Discrete duality for rough relation algebras

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November 5, 2015

Dedicated to Andrzej Skowron on

the occasion of his 70th birthday

Abstract

Rough relation algebras are a generalization of relation algebras such that the underlying lattice structure is a regular double Stone algebra. Standard models are algebras of rough relations. A discrete duality is a relationship between classes of algebras and classes of relational systems (frames). In this paper we prove a discrete duality for a class of rough relation algebras and a class of frames.

1 Introduction

Rough sets [16] and rough relations [3] were introduced in connection with reasoning with incomplete information. With such information we may not be able to distinguish between members and non-members of a set or a relation in a definite way. As a consequence sets or relations can be defined only approximately: An *approximation space* is a system $\langle U, \theta \rangle$ where U is a nonempty

set and θ is an equivalence relation on U. Here, the assumption is that we know the world only up to the classes of θ: Given a subset X of U, the *lower approximation* of X in an approximation space $\langle U, \theta \rangle$ is defined as $l(X) = \{x \in U : \theta(x) \subseteq X\}$, where $\theta(x) = \{y \in U : x\theta y\}$. Similarly, the *upper approximation* of X in U is $u(X) = \{x \in U : \theta(x) \cap X \neq \emptyset\}$. A *rough set* in an approximation space $\langle U, \theta \rangle$ is any pair $\langle l(X), u(X) \rangle$ where $X \subseteq U$. The collection of rough subsets of $\langle U, \theta \rangle$ is denoted by $Rough(U) = \{\langle l(X), u(X) \rangle : X \subseteq U\}$. In [17] it is shown that for every approximation space $\langle U, \theta \rangle$, $Rough(U)$ can be made into a Stone algebra. Their result was extended by Comer [3] who showed that $Rough(U)$ can even be made into a regular double Stone algebra by the following operations:

$$\langle l(X), u(X) \rangle + \langle l(Y), u(Y) \rangle = \langle l(X) \cup l(Y), u(X) \cup u(Y) \rangle \qquad \text{join}$$
$$\langle l(X), u(X) \rangle \cdot \langle l(Y), u(Y) \rangle = \langle l(X) \cap l(Y), u(X) \cap u(Y) \rangle \qquad \text{meet}$$
$$\langle l(X), u(X) \rangle^* = \langle U \setminus u(X), U \setminus u(X) \rangle \qquad \text{pseudocomplement}$$
$$\langle l(X), u(X) \rangle^+ = \langle U \setminus l(X), U \setminus l(X) \rangle \qquad \text{dual pseudocomplement}$$
$$0 = \langle \emptyset, \emptyset \rangle$$
$$1 = \langle U, U \rangle$$

In the same paper he proved a representation theorem for regular double Stone algebras showing that every such algebra is embeddable into the algebra of all rough subsets of an approximation space.

In related work, various algebras derived from approximation spaces are known in the literature and shown to be representation algebras for some algebras which provide semantics for non-classical logics, for example, in [6, 10, 15] monadic algebras and three-valued Łukasiewicz algebras derived from approximation spaces are considered and representation theorems are proved. In [15] a representation theorem for finite Nelson algebras is proved such that the representation algebra is derived from an approximation space. A survey of those and several other representation results where the representation algebras are determined by algebraic models of incomplete information can be found in [4], Chapter 14, see also [1].

Given an approximation space $\langle U, \theta \rangle$, it is easy to see that the system $\langle U^2, \theta^2 \rangle$ is an approximation space as well; here, θ^2 is the equivalence relation on U^2 induced by θ, i.e. $\langle x, y \rangle \theta^2 \langle x', y' \rangle$ if and only if $x\theta x'$ and $y\theta y'$. A *rough (binary) relation* on an approximation space $\langle U, \theta \rangle$ is defined as a rough subset of the approximation space $\langle U^2, \theta^2 \rangle$. The common relational operators can now be defined on the set of rough binary relations as below. With some abuse of language we use the same notation for operators on rough relation as for the operators on binary relations:

$$\langle l(R), u(R) \rangle \,;\, \langle l(S), u(S) \rangle = \langle l(R) \,;\, l(S), u(R) \,;\, u(S) \rangle \qquad \text{composition}$$
$$\langle l(R), u(R) \rangle^\smile = \langle l(R^\smile), u(R^\smile) \rangle \qquad \text{converse}$$
$$1' = \langle \theta, \theta \rangle \qquad \text{identity}$$

The structure $\langle Rough(U^2), +, \cdot, ^*, ^+, 0, 1, ;, \smile, 1' \rangle$ is called the *full algebra of rough relations (over $\langle U, \theta \rangle$)*. An *algebra of rough relations* is a subalgebra of some full algebra of rough relations.

In analogy to Tarski's axiomatization of relation algebras [18], Comer proposed a tentative set of axioms for *rough relation algebras*, the intended standard models of which are algebras of rough relations. These algebras were investigated in some detail in [5] and [8]; numerous examples of rough relation algebras can be found in these papers. A rough relation algebra (R2A) is called *representable* if it is isomorphic to a subdirect product of algebras of rough relations. Even though the class of representable R2As is not finitely axiomatizable, it was shown in [5] that it is finitely axiomatizable over the class of representable relation algebras.

