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Discrete fractional differences with nonsingular discrete Mittag-Leffler kernels

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Abstract

In this manuscript we propose the discrete versions for the recently introduced fractional derivatives with nonsingular Mittag-Leffler function. The properties of such fractional differences are studied and the discrete integration by parts formulas are proved. Then a discrete variational problem is considered with an illustrative example. Finally, some more tools for these derivatives and their discrete versions have been obtained.

Keywords: discrete fractional derivative; modified Mittag-Leffler function; discrete Mittag-Leffler function; discrete nabla Laplace transform; convolution; discrete *ABR* fractional derivative

1 Introduction and preliminaries

Fractional calculus has become an important mathematical tool used in several branches of science and engineering in order to describe better the properties of non-local complex systems [1–5]. Recently some authors have introduced new non-local derivatives with nonsingular kernels and they applied them successfully to some real world problems [6–14]. However, several areas where the fractional calculus can be applied successfully remain still not deeply investigated, *e.g.* the thermoelasticity of bodies with microstructure (see [15–17] for example and the references therein). Finding the discrete counterparts of these new fractional operators is an important step to apply them to model the dynamics of complex systems.

In the following we recall and prove some results in discrete fractional calculus that will be necessary in proceeding to obtain our discrete results (see [18–29]).

Definition 1 [30]

(i) Let m be a natural number, then the m rising factorial of t is written as

$$t^{\overline{m}} = \prod_{k=0}^{m-1}(t+k), \qquad t^{\overline{0}} = 1. \tag{1}$$

(ii) For any real number the α rising function becomes

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad \text{such that } t \in \mathbb{R} \setminus \{\dots,-2,-1,0\}, 0^{\overline{\alpha}} = 0. \tag{2}$$

In addition, we have

$$\nabla\big(t^{\overline{\alpha}}\big) = \alpha t^{\overline{\alpha-1}}, \tag{3}$$

hence $t^{\overline{\alpha}}$ is increasing on $\mathbb{N}_0$, where $\rho(t) = t-1$.

Definition 2 (See [31, 32]) Let $\rho(t) = t-1$ be the backward jump operator. Thus, for a function $f : \mathbb{N}_a = \{a, a+1, a+2, \ldots\} \to \mathbb{R}$, the nabla left fractional sum of order $\alpha > 0$ becomes

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} \big(t-\rho(s)\big)^{\overline{\alpha-1}} f(s), \quad t \in \mathbb{N}_{a+1}.$$

The nabla right fractional sum of order $\alpha > 0$ for $f : {}_b\mathbb{N} = \{b, b-1, b-2, \ldots\} \to \mathbb{R}$ is written as

$${}_b\nabla^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} \big(s-\rho(t)\big)^{\overline{\alpha-1}} f(s) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} \big(\sigma(s)-t\big)^{\overline{\alpha-1}} f(s), \quad t \in {}_{b-1}\mathbb{N}.$$

The nabla left fractional difference of order $\alpha > 0$ has the form

$$\nabla_a^{\alpha} f(t) = \nabla^n \nabla_a^{-(n-\alpha)} f(t) = \frac{\nabla^n}{\Gamma(n-\alpha)} \sum_{s=a+1}^{t} \big(t-\rho(s)\big)^{\overline{n-\alpha-1}} f(s), \quad t \in \mathbb{N}_{a+1},$$

and the nabla right fractional difference of order $\alpha > 0$ is defined as

$${}_b\nabla^{\alpha} f(t) = {}_{\ominus}\Delta^n {}_b\nabla^{-(n-\alpha)} f(t) = \frac{(-1)^n \Delta^n}{\Gamma(n-\alpha)} \sum_{s=t}^{b-1} \big(s-\rho(t)\big)^{\overline{n-\alpha-1}} f(s), \quad t \in {}_{b-1}\mathbb{N}.$$

The left Caputo fractional difference of order $\alpha > 0$ started by $a(\alpha) = a+n-1, n = [\alpha]+1$ is written as

$$\big({}^C\nabla_{a(\alpha)}^{\alpha} f\big)(t) = \nabla_{a(\alpha)}^{n-\alpha} \nabla^n f(t), \quad t \in \mathbb{N}_{a+n},$$

and the right Caputo fractional difference of order $\alpha > 0$ ending at $b(\alpha) = b-n+1$ has the following form:

$$\big({}_{b(\alpha)}^{C}\nabla^{\alpha} f\big)(t) = {}_{b(\alpha)}\nabla_{\ominus}^{n-\alpha} \Delta^n f(t), \quad t \in {}_{b-n}\mathbb{N}.$$

The Q-operator action, $(Qf)(t) = f(a+b-t)$, was used in [31, 32] to connect left and right fractional sums and differences. We recall the following results:

- $(\nabla_a^{-\alpha} Qf)(t) = Q_b\nabla^{-\alpha} f(t)$.
- $(\nabla_a^{\alpha} Qf)(t) = Q_b\nabla^{\alpha} f(t)$.
- $({}^C\nabla_a^{\alpha} Qf)(t) = Q_b^C\nabla^{\alpha} f(t)$.

