Discrete Empirical Interpolation for Nonlinear Model Reduction

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Introduction

- Model order reduction (MOR): Reduce the computational complexity/time of large-scale dynamical systems by approximations of much lower dimension with nearly the same input/output response characteristics.
- Goal: Construct reduced-order model for finite difference (FD) discretization of unsteady and/or parametrized nonlinear PDEs. E.g., PDE:

$$\frac{\partial y}{\partial t}(x,t) = \mathcal{L}(y(x,t)) + F(y(x,t))$$

with FD system (ODE dim n):

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{F}(\mathbf{y}(t)).$$

- Common MOR method: Galerkin projection with basis $\mathbf{V}_k \in \mathbb{R}^{n\times k}$ from Proper Orthogonal Decomposition (POD), for $k \ll n$: $\mathbf{y} \leftarrow \mathbf{V}_k\tilde{\mathbf{y}}(t)$,

$$\frac{d}{dt}\tilde{\mathbf{y}}(t) = \underbrace{\mathbf{V}_k^T\mathbf{A}\mathbf{V}_k}_{\tilde{\mathbf{A}}:k\times k}\tilde{\mathbf{y}}(t) + \underbrace{\mathbf{V}_k^T\mathbf{F}(\mathbf{V}_k\tilde{\mathbf{y}}(t))}_{\tilde{\mathbf{N}}(\tilde{\mathbf{y}})}.$$

- Effectiveness of POD-Galerkin approach is limited to the linear or bi-linear terms; the projected nonlinear term still *depends on the dimension of the original system*:

$$\tilde{\mathbf{N}}(\tilde{\mathbf{y}}) = \underbrace{\mathbf{V}_k^T}_{k\times n}\underbrace{\mathbf{F}(\mathbf{V}_k\tilde{\mathbf{y}}(t))}_{n\times 1}.$$

- To remove this inefficiency, we proposed **"Discrete Empirical Interpolation Method (DEIM)"** for nonlinear approximation. For $m \ll n$,

$$\tilde{\mathbf{N}}(\tilde{\mathbf{y}}) \approx \underbrace{\mathbf{V}_k^T\mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}}_{\text{precomputed}:k\times m}\underbrace{\mathbf{F}(\mathbf{P}^T\mathbf{V}_k\tilde{\mathbf{y}}(t))}_{m\times 1}.$$

Discrete Empirical Interpolation Method (DEIM)

DEIM: A discrete variation of the EIM proposed in [1].

- Let $\mathbf{f}: \mathscr{D} \mapsto \mathbb{R}^n$ with $\mathscr{D} \subset \mathbb{R}^d$, $d \in \mathbb{Z}_+$.
- Let $\{\mathbf{u}_\ell\}_{\ell=1}^m \subset \mathbb{R}^n$ be a linearly independent set, for $m = 1,\dots,n$.
- For $\tau \in \mathscr{D}$, the DEIM approximation of order m for $\mathbf{f}(\tau)$ in the space spanned by $\{\mathbf{u}_\ell\}_{\ell=1}^m$ is given by

$$\hat{\mathbf{f}}(\tau) := \mathbf{U}\mathbf{c},$$

  - $(\mathbf{P}^T\mathbf{U})\mathbf{c} = \mathbf{P}^T\mathbf{f}(\tau)$
  - $\mathbf{U} = [\mathbf{u}_1,\dots,\mathbf{u}_m]$, $\mathbf{P} = [\mathbf{e}_{\wp_1},\dots,\mathbf{e}_{\wp_m}] \in \mathbb{R}^{n\times m}$
  - $\{\wp_1,\dots,\wp_m\}$ = output from DEIM Algorithm with the input basis $\{\mathbf{u}_i\}_{i=1}^m$.
- DEIM algorithm is well-defined
  - $\mathbf{P}^T\mathbf{U}$ nonsingular, $\{\wp_1,\dots,\wp_m\}$ non-repeated
- That is,

$$\hat{\mathbf{f}}(\tau) = \mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}\mathbf{P}^T\mathbf{f}(\tau).$$

