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Discrete Exterior Calculus for Variational Problems in Computer Vision and Graphics

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Abstract— **This paper demonstrates how discrete exterior calculus tools may be useful in computer vision and graphics. A variational approach provides a link with mechanics.**

I. Introduction

Many problems in computer vision, image processing and computer graphics can be posed naturally as *variational problems*. Examples are template matching, image restoration, image segmentation, computation of minimal distortion maps and many others. See for example the special issue of the International Journal of Computer Vision (December 2002, Volume 50, Issue 3).

Therefore, new mathematical and computational tools that have been developed for solving variational problems in other areas can be usefully employed in computer vision, image processing and computer graphics.

An emerging general tool which we believe will be useful for the computational solution of variational problems in computer vision and related fields is the development of a calculus on discrete manifolds. We have made some progress in this field (Hirani [15] and Desbrun et al [7]) by developing a *discrete exterior calculus* (DEC). Bits and pieces of such a calculus have been appearing in literature but we believe [15] and [7] introduce several key new concepts in a systematic way. Our development of DEC includes discrete differential forms, discrete vector fields and the operators acting on these. In related work, a discrete calculus leads for example, to the proper definition of discrete divergence and curl operators which has applications such as a discrete Hodge decomposition of 3D vector fields on irregular grids (Tong et al [35]).

A closely related area in which a great deal of progress has been made, is discrete mechanics (see, e.g., Marsden and West [29] and Lew et al [27]). The main idea there is to discretize the variational principle itself rather than the Euler-Lagrange equations. In these and related references, the discretization is in time only, but one of the most promising areas in which DEC methods can be used is in spatially extended mechanics; that is, in classical field theory or networks of interconnected systems, where discrete methods have been developed and applied; see, for instance Lew et. al. [26]. We believe that there are important lessons to be learned from discrete mechanics when solving variational problems in computer vision and graphics.

This development of a discrete calculus, when combined with the methods of discrete mechanics and other recent work (e.g., [4], [5], [13], [14], [21], [22], [30], [34]) is likely to have promising applications in a field like computer vision which offers such a rich variety of challenging variational problems to be solved computationally. As a specific example we consider the problem of template matching and show how numerical methods derived from a discrete exterior calculus are starting to play an important role in solving the equations of averaged template matching. We also show some example applications using variational problems from computer graphics and mechanics to demonstrate that formulating the problem discretely and using discrete methods for solution can lead to efficient algorithms.

II. Template Matching and DEC

We start with a concrete example of a vision problem in which discrete exterior calculus turns out to play an important role in computations. This is the problem of deformable template matching. Deformable template matching is a technique for comparing images with applications in computer vision, medical imaging and other fields. It has been reported on extensively in the literature. See for example, Younes [39], Trouvé [36], [37], Grenander and Miller [10] and the references therein.

Template matching is based on computing a deformation induced distance between two images. The “energy” required to do a deformation that takes one image to the other defines the distance between them. The deformations are often taken to be diffeomorphisms of the image rectangle, i.e smooth maps with smooth inverse. The energy can be defined using various metrics on the space of diffeomorphisms.

In this way of posing the problem, template matching is similar to the way fluid mechanics is formulated. In fluid mechanics, averaged equations have been shown to have the property that length scales smaller than a certain parameter in the equation are averaged over correctly and don’t need to be resolved in a numerical solution. See Marsden and Shkoller [28] for details. Motivated by this, in Hirani et al [16] we derived the partial differential equation that we call the Averaged Template Matching Equation (ATME). This equation can also be written in a div, grad, curl form or in Lie derivative form. The unknown in ATME is the time dependent vector field that makes the initial image flow to the final image while minimizing the kinetic energy.

Our hope in deriving the ATME was that it would allow matching while ignoring features smaller than a fixed size. This property has not yet been verified but some progress has been made in the analysis of the equation in one and two spatial dimensions. For example, in [6] we show how natural boundary conditions leads to the reduction of the boundary value problem of template matching into a parameterized initial value formulation. Specifically we derive the form that the initial velocity must take to distort one image to the other while satisfying the ATME. This initial condition is a piecewise smooth, continuous function with a jump in the derivative at edges of the image.