The mixing of nabla and delta operators in defining right fractional differences plays a crucial role in obtaining the above dual identities.

In our manuscript, we use the properties of the discrete version of Q-operator to define and confirm our definitions of fractional differences with discrete Mittag-Leffler function kernels.

Definition 3 [33] Let a function f be defined on $\mathbb{N}_0$. Then the nabla discrete Laplace transform has the form

$$\mathcal{N}f(z) = \sum_{t=1}^{\infty}(1-z)^{t-1}f(t). \tag{4}$$

More generally for a function f it is defined on $\mathbb{N}_a$ by

$$\mathcal{N}_a f(z) = \sum_{t=a+1}^{\infty}(1-z)^{t-1}f(t). \tag{5}$$

If $f(t,s)$ denotes a function of two variables, we have explicitly to show to which parameter we use the transform.

Lemma 1 [30] *For any* $\alpha \in \mathbb{R} \setminus \{\ldots,-2,-1,0\}$,
(i) $\mathcal{N}(t^{\overline{\alpha-1}})(z) = \frac{\Gamma(\alpha)}{z^\alpha}$, $|1-z|<1$,
(ii) $\mathcal{N}(t^{\overline{\alpha-1}}b^{-t})(z) = \frac{b^{\alpha-1}\Gamma(\alpha)}{(z+b-1)^\alpha}$, $|1-z|<b$.

Remark 1 We can generalize (i) of Lemma 1 to $(\mathcal{N}_a(t-a)^{\overline{\alpha-1}})(s) = (1-s)^a\frac{\Gamma(\alpha)}{s^\alpha}$. Here we accept $\mathcal{N}_0 = \mathcal{N}$.

Definition 4 [2] The Mittag-Leffler function of one parameter has the following form:

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \quad \bigl(z\in\mathbb{C}; \operatorname{Re}(\alpha)>0\bigr), \tag{6}$$

and the one with two parameters α and β becomes

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \quad \bigl(z,\beta\in\mathbb{C}; \operatorname{Re}(\alpha)>0\bigr), \tag{7}$$

where $E_{\alpha,1}(z) = E_\alpha(z)$.

Since in general it is not true that $(t^\alpha)^{\overline{\beta}} = t^{\overline{\alpha\beta}}$ and $(ab)^{\overline{\alpha}} = a^{\overline{\alpha}}b^{\overline{\alpha}}$ in general, we define for the sake of discretization the following (modified) versions of Mittag-Leffler functions which also agree with the time scale calculus notations.

Definition 5 (Modified classical Mittag-Leffler functions) The Mittag-Leffler function of one parameter is defined by

$$E_\alpha(\lambda,z) = E_\alpha\bigl(\lambda z^\alpha\bigr) = \sum_{k=0}^{\infty}\lambda^k\frac{z^{\alpha k}}{\Gamma(\alpha k+1)} \quad \bigl(0\neq\lambda\in\mathbb{R}, z\in\mathbb{C}; \operatorname{Re}(\alpha)>0\bigr), \tag{8}$$

and the one with two parameters α and β by

$$E_{\alpha,\beta}(\lambda,z) = z^{\beta-1}E_{\alpha,\beta}\left(\lambda z^{\alpha}\right) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} \quad \left(0 \neq \lambda \in \mathbb{R}, z, \beta \in \mathbb{C}; \operatorname{Re}(\alpha) > 0\right), \tag{9}$$

where $E_{\alpha,1}(\lambda,z) = E_{\alpha}(\lambda,z)$.

Agreeing with Definition 5, the author in [31, 32] defined the following discrete versions of Mittag-Leffler functions.

Definition 6 (Nabla discrete Mittag-Leffler) (see [31–33]) For $\lambda \in \mathbb{R}$, $|\lambda| < 1$, and $\alpha, \beta, z \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, the nabla discrete Mittag-Leffler functions is

$$E_{\overline{\alpha,\beta}}(\lambda,z) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{\overline{k\alpha+\beta-1}}}{\Gamma(\alpha k+\beta)}. \tag{10}$$

For $\beta = 1$, we have

$$E_{\overline{\alpha}}(\lambda,z) \triangleq E_{\overline{\alpha,1}}(\lambda,z) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{\overline{k\alpha}}}{\Gamma(\alpha k+1)}, \quad |\lambda| < 1. \tag{11}$$

The generalized *ML* of three parameters was defined in the literature by

$$E_{\alpha,\beta}^{\rho}(z) = \sum_{k=0}^{\infty} (\rho)_k \frac{z^k}{k!\Gamma(\alpha k+\beta)}, \tag{12}$$

where $(\rho)_k = \rho(\rho+1)\cdots(\rho+k-1)$. Notice that $(1)_k = k!$ so that $E_{\alpha,\beta}^{1}(z) = E_{\alpha,\beta}(z)$.