- Error bound:

$$\|\mathbf{f}(\tau) - \hat{\mathbf{f}}(\tau)\|_2 \le C\mathcal{E}_*(\tau),$$

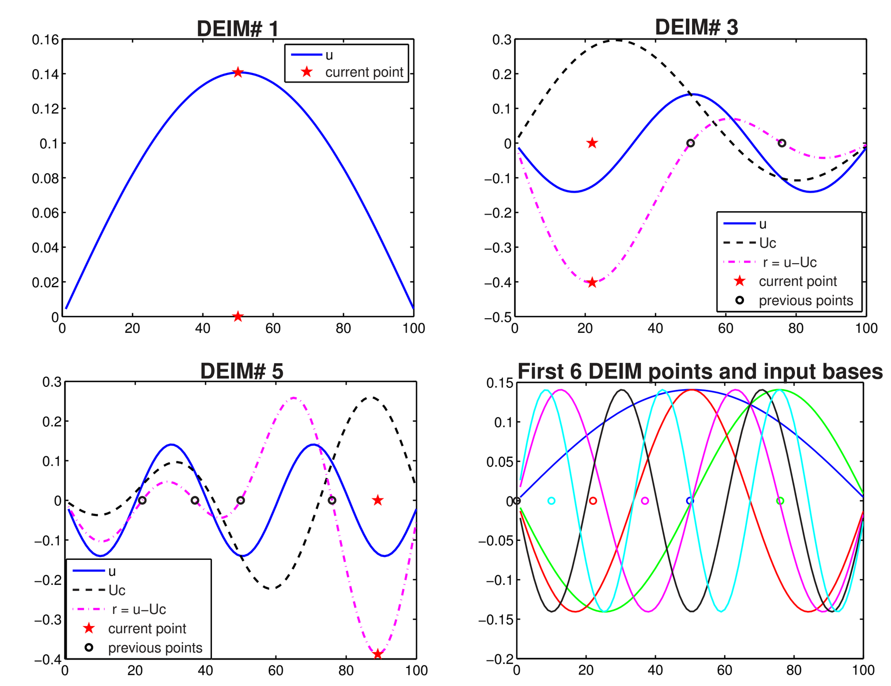
$\mathcal{E}_*(\tau) = \|(\mathbf{I} - \mathbf{U}\mathbf{U}^T)\mathbf{f}(\tau)\|_2$, for $m = 1$,
$C = 1/|\mathbf{e}_{\wp_1}^T\mathbf{u}_1|$, and for $m = 2,\dots,n$
$C = \sqrt{m}\,(1+\|\mathbf{a}\|_1)\max\left\{\|\bar{\mathbf{M}}^{-1}\|_1, \frac{1}{\rho}(1+\|\mathbf{c}\|_1)\right\}$.

DEIM Algorithm: Interpolation Indices

INPUT: $\{\mathbf{u}_\ell\}_{\ell=1}^m \subset \mathbb{R}^n$ (linearly independent)

OUTPUT: $\wp_1,\dots,\wp_m$

- $[\rho\ \ \wp_1] = max|\mathbf{u}_1|$
  $\mathbf{U} = [\mathbf{u}_1]$, $\vec{\wp} = [\wp_1]$,
- for $\ell = 2$ to m
  $\mathbf{u} \leftarrow \mathbf{u}_\ell$
  Solve $\mathbf{U}_{\vec{\wp}}\mathbf{c} = \mathbf{u}_{\vec{\wp}}$ for $\mathbf{c}$
  $\mathbf{r} = \mathbf{u} - \mathbf{U}\mathbf{c}$
  $[\rho\ \ \wp_i] = max\{|\mathbf{r}|\}$
  $\mathbf{U} \leftarrow [\mathbf{U}\ \ \mathbf{u}]$, $\vec{\wp} \leftarrow \begin{bmatrix}\vec{\wp}\\ \wp_i\end{bmatrix}$