Holm's group has analyzed and computed the solutions of the ATME and related equations in one and two-spatial dimensions. In 1D, Holm et al [12], [17], [18] found that the initial condition that they called a peakon leads to stable solutions in which the initial peakons move like solitons. Other initial conditions broke up into peakons that moved around and collided elastically. Interestingly, the initial condition for the 1D problem that we derived from the natural boundary conditions is a peakon. Recently Holm's group has discovered solutions to the two spatial dimension case. These are collections of peakons along one dimensional curves that move and collide in very interesting soliton-like ways. In the 2D case (see also [19]), the crucial step in the numerical solution was the use of mimetic discretization of the ATME written using div, grad and curl. Mimetic discretization (see, for example, [23]) is related to a basic form of discrete exterior calculus involving discrete forms. This suggests to us that DEC should be highly relevant to template matching.

III. Discrete Exterior Calculus

In [15] and [7] we present a theory of discrete exterior calculus motivated by potential applications in computational methods for field theories (elasticity, fluids, electromagnetism) as well as in areas such as vision and graphics. This theory has a long history but we have aimed at a comprehensive, systematic, applicable, treatment. Many previous works, are incomplete both in terms of the objects that they treat as well as the types of meshes that they allow. For more details on the large body of work on DEC, see [15].

Our vision of this theory is that it should proceed ab initio as a discrete theory that parallels the continuous one. General views of the subject area of DEC are common in the literature, but they usually stress the process of discretizing a continuous theory and the overall approach is tied to this goal. However, if one takes the point of view that the discrete theory can stand in its own right, then the range of application areas is naturally enriched and increases. Applications to graphics will illustrate this point.

Applications to Variational Problems. A major application areas we envision is to variational problems, be they in mechanics, optimal control, vision or graphics. A key ingredients in this direction that should play a key role is that of AVI's (asynchronous variational integrators) designed for the numerical integration of mechanical systems, as in Lew et al [26]. These integration algorithms respect key features of the continuous theory, such as their (multi)symplectic nature and exact conservation laws. They do so by discretizing the underlying variational principles of mechanics rather than discretizing the equations. It is well-known (see the reference just mentioned for some of the literature) that variational problems come equipped with a rich exterior calculus structure and so on the discrete level, such structures will be enhanced by the availability of a discrete exterior calculus.

There are other variational problems that motivate DEC. For instance, in many problems one requires a hierarchical and network structure for solving large systems (eg, to simulate a swarm of complex agents or power or internet systems) to enable one to simulate at a variety of resolutions and also to make use of parallel strategies together with message and information passing. In such a network setting, many problems involve optimization (such as optimal throughput in an internet setting) and optimization problems are also variational and so have an associated DEC structure that comes along with the problem. In such a spatially distributed problem, one has a discrete system ab initio, and so it is natural to begin with the discrete problem from the start. We have already mentioned the variational nature of several problems in computer vision and graphics in the Introduction.

Structured Constraints. Many constraints in numerical algorithms involve differential forms, such as the divergence constraint for incompressibility of fluids as well as the fact that differential forms are naturally the fields in electromagnetism and some of Maxwell's equations are expressed in terms of the divergence and curl operations on these fields. Preserving, as in the mimetic differencing literature, such features directly on the discrete level is another one of the goals, overlapping with our goals for variational problems.

The Objects in DEC. To develop a discrete theory, one must define discrete differential forms along with vector fields and operators involving these. We define these on discrete manifolds that are piecewise affine simplicial complexes (such as a triangle mesh embedded in 3D). Once discrete forms and vector fields are defined, a calculus can be developed by defining the discrete exterior derivative ($\mathbf{d}$), codifferential ($\boldsymbol{\delta}$) and Hodge star ($*$) for operating on forms, discrete wedge product ($\wedge$) for combining forms, discrete flat ($\flat$) and sharp ($\sharp$) operators for going between vector fields and one forms and discrete contraction operator ($\mathbf{i}_X$) for combining forms and vector fields. Once these are done one can then define other useful operators. For example a discrete Lie derivative ($\pounds_X$) can be *defined* by requiring that the Cartan magic (or homotopy) formula hold. A discrete divergence in any dimension can also be defined. A discrete Laplace-deRham operator (Δ) can be defined using the usual definition of $\mathbf{d}\boldsymbol{\delta} + \boldsymbol{\delta}\mathbf{d}$. When applied to functions this is the same as the discrete Laplace-Beltrami operator (∇^2) which is the defined as $\operatorname{div} \circ \operatorname{curl}$. We define all these operators in DEC.

