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DISCRETE EQUATION ON A SQUARE LATTICE WITH A NONSTANDARD STRUCTURE OF GENERALIZED SYMMETRIES

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We clarify the integrability nature of a recently found discrete equation on the square lattice with a nonstandard symmetry structure. We find its L–A pair and show that it is also nonstandard. For this discrete equation, we construct the hierarchies of both generalized symmetries and conservation laws. This equation yields two integrable systems of hyperbolic type. The hierarchies of generalized symmetries and conservation laws are also nonstandard compared with known equations in this class.

Keywords: discrete integrable equation, generalized symmetry, conservation law, L–A pair

1. Introduction

The equation

$$u_{n+1,m+1}(u_{n,m}-u_{n,m+1})-u_{n+1,m}(u_{n,m}+u_{n,m+1})+2=0, \tag{1}$$

where n and m are arbitrary integers, was found in [1]. It was shown that its generalized symmetry in the direction m has the form

$$\frac{d}{dt_2}u_{n,m}=(-1)^n\frac{u_{n,m+1}u_{n,m-1}+u_{n,m}^2}{u_{n,m+1}+u_{n,m-1}}. \tag{2}$$

The simplest generalized symmetry in the direction n turns out to be

$$\frac{d}{dt_1}u_{n,m}=h_{n,m}h_{n-1,m}(a_nu_{n+2,m}-a_{n-1}u_{n-2,m}), \tag{3}$$

where

$$h_{n,m}=u_{n+1,m}u_{n,m}-1, \qquad a_{n+2}=a_n.$$

This symmetry depends on an arbitrary double-periodic function a_n, which can be represented as

$$a_n=\tilde{a}+\hat{a}(-1)^n, \tag{4}$$

where $\tilde{a}$ and $\hat{a}$ are arbitrary complex numbers. We here have both the autonomous particular case $a_n=1$ and the nonautonomous case $a_n=(-1)^n$; all other possible particular cases are linear combinations of them. For instance, we can obtain

$$a_n=\frac{1+(-1)^n}{2}=\begin{cases}0, & n=2k+1,\\ 1, & n=2k.\end{cases} \tag{5}$$

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Generalized symmetries (2) and (3) are themselves integrable equations. More precisely, for any fixed n in case (2) and for any fixed m in case (3), we have an integrable equation with one continuous and one discrete variable. Equation (1) generates the chains of the auto-Bäcklund transformations for each of Eqs. (2) and (3) (see [2] for more details). Symmetries (2) and (3) are compatible not only with (1) but also with each other on solutions of discrete equation (1). On the other hand, Eq. (1) can be obtained as the compatibility condition of generalized symmetries (2) and (3). For these and many other reasons, we here consider the triad of Eqs. (1)–(3) as a whole instead of the single discrete equation (1). This approach allows obtaining some important results, which are presented in Secs. 2 and 3.

Almost all known integrable discrete equations of the form

$$F_{n,m}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = 0 \tag{6}$$

with symmetries in both the directions n and m have generalized symmetries of the form [2]–[5]

$$\begin{aligned} \frac{d}{dt_1} u_{n,m} &= \Phi_{n,m}(u_{n+1,m}, u_{n,m}, u_{n-1,m}), \\ \frac{d}{dt_2} u_{n,m} &= \Psi_{n,m}(u_{n,m+1}, u_{n,m}, u_{n,m-1}). \end{aligned} \tag{7}$$

Along with Eq. (1) we know only a few exceptions obtained in [6]–[8]. In those examples, the simplest generalized symmetries in both directions have a more complicated structure than (7): they also depend on $u_{n\pm 2,m}$ or $u_{n,m\pm 2}$.

From the standpoint of the structure of generalized symmetries, Eq. (1) is the only one of its kind, and we therefore study it in more detail. In Sec. 2, we construct the L–A pairs for each of Eqs. (1)–(3) and show that the L–A pair for Eq. (1) is also nonstandard. In Sec. 3, using the obtained triad of L–A pairs, we find two hierarchies of conservation laws for discrete equation (1). In Sec. 4, we construct a master symmetry for Eq. (2) and the recursion operator for Eq. (3), thus obtaining the hierarchies of generalized symmetries for the discrete equation in each of the directions n and m. In Sec. 5, using the triad of Eqs. (1)–(3), we construct two examples of continuous integrable hyperbolic systems.

2. The *L*–*A* pairs

Constructing the L–A pair for Eq. (1), we use the following interesting property of symmetry (3). This symmetry turns out to be equivalent to the known system of two equations found by Tsuchida (see (3.13) in [9]). The L–A pair for this system is known and is given in that paper.

Applying the transformation

$$v_k \to (-1)^k v_k, \qquad w_k \to (-1)^{k+1} w_k$$

and appending the point symmetry, we can write the Tsuchida system in the form

$$\begin{aligned} \frac{d}{dt_1} v_k &= (\alpha v_{k+1} - \beta v_{k-1})(v_k w_k - 1)(v_k w_{k+1} - 1), \\ \frac{d}{dt_1} w_k &= (\beta w_{k+1} - \alpha w_{k-1})(v_k w_k - 1)(v_{k-1} w_k - 1). \end{aligned} \tag{8}$$

This system is an integrable discretization of one of the nonlinear Schrödinger equations with derivative, introduced in [10] (also see [11]). For any fixed m, Eq. (3) is related to system (8) as

$$v_k = u_{2k,m}, \qquad w_k = u_{2k-1,m}, \qquad \alpha = a_{2k}, \qquad \beta = a_{2k-1}. \tag{9}$$

We note that system (8) is a linear combination of two compatible systems corresponding to the particular cases $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$. In this sense, system (8) is an analogue of the known Ablowitz–Ladik chain, which is also a linear combination of two commuting equations of the relativistic Toda type (see, e.g., Sec. 5.2 in [12]).

The equations related by transformation (9) are equivalent, and we can transfer the generalized symmetries and conservation laws from one equation to the other (see [13]). We obtain the L–A pair for (3) using transformation (9) and rewriting the known L–A pair for (8) obtained in [9]. This L–A pair is standard and is represented by the pair of compatible linear equations

$$T_k\Phi_k = U_k\Phi_k, \qquad D_{t_1}\Phi_k = V_k\Phi_k, \tag{10}$$

where Φ_k is a two-component vector function, U_k and V_k are 2×2 matrices depending on the spectral parameter, and T_k is the shift operator in k, $T_k h_k = h_{k+1}$.