In this paper we take a different approach to the representation of R2As, namely, via a *discrete duality*. The term discrete duality is the name of a system $\langle(\mathsf{Alg}, \mathsf{Frm}, \mathfrak{Cm}, \mathfrak{Cf}\rangle$, where

- Alg is a class of lattices,

- Frm is a class of relational systems, referred to as *frames* according to the tradition in the field of non–classical logics,

- $\mathfrak{Cm} : \mathsf{Frm} \to \mathsf{Alg}$ is a mapping assigning to every frame $\mathscr{X} \in \mathsf{Frm}$ its complex algebra $\mathfrak{Cm}(\mathscr{X})$ in such a way that $\mathfrak{Cm}(X) \in \mathsf{Alg}$,

- $\mathfrak{Cf} : \mathsf{Alg} \to \mathsf{Frm}$ is a mapping assigning to every algebra $\mathfrak{L} \in \mathsf{Alg}$ its canonical frame $\mathfrak{Cf}(\mathfrak{L})$ in such a way that $\mathfrak{Cf}(\mathfrak{L}) \in \mathsf{Frm}$,

and the following two representation theorems hold:

- Every $\mathfrak{L} \in \mathsf{Alg}$ is embeddable into the complex algebra of its canonical frame $\mathfrak{Cm}(\mathfrak{Cf}(\mathfrak{L}))$.

- Every $\mathscr{X} \in \mathsf{Frm}$ is embeddable into the canonical frame of its complex algebra $\mathfrak{Cf}(\mathfrak{Cm}(\mathscr{X}))$.

In this case we say that the system $\langle(\mathsf{Alg}, \mathsf{Frm}, \mathfrak{Cm}, \mathfrak{Cf}\rangle$ is a *discrete duality* between the classes Alg and Frm. Frames, being semantic structures of formal languages, do not involve topology and therefore, such duality is said to be discrete. Discrete duality differs from the classical Stone, Jonssón-Tarski or Priestley dualities in that it requires that the complex algebra of any frame belongs to the class Alg, not only the complex algebra of a canonical frame $\mathfrak{Cf}(\mathfrak{L})$. Furthermore, discrete duality representation theorems establish an embedding of the underlying structures, not necessarily an isomorphism. A general outline of discrete duality can be found in [13].

The paper is organized as follows: First, we recall the definition of regular double Stone algebras and present some of their relevant properties. Then we introduce rough relation algebras and rough relation frames and prove the duality theorems.

2 Regular double Stone algebras

If $\langle P, \leq \rangle$ is a partially ordered set and $Q \subseteq P$ we let $\uparrow Q = \{x \in P : (\exists y \in P)[y \in Q \text{ and } y \leq x]\}$ and $\downarrow Q = \{x \in P : (\exists y \in P)[y \in Q \text{ and } y \geq x]\}$.

Suppose that $\langle L, +, \cdot, 0, 1 \rangle$ is a bounded distributive lattice with natural order $\leq$. An *ideal* of L is a nonempty subset I of L such that I is closed under join and $I = \downarrow$. Dually, a *filter* of L is a nonempty subset F of L such that F is closed under meet and $F = \uparrow F$. A filter F is called a *prime filter* if $a + b \in F$ implies $a \in F$ or $b \in F$.

Next, we briefly review some properties of regular double Stone algebras and their ordered sets of prime filters. A *double Stone algebra* (DSA) is a structure $\langle L, +, \cdot, {}^{*}, {}^{+}, 0, 1 \rangle$ of type $2, 2, 1, 1, 0, 0$ such that

> DSA$_1$. $\langle L, +, \cdot, 0, 1 \rangle$ is a bounded distributive lattice.
>
> DSA$_2$. For all $a, b \in L$, $a \cdot b = 0$ if and only if $b \leq a^*$.
>
> DSA$_3$. For all $a, b \in L$, $a + b = 1$ if and only if $a^+ \leq b$.
>
> DSA$_4$. For all $a \in L$, $a^* + a^{**} = 1$.
>
> DSA$_5$. For all $a \in L$, $a^+ \cdot a^{++} = 0$.

With some abuse of notation, in the sequel we will usually denote algebras by their base set.

a^* is called the *pseudocomplement of a*, and a^+ is called the *dual pseudocomplement of a*. An element a of L is called *dense*, respectively, *dually dense*, if it is of the form $a = b + b^*$ ($a = b \cdot b^+$) for some $b \in L$. The set of dense elements, respectively, dually dense elements, is denoted by D^*, respectively, D^+. It is well known that D^* is a filter and D^+ is an ideal of L. The set $B(L) = \{a^* : a \in L\}$ is called the *centre of L*. It is well known that $\langle B(L), +, \cdot, {}^{*}, 0, 1 \rangle$ is a subalgebra of $\langle L, +, \cdot, {}^{*}, 0, 1 \rangle$ and a Boolean algebra; in particular, $a + a^* = 1$ for all $a \in B(L)$. Furthermore, $B(L) = \{a^+ : a \in L\}$ and

$$(2.1) \qquad\qquad a^{*+} = a^{**}, \; a^{+*} = a^{++},$$

so that the restrictions of the operators * and $^+$ to $B(L)$ coincide.

It is well known (see e.g. [9], II.6, Exercise 11) that for a pseudocomplemented and dually pseudocomplemented distributive lattice

$$(2.2) \qquad\qquad (a \cdot b)^{**} = a^{**} \cdot b^{**} \text{ and } (a + b)^{++} = a^{++} + b^{++}.$$

In the sequel, let L be a double Stone algebra and let $\langle \mathrm{Prim}(L), \subseteq \rangle$ be its set of prime filters, partially ordered by set inclusion.

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Lemma 2.1. *(See [9], II.6, Theorem 6) For each $F \in \mathrm{Prim}(L)$ there are a unique minimal prime filter G and a unique maximal prime filter H such that $G \subseteq F \subseteq H$.*

In the sequel, let $\underline{F}$ be the minimal prime filter below F and let $\overline{F}$ be the maximal prime filter above F.