To pass to the discrete process we define the following version of *ML* function of three parameters:

$$E_{\alpha,\beta}^{\rho}(\lambda,z) = \sum_{k=0}^{\infty} \lambda^k (\rho)_k \frac{z^{\alpha k+\beta-1}}{k!\Gamma(\alpha k+\beta)}. \tag{13}$$

Definition 7 The (nabla) discrete general *ML* function of three parameters α, β, and ρ is defined by

$$E_{\overline{\alpha,\beta}}^{\rho}(\lambda,z) = \sum_{k=0}^{\infty} \lambda^k (\rho)_k \frac{z^{\overline{k\alpha+\beta-1}}}{k!\Gamma(\alpha k+\beta)}. \tag{14}$$

Notice that $E_{\overline{\alpha,\beta}}^{1}(\lambda,z) = E_{\overline{\alpha,\beta}}(\lambda,z)$.

Proposition 1 (Summation and difference of discrete *ML* functions)

- $\nabla_t E_{\overline{\alpha}}(\lambda,z) = \lambda E_{\overline{\alpha,\alpha}}(\lambda,z)$.
- $\nabla_t E_{\overline{1,\beta}}(\lambda,z) = \lambda E_{\overline{1,\beta+1}}(\lambda,z)$.
- $\nabla_t E_{\overline{\alpha,\beta}}^{\gamma}(\lambda,z) = E_{\overline{\alpha,\beta-1}}^{\gamma}(\lambda,z)$.
- $\sum_{t=a+1}^{z} E_{\overline{\alpha,\beta}}(\lambda,t-a) = E_{\overline{\alpha,\beta+1}}(\lambda,z-a)$.

Definition 8 [31, 32] Let a function f be defined on $\mathbb{N}_0$. Thus, for $0 < \alpha \leq 1$ its α-order Caputo fractional derivative is

$$
\begin{aligned}
{}^C\nabla_0^\alpha f(t) &= \nabla_0^{-(1-\alpha)} \nabla f(t) \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{t} \bigl(t-\rho(s)\bigr)^{\overline{-\alpha}} \nabla f(s),
\end{aligned}
$$

where $\rho(s) = s-1$ and $\nabla_0^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s)$ is the nabla left fractional sum of order α.

We recall that if f is defined on $\mathbb{N}_0$, then ${}^C\nabla_0^\alpha f(t)$ is defined on $\mathbb{N}_1 = \{1, 2, 3, \ldots\}$.

For the Caputo fractional difference of order $n-1 < \alpha \leq n$ starting from $a(\alpha) = a+n-1$ we refer to Section 5 in [31].

Example 1 [31, 32] Let $0 < \alpha \leq 1$, $a \in \mathbb{R}$, and consider the nabla left Caputo nonhomogeneous fractional difference equation

$$
{}^C\nabla_0^\alpha y(t) = \lambda y(t) + f(t), \qquad y(0) = a_0, \quad t \in \mathbb{N}_0. \tag{15}
$$

Thus, the solution of (15) is written as

$$
y(t) = a_0 E_{\overline{\alpha}}(\lambda, t) + \sum_{s=1}^{t} E_{\overline{\alpha,\alpha}}\bigl(\lambda, t-\rho(s)\bigr) f(s). \tag{16}
$$

Remark 2 [31, 32] The solution of (15) with $\alpha = 1$ and $a_0 = 1$ is

$$
y(t) = \sum_{k=0}^{\infty} \lambda^k \frac{t^{\overline{k}}}{k!} + \sum_{s=1}^{t} \sum_{k=0}^{\infty} \lambda^k \frac{(t-\rho(s))^{\overline{k}}}{k!} f(s).
$$

The nabla discrete exponential function $\widehat{e}_\lambda(t, 0) = (1-\lambda)^{-t}$ represents the first part of the above solution, with $|\lambda| < 1$. For more details see [34], p.118.

For the rest of this section, we summarize some facts as regards the discrete Laplace transform of Mittag-Leffler type and convolution type functions (see [33] for some details).

Definition 9 (See [33]) Let $s \in \mathbb{R}$, $0 < \alpha < 1$, and $f, g : \mathbb{N}_a \to \mathbb{R}$ be functions. The nabla discrete convolution of f with g is defined by

$$
(f * g)(t) = \sum_{s=a+1}^{t} g\bigl(t-\rho(s)\bigr) f(s). \tag{17}
$$

In the above, $\rho(s) = s-1$ is the backward jumping operator used in ∇-analysis for the time scale $\mathbb{Z}$. This operator is necessary to prove for example a discrete convolution theorem as shown below. Also it is necessary to obtain dual relations between the left and right fractional sums and differences via the Q-operator.