It is clear from transformation (9) that the shift of the functions v_k and w_k in k corresponds to the double shift of $u_{n,m}$ in n. We therefore obtain the L–A pair for (3) in a somewhat different form,

$$T_n^2\Psi_{n,m} = N_{n,m}\Psi_{n,m}, \qquad D_{t_1}\Psi_{n,m} = A_{n,m}\Psi_{n,m}. \tag{11}$$

Here, T_n is the shift operator in n, and the matrices $N_{n,m}$ and $A_{n,m}$ have the forms

$$N_{n,m} = \begin{pmatrix} h_{n,m}(1-\lambda) - 2\lambda & u_{n+1,m}(\lambda-1) \\ -2\lambda u_{n,m}h_{n+1,m} & h_{n+1,m}(\lambda-1) \end{pmatrix}, \tag{12}$$

$$A_{n,m} = \begin{pmatrix} h_{n-1,m}\left(a_{n-1}u_{n+1,m}u_{n-2,m} + a_n\dfrac{\lambda-1}{\lambda+1}\right) + a_{n-1}\dfrac{2\lambda}{\lambda-1} & -a_n u_{n-1,m}\dfrac{\lambda-1}{\lambda+1} - a_{n-1}u_{n+1,m} \\ 2\lambda h_{n-1,m}\left(\dfrac{u_{n,m}a_n}{1+\lambda} + \dfrac{u_{n-2,m}a_{n-1}}{\lambda-1}\right) - a_n u_{n,m}u_{n-1,m}\dfrac{2\lambda}{1+\lambda} & h_{n,m}(a_n u_{n-1,m}u_{n+2,m} - a_{n-1}) \end{pmatrix}. \tag{13}$$

The operator $T_n^2 - N_{n,m}$ in the first equation in (11) is a discrete analogue of the Schrödinger operator with matrix coefficients. The compatibility condition for Eqs. (11) has the form

$$D_{t_1}N_{n,m} = (T_n^2 A_{n,m})N_{n,m} - N_{n,m}A_{n,m}, \tag{14}$$

and this relation is equivalent to Eq. (3). The L–A pair can be rewritten in the standard form using 4×4 matrices,

$$T_n\begin{pmatrix}\Psi_{n,m}\\ \Psi_{n+1,m}\end{pmatrix} = \begin{pmatrix}0 & E\\ N_{n,m} & 0\end{pmatrix}\begin{pmatrix}\Psi_{n,m}\\ \Psi_{n+1,m}\end{pmatrix},$$

$$D_{t_1}\begin{pmatrix}\Psi_{n,m}\\ \Psi_{n+1,m}\end{pmatrix} = \begin{pmatrix}A_{n,m} & 0\\ 0 & A_{n+1,m}\end{pmatrix}\begin{pmatrix}\Psi_{n,m}\\ \Psi_{n+1,m}\end{pmatrix}.$$

Passing to $z_m = i^m u_{n,m}$ for any fixed $n \in \mathbb{Z}$ and using time dilation, we obtain the well-known equation [14], [15]

$$\frac{dz_m}{dt_2} = \frac{z_{m+1}z_{m-1} + z_m^2}{z_{m+1} - z_{m-1}} \tag{15}$$

for symmetry (2). The auxiliary linear problem for equations of this type is known [4]. It is the standard problem

$$T_m \Psi_{n,m} = M_{n,m} \Psi_{n,m}, \qquad D_{t_2} \Psi_{n,m} = B_{n,m} \Psi_{n,m}. \tag{16}$$

Here, $M_{n,m}(u_{n,m}, u_{n,m+1})$ and $B_{n,m}(u_{n,m-1}, u_{n,m}, u_{n,m+1})$ are 2×2 matrices, and the compatibility condition has the form

$$D_{t_2} M_{n,m} = (T_m B_{n,m}) M_{n,m} - M_{n,m} B_{n,m}. \tag{17}$$

The first linear discrete equations in (11) and (16) represent the Lax pair for discrete equation (1) if the vector functions $\Psi_{n,m}$ and the spectral parameters λ coincide in them. After the vector function is replaced using the matrix $\Omega_{n,m}$, the matrix $M_{n,m}$ changes according to the gauge transformation

$$\widetilde{M}_{n,m} = \Omega^{-1}_{n,m+1} M_{n,m} \Omega_{n,m}.$$

The matrix $\Omega_{n,m}$ and the change of the spectral parameter can be found by direct calculation, but we prefer another simpler way. We use the relation

$$(T_n^2 M_{n,m}) N_{n,m} = (T_m N_{n,m}) M_{n,m}, \tag{18}$$

(which is equivalent to (1)) and the known matrix $N_{n,m}$ and seek the matrix $M_{n,m}(u_{n,m}, u_{n,m+1})$. As a result, we obtain

$$M_{n,m} = \begin{pmatrix} \lambda & \dfrac{1-\lambda}{u_{n,m} + u_{n,m+1}} \\ \lambda(u_{n,m+1} - u_{n,m}) & \dfrac{\lambda(u_{n,m} - u_{n,m+1})}{u_{n,m} + u_{n,m+1}} \end{pmatrix}. \tag{19}$$

The corresponding matrix $B_{n,m}$ determining the L–A pair for (2) can be constructed using condition (17):

$$B_{n,m} = \frac{(-1)^n}{u_{n,m+1} + u_{n,m-1}} \begin{pmatrix} (1-\lambda)(u_{n,m-1} - u_{n,m}) & 1-\lambda \\ \lambda(u_{n,m}^2 - u_{n,m+1}^2) & \lambda(u_{n,m} + u_{n,m+1}) - (u_{n,m} + u_{n,m-1}) \end{pmatrix}. \tag{20}$$

We note that discrete L–A pair (18) can also be rewritten in the standard form:

$$(T_n \widetilde{M}_{n,m}) \widetilde{N}_{n,m} = (T_m \widetilde{N}_{n,m}) \widetilde{M}_{n,m} \tag{21}$$

in terms of the 4×4 matrices $\widetilde{M}_{n,m}$ and $\widetilde{N}_{n,m}$ with a block structure. We formulate the obtained results.

Theorem 1. *The L–A pairs for Eqs.* (1)–(3) *are given by the corresponding relations* (18), (17), *and* (14), *where the* 2×2 *matrices have forms* (12), (13), (19), *and* (20).

For almost all known integrable discrete equations of form (6), the L–A pair is represented by relation (21) with the 2×2 matrices $\widetilde{M}_{n,m}$ and $\widetilde{N}_{n,m}$. Two exceptions are known [6], [7] where the L–A pair has the same form (21) but is given by 3×3 matrices. These equations are discrete analogues of the Tzitzéica equation [16]

$$u_{xy} = e^u + e^{-2u},$$

whose L–A pair is given by the equations

$$D_x \Psi = L\Psi, \qquad D_y \Psi = A\Psi$$

with the 3×3 matrices L and A [17]. Our Eq. (1) is one more exception: its L–A pair (18) is defined by 2×2 matrices and can be rewritten in form (21) with 4×4 matrices.

We have verified that discrete equation (1) has no standard L–A pair (21) in terms of 2×2 matrices. More precisely, we fixed the matrix $\widetilde{M}_{n,m} = M_{n,m}$ in (19) and sought the 2×2 matrix

$$\widetilde{N}_{n,m}(u_{n-1,m}, u_{n,m}, u_{n+1,m}, u_{n+2,m}).$$

Using the equivalence of L–A pair (21) to discrete equation (1), we showed that there is no matrix $\widetilde{N}_{n,m}$ of this form.