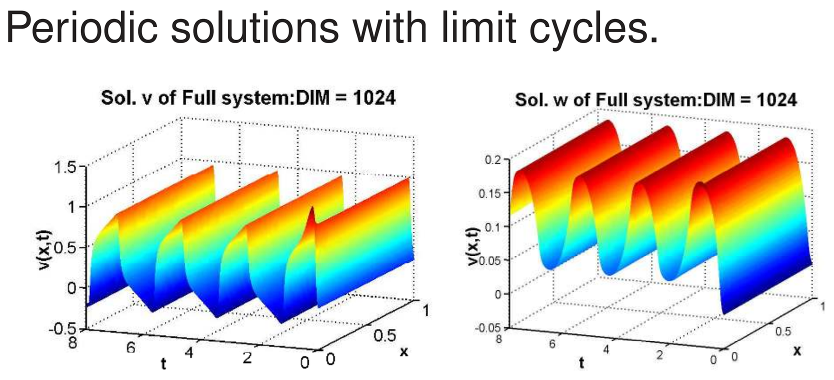
Numerical Results: The FitzHugh-Nagumo System (1-D Unsteady Nonlinear PDE)

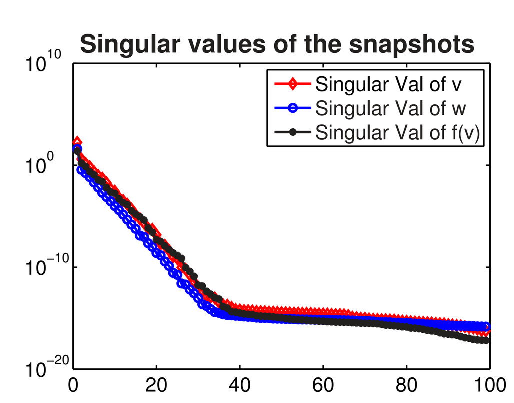
Let $x \in [0,L]$, $t \ge 0$,

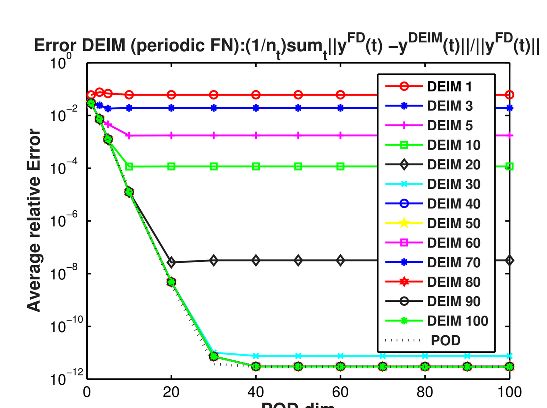
$$\varepsilon v_t(x,t) = \varepsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + c$$
$$w_t(x,t) = bv(x,t) - \gamma w(x,t) + c,$$

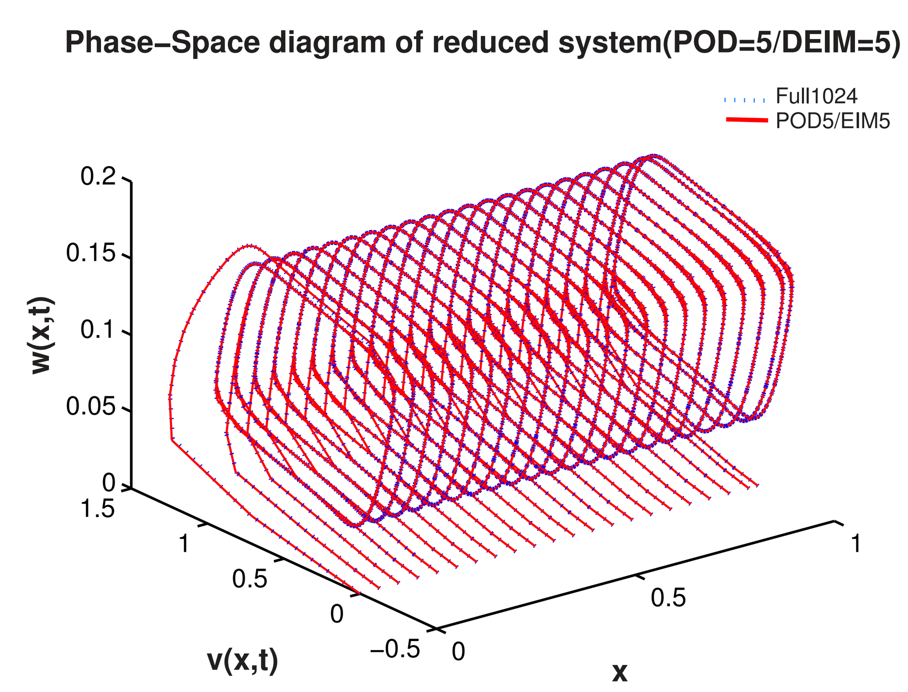
where $f(v) = v(v-0.1)(1-v)$ with initial conditions and boundary conditions:
$v(x,0) = 0$, $w(x,0) = 0$, $x \in [0,L]$
$v_x(0,t) = -i_0(t)$, $v_x(L,t) = 0$, $t \ge 0$ where
$L = 1$, $\varepsilon = 0.015$, $b = 0.5$, $\gamma = 2$,
$i_0(t) = 50000t^3 exp(-15t)$, $c = 0.05$, $t \in [0,8]$.