Proposition 2 *For any* $\alpha \in \mathbb{R} \setminus \{\dots,-2,-1,0\}$, $s \in \mathbb{R}$, *and* f, g *defined on* $\mathbb{N}_a$ *we have*

$$\big(\mathcal{N}_a(f * g)\big)(z) = (\mathcal{N}_a f)(z)(\mathcal{N} g)(z). \tag{18}$$

Proof

$$\begin{aligned}\big(\mathcal{N}_a(f * g)\big)(z) &= \sum_{t=a+1}^{\infty}(1-z)^{t-1}\sum_{a+1}^{t} f(s)g\big(t-\rho(s)\big)\\ &= \sum_{s=a+1}^{\infty}\sum_{t=s}^{\infty}(1-z)^{t-1}\sum_{a+1}^{t} f(s)g\big(t-\rho(s)\big)\\ &= \sum_{s=a+1}^{\infty}\sum_{r=1}^{\infty}(1-z)^{r-1}(1-z)^{s-1}f(s)g(r)\\ &= (\mathcal{N}_a f)(z)(\mathcal{N} g)(z),\end{aligned}$$

where the change of variable $r = t - \rho(s)$ was used. □

For the case $a = 0$ and $g(t) = t^{\overline{\alpha}}$ we refer to [33].

Lemma 2 [33] *Let* f *be a function defined on* $\mathbb{N}_0$. *Thus,*

$$\big(\mathcal{N}\nabla\big(f(t)\big)\big)(z) = z(\mathcal{N} f)(z) - f(0). \tag{19}$$

We can generalize Lemma 2 as follows.

Lemma 3 *Let* f *be a function defined on* $\mathbb{N}_a$. *The following result holds*:

$$\big(\mathcal{N}_a\nabla\big(f(t)\big)\big)(z) = z(\mathcal{N}_a f)(z) - (1-z)^a f(a). \tag{20}$$

More generally,

$$\big(\mathcal{N}_{a(\alpha)}\nabla^n f\big)(z) = z^n(\mathcal{N}_{a(\alpha)} f)(z) - (1-z)^{a(\alpha)}\sum_{i=0}^{n-1} z^{n-1-i}\nabla^i f(a+1). \tag{21}$$

Lemma 4 [30] *For any positive real number* ν,

$$\big(\mathcal{N}_{a-1}\nabla_{a-1}^{-\nu}\big)f(s) = s^{-\nu}(\mathcal{N}_{a-1} f)(s).$$

For the following lemma we will present an alternative proof without using convolutions as was done in [33].

Lemma 5 [33] *Let* f *be defined on* $\mathbb{N}_0$ *and* $0 < \alpha \leq 1$. *Then*

$$\big(\mathcal{N}\,{}^C\nabla_0^\alpha f\big)(z) = z^\alpha(\mathcal{N} f)(z) - z^{\alpha-1} f(0). \tag{22}$$

Proof From the definition and Lemma 3.2 in [35] we have

$$\left({}^{C}\nabla_0^{\alpha} f\right)(t) = \left(\nabla_0^{-(1-\alpha)}\nabla f\right)(t) = \nabla\left(\nabla_0^{-(1-\alpha)} f\right)(t) - \frac{t^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(0).$$

Apply the nabla discrete Laplace transform and make use of Lemma 2 and Lemma 1 to get

$$\left(\mathcal{N}^{C}\nabla_0^{\alpha} f\right)(z) = z\left(\mathcal{N}\nabla_0^{-(1-\alpha)} f\right)(z) - \left(\nabla_0^{-(1-\alpha)} f\right)(0) - \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha)z^{1-\alpha}} f(0).$$

Then the result follows by Lemma 4 with $a = 1$ and $(\nabla_0^{-(1-\alpha)} f)(0) = 0$. □

Remark 3 Lemma 5 can be generalized as follows. For f defined on $\mathbb{N}_a$ and $0 < \alpha \leq 1$, we have

$$\left(\mathcal{N}_a{}^{C}\nabla_a^{\alpha} f\right)(z) = z^{\alpha}(\mathcal{N}_a f)(z) - (1-z)^a z^{\alpha-1} f(a). \tag{23}$$

This can be proved by making use of Remark 1.

Lemma 6 [33] *Let* $0 < \alpha \leq 1$ *and* f *be defined on* $\mathbb{N}_0$. *Then*:
(i) $(\mathcal{N}E_{\overline{\alpha}}(\lambda, t))(z) = \frac{z^{\alpha-1}}{z^{\alpha}-\lambda}$.
(ii) $(\mathcal{N}E_{\overline{\alpha,\alpha}}(\lambda, t))(z) = \frac{1}{z^{\alpha}-\lambda}$.