Equation (2) with $a_n \equiv 1$ for any fixed $m \in \mathbb{Z}$ is an autonomous chain of the form

$$\frac{d}{dt}u_n = G(u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}). \tag{22}$$

In this class of equations, it has a special place analogous to the place of Eq. (1) in class (6). The known integrable equations (22) are mostly generalized symmetries of Volterra-type equations [14], [15] and have L–A pairs of the form

$$D_t L = (T_n A)L - LA \tag{23}$$

with 2×2 matrices L and A. The rest of the known integrable chains are analogues of the Itoh–Narita–Bogoyavlensky equation (see, e.g., [18]–[20])

$$\frac{d}{dt}u_n = u_n(u_{n+2} + u_{n+1} - u_{n-1} - u_{n-2}), \tag{24}$$

whose L–A pair (23) is given by 3×3 matrices. Our Eq. (2) with $a_n \equiv 1$ has the L–A pair of the form

$$D_t L = (T_n^2 A)L - LA \tag{25}$$

with 2×2 matrices. It can be rewritten as (23) with 4×4 matrices.

3. Conservation laws

In this section, we construct the conservation laws for discrete equation (1) using the scheme proposed by Mikhailov [21]. This approach assumes the substantial use of the triad of compatible L–A pairs (18), (17), and (14) for discrete equation (1) and its generalized symmetries (2) and (3). There is an alternative approach that also allows solving this problem [22].

The method we use to construct the conservation laws assumes the formal diagonalization of all matrices determining the triad of L–A pairs (18), (17) and (14). It is hence convenient to proceed to this diagonalization with the linear differential equations in (11) and (16). For this, we use known results in the theory of linear differential equations [23] (also see [24]).

The vector function $\Psi_{n,m}$ can be transformed using the matrix $\Omega_{n,m}$:

$$\widetilde{\Psi}_{n,m} = \Omega_{n,m}\Psi_{n,m}.$$

The matrices defining L–A pairs (14), (17), and (18) are then transformed according to the formulas

$$\begin{aligned}
\widetilde{B}_{n,m} &= \Omega_{n,m}^{-1} B_{n,m}\Omega_{n,m} - \Omega_{n,m}^{-1}\partial_{t_2}\Omega_{n,m},\\
\widetilde{M}_{n,m} &= \Omega_{n,m+1}^{-1} M_{n,m}\Omega_{n,m},\\
\widetilde{N}_{n,m} &= \Omega_{n+2,m}^{-1} N_{n,m}\Omega_{n,m},\\
\widetilde{A}_{n,m} &= \Omega_{n,m}^{-1} A_{n,m}\Omega_{n,m} - \Omega_{n,m}^{-1}\partial_{t_1}\Omega_{n,m}.
\end{aligned} \tag{26}$$

The formal diagonalization can be effectively constructed in the neighborhood of the poles of the matrices of linear differential operators. The matrices $B_{n,m}$ and $A_{n,m}$ have poles at the respective points $\lambda = \infty$ and $\lambda = \pm 1$. We formulate the lemma on the diagonalization of the matrix $B_{n,m}$ in the neighborhood of its pole (see, e.g., p. 86 in [24]).

Lemma 1. *If the principal part $\partial B_{n,m}/\partial\lambda$ of the matrix $B_{n,m}$ has distinct eigenvalues, then there is a formal series*

$$\Omega_{n,m} = \Omega^*_{n,m}\left(E + \sum_{j=1}^{\infty} \lambda^{-j}\Omega^{(-j)}_{n,m}\right)$$

such that E is the identity matrix and $\Omega^{(-j)}_{n,m}$, $j \geq 1$, are antidiagonal matrices. The matrix $\widetilde{B}_{n,m}$ obtained using the first formula in (26) *has the form*

$$\widetilde{B}_{n,m} = \lambda B^{(1)}_{n,m} + \sum_{j=0}^{\infty} \lambda^{-j} B^{(-j)}_{n,m},$$

where $B^{(l)}_{n,m}$ are diagonal matrices.

Here, we deal with formal series and do not discuss their convergence. To construct any finite number of conservation laws, it suffices to know that finite number of coefficients of the formal series.

The matrix $\partial B_{n,m}/\partial\lambda$ has different eigenvalues. Therefore, there is a transformation reducing it to the diagonal form. Any matrix of this transformation can be taken as the matrix $\Omega^*_{n,m}$. In particular,

$$\Omega^*_{n,m} = \begin{pmatrix} 1 & 1 \\ u_{n,m} - u_{n,m-1} & u_{n,m} + u_{n,m-1} \end{pmatrix}. \tag{27}$$

We then find $B^{(1)}_{n,m}$:

$$B^{(1)}_{n,m} = \begin{pmatrix} 0 & 0 \\ 0 & (-1)^{n+1} \end{pmatrix}. \tag{28}$$

To obtain the matrices $\Omega^{(l)}_{n,m}$ and $B^{(l)}_{n,m}$ for $l \leq 0$, we rewrite the first equation in (26) as

$$\Omega_{n,m}\widetilde{B}_{n,m} = B_{n,m}\Omega_{n,m} - \partial_{t_2}\Omega_{n,m}$$

and equate the coefficients of like powers of λ to zero. The relation corresponding to λ^1 is an identity by virtue of the choice of $\Omega^*_{n,m}$ and $B^{(1)}_{n,m}$. The relations at other powers of λ yield recurrence relations for the other coefficients $\widetilde{B}_{n,m}$ and $\Omega_{n,m}$, whence they can be determined explicitly. In particular,

$$\Omega^{(-1)}_{n,m} = \begin{pmatrix} 0 & \dfrac{(u_{n,m+1} + u_{n,m+2})(u_{n,m} - u_{n,m-1})}{(u_{n,m} + u_{n,m-2})(u_{n,m+1} + u_{n,m-1})} \\ \dfrac{(u_{n,m} - u_{n,m-1})(u_{n,m-1} - u_{n,m-2})}{(u_{n,m} + u_{n,m-2})(u_{n,m+1} + u_{n,m-1})} & 0 \end{pmatrix}.$$

We do not present the other coefficients because they are too cumbersome.