Lemma 2.2. *[7] Let F be a prime filter of L.*

1. *F is maximal if and only if $D^* \subseteq F$.*

2. *F is minimal if and only if $D^+ \cap F = \emptyset$.*

Recalling that $D^* = \{a + a^* : a \in L\}$ we obtain

Corollary 2.1. *Let $F \in \mathrm{Prim}(L)$.*

1. *F is maximal if and only if either $a \in F$ or $a^* \in F$ for all $a \in L$.*

2. *$\overline{F}$ is the filter generated by $F \cup D^*$.*

Lemma 2.3. *If $F, F' \in \mathrm{Prim}(L)$ and $F \subseteq F'$, then $F \cap B(L) = F' \cap B(L)$.*

Proof. "$\subseteq$": This follows immediately from $F \subseteq F'$.

"$\supseteq$": Let $a \in F' \cap B(L)$. Then, $a^* \notin F'$, and therefore, $F \subseteq F'$ implies $a^* \notin F$. It follows from $a \in B(L)$ that $a + a^* = 1$, and therefore, $a \in F$ since F is prime. $\square$

Lemma 2.4. *Let $F \in \mathrm{Prim}(L)$. Then, $\underline{F}$ is the filter generated by $F \cap B(L)$.*

Proof. Let G be the filter generated by $F \cap B(L)$, and let $a + b \in G$. Then, there is some $c \in F \cap B(L)$ such that $c \le a + b$; since $c \in B(L)$ we may suppose w.l.o.g. that $c = c^{++}$. Now, $c = c^{++} \le (a + b)^{++} = a^{++} + b^{++}$, the latter by (2.2), and $c \in F$ show that $a^{++} \in F$ or $b^{++} \in F$. Since $a^{++} \le a$ and $b^{++} \le b$ we have $a \in G$ or $b \in G$. Thus, G is a prime filter. By Lemma 2.3 and the definition of G we have $G \subseteq \underline{F}$, and thus, $G = \underline{F}$ since $\underline{F}$ is minimal. $\square$

A DSA L is called *regular* if

$$(2.3) \qquad\qquad\qquad a \cdot a^+ \le b + b^*$$

for all $a, b \in L$.

Lemma 2.5. *[20, 21] Let L be a double Stone algebra. Then, the following are equivalent:*

1. *L is regular*

2. *For all $a, b \in L$, $a^* = b^*$ and $a^+ = b^+$ implies $a = b$.*

3. *Each chain of prime filters has at most two elements.*

Lemma 2.5(3) together with Lemma 2.1 shows that in a regular DSA $\mathrm{Prim}(L)$ is the disjoint union of chains of length at most 2.

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3 Rough relation algebras and rough relation frames

In analogy to Tarski's relation algebras, Comer [3] has introduced a class of algebras which are intended to serve as an abstract counterpart to algebras of rough relations: A *rough relation algebra* (R2A) is a structure $\langle L, +, \cdot, {}^*, {}^+, 0, 1, ;, {}^\smile, 1' \rangle$ of type $2, 2, 1, 1, 0, 0, 2, 1, 0$ satisfying the following axioms:

R2A$_0$. $\langle L, +, \cdot, {}^*, {}^+, 0, 1 \rangle$ is a regular double Stone algebra.

R2A$_1$. $\langle L, ;, 1' \rangle$ is a semigroup with identity $1'$.

R2A$_2$. $a ; (b+c) = (a ; b) + (a ; c)$, $(b+c) ; a = (b ; a) + (c ; a)$.

R2A$_3$. $a^{\smile\smile} = a$.

R2A$_4$. $(a+b)^\smile = a^\smile + b^\smile$.

R2A$_5$. $(a ; b)^\smile = b^\smile ; a^\smile$.

R2A$_6$. $a^\smile ; (a ; b)^* \leq b^*$.

R2A$_7$. $(a^* ; b^*)^{**} = a^* ; b^*$.

R2A$_8$. $1'^{**} = 1'$.

In the sequel, L will denote an R2A. The original axiom system of [3] included the property

R2A$_{6a}$. $(a ; b) \cdot c \leq a ; a^\smile ; c$.

The inequality R2A$_{6a}$ is not required to characterize representability of R2As, since it was shown in [5] without R2A$_{6a}$ that an R2A L is representable if and only if it satisfies the equation

$$(a ; b)^{++} = a^{++} ; b^{++},$$

and $B(L)$ is a representable relation algebra.

The following properties of R2As can be found in [5]; we have checked that R2A$_{6a}$ was not used in the proofs:

Lemma 3.1. *Let $a, b, c \in L$. Then,*

1. $a \leq b \iff a^\smile \leq b^\smile$.

2. $(a \cdot b)^\smile = a^\smile \cdot b^\smile$.

3. $a^{*\smile} = a^{\smile*}$, $a^{\smile+} = a^{+\smile}$.

4. $(a ; b)^{**} = a^{**} ; b^{**}$.

5. $a \le b$ implies $a\,;c \le b\,;c$ and $c\,;a \le c\,;b$.

6. $(a\,;b)^* \cdot b^{\smile} \le a^*$.

7. D^* is closed under $;$ and $\smile$.

8. $(a\,;b) \cdot c = 0 \Longleftrightarrow (a^{\smile}\,;c) \cdot b = 0 \Longleftrightarrow (c\,;b^{\smile}) \cdot a = 0$.