Proof We just repeat the proof of (ii) due to the calculation error in Lemma 4(ii) in [33].
(ii) First it is easy to see that $\nabla E_{\overline{\alpha}}(\lambda, t) = \lambda E_{\overline{\alpha,\alpha}}(\lambda, t)$. Indeed,

$$\nabla E_{\overline{\alpha}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{k\alpha z^{\overline{k\alpha-1}}}{\Gamma(\alpha k+1)}.$$

Since dividing by balls of Gamma function leads to zero, we then have $\nabla E_{\overline{\alpha}}(\lambda, t) = \sum_{k=1}^{\infty} \lambda^k \frac{z^{\overline{k\alpha-1}}}{\Gamma(\alpha k)} = \lambda \sum_{k=0}^{\infty} \lambda^k \frac{z^{\overline{k\alpha+\alpha-1}}}{\Gamma(\alpha k+\alpha)} = \lambda E_{\overline{\alpha,\alpha}}(\lambda, t)$. If we use $\mathcal{N}$ together with (i) and Lemma 2, we conclude that

$$\begin{aligned}
\left(\mathcal{N}E_{\overline{\alpha,\alpha}}(\lambda, t)\right)(z) &= \lambda^{-1}\left[z\left(\mathcal{N}E_{\overline{\alpha}}(\lambda, t)\right)(z) - E_{\overline{\alpha}}(\lambda, 0)\right] \\
&= \lambda^{-1}\left[\frac{z^{\alpha}}{z^{\alpha}-\lambda} - 1\right] \\
&= \frac{1}{z^{\alpha}-\lambda}.
\end{aligned}$$

□

2 Discrete fractional differences with discrete Mittag-Leffler kernels

Definition 10 Let f be defined on $\mathbb{N}_a \cap {}_b\mathbb{N}, a < b, \alpha \in [0,1]$, then the nabla discrete new (left Caputo) fractional difference in the sense of Atangana and Baleanu is defined by

$$\begin{aligned}
\left({}^{ABC}_{a}\nabla^{\alpha} f\right)(t) &= \frac{B(\alpha)}{1-\alpha} \sum_{s=a+1}^{t} \nabla_s f(s) E_{\overline{\alpha}}\left(\frac{-\alpha}{1-\alpha}, t-\rho(s)\right) \\
&= \frac{B(\alpha)}{1-\alpha}\left[\nabla f(t) * E_{\overline{\alpha}}\left(\frac{-\alpha}{1-\alpha}, t\right)\right]
\end{aligned} \tag{24}$$

and in the left Riemann sense by

$$\begin{aligned}\left({}_{a}^{ABR}\nabla^{\alpha}f\right)(t) &= \frac{B(\alpha)}{1-\alpha}\nabla_t \sum_{s=a+1}^{t} f(s)E_{\overline{\alpha}}\left(\frac{-\alpha}{1-\alpha}, t-\rho(s)\right) \\ &= \frac{B(\alpha)}{1-\alpha}\nabla_t\left[f(t) * E_{\overline{\alpha}}\left(\frac{-\alpha}{1-\alpha}, t\right)\right].\end{aligned} \tag{25}$$

It is be noted that since for $0 < \alpha < \frac{1}{2}$ we have $-1 < \lambda = -\frac{\alpha}{1-\alpha} < 0$, then $E_{\overline{\alpha}}(\lambda, t)$ is convergent for any $t \in \mathbb{N}$. For example, $E_{\overline{\alpha}}(\lambda, 1) = (1-\alpha)$ provided that $0 < \alpha < \frac{1}{2}$. Hence, all the *AB*-type fractional differences will converge under the restriction $0 < \alpha < \frac{1}{2}$. Also note that since $t^{\overline{\alpha}}$ is increasing on $\mathbb{N}_0$, $E_{\overline{\alpha}}(\lambda, t)$ is monotone decreasing for $0 < \alpha < \frac{1}{2}, t > 0$, and $\lambda = \frac{-\alpha}{1-\alpha} < 0$ (see [36] for the continuous case $E_\alpha(-t^\alpha)$). We can show that $\lim_{\sigma\to 0}\frac{1}{\sigma}E_{\overline{\alpha}}(\frac{-1}{\sigma}, t-\rho(s)) = \delta_s(t) = \begin{cases}1, & t = s,\\ 0, & t \neq s,\end{cases}$ $\alpha = 1$, which is the delta Dirac function on the time scale $\mathbb{Z}$, and hence as in [6] we can show that, for $\alpha \to 0$, we have $({}_{a}^{ABR}\nabla^{\alpha}f)(t) \to f(t)$ and for $\alpha \to 1$, we have $({}_{a}^{ABR}\nabla^{\alpha}f)(t) \to \nabla f(t)$. Notice that $E_{\overline{1}}(\frac{-1}{\sigma}, t-\rho(s)) = (1-\alpha)^{t-\rho(s)}$, $\sigma = \frac{1-\alpha}{\alpha}$, and hence for example

$$\lim_{\alpha\to 1}\left({}_{a}^{ABR}\nabla^{\alpha}f\right)(t) = \lim_{\alpha\to 1} B(\alpha)\nabla_t \sum_{s=a+1}^{t} f(s)(1-\alpha)^{t-s} = \nabla f(t).$$

Above we have made used of the fact that the nabla discrete exponential function has the form $e_\lambda(t, \rho(s)) = (\frac{1}{1-\lambda})^{t-\rho(s)}$ and $E_{\overline{1}}(\lambda, t-\rho(s)) = e_\lambda(t, \rho(s))$.