Using Eq. (17) and formula (28) for $B^{(1)}_{n,m}$, we can use induction to prove that the matrix $\widetilde{M}_{n,m}$ is diagonal. From (26), we obtain

$$\begin{aligned} \widetilde{M}_{n,m} = &- \lambda \frac{u_{n,m+1} + u_{n,m-1}}{u_{n,m} + u_{n,m+1}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \\ &- \begin{pmatrix} \dfrac{(u_{n,m+2} + u_{n,m+1})u_{n,m-1} - u_{n,m}(u_{n,m+1} + u_{n,m+2})}{(u_{n,m+2} + u_{n,m})(u_{n,m} + u_{n,m+1})} & 0 \\ 0 & \dfrac{u_{n,m+1} - u_{n,m}}{u_{n,m} + u_{n,m+2}} \end{pmatrix} + \dots. \end{aligned}$$

Using equation (18) and the coefficient of λ in the series $\widetilde{M}_{n,m}$, we can use induction to prove that the matrix $\widetilde{N}_{n,m}$ is also diagonal. From (26), we obtain

$$\widetilde{N}_{n,m} = -\lambda \begin{pmatrix} 1 + u_{n+1,m}u_{n,m-1} & 0 \\ 0 & 1 - u_{n+1,m}u_{n,m+1} \end{pmatrix} - \\ - \frac{u_{n,m+1} - u_{n,m-1}}{u_{n,m+1} + u_{n,m-1}} \begin{pmatrix} 1 + u_{n+1,m}u_{n,m-1} & 0 \\ 0 & -1 + u_{n+1,m}u_{n,m+1} \end{pmatrix} + \dots .$$

The matrices $\widetilde{M}_{n,m}$ and $\widetilde{N}_{n,m}$ are diagonal, and their elements are formal power series in λ^{-1}. Equation (18) for these matrices can be rewritten as

$$(T_n^2 - 1)\log \widetilde{M}_{n,m} = (T_m - 1)\log \widetilde{N}_{n,m}, \tag{29}$$

where we use the notation

$$\log \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \log\alpha & 0 \\ 0 & \log\beta \end{pmatrix}.$$

The diagonal elements

$$(\log \widetilde{M}_{n,m})_{1,1} = \log\lambda + \sum_{j=0}^{\infty} \lambda^{-j} p_{n,m}^{(j)}, \qquad (\log \widetilde{N}_{n,m})_{1,1} = \log\lambda + \sum_{j=0}^{\infty} \lambda^{-j} q_{n,m}^{(j)}$$

can be expanded in formal series, and we obtain the hierarchy of conservation laws

$$(T_n^2 - 1)p_{n,m}^{(j)} = (T_m - 1)q_{n,m}^{(j)}, \quad j \geq 0. \tag{30}$$

These conservation laws can be rewritten in the standard form

$$(T_n - 1)p_{n,m} = (T_m - 1)q_{n,m} \tag{31}$$

because $T_n^2 - 1 = (T_n - 1)(T_n + 1)$.

The first two conservation laws are

$$p_{n,m}^{(0)} = \log \frac{u_{n,m+1} + u_{n,m-1}}{u_{n,m} + u_{n,m+1}}, \qquad q_{n,m}^{(0)} = \log(1 + u_{n+1,m}u_{n,m+1}).$$

$$p_{n,m}^{(1)} = \frac{(u_{n,m+1} + u_{n,m+2})(u_{n,m} - u_{n,m-1})}{(u_{n,m+2} + u_{n,m})(u_{n,m+1} + u_{n,m-1})}, \qquad q_{n,m}^{(1)} = \frac{u_{n,m+1} - u_{n,m-1}}{u_{n,m+1} + u_{n,m-1}}.$$

From the second diagonal elements, we obtain the same conservation laws (in accordance with the comment below) except in the first step, where we obtain

$$\hat{p}_{n,m}^{(0)} = \log \frac{u_{n,m} - u_{n,m+1}}{u_{n,m} + u_{n,m+2}}, \qquad \hat{q}_{n,m}^{(0)} = \log(1 - u_{n+1,m}u_{n,m+1}).$$

All conservation law densities $p_{n,m}^{(j)}$ of this hierarchy depend on a finite number of functions from the set

$$u_{n,m+k}, \quad k \in \mathbb{Z}. \tag{32}$$

For this type of conservation law, we formulate Lemma 2 and then define the concept of the order of a conservation law. This allows distinguishing the conservation laws and selecting the trivial laws.

We first define the formal variational derivative of the density $p_{n,m}$ in the direction m:

$$\frac{\delta_m p_{n,m}}{\delta_m u_{n,m}} = \sum_{k\in\mathbb{Z}} T_m^{-k} \frac{\partial p_{n,m}}{\partial u_{n,m+k}}.$$

This sum is always finite because $p_{n,m}$ depends on a finite number of functions (32). For any conserved density $p_{n,m}$, we have

$$\frac{\delta_m p_{n,m}}{\delta_m u_{n,m}} = P_{n,m}(u_{n,m-M}, u_{n,m-M+1}, \ldots, u_{n,m+M-1}, u_{n,m+M}).$$

Using the function $r_{n,m}$ depending on a finite number of functions (32), we can pass to the *equivalent* conservation law

$$(T_n^2 - 1)\tilde{p}_{n,m} = (T_m - 1)\tilde{q}_{n,m}$$

according to the rule

$$\tilde{p}_{n,m} = p_{n,m} + (T_m - 1)r_{n,m}, \qquad \tilde{q}_{n,m} = q_{n,m} + (T_n^2 - 1)r_{n,m}. \tag{33}$$

Lemma 2. *Using equivalence transformation* (33), *we can reduce the density of any conservation law* (30) *to one of the three forms*

1. $\tilde{p}_{n,m} = 0 \Leftrightarrow P_{n,m} = 0$,
2. $\tilde{p}_{n,m} = \tilde{p}_{n,m}(u_{n,m})$, $\tilde{p}'_{n,m} \not\equiv 0 \Leftrightarrow M = 0$, $P_{n,m}(u_{n,m}) \not\equiv 0$, *or*
3. $\tilde{p}_{n,m} = \tilde{p}_{n,m}(u_{n,m}, u_{n,m+1}, \ldots, u_{n,m+M})$, $\partial^2\tilde{p}_{n,m}/\partial u_{n,m}\partial u_{n,m+M} \not\equiv 0 \Leftrightarrow M > 0$, $\partial P_{n,m}/\partial u_{n,m-M} \not\equiv 0$, $\partial P_{n,m}/\partial u_{n,m+M} \not\equiv 0$.

In case 1 of Lemma 2, the conservation law is called a *trivial* law. In cases 2 and 3, we call the law a *nontrivial conservation law of order* M and write $\operatorname{ord} p_{n,m} = M$. It can be seen that conservation laws of different orders are not equivalent in the sense of transformation (33). For instance, we have the property that the number of variables in the function $p_{n,m}(u_{n,m}, u_{n,m+1}, \ldots, p_{n,m+k})$, $k > 0$, can be reduced using transformation (33) if and only if $\partial^2 p_{n,m}/\partial u_{n,m}\partial u_{n,m+k} \equiv 0$ for all $n, m \in \mathbb{Z}$.

This theory is completely analogous to that for the discrete–differential conservation laws of the form

$$D_t p_m = (T_m - 1)q_m.$$

A detailed discussion of this theory together with the proofs can be found, for instance, in [13], [15]. The corresponding theory for discrete conservation laws of form (31) was discussed in [25].