If $F, G \subseteq L$, we define $F\,;G = \{c : (\exists a, b)[a \in F, b \in T, \text{ and } a\,;b \le c]\}$. Furthermore, we set $F^{\smile} = \{a^{\smile} : a \in F\}$.

Lemma 3.2. *[19] Suppose that F, G are filters of L.*

1. *$F\,;G$ is a (not necessarily proper) filter of L.*

2. *if H is a prime filter of L and $F\,;G \subseteq H$, then there are prime filters F', G' such that $F \subseteq F'$, $G \subseteq G'$ and $F'\,;G' \subseteq H$.*

Proof. 1. Let $c \in F\,;G$; there are $a \in F, b \in G$ such that $a\,;b \le c$. If $c \le c'$, then $a\,;b \le c'$ showing that $c' \in F\,;G$.

Let $c_0, c_1 \in F\,;G$; then, there are $a_0, a_1 \in F, b_0, b_1 \in G$ such that $a_0\,;b_0 \le c_0$ and $a_1\,;b_1 \le c_1$. From Lemma 3.1(5) it follows that $(a_0 \cdot a_1)\,;(b_0 \cdot b_1) \le (a_0\,;b_0) \cdot (a_1\,;b_1) \le c_0 \cdot c_1$. Since F, G are filters, $a_0 \cdot a_1 \in F$, $b_0 \cdot b_1 \in G$, and thus, $(a_0 \cdot a_1)\,;(b_0 \cdot b_1) \le c_0 \cdot c_1 \in F\,;G$.

If $a \in F$, $b \in G$ such that $a\,;b = 0$, then $F\,;G = L$.

2. Suppose that $\mathfrak{F}$ is the collection of all filters Q of L such that $F \subseteq Q$ and $Q\,;G \subseteq H$. Note that $\mathfrak{F} \ne \emptyset$, since $F \in \mathfrak{F}$. Furthermore, $\mathfrak{F}$ is closed under unions of chains, and thus, it contains a maximal element, say, F'. All that is left to show is that F' is prime. Let $a + b \in F'$ and assume that $a, b \notin F'$. Suppose that $F'(a)$ is the filter of L generated by $F' \cup \{a\}$. Since $a \notin F'$ we have $F' \subsetneq F'(a)$, and the maximality of F' implies that $F'(a)\,;G \not\subseteq H$. Thus, there exist $a_0 \in F'$, $a_1 \in G$ such that $(a_0 \cdot a)\,;a_1 \notin H$. Similarly, if $F'(b)$ is the filter generated by $F' \cup \{b\}$, there are $b_0 \in F', b_1 \in G$ such that $(b_0 \cdot b)\,;b_1 \notin H$. Since H is prime, it follows that $((a_0 \cdot a)\,;a_1) + ((b_0 \cdot b)\,;b_1) \notin H$. Now,

$$(a_0 \cdot b_0 \cdot a)\,;(a_1 \cdot b_1) + (a_0 \cdot b_0 \cdot b)\,;(a_1 \cdot b_1) = (a_0 \cdot b_0 \cdot a + a_0 \cdot b_0 \cdot b)\,;(a_1 \cdot b_1) \qquad \text{by R2A}_2$$
$$= (a_0 \cdot b_0 \cdot (a + b))\,;(a_1 \cdot b_1).$$

Since $a_0, b_0, a + b \in F'$ and $a_1, b_1 \in G$ it follows that $(a_0 \cdot b_0 \cdot a)\,;(a_1 \cdot b_1) + (a_0 \cdot b_0 \cdot b)\,;(a_1 \cdot b_1) \in F'\,;G \subseteq H$. On the other hand, $(a_0 \cdot b_0 \cdot a)\,;(a_1 \cdot b_1) \le (a_0 \cdot a)\,;a_1$ and $(a_0 \cdot b_0 \cdot b)\,;(a_1 \cdot b_1) \le (b_0 \cdot b)\,;b_1$. Hence, $(a_0 \cdot b_0 \cdot a)\,;(a_1 \cdot b_1) + (a_0 \cdot b_0 \cdot b)\,;(a_1 \cdot b_1) \le ((a_0 \cdot a)\,;a_1) + ((b_0 \cdot b)\,;b_1) \in H$, a contradiction.

The existence of G' is shown analogously. $\square$

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A *rough relation frame* is a structure $\mathscr{X} = \langle X, \leq, R, f, I \rangle$ where $\leq$ is a partial order on X which is the disjoint union of chains of length at most 2. If $x \in X$, we denote the minimal element below x by $\underline{x}$, and the maximal element above x by $\bar{x}$. Furthermore R is a ternary relation on X, $f : X \to X$ a function, and $I \subseteq X$ for which the following axioms hold:

R2F$_1$. $R(x,y,z)$, $x' \leq x$, $y' \leq y$, $z \leq z' \Rightarrow R(x',y',z')$.

R2F$_2$. $x \leq y \Rightarrow f(x) \leq f(y)$.

R2F$_3$. I is $\leq$–closed and $\geq$–closed.

R2F$_4$. $R(x,y,z)$ and $R(z,v,w) \Rightarrow (\exists u)[R(x,u,w)$ and $R(y,v,u)]$.

R2F$_5$. $R(x,y,z)$ and $R(v,z,w) \Rightarrow (\exists u)[R(u,y,w)$ and $R(v,x,u)]$.

R2F$_6$. $f(f(x)) = x$.

R2F$_7$. $f(\bar{x}) = \overline{f(x)}, f(\underline{x}) = \underline{f(x)}$.

R2F$_8$. $R(x,y,z) \Rightarrow R(f(x),z,\bar{y})$.