- Modeling in cardiac electrical activity: v = voltage variable, w= recovery variable.
- Periodic solutions with limit cycles.









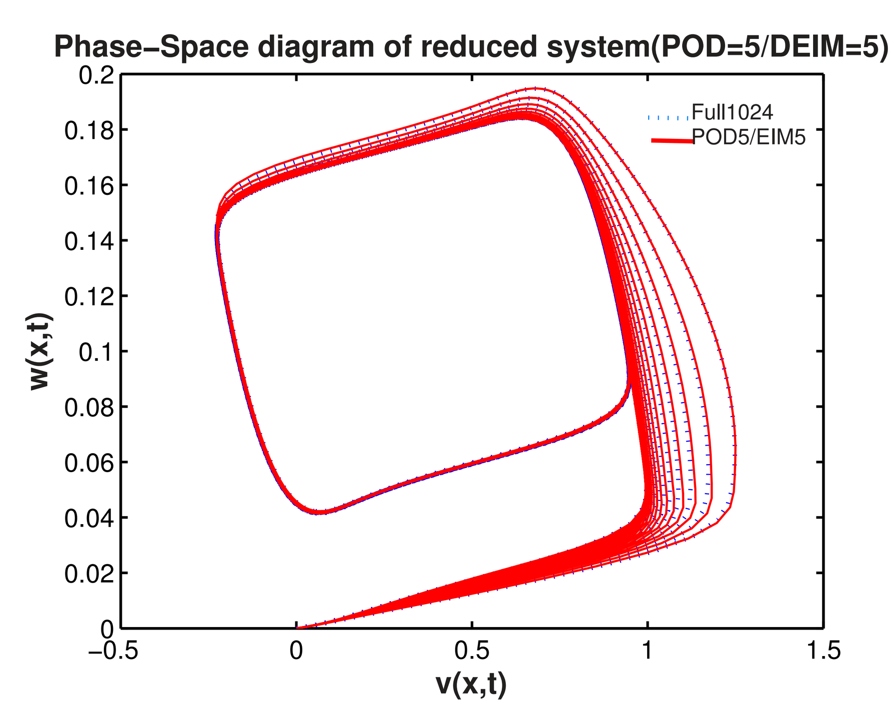


Figure: **DIM: Full = 1024, POD=5/DEIM=5**

Numerical Results: Highly Nonlinear 2-D Steady State Problem

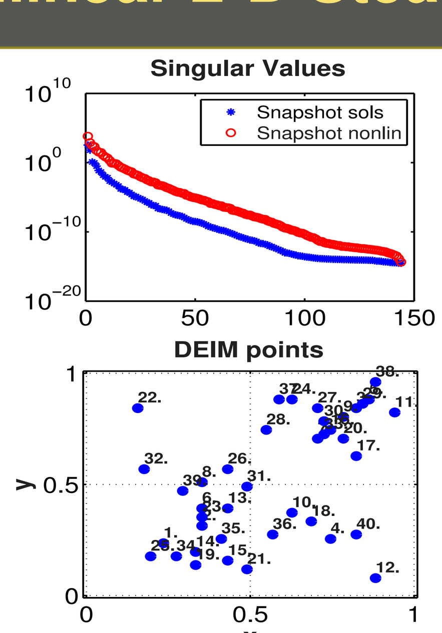
Let $(x,y) \in \Omega = (0,1)^2$.