We see that

$$\operatorname{ord} p^{(0)}_{n,m} = 2, \qquad \operatorname{ord} p^{(1)}_{n,m} = 3, \qquad \operatorname{ord} \hat{p}^{(0)}_{n,m} = 2.$$

Hence, the density $p^{(1)}_{n,m}$ differs from the other two. The densities $p^{(0)}_{n,m}$ and $\hat{p}^{(0)}_{n,m}$ might, in principle, be equivalent up to the total difference and multiplication by a constant. But if we construct a new conservation law as

$$\begin{aligned} \check{p}^{(0)}_{n,m} &= \hat{p}^{(0)}_{n,m} + p^{(0)}_{n,m} + (T_m - 1)(p^{(0)}_{n,m} + \log(u_{n,m} + u_{n,m+1})) = \log\frac{u_{n,m} - u_{n,m+1}}{u_{n,m} + u_{n,m+1}}, \\ \check{q}^{(0)}_{n,m} &= \hat{q}^{(0)}_{n,m} + q^{(0)}_{n,m} + (T_n^2 - 1)(p^{(0)}_{n,m} + \log(u_{n,m} + u_{n,m+1})) = \log(h_{n,m}h_{n+1,m}), \end{aligned} \tag{34}$$

then we see that $\operatorname{ord}\check{p}^{(0)}_{n,m} = 1$. We therefore have three different nontrivial conservation laws of the respective orders 1, 2, and 3.

To construct the conservation laws in the direction n, we use the linear differential equation in (11) and proceed to diagonalizing the matrix $A_{n,m}$. The matrix $A_{n,m}$ in (13) has poles at $\lambda = \pm 1$, and we can formally diagonalize it in terms of formal power series in $\lambda + 1$ or $\lambda - 1$ using an analogue of Lemma 1. The results in the two cases coincide, and we restrict ourself to the case $\lambda = -1$. In this case,

$$\Omega_{n,m} = \begin{pmatrix} u_{n-1,m} & 1 \\ -h_{n-1,m} & u_{n,m} \end{pmatrix} \begin{pmatrix} 1 & \dfrac{\lambda+1}{2}\dfrac{1 - h_{n,m}h_{n+1,m}}{u_{n+1,m}} + \cdots \\ \dfrac{\lambda+1}{2} u_{n-3,m}h_{n-1,m}h_{n-2,m} + \cdots & 1 \end{pmatrix},$$

$$\widetilde{N}_{n,m} = \begin{pmatrix} 2 + (\lambda+1)(h_{n,m}u_{n-1,m}u_{n+2,m} - h_{n-1,m} - 2) & 0 \\ 0 & -(\lambda+1)h_{n,m}h_{n+1,m} \end{pmatrix} + \ldots,$$

$$\widetilde{M}_{n,m} = \begin{pmatrix} 1 - (\lambda+1)\dfrac{u_{n,m}(1 + u_{n,m+1}u_{n-1,m})}{u_{n,m} + u_{n,m+1}} & 0 \\ 0 & \dfrac{u_{n,m+1} - u_{n,m}}{u_{n,m+1} + u_{n,m}}\left(1 + (\lambda+1)\dfrac{h_{n-1,m}u_{n,m+1}}{u_{n,m+1} + u_{n,m}}\right) \end{pmatrix} + \ldots.$$

In the leading term (of the order $(\lambda + 1)^0$) of the expansion, we obtain conservation law (34). In the subsequent orders $(\lambda + 1)^1$ and $(\lambda + 1)^2$, we obtain the conservation laws

$$\begin{aligned}
\check{p}^{(1)}_{n,m} &= \frac{2h_{n-1,m}u_{n,m+1}}{u_{n,m+1} + u_{n,m}}, \\
\check{q}^{(1)}_{n,m} &= u_{n-1,m}(u_{n+2,m}h_{n,m} - u_{n,m}), \\
\check{p}^{(2)}_{n,m} &= 4h_{n-1,m}\left(\frac{h_{n-1,m}(u_{n+2,m}h_{n,m} + u_{n,m})}{u_{n,m} + u_{n,m+1}} - \frac{u^2_{n,m}h_{n-1,m}}{(u_{n,m} + u_{n,m+1})^2}\right) + \\
&\quad + 4h_{n-1,m} - 4u_{n-1,m}u_{n+2,m}h_{n-1,m}h_{n,m}, \\
\check{q}^{(2)}_{n,m} &= 2u_{n-1,m}u_{n+4,m}h_{n,m}h_{n+1,m}h_{n+2,m} + (h_{n-1,m}h_{n+1,m} - u_{n-1,m}u_{n+2,m} - 1)^2 - \\
&\quad - 2h_{n-1,m}h_{n+1,m} - (h_{n+1,m} + 2)^2.
\end{aligned}$$

We can thus obtain a hierarchy of conservation laws of form (30), where the function $q^{(j)}_{n,m}$ depends on a finite number of functions in the set

$$u_{n+k,m}, \quad k \in \mathbb{Z}. \tag{35}$$

In this case, the theory is analogous to that discussed above for the conservation laws in the direction m. It is convenient to rewrite conservation laws (30) in standard form (31), where

$$p_{n,m} = (T_n + 1)p^{(j)}_{n,m}, \qquad q_{n,m} = q^{(j)}_{n,m}.$$

In this case, the function $q_{n,m}$ plays the role of the conservation law density. We introduce the formal variational derivative in the direction n,

$$\frac{\delta_n q_{n,m}}{\delta_n u_{n,m}} = \sum_{k\in\mathbb{Z}} T_n^{-k}\frac{\partial q_{n,m}}{\partial u_{n+k,m}} = Q(u_{n-N,m}, u_{n-N+1,m}, \ldots, u_{n+N,m}),$$

and define the order of a conservation law as in the preceding case. The equivalence transformation now has the form

$$\tilde{q}_{n,m} = q_{n,m} + (T_n - 1)s_{n,m}, \qquad \tilde{p}_{n,m} = p_{n,m} + (T_m - 1)s_{n,m}, \tag{36}$$

where $s_{n,m}$ depends on a finite number of functions (35).

For the conservation laws given above, we have

$$\operatorname{ord} \breve{q}^{(0)}_{n,m} = 1, \qquad \operatorname{ord} \breve{q}^{(1)}_{n,m} = 3, \qquad \operatorname{ord} \breve{q}^{(2)}_{n,m} = 5.$$

The function $\breve{q}^{(0)}_{n,m}$ depends on three variables. We can therefore reduce their number and obtain

$$\breve{p}^{(0)}_{n,m} = \log\left(\frac{u_{n,m} - u_{n,m+1}}{u_{n,m} + u_{n,m+1}}\right)^2, \qquad \breve{q}^{(0)}_{n,m} = 2\log h_{n,m}.$$

This can be further simplified by introducing the n-dependent square root of unity:

$$\bar{p}^{(0)}_{n,m} = \log\left((-1)^n \frac{u_{n,m} - u_{n,m+1}}{u_{n,m} + u_{n,m+1}}\right), \qquad \bar{q}^{(0)}_{n,m} = \log h_{n,m}. \tag{37}$$

We note that we can pass from conservation law (34) to (37) by applying the operator $(T_n + 1)^{-1}$ to both its sides and taking into account that $(-1)^n$ belongs to the kernel of the operator $T_n + 1$.