R2F$_9$. $R(x,y,z) \Rightarrow R(z,f(y),\bar{x})$.

R2F$_{10}$. $R(x,y,z) \Rightarrow R(\underline{x},\underline{y},\underline{z})$.

R2F$_{11}$. $x \leq y \Longleftrightarrow (\exists z)[z \in I$ and $R(x,z,y)]$.

R2F$_{12}$. $z \leq y \Longleftrightarrow (\exists x)[x \in I$ and $R(x,z,y)]$.

Lemma 3.3. *For all $x,y,z \in X$, $R(x,y,z) \Rightarrow R(\bar{x},\bar{y},\bar{z})$.*

Proof. Suppose that $R(x,y,z)$. Then, $R(f(x),z,\bar{y})$ by R2F$_8$, $R(x,\bar{y},\bar{z})$ by R2F$_6$ and R2F$_8$, $R(\bar{z},f(\bar{y}),\bar{x})$ by R2F$_9$, and, finally, $R(\bar{x},\bar{y},\bar{z})$ by R2F$_6$ and R2F$_9$. $\square$

4 The duality theorems

The *canonical frame of L* has the ordered set $\mathrm{Prim}(L)$ as its base set. For $F,G,H \in \mathrm{Prim}(L)$ we let $R_{\mathfrak{L}}(F,G,H)$ if and only if $F \, ; G \subseteq H$, $f_{\mathfrak{L}}(F) = F^{\smile}$, and $I_{\mathfrak{L}} = \{F \in \mathrm{Prim}(L) : 1' \in F\}$.

Theorem 4.1. *The canonical frame of a rough relation algebra satisfies* R2F$_1$ – R2F$_{12}$.

Proof. R2F$_1$ – R2F$_3$ and R2F$_6$ are immediate from the definitions and Lemma 3.1. R2F$_4$ and R2F$_5$ can be proved with Lemma 3.2 as in [14].

R2F$_7$: Let $F \in \mathrm{Prim}(L)$, and $\bar{F}$ be the maximal prime filter containing F. Let $a \in \bar{F}^{\smile}$; then, $a^{\smile} \in \bar{F}$. Since $\bar{F}$ is generated by $F \cup D^*$, there are $b \in F, d \in D^*$ such that $b \cdot d \leq a^{\smile}$. Now, $b^{\smile} \in F^{\smile}$ and $d^{\smile} \in D^*$ by Lemma 3.1(7) shows that $a \in \overline{F^{\smile}}$. Conversely, let $a \in \overline{F^{\smile}}$; there are some $b \in F^{\smile}, d \in D^*$ such that $b \cdot d \leq a$, so that $b^{\smile} \cdot d^{\smile} \leq a^{\smile}$. Since $b^{\smile} \in F$ and $d^{\smile} \in D^*$ we obtain $a^{\smile} \in \bar{F}$, and therefore, $a \in \bar{F}^{\smile}$.

Next, let $\underline{F}$ be the minimal prime filter containing F. Let $a \in \underline{F}^\smile$; then, $a^\smile \in \underline{F}$. By Lemma 2.4, there is some $b \in F \cap B(L)$ such that $b \leq a^\smile$. Now, $b^\smile \in F^\smile \cap B(L)$ and $b^\smile \leq a$ show that $a \in \underline{F^\smile}$. The other direction is similar.

R2F$_8$: Let $F,G,H \in \mathrm{Prim}(L)$, $F\,;G \subseteq H$, and $\overline{G}$ be the maximal prime filter containing G. Suppose that $z \in F^\smile\,;H$. Then, there are $a \in F$, $b \in H$ such that $a^\smile\,;b \leq z$. Assume that $a^\smile\,;b \notin \overline{G}$. Since $\overline{G}$ is maximal we have $(a^\smile\,;b)^* \in \overline{G}$, and therefore, $(a^\smile\,;b)^* \in G$ since $G \cap B(L) = \overline{G} \cap B(L)$. Now, $a\,;(a^\smile\,;b)^* \in F\,;G \subseteq H$, and thus, $b^* \in H$ by R2A$_6$. This contradicts $b \in H$.

R2F$_9$: Let $F,G,H \in \mathrm{Prim}(L)$, $F\,;G \subseteq H$, and $\overline{F}$ be the maximal prime filter containing F. Suppose that $z \in H\,;G^\smile$. Then, there are $a \in H, b \in G$ such that $a\,;b^\smile \leq z$. Assume that $a\,;b^\smile \notin \overline{F}$. Since $\overline{F}$ is maximal, this implies that $(a\,;b^\smile)^* \in \overline{F}$, hence, $(a\,;b^\smile)^* \in F$, since $(a\,;b^\smile)^*$ is Boolean. Therefore, $(a\,;b^\smile)^*\,;b \in F\,;G \subseteq H$. Now, Lemma 3.1(6) shows that $(a\,;b^\smile)^*\,;b \leq a^*$. It follows that $a^* \in H$ which contradicts $a \in H$.

R2F$_{10}$: Let $F,G,H \in \mathrm{Prim}(L)$, $F\,;G \subseteq H$, $\underline{F}$ be the minimal prime filter below F, $\underline{G}$ be the minimal prime filter below G, and $\underline{H}$ the minimal prime filter below H. Suppose that $a \in \underline{F}\,;\underline{G}$; then, there are $b \in \underline{F}, c \in \underline{G}$ such that $b\,;c \leq a$. Since $\underline{F},\underline{G}$ are minimal, by Lemma 2.4 we may suppose that $b,c \in B(L)$. From Lemma 3.1(4) we obtain $b\,;c \in B(L)$, and now $F\,;G \subseteq H$ implies $b\,;c \in B(L) \cap H$. It follows from Lemma 2.4 that $b\,;c \in \underline{H}$, and $b\,;c \leq a$ implies $a \in \underline{H}$.