IV. Discrete Mechanics

Many standard integrators used for simulating mechanical systems fail to respect the mechanical structure (like momenta, energy, symplectic form etc.) and often have misleading numerical dissipation. Recently progress has been made in the development of variational discrete mechanics, both in the fundamental theory and in the applications to challenging problems. These include collision algorithms and the development of AVI's (Asynchronous Variational Integrators). Variational integrators are based on a discretization of Hamilton's principle (or the Lagrange-d'Alembert principle if there is dissipation or external forces present), which underlies essentially *all* of mechanics, from particle mechanics to continuum mechanics. Preserving the basic variational structure in the algorithm retains the structure of mechanics (such as conservation laws) at the algorithmic level. This avoids many of the problems with existing integrators, such as spurious dissipation, which may take very expensive runs to eliminate by standard techniques. With an appropriate development of the connection between mechanics and geometry in the discrete setting, one will be able to use for geometry the technology described in this section. The theory of variational integrators has its roots

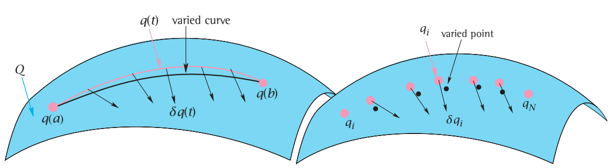


Fig. 1: *Continuous vs. discrete variational principles.*

in Hamilton-Jacobi theory and parts of the basic theory go back to, e.g., Moser and Veselov and its numerical analysis is due to a variety of groups, such as Suris, Marsden and Wendtlandt. The strategy is to start with a Lagrangian $L(q, \dot{q})$ and requires Hamilton's principle, namely to make the action integral stationary: $\delta \int_a^b L(q, \dot{q})\, dt = 0$ for fixed endpoints, which leads to the Euler-Lagrange equations. This theory has a PDE counterpart in which one makes a space-time integral stationary which is appropriate for elasticity and fluids, for example.

In Discrete Mechanics, one approximates the action integral for time Δt which gives the *discrete Lagrangian*: $L_d(q_0, q_1, \Delta t)$. The discrete action sum, defined by $S_d = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}, \Delta t_k)$ is required to satisfy *discrete variational principle*: Extremize S_d given fixed end points, q_0 and q_N. This yields the *DEL* (discrete Euler–Lagrange) equations: $D_2 L_d(q_{i-1}, q_i, \Delta t_{i-1}) + D_1 L_d(q_i, q_{i+1}, \Delta t_i) = 0$ (where D_2 denotes the second slot derivative, etc.). This gives an update rule $(q_{i-1}, q_i) \mapsto (q_i, q_{i+1})$, which is called a *variational integrator*. To get energy conservation, one can vary the time steps and get: $D_3 L_d(q_{i-1}, q_i, \Delta t_{i-1}) - D_3 L_d(q_i, q_{i+1}, \Delta t_i) = 0$, which encodes the discrete version of conservation of energy.

For example, let $L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M \dot{q} - V(q)$, where M is a symmetric positive-definite mass matrix and V is a potential function; the Euler-Lagrange equations are the standard ones: $M\ddot{q} = -\nabla V(q)$. Approximating the action integral $\int_0^T L\, dt$ using the rectangle rule yields the finite difference equation $M\left((q_{k+1} - 2q_k + q_{k-1})/\Delta t^2\right) = -\nabla V(q_k)$ while another choice gives the well-known Newmark method for the parameters $\gamma = \frac{1}{2}$ and $\beta = 0$. Thus, this simple example is closely connected with simple symplectic integrators such as the mid-point rule and the Newmark algorithm. More sophisticated quadrature rules lead to more accurate integrators.