We can thus construct conservation laws in the direction m of any natural order, and in the case of direction n, we obtain conservation laws of odd orders. Conservation laws of even orders apparently do not exist, and we can prove this statement in the case of order 2. More precisely, we consider a conserved density of the form $q_{n,m}(u_{n,m}, u_{n+1,m}, u_{n+2,m})$ and assume its explicit dependence on n and m. The corresponding function $p_{n,m}$ must depend on $u_{n,m}, u_{n+1,m}, u_{n,m+1}$. We use the weakest assumption that $\partial^2 q_{n,m}/\partial u_{n,m}\partial u_{n+2,m} \neq 0$ at least at one point n, m and prove that such a conservation law does not exist.

4. The master symmetry and recursion operator for generalized symmetries

In this section, we discuss the problem of constructing generalized symmetries for discrete equation (1). There are two hierarchies of symmetries in the directions n and m. We first consider the case of direction m and construct symmetries analogous to (2).

As noted above, symmetry (2) is equivalent to the known Volterra-type equation (15). Equations analogous to (15) have master symmetries generating both generalized symmetries and conservation laws for them [12] (a more detailed discussion can be found in [2], [15], and concrete examples are in [2]). Here, we merely rewrite the known master symmetry of (15) in terms of the variables of (2). For this kind of master symmetry, we must first introduce a generalization of Eq. (2) that depends on a parameter τ, which plays the role of time in the master symmetry:

$$\begin{aligned} \frac{d}{dt^{(1)}_2} u_{n,m} = \frac{(-1)^n}{\cosh\tau} \frac{u^2_{n,m} + u_{n,m-1}u_{n,m+1}}{u_{n,m-1} + u_{n,m+1}} &+ \\ + \tanh\tau \frac{u_{n,m}(u_{n,m+1} - u_{n,m-1})}{u_{n,m-1} + u_{n,m+1}} &= \Psi^{(1)}_{n,m}(\tau). \end{aligned} \tag{38}$$

Equation (38) at $\tau = 0$ coincides with (2). The corresponding master symmetry has the form

$$\frac{d}{d\tau} u_{n,m} = m\Psi^{(1)}_{n,m}(\tau) = \Psi^*_{n,m}. \tag{39}$$

It generates the hierarchy of symmetries

$$\frac{d}{dt_2^{(j)}}u_{n,m} = \Psi^{(j)}_{n,m}(\tau) \tag{40}$$

according to the formula

$$\Psi^{(j+1)}_{n,m}(\tau) = [\Psi^*_{n,m}, \Psi^{(j)}_{n,m}(\tau)] = D_\tau \Psi^{(j)}_{n,m}(\tau) - D_{t_2^{(j)}}\Psi^*_{n,m} =$$

$$= \frac{\partial \Psi^{(j)}_{n,m}(\tau)}{\partial \tau} + \sum_{k\in\mathbb{Z}}\left(\Psi^*_{n,m+k}\frac{\partial \Psi^{(j)}_{n,m}(\tau)}{\partial u_{n,m+k}} - \Psi^{(j)}_{n,m+k}(\tau)\frac{\partial \Psi^*_{n,m}}{\partial u_{n,m+k}}\right).$$

The quantity τ is an external parameter for all Eqs. (40), and these equations are compatible for any value of this parameter. We can therefore set $\tau = 0$ in (40) and obtain a hierarchy of generalized symmetries for (2). The obtained equations are also generalized symmetries of discrete equation (1).

For instance, for $j = 2$ and $\tau = 0$,

$$\frac{d}{dt_2^{(2)}}u_{n,m} = \Psi^{(2)}_{n,m}(0) = \frac{(u_{n,m+2} - u_{n,m-2})(u^2_{n,m+1} - u^2_{n,m})(u^2_{n,m} - u^2_{n,m-1})}{(u_{n,m} + u_{n,m-2})(u_{n,m+1} + u_{n,m-1})^2(u_{n,m+2} + u_{n,m})}.$$

By direct calculation, we can verify that this equation is compatible not only with (2) but also with discrete equation (1).

In the case of direction n, we use the recursion operator to construct the generalized symmetries. Equation (3) is equivalent to known system (8), whose recursion operator was constructed in [26]. Using identities (9), we merely rewrite this operator in the scalar form suitable for (3). In this case, the recursion operator R can be constructed in the convenient form

$$R = H \circ S, \tag{41}$$

where the operator H is a Hamiltonian and S is a symplectic operator. These operators have the forms

$$S = (-1)^n\left(\frac{1}{h_{n,m}}T_n + \frac{1}{h_{n-1,m}}T_n^{-1}\right), \tag{42}$$

$$\begin{aligned} H = {} & h_{n,m}h_{n-1,m}(c_n u_{n+2,m} - c_{n-1}u_{n-2,m})(T_n - 1)^{-1}(-1)^n u_{n,m} + \\ & + (-1)^n u_{n,m}T_n(T_n-1)^{-1}h_{n,m}h_{n-1,m}(c_n u_{n+2,m} - c_{n-1}u_{n-2,m}) - \\ & - (-1)^n h_{n-1,m}h_{n,m}(c_n h_{n+1,m}T_n + c_{n-1}h_{n-2,m}T_n^{-1}), \end{aligned} \tag{43}$$

where c_n is an arbitrary double-periodic function depending on n. These operators satisfy the equations

$$\begin{aligned} &\frac{dS}{dt_1} + S\circ f^*_{n,m} + f^{*\perp}_{n,m}\circ S = 0, \\ &\frac{dH}{dt_1} = f^*_{n,m}\circ H + H\circ f^{*\perp}_{n,m}. \end{aligned} \tag{44}$$

Here, $f^*_{n,m}$ and $f^{*\perp}_{n,m}$ are the operators defined by the right-hand side $f_{n,m}$ of (3):

$$\frac{d}{dt_1}u_{n,m} = f_{n,m}. \tag{45}$$

The discrete analogue of the Fréchet derivative $f^*_{n,m}$ of $f_{n,m}$ has the form

$$f^*_{n,m}=\sum_{k=-2}^{2}\frac{\partial f_{n,m}}{\partial u_{n+k,m}}T_n^k,$$

and its adjoint operator $f^{*\perp}_{n,m}$ is defined by

$$f^{*\perp}_{n,m}=\sum_{k=-2}^{2}\frac{\partial f_{n+k,m}}{\partial u_{n,m}}T_n^k.$$

It follows from (44) that the operator $R=H\circ S$ satisfies the Lax equation

$$\frac{dR}{dt_1}=[f^*_{n,m},R], \tag{46}$$

where $[A,B]=A\circ B-B\circ A$. All these formulas are standard and can be found, for example, in [15] in the case of autonomous equations (45) and in [13] in the nonautonomous case.

