 3c^{8/3}(5^{5/3})(2^{-7/3})$ we find that $M'(0) < 0$ is guaranteed when

$$\lambda > \frac{9}{\lambda_1^{1/3}}\Big(\frac{2cK^{1/2} + cR^{1/2}}{\nu}\Big)^{8/3}.$$

Now choose $t^* > 0$ such that $M(h) < 1$ for all $h \in (0, t^*]$. Note that t^* only depends on K, $\|\delta(t_0)\|$, ν, Ω and λ. We now show the approximating solution u given by (1.1) with $t_n = hn$ converges to the reference solution U of (3.1) as $t \to \infty$.

For induction on n we suppose that $\|\delta(t_n)\| \le R$. In the case $n = 0$ the induction hypothesis is true by definition. Take inner product of $AQ_\lambda\delta$ with (3.6) to obtain

$$\frac{1}{2}\frac{d}{dt}\|Q_\lambda\delta\|^2 + \nu|AQ_\lambda\delta|^2 \le B_1 + B_2 + B_3$$

where

$$B_1 = |(B(\delta, U), AQ_\lambda\delta)|, \quad B_2 = |(B(U, \delta), AQ_\lambda\delta)| \quad \text{and} \quad B_3 = |(B(\delta,\delta), AQ_\lambda\delta)|.$$

Estimate using Theorem 3.5 followed by Theorem 3.3 to obtain

$$\begin{aligned} B_1 &\le c\lambda_1^{-1/8}\|\delta\|^{3/4}|A\delta|^{1/4}\|U\||AQ_\lambda\delta| \\ &\le c\Big(\frac{2\lambda}{\lambda_1}\Big)^{1/8}\|\delta\|\|U\||AQ_\lambda\delta| + c\Big(\frac{2}{\lambda_1}\Big)^{1/8}\|\delta\|^{3/4}\|U\||AQ_\lambda\delta|^{5/4} \end{aligned}$$

similarly

$$B_2 \le c\Big(\frac{2\lambda}{\lambda_1}\Big)^{1/8}\|\delta\|\|U\||AQ_\lambda\delta| + c\Big(\frac{2}{\lambda_1}\Big)^{1/8}\|\delta\|^{3/4}\|U\||AQ_\lambda\delta|^{5/4}$$

and

$$B_3 \le c\Big(\frac{2\lambda}{\lambda_1}\Big)^{1/8}\|\delta\|^2|AQ_\lambda\delta| + c\Big(\frac{2}{\lambda_1}\Big)^{1/8}\|\delta\|^{7/4}|AQ_\lambda\delta|^{5/4}.$$

It follows that

$$\sum_{i=1}^{3} B_i = J_1 + J_2$$

where by Theorem 2.3 with $p = q = 2$ we have

$$\begin{aligned} J_1 &= c\Big(\frac{2\lambda}{\lambda_1}\Big)^{1/8}\|\delta\|(\|\delta\| + 2\|U\|)|AQ_\lambda\delta| \\ &\le \frac{C_2}{2}\Big(\frac{\lambda}{\nu^4\lambda_1}\Big)^{1/4}\|\delta\|^2(\|\delta\| + 2\|U\|)^2 + \frac{\nu}{4}|AQ_\lambda\delta|^2 \end{aligned}$$

and Theorem 2.3 with $p = 8/3$ and $q = 8/5$ we have

$$\begin{aligned} J_2 &= c\Big(\frac{2}{\lambda_1}\Big)^{1/8}\|\delta\|^{3/4}(\|\delta\| + 2\|U\|)|AQ_\lambda\delta|^{5/4} \\ &\le \frac{C_3}{2}\Big(\frac{1}{\nu^5\lambda_1}\Big)^{1/3}\|\delta\|^2(\|\delta\| + 2\|U\|)^{8/3} + \frac{\nu}{4}|AQ_\lambda\delta|^2. \end{aligned}$$