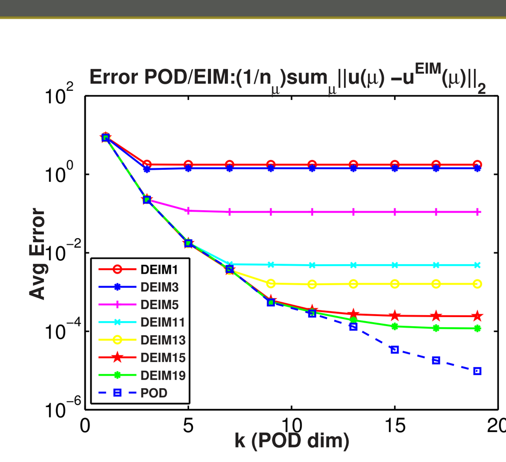
$$-\nabla^2 u(x,y) + s(u(x,y);\mu) = f(x,y),$$

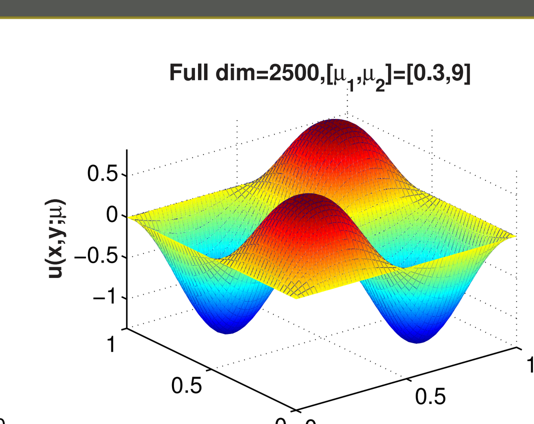
$$s(u;\mu) = \frac{\mu_1}{\mu_2}(e^{\mu_2 u} - 1),$$

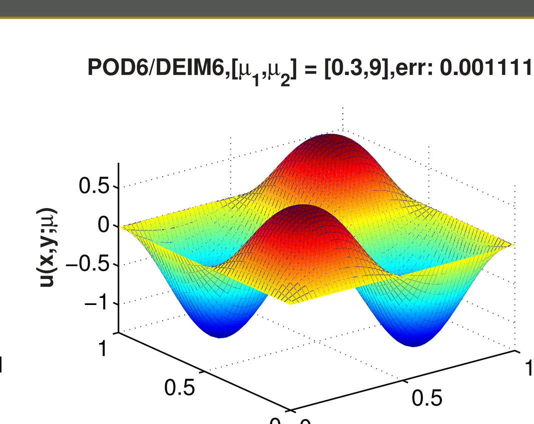
$f(x,y) = 100\sin(2\pi x)\sin(2\pi y)$,
$\mu = (\mu_1,\mu_2) \in \mathscr{D} = [0.01, 10]^2 \subset \mathbb{R}^2$, and the homogeneous Dirichlet boundary condition. Full dimension: $n = 2500$.

- FD system solved by Newton's method.









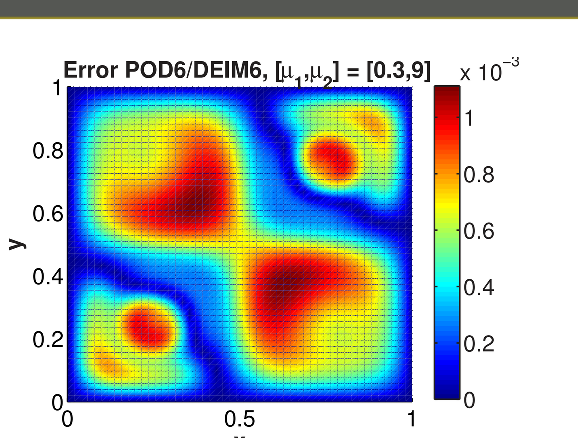


Figure: **DIM: Full = 2500, POD=6/DEIM=6**

[1] M. Barrault, Y. Maday, N. C. Nguyen, and A. T. Patera, An 'Empirical Interpolation' Method: Application to Efficient Reduced-Basis Discretization Of Partial Differential Equations, *Comptes Rendus Mathematique* , vol.339, 2004, pp 667-672.