R2F$_{11}$: Let $F \subseteq G$ and let $\uparrow \{1'\}$ be the principal filter of L generated by $\{1'\}$. Then, $F\,;\uparrow\{1'\} \subseteq G$ and by Lemma 3.2 there is some prime filter H such that $1' \in H$ and $F\,;H \subseteq G$.

R2F$_{12}$: Analogous. $\qquad\square$

Let $\mathscr{X} = \{X,\leq,R,f,I\}$ be a rough relation frame. The *complex algebra* $\mathfrak{Cm}(X)$ of $\mathscr{X}$ has as its universe the set $L(X)$ of all increasing subsets of X. For $Y \in L(X)$ we let Y^* be the largest increasing subset contained in $-Y$, and Y^+ be the smallest increasing subset containing $-Y$, so that

$$Y^* = \{y :\uparrow \{y\} \cap Y = \emptyset\},$$
$$Y^+ = \{y :\downarrow y \cap -Y \neq \emptyset\}.$$

Furthermore, we set $Y\,;_{\mathscr{X}} Z = \{t \in X : (\exists y,z)[y \in Y, z \in Z, \text{ and } R(y,z,t)]\}$, $Y^{\smile_{\mathscr{X}}} = \{f(x) : x \in Y\}$, and $1'_{\mathscr{X}} = I$.

Lemma 4.1. *Let $Y \subseteq X$ be increasing. Then, $Y^{**} = \{x \in X : (\exists y)[y \in Y \text{ and } x \leq y]\}$.*

Proof. "$\subseteq$": Let $x \in Y^{**}$; then, $\uparrow \{x\} \cap Y^* = \emptyset$, in particular, $x \notin Y^*$. Hence, $\uparrow \{x\} \cap Y \neq \emptyset$.

"$\supseteq$": Let $x \leq y$ and $y \in Y$. Assume that $\uparrow \{x\} \cap Y^* \neq \emptyset$. Then, there is some $z \in X$ such that $x \leq z$ and $\uparrow \{z\} \cap Y = \emptyset$. Since each chain in X has at most two elements, we must have $z \in \{x,y\}$, contradicting that $y' \in Y$. $\qquad\square$

Corollary 4.1. $Y = Y^{**}$ *if and only if* $Y =\downarrow\uparrow Y$.

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Lemma 4.2. *If $Y, Z \subseteq X$ are increasing, so are $Y \;_{\mathscr{X}} Z$ and $Y^{\smile \mathscr{X}}$.*

Proof. Let $x \in Y \;_{\mathscr{X}} Z$, and $x \leq x'$. Then, there are $y \in Y, z \in Z$ such that $R(y,z,x)$, and from R2F$_1$ we obtain $R(y,z,x')$, hence, $x' \in Y \;_{\mathscr{X}} Z$.

Let $x \in f(Y)$ and $x \leq x'$. Then, $f(x) \leq f(x')$ by R2F$_2$, and $f(x) \in Y$ by R2F$_6$. Since Y is increasing, we have $f(x') \in Y$, and R2F$_6$ implies $x' \in f(Y)$. □

Theorem 4.2. *The complex algebra of a rough relation frame is a rough relation algebra.*

Proof. We have shown in [7] that $\langle L(X), \cap, \cup, {}^*, {}^+, \emptyset, X \rangle$ is a regular double Stone algebra.

Next, we show that $1'_{\mathscr{X}}$ is the identity element of $\mathfrak{Cm}(\mathscr{X})$: Let $A \subseteq X$ be increasing. We first show $A \;_{\mathscr{X}} 1'_{\mathscr{X}} = A$:

"$\subseteq$": Let $z \in A \;_{\mathscr{X}} 1'_{\mathscr{X}}$. Then, there are $x \in A$, $y \in I$ such that $R(x,y,z)$. By the $\Leftarrow$ direction of R2F$_{11}$ we obtain $x \leq z$. Since A is increasing and $x \in A$ it follows that $z \in A$.

"$\supseteq$": Let $z \in A$. Since $\leq$ is reflexive we have $z \leq z$ and by the $\Rightarrow$ direction of R2F$_{11}$ there is some $y \in I$ such that $R(z,y,z)$. This implies $z \in A \;_{\mathscr{X}} 1'_{\mathscr{X}}$.

Next, we show that $1'_{\mathscr{X}} \;_{\mathscr{X}} A = A$:

"$\subseteq$": Let $t \in 1'_{\mathscr{X}} \;_{\mathscr{X}} A$. Then, there are $x \in I$ and $z \in A$ such that $R(x,z,t)$. By the $\Leftarrow$ direction of R2F$_{12}$ we obtain $z \leq t$. Since A is increasing and $z \in A$ it follows that $t \in A$.

"$\supseteq$": Let $z \in A$. Since $\leq$ is reflexive we have $z \leq z$ and by the $\Rightarrow$ direction of R2F$_{12}$ there is some $y \in I$ such that $R(y,z,z)$. This implies $z \in 1'_{\mathscr{X}} \;_{\mathscr{X}} A$.

The proofs that $\mathscr{X}$ satisfies R2A$_1$ – R2A$_5$ are the same as in the classical case, and can be found in [14], Theorem 6.