Equations (44) and (46) can be regarded as definitions of the symplectic, Hamiltonian, and recursion operators (see, e.g., [25]). The Hamiltonian operator H and the symplectic operator S provide a relation between the conservation laws and generalized symmetries of (3). The operator R satisfying (46) allows constructing the conservation laws and generalized symmetries of (3). For instance, from (46), we find that

$$\frac{\partial}{\partial t_1^{(k)}}u_{n,m}=R^{k-1}(f_{n,m})=f^{(k)}_{n,m},\quad k\geq 2, \tag{47}$$

are generalized symmetries of Eqs. (3) and (45). In the case $k=2$,

$$\begin{aligned}
f^{(2)}_{n,m}&=\hat f^{(2)}_{n,m}-(c_n+c_{n-1})f_{n,m}+(c_na_{n-1}-c_{n-1}a_n)(-1)^nu_{n,m},\\
\hat f^{(2)}_{n,m}&=h_{n,m}h_{n-1,m}(b_nh_{n+1,m}h_{n+2,m}u_{n+4,m}-b_{n-1}h_{n-2,m}h_{n-3,m}u_{n-4,m}+\\
&\quad+u_{n,m}(b_nu_{n+2,m}h_{n-2,m}u_{n-3,m}-b_{n-1}u_{n-2,m}h_{n+1,m}u_{n+3,m})+\\
&\quad+(u_{n-1,m}h_{n,m}-u_{n+1,m})(b_nu^2_{n+2,m}-b_{n-1}u^2_{n-2,m})-\\
&\quad-u_{n,m}(b_nu_{n-1,m}u_{n+2,m}-b_{n-1}u_{n+1,m}u_{n-2,m})),
\end{aligned} \tag{48}$$

where $b_n=a_nc_n$. This symmetry was found in [1]. We can verify that it is also a generalized symmetry of discrete equation (1).

Using formula (47), we construct symmetries of the form

$$\frac{\partial}{\partial t_1^{(k)}}u_{n,m}=f^{(k)}_{n,m}(u_{n+2k,m},u_{n+2k-1,m},\ldots,u_{n-2k+1,m},u_{n-2k,m}),$$

which can be called symmetries of even orders $2k$. It was shown in [1] that a first-order symmetry does not exist. Perhaps, there are no odd-order generalized symmetries in this case.

We see that generalized symmetries for Eqs. (3) and (1) depend on arbitrary double-periodic functions of n. The same holds for the Hamiltonian and recursion operators. This situation is unusual in the case of scalar discrete–differential equations of type (3) and perhaps appears for the first time. Therefore, Tsuchida system (8) can be regarded as an analogue of the relativistic Toda chain whose symmetries and operators are also depend on two parameters. The detailed symmetry properties were discussed in [12], and the properties of operators were studied in [27].

5. The hyperbolic systems of equations

In this section, we derive two integrable hyperbolic systems of equations together with their L–A pairs from the generalized symmetries of discrete equation (1).

We consider two compatible symmetries of form (3) with $a_n = \chi_n$ and $a_n = \chi_{n-1}$ (where $\chi_n = (1+(-1)^n)/2$), namely, the equations

$$\begin{aligned} \partial_x u_{n,m} &= h_{n,m}h_{n-1,m}(\chi_n u_{n+2,m} - \chi_{n-1}u_{n-2,m}), \\ \partial_y u_{n,m} &= h_{n,m}h_{n-1,m}(\chi_{n-1} u_{n+2,m} - \chi_{n}u_{n-2,m}). \end{aligned} \tag{49}$$

From (11) and (13), we can obtain the systems of linear equations

$$D_x\Psi_{n,m} = A^{(1)}_{n,m}\Psi_{n,m}, \qquad D_y\Psi_{n,m} = A^{(2)}_{n,m}\Psi_{n,m}, \tag{50}$$

where $A^{(1)}_{n,m}$ and $A^{(2)}_{n,m}$ coincide with the matrix $A_{n,m}$ with the respective substitutions $a_n = \chi_n$ and $a_n = \chi_{n-1}$. This system of linear equations is compatible on solutions of system (49). We now change the matrices $A^{(1)}_{n,m}$ and $A^{(2)}_{n,m}$, expressing the functions $u_{n\pm2,m}$ in terms of $u_{n,m}$ and $u_{n\pm1,m}$ and either $\partial_x u_{n,m}$ or $\partial_y u_{n,m}$. We can do this using the consequences of system (49)

$$\begin{aligned} \chi_n u_{n+2,m} &= \frac{\chi_n\partial_x u_{n,m}}{h_{n,m}h_{n-1,m}}, \qquad & \chi_{n-1}u_{n-2,m} &= -\frac{\chi_{n-1}\partial_x u_{n,m}}{h_{n,m}h_{n-1,m}}, \\ \chi_{n-1} u_{n+2,m} &= \frac{\chi_{n-1}\partial_y u_{n,m}}{h_{n,m}h_{n-1,m}}, \qquad & \chi_{n}u_{n-2,m} &= -\frac{\chi_{n}\partial_y u_{n,m}}{h_{n,m}h_{n-1,m}}. \end{aligned} \tag{51}$$

As a result, we obtain matrices depending only on $u_{n,m}$, $u_{n+1,m}$, and $u_{n-1,m}$. Using these matrices, we can obtain a system of three equations for three unknown functions from (50).

To avoid the explicit dependence on n, we pass to either odd or even n. In the case of odd $n = 2k-1$, we introduce the notation

$$p = u_{2k-1,m}, \qquad q = u_{2k,m}, \qquad r = u_{2k-2,m}. \tag{52}$$

The matrices then become

$$A^{(1)} = \begin{pmatrix} \dfrac{qp_x}{1-pq} + \dfrac{2\lambda}{\lambda-1} & -q \\ \dfrac{2\lambda p_x}{(\lambda-1)(1-pq)} & 1-pq \end{pmatrix}, \tag{53}$$

$$A^{(2)} = \begin{pmatrix} \dfrac{(\lambda-1)(pr-1)}{1+\lambda} & \dfrac{(1-\lambda)r}{1+\lambda} \\ \dfrac{2\lambda p(pr-1)}{1+\lambda} & \dfrac{rp_y}{pr-1} - \dfrac{2\lambda pr}{1+\lambda} \end{pmatrix}. \tag{54}$$

The corresponding linear equations can be written as

$$D_x\Psi = A^{(1)}\Psi, \qquad D_y\Psi = A^{(2)}\Psi,$$

and their compatibility condition is

$$D_xA^{(2)} - D_yA^{(1)} = [A^{(1)}, A^{(2)}]. \tag{55}$$

This matrix equation is equivalent to the hyperbolic system

$$
\begin{aligned}
&\frac{\partial^2 \log p}{\partial x\,\partial y} + \frac{p_x p_y}{p^2(pq-1)(pr-1)} + (pq-1)(pr-1) = 0,\\
&(pr-1)q_y + qrp_y - r(pq-1)(pr-1) = 0,\\
&(pq-1)r_x + qrp_x + q(pq-1)(pr-1) = 0.
\end{aligned}
\tag{56}
$$

We thus obtain integrable system (56) of three hyperbolic equations together with L–A pair (53)–(55). If $u_{n,m}(x,y)$ is the general solution of Eqs. (49), then functions (52) satisfy system (56) for any k and m.