Here, C_2 and C_3 are as defined in (3.10). It follows that

$$\frac{d}{dt}\|Q_\lambda\delta\|^2 + \nu|AQ_\lambda\delta|^2 \le J_3 + J_4$$

where by Theorem 3.8

$$\begin{aligned} J_3 &= C_2\Big(\frac{\lambda}{\nu^4\lambda_1}\Big)^{1/4}\|\delta\|^2(\|\delta\| + 2\|U\|)^2 \\ &\le C_2\Big(\frac{\lambda}{\nu^4\lambda_1}\Big)^{1/4}\|\delta(t_n)\|^2 e^{\beta(t-t_n)}\big(\|\delta(t_n)\|e^{\beta(t-t_n)/2} + 2\|U\|\big)^2 \\ &\le C_2\Big(\frac{\lambda}{\nu^4\lambda_1}\Big)^{1/4}\|\delta(t_n)\|^2 e^{\beta(t-t_n)}\big(R^{1/2}e^{\beta(t-t_n)/2} + 2K^{1/2}\big)^2 \end{aligned}$$

and

$$\begin{aligned} J_4 &= C_3\Big(\frac{1}{\nu^5\lambda_1}\Big)^{1/3}\|\delta\|^2(\|\delta\| + 2\|U\|)^{8/3} \\ &\le C_3\Big(\frac{1}{\nu^5\lambda_1}\Big)^{1/3}\|\delta(t_n)\|^2 e^{\beta(t-t_n)}\big(\|\delta(t_n)\|e^{\beta(t-t_n)/2} + 2\|U\|\big)^{8/3} \\ &\le C_3\Big(\frac{1}{\nu^5\lambda_1}\Big)^{1/3}\|\delta(t_n)\|^2 e^{\beta(t-t_n)}\big(R^{1/2}e^{\beta(t-t_n)/2} + 2K^{1/2}\big)^{8/3}. \end{aligned}$$

Then by Theorem 3.2 and the fact that $P_\lambda\delta(t_n)=0$ we have

$$\frac{d}{dt}\|Q_\lambda\delta\|^2+\nu\lambda\|Q_\lambda\delta\|^2\le\|Q_\lambda\delta(t_n)\|^2g(t-t_n)$$

where g is the function defined in (3.9). Multiply by the integrating factor $e^{\nu\lambda(t-t_n)}$ and integrate from t_n to t to obtain

$$\|Q_\lambda\delta(t)\|^2\le M(t-t_n)\|Q_\lambda\delta(t_n)\|^2$$

where M is the function defined in (3.11). Let $\gamma=M(h)$. By our choice of t^* and h we have $\gamma<1$. It follows that

$$\begin{aligned}\|\delta(t_{n+1})\|^2=\|Q_\lambda\delta(t_{n+1})\|^2&\le\lim_{t\nearrow t_{n+1}}\|Q_\lambda\delta(t)\|^2\\&\le\lim_{t\nearrow t_{n+1}}M(t-t_n)\|Q_\lambda\delta(t_n)\|^2=M(h)\|\delta(t_n)\|^2=\gamma\|\delta(t_n)\|^2.\end{aligned}$$

Therefore $\|\delta(t_{n+1})\|^2\le R$, which completes the induction.

To finish the proof note that under these hypothesis we have, in fact, proven

$$\|\delta(t_n)\|^2\le\gamma^nR.$$

The proof now finishes as in (2.14). □

Corollary 3.10. Given any $t^*>0$ there exists λ large enough depending only on K, $\|\delta(t_0)\|$, ν, Ω and t^* such that for any $h\in(0,t^*]$ the approximating solution u given by (1.1) with $t_n=hn$ converges to the reference solution U of (3.1) as $t\to\infty$.

Proof: First estimate $g(\tau)$ from (3.9) as

$$\begin{aligned}g(\tau)&\le C_2\Big(\frac{\lambda}{\nu^4\lambda_1}\Big)^{1/4}L_1^2e^{2\beta\tau}+C_3\Big(\frac{1}{\nu^5\lambda_1}\Big)^{1/3}L_1^{8/3}e^{(7\beta/3)\tau}\\&\le\nu\lambda^{1/4}L_2e^{(7\beta/3)\tau}\end{aligned}$$

where

$$L_1=R^{1/2}+2K^{1/2}\quad\text{and}\quad L_2=\frac{C_2}{\lambda_1^{1/4}}\Big(\frac{L_1}{\nu}\Big)^2+\frac{C_3}{\lambda_1^{7/12}}\Big(\frac{L_1}{\nu}\Big)^{8/3}.$$