R2A$_6$: Let $Y, Z \in L(X)$ and $z \in Y^{\smile \mathscr{X}} \;_{\mathscr{X}} (Y \;_{\mathscr{X}} Z)^*$. Then, there are $x \in Y, y \in (Y \;_{\mathscr{X}} Z)^*$ such that $R(f(x),y,z)$. By R2F$_1$, we have $R(fx),y,\bar{z})$ and $R(x,\bar{z},\bar{y})$ by R2F$_8$. Now,

(4.1) $$y \in (Y \;_{\mathscr{X}} Z)^* \Longleftrightarrow \uparrow \{y\} \cap (Y \;_{\mathscr{X}} Z) = \emptyset,$$

in particular, $\bar{y} \notin Y \;_{\mathscr{X}} Z$.

Assume $\uparrow \{z\} \cap Z \neq 0$; then, $\bar{z} \in Z$ since Z is increasing. Since $x \in Y$ and by $R(x,\bar{z},\bar{y})$ we have $\bar{y} \in Y \;_{\mathscr{X}} Z$, a contradiction.

R2A$_7$: Let $Y, Z \in L(X)$ and $x \in (Y^* \;_{\mathscr{X}} Z^*)^{**}$. By Lemma 4.1, there is some $t \in Y^* \;_{\mathscr{X}} Z^*$ such that $x \leq t$; observe that $\underline{t} \leq x$. Suppose that $r,s \in X$ such that $\uparrow \{r\} \cap Y = \emptyset$, $\uparrow \{s\} \cap Z = \emptyset$, and $R(r,s,t)$. If r is minimal. then $r = \underline{r}$, and therefore, $\uparrow \{\underline{r}\} \cap Y = \emptyset$. If r is not minimal, then $\underline{r} < r$, and $\uparrow \{\underline{r}\} = \{\underline{r},r\}$. Now, $r \notin Y$, together with the fact that Y is increasing, imply $\uparrow \{\underline{r}\} \cap Y = \emptyset$. Similarly, $\uparrow \{\underline{s}\} \cap Z = \emptyset$, so that $\underline{r} \in Y^*$ and $\underline{s} \in Z^*$. Now, $R(r,s,t)$ and R2F$_{10}$ imply $R(\underline{r},\underline{s},\underline{t})$, and it follows from $\underline{t} \leq x$ and R2F$_1$ that $R(\underline{r},\underline{s},x)$. Hence, $x \in Y^* \;_{\mathscr{X}} Z^*$.

R2A$_8$: This is R2F$_3$ together with Corollary 4.1. □

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Let $h : L \to \mathfrak{Cm}(\mathfrak{Cf}(L))$ be defined by $h(x) = \{F \in \mathrm{Prim}(L) : x \in F\}$.

Theorem 4.3. *h is an embedding of R2As.*

Proof. We have shown in [7] that h preserves the operations $+, \cdot, {}^{*}, {}^{+}, 0, 1$. Let $a, b \in L$; we show that $h(a \, ; b) = h(a) \; ;_{\mathscr{X}} h(b)$:

"$\subseteq$": Let $H \in h(a \, ; b)$, i.e. $a \, ; b \in H$. Since H is a filter it follows that $\uparrow \{a\} \, ;\uparrow \{b\} \subseteq H$. By Lemma 3.2, there are prime filters F, G such that $\uparrow \{a\} \subseteq F, \uparrow \{b\} \subseteq G$, and $F \, ; G \subseteq H$. Since $a \in F$ and $b \in G$ we have $a \, ; b \in H$, i.e. $H \in h(a) \; ;_{\mathscr{X}} h(b)$.

"$\supseteq$": Suppose that $H \in h(a) \; ;_{\mathscr{X}} h(b)$. Then, there are $F \in h(a), G \in h(b)$ such that $R(F, G, H)$, i.e. $F \, ; G \subseteq H$. It follows from $a \in F, b \in G$ that $a \, ; b \in H$, i.e. $H \in h(a \, ; b)$.

Next,

$$H \in h(a^{\smallsmile}) \Longleftrightarrow a^{\smallsmile} \in H \Longleftrightarrow a^{\smallsmile\smallsmile} \in f(H) \Longleftrightarrow a \in f(H)$$
$$\Longleftrightarrow f(f(H)) \in h(a)^{\smallsmile_{\mathscr{X}}} \Longleftrightarrow H \in h(a)^{\smallsmile_{\mathscr{X}}}.$$

Finally,

$$H \in h(1') \Longleftrightarrow 1' \in H \Longleftrightarrow H \in I \Longleftrightarrow H \in 1'_{\mathscr{X}}.$$

This completes the proof. $\square$

Let $\mathscr{X} = \langle X, \leq, R, f, I \rangle$ be a rough relation frame, and let $k : X \to \mathfrak{Cf}(\mathfrak{Cm}(X))$ be defined by $k(x) = \{A \in L(X) : x \in A\}$; note that $k(x)$ is the (principal) prime filter of $L(X)$ generated by $\uparrow \{x\}$.