For even values $n = 2k$, we obtain the same hyperbolic system up to the involution $x \leftrightarrow y$. In both cases, regardless of the parity of n, the first equation in (56) can be written as

$$
\begin{aligned}
\frac{\partial^2 \log u_{n,m}}{\partial x\,\partial y} &+ \frac{\partial_x u_{n,m}\partial_y u_{n,m}}{u_{n,m}^2(u_{n,m}u_{n-1,m}-1)(u_{n,m}u_{n+1,m}-1)} +\\
&+ (u_{n,m}u_{n-1,m}-1)(u_{n,m}u_{n+1,m}-1) = 0,
\end{aligned}
\tag{57}
$$

and this is just a (2+1)-dimensional chain (m is an external parameter) similar to the (2+1)-dimensional generalization of the Toda chain [17]. Any compatible solution $u_{n,m}(x,y)$ of Eqs. (49) is also a solution of chain (57). The integrability problem of chain (57) is not studied in this paper and remains open.

To obtain the second hyperbolic system, we consider the pair of generalized symmetries

$$
\begin{aligned}
&\partial_x u_{n,m} = h_{n,m}h_{n-1,m}(\chi_n u_{n+2,m} - \chi_{n-1}u_{n-2,m}),\\
&\partial_z u_{n,m} = (-1)^n \frac{u_{n,m}^2 + u_{n,m+1}u_{n,m-1}}{u_{n,m+1}+u_{n,m-1}},
\end{aligned}
\tag{58}
$$

which are compatible on solutions of discrete equation (1). The corresponding auxiliary linear problem has the form

$$
D_x\Psi_{n,m} = A^{(1)}_{n,m}\Psi_{n,m}, \qquad D_z\Psi_{n,m} = B_{n,m}\Psi_{n,m}.
$$

Here, the matrix $A^{(1)}_{n,m}$ is defined as in the preceding case. The matrix $B_{n,m}$ is given by formula (20).

The matrix $A^{(1)}_{n,m}$ is transformed using formulas (51). We eliminate $u_{n,m-1}$ from the matrix $B_{n,m}$ using the second equation in (58). To eliminate the explicit dependence on n, we pass to odd $n = 2k-1$ and introduce the notation

$$
p = u_{2k-1,m}, \qquad q = u_{2k,m}, \qquad r = u_{2k-1,m+1}. \tag{59}
$$

As a result, we obtain the same matrix $A^{(1)}$ given by (53) and the matrix

$$
B = \frac{1}{p-r}\begin{pmatrix} (\lambda-1)(p_z+p) & \dfrac{(1-\lambda)(p_z+r)}{p+r} \\ \lambda(p+r)(p_z+p) & \dfrac{(p-r)(p_z-p)}{p+r} - \lambda(p_z+r) \end{pmatrix}.
$$

In this case, instead of (55), we obtain the matrix relation

$$
D_xB - D_zA^{(1)} = [A^{(1)}, B]. \tag{60}
$$

Relation (60) is equivalent to the hyperbolic system

$$
\begin{aligned}
&\frac{\partial^2 \log p}{\partial x\,\partial z} + \frac{(p_z - p)(p-r)p_x}{p^2(p+r)(pq-1)} + \frac{(p_z+p)(p+r)(pq-1)}{(p-r)p} = 0,\\
&(p^2-r^2)q_z - 2(qr-1)p_z - q(p^2+r^2) + 2r = 0,\\
&(pq-1)r_x - (qr-1)p_x - (p+r)(pq-1)(qr-1) = 0.
\end{aligned}
\tag{61}
$$

Depending on the choice of either even or odd n and on whether the function $u_{n,m+1}$ or $u_{n,m-1}$ is eliminated from the matrix $B_{n,m}$, there are four variants of the hyperbolic systems, but they are equivalent up to simple point transformations.

The case of generalized symmetries (49) compatible with each other without any additional assumption is analogous to the examples considered in [12], [28]. The second case of generalized symmetries (58) compatible on solutions of discrete equation (1) is apparently new.

In the second case, we can also construct a (2+1)-dimensional chain similar to (57), but it is nonautonomous and rather cumbersome, and we therefore do not write it here.

We can eliminate r or q from systems (56) and (61) using the respective second or third equation. In this case, we obtain hyperbolic systems of two equations analogous to the systems presented in [12], [28], but these systems contain square roots and are complicated.

6. Conclusion

We have constructed the L–A pair for discrete equation (1) and have shown that this equation differs from the known examples from the standpoint of L–A pairs.

Most of the known integrable discrete equations of form (6) have L–A pairs of the form

$$
(T_n - N_{n,m})\Psi_{n,m} = 0, \qquad (T_m - M_{n,m})\Psi_{n,m} = 0 \tag{62}
$$

with 2×2 matrices $N_{n,m}$ and $M_{n,m}$. There are also discrete analogues of the Tzitzéica equation with 3×3 matrices in L–A pair (62). The L–A pair of our Eq. (1) has the form

$$
(T_n^2 - N_{n,m})\Psi_{n,m} = 0, \qquad (T_m - M_{n,m})\Psi_{n,m} = 0 \tag{63}
$$

with the 2×2 matrices.

In Sec. 4, we constructed two hierarchies of generalized symmetries in both the directions m and n for Eq. (1). These hierarchies turn out to be of different types: there are symmetries of any natural order in the direction m, while we have only even-order symmetries in the direction n. For other known examples, both hierarchies are of the same type.

The hierarchies of the conservation laws constructed in Sec. 3 also turn out to be of different types. There are conservation laws of any order in the direction m, while there are only odd-order conservation laws in the direction n.

Equation (3) with $a_n = 1$ belongs to the class of autonomous equations of form (22) and is the lowest representative of its hierarchy. We could expect that it is an analogue of Itoh–Narita–Bogoyavlensky chain (24). Our study shows that Eq. (3) with $a_n = 1$ is closely related to the known integrable Tsuchida system (8), whose symmetry properties are close to those of the relativistic Toda chain.

Other known integrable chains of form (22) have L–A pairs (23) with either 2×2 or 3×3 matrices. Our example of Eq. (3) with $a_n = 1$ differs from them because the L–A pair has form (25) with 2×2 matrices.

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