Therefore

$$\begin{aligned}M(\tau)&\le e^{-\nu\lambda\tau}\Big(1+\nu\lambda^{1/4}L_2\int_0^\tau e^{(7\beta/3+\nu\lambda)s}ds\Big)\\&=e^{-\nu\lambda\tau}\Big(1+\frac{\nu\lambda^{1/4}L_2}{7\beta/3+\nu\lambda}(e^{(7\beta/3+\nu\lambda)\tau}-1)\Big)\le m(\tau)\end{aligned}$$

where

$$m(\tau)=\big(1-L_2\lambda^{-3/4}\big)e^{-\nu\lambda\tau}+L_2\lambda^{-3/4}e^{(7\beta/3)\tau}.$$

Differentiating yields

$$m'(\tau) = -\nu\lambda\big(1 - L_2\lambda^{-3/4}\big)e^{-\nu\lambda\tau} + \frac{7\beta}{3}L_2\lambda^{-3/4}e^{(7\beta/3)\tau} \quad \text{and}$$
$$m''(\tau) = \nu^2\lambda^2\big(1 - L_2\lambda^{-3/4}\big)e^{-\nu\lambda\tau} + \frac{49\beta^2}{9}L_2\lambda^{-3/4}e^{(7\beta/3)\tau}.$$

Given $t^* > 0$ choose λ large enough such that $m'(t^*) < 0$. Clearly $m'' > 0$ for this value of λ. Thus m' is a strictly increasing function. It follows that $m'(s) < 0$ for $s \in [0, t^*]$ and therefore

$$M(h) \le m(h) = 1 + \int_0^h m'(s)ds < 1$$

for $h \in (0, t^*]$. Hence, the approximating solution u given by (1.1) with $t_n = hn$ converges to the reference solution U of (3.1) as $t \to \infty$. □

Corollary 3.11. If we take $\eta = 0$ and

$$\lambda > \frac{9}{\lambda_1^{5/3}}\Big(\frac{3c|f|}{\nu^2}\Big)^{8/3},$$

then the results of Theorem 3.9 hold where t^* may be chosen independent of $\|\delta(t_0)\|$.

Proof: When $\eta = 0$ then $\|\delta(t_0)\|^2 = \|QU(t_0)\|^2 \le K$. □

The bound in Corollary 3.11 would be comparable to the bound in Theorem 3.7 if the exponent of 8/3 were instead 2. The power 8/3 comes as a result of using the $L^{8/3}$ norm and the particular form of Agmon's inequality we have used in estimating the nonlinear term. Using the same proof technique with different interpolation inequalities in place of Theorem 3.4, this exponent could be reduced to as near 2 as one might like at the expense of increasing the constant c.

Next we state without proof the analog of Corollary 2.7 for the incompressible two-dimensional Navier–Stokes equations.

Corollary 3.12. Let t^* be the bound given in Corollary 3.10. Suppose $t_{n+1} - t_n \le t^*$ where $t_n \to \infty$ as $n \to \infty$. Then the approximating solution u given by (1.1) converges to the reference solution U of (3.1) as $t \to \infty$.

We finish with the equivalent of Theorem 2.8 for the Lorenz system. This result is interesting because it shows that even if the approximating solution doesn't converge to the reference solution it is still bounded. This is striking because in the case of continuous data assimilation it is unknown whether the approximating solution is in general bounded or not. In particular, the comments before Theorem 3.5 in Olson and Titi [11] indicate that the approximate solution obtained by continuous data assimilation is not known to be bounded if it does not converge to the reference solution.

Theorem 3.13. *There exists M_5 independent of h and depending only on $|f|$, Ω and ν such that $\|u(t)\|^2 \le M_5/(1 - e^{-\nu\lambda_1 h})$ for all t.*

Proof: Multiply (3.1) by Au, integrate over Ω and apply the inequalities of Cauchy–Schwartz and Young to obtain

$$\frac{1}{2}\frac{d\|u\|^2}{dt} + \nu|Au|^2 = (f, Au) \le |f||Au| \le \frac{\nu}{2}|Au|^2 + \frac{1}{2\nu}|f|^2.$$

It follows from Theorem 3.2 that

$$\frac{d\|u\|^2}{dt} + \nu\lambda_1\|u\|^2 \le \frac{1}{\nu}|f|^2.$$

The rest of the proof is similar to the proof of Theorem 2.8. □

Corollary 3.14. If

$$\lambda > \frac{9}{\lambda_1^{1/3}}\Big(\frac{2cK^{1/2} + c\|\delta(t_0)\|}{\nu}\Big)^{8/3}$$

then there exists a bound M_6 depending only on K, $\|\delta(t_0)\|$, ν, Ω and λ such that the approximating solution u obtained with $t_n = hn$ is bounded by M_6 for any $h > 0$.