To derive the proper fractional difference for the above proposed fractional difference we consider the equation

$$\left({}_{a}^{ABR}\nabla^{\alpha}f\right)(t) = u(t). \tag{26}$$

Apply $\mathcal{N}_a$ to (26) above and make use of Lemma 3, Proposition 2 with $g(t) = E_{\overline{\alpha}}(\lambda, t)$ with $\lambda = \frac{-\alpha}{1-\alpha}$, and Lemma 6,

$$\begin{aligned}\frac{B(\alpha)}{1-\alpha}\mathcal{N}_a\left(\nabla f(t) * E_{\overline{\alpha}}(\lambda, t)\right)(z) &= \frac{B(\alpha)}{1-\alpha}z\left(\mathcal{N}_a f(t) * E_{\overline{\alpha}}(\lambda, t)\right)(z) - 0 \\ &= \frac{B(\alpha)}{1-\alpha}z\left[(\mathcal{N}_a f)(z)\cdot\frac{z^{\alpha-1}}{z^\alpha - \lambda}\right] \\ &= \left(\mathcal{N}_a u(t)\right)(z).\end{aligned} \tag{27}$$

That is,

$$(\mathcal{N}_a f)(z) = \frac{1-\alpha}{B(\alpha)}\left(\mathcal{N}_a u(t)\right)(z) - \frac{1-\alpha}{B(\alpha)}\frac{\lambda}{z^\alpha}\left(\mathcal{N}_a u(t)\right)(z). \tag{28}$$

Apply the inverse of $\mathcal{N}_a$ and use of Proposition 2 and Lemma 1 to conclude that

$$f(t) = \frac{1-\alpha}{B(\alpha)}u(t) + \frac{\alpha}{B(\alpha)}\left(\nabla_a^{-\alpha}u\right)(t). \tag{29}$$

This suggests the following definition for the fractional sum corresponding to the fractional difference with discrete Mittag-Leffler function kernel.

Definition 11 The fractional sum associated to $({}_{a}^{ABR}\nabla^{\alpha}f)(t)$ with order $0 < \alpha < 1$ is defined by

$$\left({}_{a}^{AB}\nabla^{-\alpha}f\right)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\left(\nabla_{a}^{-\alpha}f\right)(t). \tag{30}$$

It is clear that $\alpha = 0$ gives the function f and $\alpha = 1$ gives $\sum_{s=a+1}^{t} f(s)$.

From the definition of the discrete fractional integral we have

$$\left({}_{a}^{ABR}\nabla^{\alpha}{}_{a}^{ABR}\nabla^{-\alpha}f\right)(t) = f(t).$$

On the other hand we have the following.

Theorem 1 *For any* $0 < \alpha \leq 1$ *and* f *defined on* $\mathbb{N}_a$, $({}_{a}^{ABR}\nabla^{\alpha}f)(t)$ *satisfies the equation*

$$\left({}_{a}^{ABR}\nabla^{-\alpha}g\right)(t) = f(t). \tag{31}$$

Proof From the definition of fractional sum the equation in the statement of the theorem is equivalent to

$$\frac{1-\alpha}{B(\alpha)}g(t) + \frac{\alpha}{B(\alpha)}\left(\nabla_{a}^{-\alpha}g\right)(t) = f(t). \tag{32}$$

Apply the discrete Laplace transform $\mathcal{N}_a$ and make use of Lemma 4 to obtain

$$\frac{1-\alpha}{B(\alpha)}G(s) + \frac{\alpha}{B(\alpha)}s^{-\alpha}G(s) = F(s), \tag{33}$$

where $G(s) = (\mathcal{N}_a g)(s)$ and $F(s) = (\mathcal{N}_a f)(s)$. From this it follows that

$$G(s) = \frac{s^{\alpha}B(\alpha)}{(1-\alpha)s^{\alpha} + \alpha}F(s) = \frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha}}{s^{\alpha} - \lambda}F(s), \tag{34}$$

where $\lambda = \frac{-\alpha}{1-\alpha}$. Finally, apply the inverse of $\mathcal{N}_a$ and use the discrete convolution theorem, Proposition 2, or (27) to conclude that $g(t) = ({}_{a}^{ABR}\nabla^{\alpha}f)(t)$. □