Theorem 4.4. *k preserves R, f, and I.*

Proof. First, we will show that $R(x, y, z) \Longleftrightarrow R_{\mathfrak{Cf}(\mathfrak{Cm}(X))}(k(x), k(y), k(z))$. If $F, G, H \in \mathrm{Prim}(\mathfrak{Cm}(X))$, then, by the definition,

$R_{\mathfrak{Cf}(\mathfrak{Cm}(X))}(F, G, H) \Longleftrightarrow F \; ;_{\mathscr{X}} G \subseteq H \quad$ by definition of $R_{\mathfrak{Cf}(\mathfrak{Cm}(X))}$
$\qquad\qquad \Longleftrightarrow \{C : (\exists A \in F, B \in G) A \; ;_{\mathscr{X}} B \subseteq C\} \subseteq H \quad$ by definition of complex $\; ;_{\mathscr{X}}$
$\qquad\qquad \Longleftrightarrow (A \in F \text{ and } B \in G \Rightarrow A \; ;_{\mathscr{X}} B \in H) \quad$ since H is a filter
$\qquad\qquad \Longleftrightarrow (A \in F \text{ and } B \in G \Rightarrow \{u \in X : (\exists s \in A, t \in B) R(s, t, u)\} \in H)$

by definition of $\; ;_{\mathscr{X}}$, so that

$$R_{\mathfrak{Cf}(\mathfrak{Cm}(X))}(k(x), k(y), k(z)) \Longleftrightarrow (x \in A \text{ and } y \in B \Rightarrow z \in \{c \in X : (\exists a \in A, b \in B) R(a, b, c)\}).$$

"$\Rightarrow$": Suppose that $R(x, y, z)$, and let $x \in A$ and $y \in B$. Since $R(x, y, z)$ it follows that $z \in \{u \in X : (\exists s \in A, t \in B) R(s, t, u)\})$, and therefore, $R_{\mathfrak{Cf}(\mathfrak{Cm}(X))}(k(x), k(y), k(z))$.

"$\Leftarrow$": Suppose that $R_{\mathfrak{Cf}(\mathfrak{Cm}(X))}(k(x), k(y), k(z))$, and set $A = \uparrow \{x\}$, $B = \uparrow \{y\}$.

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1. $x = \underline{x}$, $y = \underline{y}$: Then, $A = \{\underline{x}, \bar{x}\}, B = \{\underline{y}, \bar{y}\}$. Let $s \in A, t \in B$ such that $R(s,t,z)$. By R2F$_1$, $R(\underline{s},\underline{t},z)$ and thus, $R(x,y,z)$.

2. $x = \bar{x}$, $y = \bar{y}$: Then, $A = \{x\}, B = \{y\}$, and therefore, $R(x,y,z)$.

3. $x = \underline{x}$, $y = \bar{y}$: Then, $A = \{\underline{x}, \bar{x}\}, B = \{y\}$. Let $s \in A, t \in B$ such that $R(s,t,z)$; then, $t = y$. By R2F$_1$, $R(\underline{s},y,z)$ and thus, $R(x,y,z)$.

4. $x = \bar{x}$, $y = \underline{y}$: Analogous.

$k(f(x)) = f_{\mathfrak{Cf}(\mathfrak{Cm}(X))}(k(x))$: Using R2F$_6$ repeatedly we obtain

$$A \in k(f(x)) \iff f(x) \in A \iff x \in A^{\smile \mathscr{x}} \iff A^{\smile \mathscr{x}} \in k(x) \iff A \in f_{\mathfrak{Cf}(\mathfrak{Cm}(X))}(k(x).$$

Finally we show that $x \in I \iff k(x) \in I_{\mathfrak{Cf}(\mathfrak{Cm}(X))}$:

$$x \in I \iff I \in k(x) \iff 1'_{\mathscr{x}} \in k(x) \iff k(x) \in I_{\mathfrak{Cf}(\mathfrak{Cm}(X))}.$$

This completes the proof. □

5 Conclusion

In this paper we extended discrete dualities for relation algebras [14] and regular double Stone algebras [7] to rough relation algebras. Rough relation algebras differ from relation algebras in that the underlying Boolean algebras are replaced by regular double Stone algebras. As a consequence, the axiom $a\check{\ } ; -(a ; b) \le -b$ of relation algebras which is equivalent to De Morgan theorem K and involves the Boolean complement was replaced by the axiom R2A$_6$ obtained from it by using the pseudocomplement * instead of the Boolean complement $-$. Moreover, the axioms R2A$_7$ and R2A$_8$ were added which further characterize the action of the pseudocomplement with composition of rough relations and with the unit element of the monoid, respectively. As shown in [7], the frames associated with regular double Stone algebras are obtained from the frames of double Stone algebras by postulating that their partial order is the disjoint union of chains of length at most 2. This specific feature implied that properties of relation R – being a frame counterpart to the algebraic operation of composition of rough relations, and of the function f – corresponding to the operation of converse of rough relations – had to be confronted with this particular ordering, and tuned accordingly. The frame axioms R2F$_8$ – R2F$_{10}$ were postulated in response to that requirement. Furthermore, the RA frame axiom characterizing the set I, which is a counterpart to the unit element of the monoid, is split into two axioms R2F$_{11}$ and R2F$_{12}$. It was shown in the present paper that these frames are an adequate counterpart to rough relation algebras, so that a discrete duality between the two classes of structures holds.

A discrete duality is a step towards a duality via truth [12]. Duality via truth holds between a class of algebras and a class of frames whenever they are equivalent as semantic structures of a formal language whose signature coincides with the signature of the algebras in question. This amounts to saying that the notions of truth determined by these classes of structures are the same in that the structures validate the same formulas of the language. Development of duality via truth for rough relation algebras and their associated frames is an open problem. Once a frame semantics of a logic is given, a dual tableaux proof system for the logic can be constructed following the methods presented in [11]. Thus providing duality via truth for a logic of rough relations would enable us to contribute to automated deduction for the logic.

Further work is also needed on the status of the inequality $R2A_{6a}$ which was postulated in [3] as an axiom. As shown in [2] it is true in relation algebras. Neither the conditions of representability of rough relation algebras nor the discrete duality developed in the paper depend on that axiom.

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