Proof: The proof is the same as the proof of Corollary 2.9 for the Lorenz equations. Note that if $\eta = 0$ as in Corollary 3.11 then λ and M_6 may be chosen independent of $\|\delta(t_0)\|$. □

4. Concluding Remarks

We have studied discrete data assimilation for the Lorenz system and the incompressible two-dimensional Navier–Stokes equations. Comparing the results we have obtained for discrete data assimilation to prior studies of continuous data assimilation we find the following. For the Lorenz system Pecora and Carroll [14] showed that continuous data assimilation of the X variable lead to convergence of the approximating solution to the reference solution as time tends to infinity. In Theorem 2.5 we provide a similar result for discrete data assimilation provided the update time interval h is sufficiently small. For the incompressible two-dimensional Navier–Stokes equations Olson and Titi [11] obtained conditions on the resolution parameter λ under which the approximating solution converged to the reference solution. Theorem 3.9 states a similar condition on λ that leads to convergence of the approximating solution to the reference solution provided h is sufficiently small. In Corollary 3.10 we also show that for any $h > 0$ there is λ large enough such that the approximating solution converges to the reference solution as time tends to infinity. Thus, discrete data assimilation has been shown to work under similar conditions as continuous data assimilation.

A striking difference between discrete and continuous data assimilation is given by Theorem 2.8 and Theorem 3.13 which show that the approximating solution obtained by discrete data assimilation is bounded even if it doesn't converge to the reference solution. Although the approximating solution appears bounded in the case of continuous data assimilation for all numerical experiments performed to date, there does not yet exist an analytic proof of this property. Boundedness of an approximating solution that does not converge to the reference solution remains a conjecture for continuous data assimilation.

In this work we have found analytic bounds on the update time interval h for discrete data assimilation which guarantee that the approximating solution converges to the reference solution. It is natural to compare our analytic bound with numerical simulation. For the Lorenz system Corollary 2.6 indicates that for $t^* \approx 0.000129$ for the standard parameter values $\sigma = 10$, $b = 8/3$ and $r = 28$. For these same parameter values Hayden [6] performed a numerical simulation of discrete data assimilation using the 150-digit-precision variable-step-size variable-order Taylor-method integrator [9]. In this work the maximum value for t^* was found numerically to lie in the interval $[0.175, 0.1875]$. Thus, convergence of the approximating solution to the reference solution numerically occurs for values of h three orders of magnitude larger than those guaranteed by our analysis.

For the incompressible two-dimensional Navier–Stokes equations bounds on λ for continuous data assimilation appear in [11] and [12]. That work shows that the approximating solution converges to the reference solution for values of λ smaller than expected from the analysis. As similar techniques are used to treat the discrete assimilation in this paper, we expect our bounds on λ to be similarly conservative and that the approximating solution obtained by discrete data assimilation will converge numerically for much smaller values of λ than given by Theorem 3.9 and Corollary 3.11. We also expect our bounds on h to be conservative. A computational study of discrete data assimilation for the incompressible two-dimensional Navier–Stokes equations is currently in progress.

We conclude by returning to the motivating problem of using satellite imaging data to initialize a weather forecasting model. In applications the value of h governing the time interval between consecutive observational measurements is generally much larger than the value of Δt used by the numerical integrator. Therefore, it is more realistic to treat the observational data as measurements occurring at a sequence of times t_n as was done in this paper rather than as measurements occurring continuously in time. In the context of the Lorenz equations and the incompressible two-dimensional Navier–Stokes equations we have obtained similar theoretical results for discrete data assimilation as for continuous data assimilation. We hope that these results will shed light on the differences and similarities between discrete and continuous data assimilation and help guide future work in understanding more complicated problems.

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