Theorem 2 (The relation between the Caputo and Riemann fractional differences with *ML* kernels) *We have*

$$\left({}_{a}^{ABC}\nabla^{\alpha}f\right)(t) = \left({}_{a}^{ABR}\nabla^{\alpha}f\right)(t) - f(a)\frac{B(\alpha)}{1-\alpha}E_{\overline{\alpha}}(\lambda, t-a). \tag{35}$$

Proof From (27) we have

$$\left(\mathcal{N}_{a}{}_{a}^{ABR}\nabla^{\alpha}f\right)(z) = \frac{B(\alpha)}{1-\alpha}\left[(\mathcal{N}_a f)(z)\cdot\frac{z^{\alpha}}{z^{\alpha}-\lambda}\right], \tag{36}$$

where $\lambda = \frac{-\alpha}{1-\alpha}$. On the other hand, we have

$$\begin{aligned}\left(\mathcal{N}_{a}{}_{a}^{ABC}\nabla^{\alpha}f\right)(z) &= \frac{B(\alpha)}{1-\alpha}\left(\mathcal{N}_a \nabla f(t) * E_{\overline{\alpha}}(\lambda, t)\right)(z)\\ &= \frac{B(\alpha)}{1-\alpha}(\mathcal{N}_a \nabla f)(z)\cdot\left(\mathcal{N}E_{\overline{\alpha}}(\lambda, t)\right)(z)\end{aligned}$$

$$= \frac{B(\alpha)}{1-\alpha}\big[z(\mathcal{N}_a f)(z) - (1-z)^a f(a)\big] \cdot \left[\frac{z^{\alpha-1}}{z^\alpha - \lambda}\right]$$

$$= \frac{B(\alpha)}{1-\alpha}\left[(\mathcal{N}_a f)(z) \cdot \frac{z^\alpha}{z^\alpha - \lambda}\right] - (1-z)^a f(a)\frac{B(\alpha)}{1-\alpha}\left[\frac{z^{\alpha-1}}{z^\alpha - \lambda}\right]. \tag{37}$$

From (36) and (37), we see that

$$\big(\mathcal{N}_a {}_a^{ABC}\nabla^\alpha f\big)(z) = \big(\mathcal{N}_a {}_a^{ABR}\nabla^\alpha f\big)(z) - (1-z)^a f(a)\frac{B(\alpha)}{\alpha}\left[\frac{z^{\alpha-1}}{z^\alpha - \lambda}\right]. \tag{38}$$

Apply the inverse of $\mathcal{N}_a$ to (38) to conclude (35). The fact that $(\mathcal{N}_a f(t-a))(z) = (1-z)^a(\mathcal{N} f(t))(z)$ was used above. □

By means of the action of the Q-operator on left and right fractional sums and differences, we can define the right fractional sums $({}^{AB}\nabla_b^{-\alpha} f)(t)$ and differences $({}^{AB}\nabla_b^{\alpha} f)(t)$ as follows.

Definition 12 (The new right fractional difference with ML kernel) For $0 < \alpha < 1$, and f defined on ${}_b\mathbb{N}$, the right fractional difference of f is defined by

$$\big({}^{ABR}\nabla_b^\alpha f\big)(t) = \frac{B(\alpha)}{1-\alpha}(-\Delta_t)\sum_{s=t}^{b-1} f(s)E_{\overline{\alpha}}\left(\frac{-\alpha}{1-\alpha}, \big(s - \rho(t)\big)\right) \tag{39}$$

and the right Caputo one by

$$\big({}^{ABC}\nabla_b^\alpha f\big)(t) = \frac{B(\alpha)}{1-\alpha}\sum_{s=t}^{b-1}(-\Delta_s f)(s)E_{\overline{\alpha}}\left(\frac{-\alpha}{1-\alpha}, \big(s - \rho(t)\big)\right). \tag{40}$$

Definition 13 (The new right fractional sum with ML kernel) For $0 < \alpha < 1$, and f defined on ${}_b\mathbb{N}$, the right fractional sum of f is defined by

$$\big({}^{AB}\nabla_b^{-\alpha} f\big)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\big({}_b\nabla^{-\alpha} f\big)(t). \tag{41}$$

Theorem 3 *For a function f defined on ${}_b\mathbb{N}$ and $0 < \alpha < 1$, we have $({}^{ABR}\nabla_b^{\alpha}{}^{AB}\nabla_b^{-\alpha} f)(t) = f(t)$ and $({}^{AB}\nabla_b^{-\alpha}{}^{ABR}\nabla_b^{\alpha} f)(t) = f(t)$.*

If we apply the Q-operator to both sides and then replace $f(t)$ by $(Qf)(t) = f(a+b-t)$, then we can state the following.