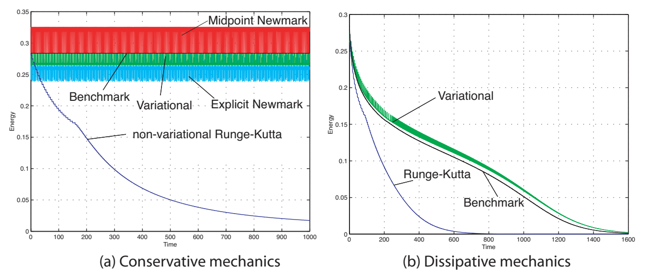


Fig. 2: *Energies: (a) non-dissipative system computed using variational and non-variational integrators vs a benchmark calculation; variational methods do not artificially dissipate energy ; (b) dissipative system computed with variational midpoint and Runge-Kutta – note the accurate energy behavior of the variational method, vs the higher order Runge-Kutta method (with artificial dissipation).*

Variational integrators preserve the symplectic structure, a classical property of mechanical systems. It is believed that this preservation is related to the good numerical properties of these integrators. These integrators also have a natural discrete Noether's theorem (preserve momenta) for systems with symmetry. They also have excellent energy behavior for mechanical systems (see, eg., [26]), even with some dissipation added, compared to conventional schemes. Figure 2 (a) and (b) show that the energy behavior of even low order variational integrators can be better than that of higher order conventional integrators for *both conservative and dissipative* mechanical systems. More relevant for shells and continua, is the development of AVI's or Asynchronous variational integrators ([26]), that allow one to take different time steps at different points, which is important for efficiency. Figure 2 (c) shows an example of energy behavior for a long-time run in a 3D dynamic elasticity calculation; one sees that the energy behavior is still excellent; this is true for both the global energy and the energy balance between modes.

V. Other Applications of Discrete Mechanics and Exterior Calculus

Parametrization is a central issue in graphics. Parameterizing a 2D mesh in 3D amounts to computing a correspondence

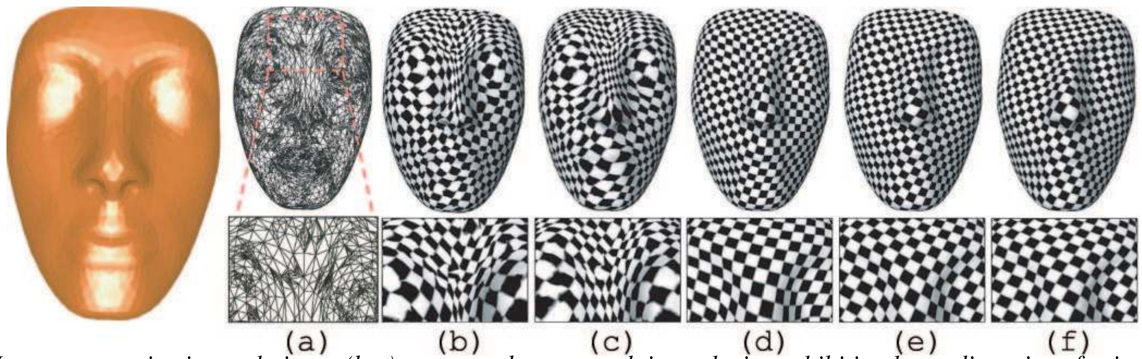


Fig. 3: *Many parametrization techniques (b-c) are not robust to mesh irregularity, exhibiting large distortions for irregular, yet geometrically smooth meshes, as in (a). Nonlinear techniques (d) achieve better results, but often require several minutes of computational time. With the same boundary conditions, our newly-developed, variationally-based technique quickly generates smooth parametrizations, regardless of the mesh irregularity (sampling quality) as shown by the two texture-mapped members (e-f) of the novel parametrization family (denoted* Intrinsic Parametrizations*) [8].*

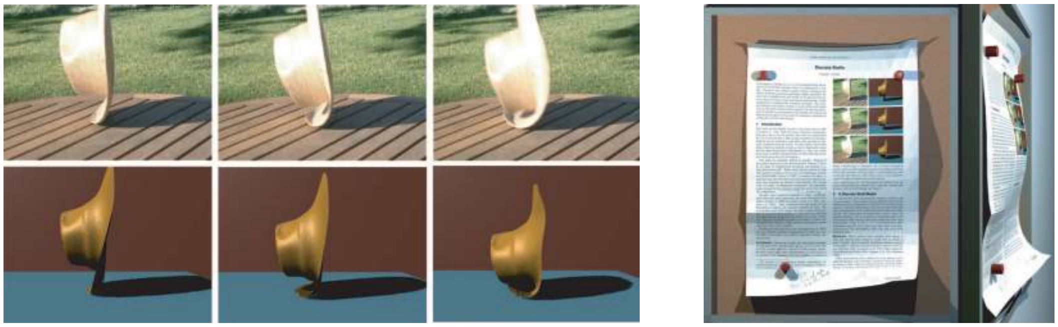
between a discrete surface patch and an isomorphic planar mesh through a piecewise linear *mapping* which assigns each mesh node a pair of coordinates (u, v) referring to its position on the planar region. Such a one-to-one mapping provides a flat parametric space, allowing one to perform any complex operation directly on the flat domain rather than on the curved surface. This facilitates most forms of mesh processing, such as surface fitting, texture mapping [33], or remeshing. This last application, for instance, is widely used both in the graphics and the mesh generation community interested in finite element computations: while graphics applications tends to minimizes the number of triangles or quadrangles used to represent a geometry, physical computations often require special element shape quality to ensure optimal accuracy. A good parametrization of a given (often scanned-in) geometry has been proven crucial for the efficiency of both isotropic and anisotropic remeshing [2], [1].

Much work on parametrization has been published over the last ten years. Almost all techniques explicitly aim at producing least-distorted parametrizations, and vary only by the distortions considered and the minimization processes used. Early work used the notion of flattening to obtain an isomorphic planar triangulation (for instance, [3]), minimizing ratio of angles between the 3D triangles and their associated 2D versions. Others (see, for instance, [24]) considered spring-like energies that can be quickly minimized by a linear system solver when the boundary has been fixed to an arbitrary contour (with the noticeable exception of [25] where only a few internal points need to be fixed by the user). More recently, a number of authors have proposed nonlinear formulations to define optimal parametrizations. The MIPS [20] method for instance finds a "natural boundary" that minimizes their highly non-linear energy [20]. Unfortunately, this requires considerable computational effort (even if hierarchical solvers can be used). Sander and co-authors [33] proposed another nonlinear energy for the very specific problem of texture stretch distortion.

Most of these techniques proposed to minimize a *continuous* energy over a piecewise linear surface. However, the choice of the energy and its discretization seems very arbitrary, and most of these techniques may visually result in non-smooth parametrizations and therefore non-smooth textured meshes as in Figure 3(b-c). It thus seems that here again, a discrete variational approach would be very appropriate and would provide the necessary robustness and guaranteed smoothness quality we need in parametrizations. It turns out that a number of independent authors have proposed a discrete harmonic parametrization (see, for instance [31]) who derived the same *linear condition* for harmonicity either using differential geometry, harmonic maps, or finite elements. A boundary condition is still needed before computing a discrete conformal mapping. We have recently generalized this approach and proposed a new, variationally-based family of parametrizations, based on Minkowski functionals to measure discrete distortion measures. Our results, depicted in Figure 3(e&f), seem to prove the validity of our approach. Unlike the previous techniques, we can find very smooth parametrizations independently of the mesh regularity. Furthermore, we have also experimented with natural boundary conditions to provide a natural, optimal **conformal parametrization** that does *not* require any specification of a boundary mapping as we present in [8]. Numerical experiments confirm the importance of this discrete variational approach for conformal geometry compared to, for instance, much more computation-intensive circle packing techniques.

Geometric Flows and Mesh Smoothing. These parametrization results using simple, additive, geometric invariants also have direct applications to mesh smoothing, and more generally, to geometric flows. We have partially studied this topic [9] by proposing a variational approach to smoothing that is equivalent to the mean curvature flow. But this time, we did use a direct discrete formulation, by simply stating the process as an area minimization. The results, compared to previous fairing methods, have again proven once again the importance of finding the "right" discretization.

Simple Constitutive Laws for Simulation. Several complex



(a) (b)

Fig. 4: *Discrete Constitutive Equations for Thin-Shells: (a) Real footage vs. Simulation of a Hat: top, a real hat is dropped on a table; bottom, our shell simulation captures the bending of the brim. (b) We can also model a curled, creased, and pinned sheet of paper. By altering dihedral angles of the flat reference configuration, we effect plastic deformation. While the rendering is texture-mapped, we use flat-shading of the triangles to emphasize the discrete structure of the underlying mesh. The final shape is fully simulated: the artist only indicates the curl radius, the crease sharpness, and the pin positions.*

Fig. 5: *Denoising of Geometry: we initially developed a variational approach to smoothing of arbitrary meshes in order to remove the spurious noise from scanned meshes even when the original sampling is highly irregular and very coarse [9]. Recent results from our parametrization work mentioned above have made clear that another family of discrete smoothing flows exists.*

physical simulations such as thin-shell or cloth animation are overly computationally-intensive for computer animation. No simpler method is known to obtain similar effects (buckling, wrinkles) at low cost. Our initial work on surface invariants helped us to design new *plausible constitutive laws*, with very good numerical properties (since they derive from variational principles). For instance, a fabric could be defined as behaving like a weighted mix between an area-preserving and a conformal surface – properties that are actually very close to reality for a number of fabrics such as denim or leather. Similarly, the Kirchhoff-Love theory of thin-shells can be substituted by a simpler constitutive law, very attractive for Computer Graphics as having similar properties (buckling for instance) but with better numerical properties and therefore significantly reduced computation times. The simplest expression of flexural energy for piecewise-linear geometry that we have considered is the *square trace of the difference between the shape operator on the undeformed surface and the pullback of the one on the current, deformed surface* [11]. As depicted in Figure 4, the visual impact of this extremely simple model matches the expected behavior. Movie clips showing an animation of the falling hat can be found at http://multires.caltech.edu/pubs/DS-CDROM. We intend to analyze the convergence properties, as well as to develop further, more complex discrete models of thin-shell and other deformable objects.

Discrete Hodge Decomposition. To develop a discrete 3D Helmholtz-Hodge decomposition, we followed the 2D approach of Polthier [32] and defined the curl-free part of a vector field ξ as the critical point of the functional: $\int_\Omega (\nabla u - \xi)^2 \, dV$ and the divergence-free part as the critical point of the functional: $\int_\Omega (\nabla \times \vec{v} - \xi)^2 \, dV$, which matches its differential analog. Use is made of two discrete operators Div and Curl. We have also introduced a multiscale representation of the projected fields, where fine-scale details are successively suppressed while main features are preserved. Such a hierarchical decomposition is interesting for numerical purposes as well as visualization. Our resulting *multiscale vector field decomposition* is a versatile computational tool: we have already explored several applications as presented in [35], from a vector field processing and visualization toolbox, to the animation of fluids and elastic objects on irregular grids (Figure 6).

VI. Conclusions

The intent of this paper is to demonstrate that a discrete exterior calculus can be a powerful and versatile tool, not only for mechanics, but also in graphics, visualization and computer vision. It has close connections with mimetic differencing and other techniques that are known to give good numerical results, such as those involving constraints that can be written in terms of differential forms, such as a divergence constraint. Even in classical field theories, such as electromagnetism, there is much activity in applying DEC-like ideas. We expect that a further development of DEC and

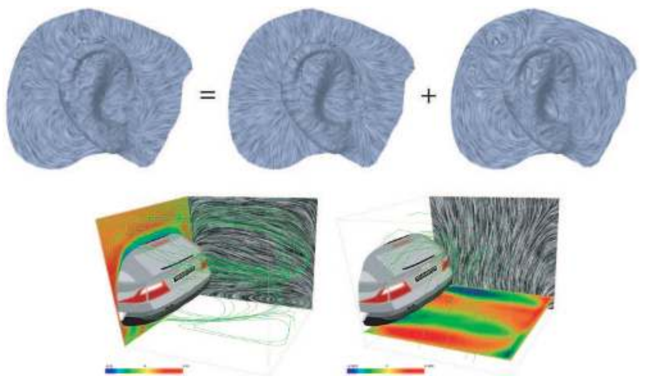


Fig. 6: *Even on curved 2-or 3-manifolds, the variationally-based vector field decomposition separates the curl-free and the divergence-free components of a vector field. The bottom figure shows the vector field of a 3D dataset (turbulence behind a moving car), decomposed using our technique. Represented is the divergence-free part $\nabla \times \vec{v}$ of the vector field, evidenced by particle tracing and a LIC cross-section of this component; the potential $\vec{v}$ is also represented on the right side. The false color cross-sections represent the magnitudes of these fields.*

its applications will also be of continued use in graphics and vision problems; we outlined a few possible application areas in this paper.

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