Theorem 4 (The relation between the new right Caputo fractional difference and the new right Riemann fractional difference) *We have*

$$\big({}^{ABC}\nabla_b^\alpha f\big)(t) = \big({}^{ABR}\nabla_b^\alpha f\big)(t) - f(b)\frac{B(\alpha)}{1-\alpha}E_{\overline{\alpha}}(\lambda, b-t). \tag{42}$$

3 Integration by parts for fractional sums and differences with discrete *ML*

First we state and prove an integration by parts formula for fractional sums.

Theorem 5 (Integration by parts for the fractional sums with *ML* kernels) *For f and g defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$, $a \equiv b \pmod 1$, and $0 < \alpha < 1$, one has*

$$\begin{aligned}\sum_{s=a+1}^{b-1} g(s)\big({}_a^{AB}\nabla^{-\alpha} f\big)(s) &= \frac{1-\alpha}{B(\alpha)} \sum_{s=a+1}^{b-1} g(s)f(s) + \frac{\alpha}{B(\alpha)} \sum_{s=a+1}^{b-1} f(s)\big({}_b\nabla^{-\alpha} g\big)(s) \\ &= \sum_{s=a+1}^{b-1} f(s)\big({}^{AB}\nabla_b^{-\alpha} g\big)(s).\end{aligned} \tag{43}$$

Similarly, we have

$$\begin{aligned}\sum_{s=a+1}^{b-1} g(s)\big({}^{AB}\nabla_b^{-\alpha} f\big)(s) &= \frac{1-\alpha}{B(\alpha)} \sum_{s=a+1}^{b-1} g(s)f(s) + \frac{\alpha}{B(\alpha)} \sum_{s=a+1}^{b-1} f(s)\big(\nabla_a^{-\alpha} g\big)(s) \\ &= \sum_{s=a+1}^{b-1} f(s)\big({}_a^{AB}\nabla^{-\alpha} g\big)(s).\end{aligned} \tag{44}$$

Proof The proof follows by the definition of the new left fractional sum, the integration by parts formula for nabla classical fractional sums (see Proposition 37 in [31]), and the definition of the new right fractional sum. □

Theorem 6 (Integration by parts for the fractional differences with *ML* kernels) *For f and g defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$, $a \equiv b \pmod 1$, and $0 < \alpha < 1$, one has*

$$\sum_{s=a+1}^{b-1} f(s)\big({}_a^{ABR}\nabla^{\alpha} g\big)(s) = \sum_{s=a+1}^{b-1} g(s)\big({}^{ABR}\nabla_b^{\alpha} f\big)(s). \tag{45}$$

Similarly,

$$\sum_{s=a+1}^{b-1} f(s)\big({}^{ABR}\nabla_b^{\alpha} g\big)(s) = \sum_{s=a+1}^{b-1} g(s)\big({}_a^{ABR}\nabla^{\alpha} f\big)(s). \tag{46}$$

Proof The proof is achieved by Theorem 5 and the previously proved fact that the new fractional sums and differences are the inverses to each other. Indeed,

$$\begin{aligned}\sum_{s=a+1}^{b-1} f(s)\big({}_a^{AB}\nabla^{\alpha} g\big)(s) &= \sum_{s=a+1}^{b-1} \big({}^{AB}\nabla_b^{-\alpha}\,{}^{AB}\nabla_b^{-\alpha} f\big)(s)\big({}_a^{AB}\nabla^{\alpha} g\big)(s) \\ &= \sum_{s=a+1}^{b-1} \big({}^{AB}\nabla_b^{-\alpha} f\big)(s)\big({}_a^{AB}\nabla^{-\alpha}\,{}_a^{AB}\nabla^{\alpha} g\big)(s) \\ &= \sum_{s=a+1}^{b-1} g(s)\big({}^{AB}\nabla_b^{\alpha} f\big)(s).\end{aligned} \tag{47}$$

□

Next, in order to present an integration by parts formula for Caputo type fractional differences with *ML* kernels, we first define the discrete versions of the (left) generalized fractional integral operator introduced and studied in [37],

$$\big(\mathbf{E}^{\gamma}_{\rho,\mu,\omega,a^+}\varphi\big)(x) = \int_a^x (x-t)^{\mu-1} E^{\gamma}_{\rho,\mu}\big[\omega(x-t)^{\rho}\big]\varphi(t)\,dt, \quad x > a, \tag{48}$$

where $E^{\gamma}_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\rho k+\mu)k!}$ is the generalized Mittag-Leffler function which is defined for complex ρ, μ, γ ($\mathrm{Re}(\rho) > 0$) [3, 37]. For our purposes we just introduce the discrete version for $\gamma = 1$.

Definition 14

- The discrete (left) generalized fractional integral operator is defined by

$$\big(\mathbf{E}^{1}_{\overline{\rho},\overline{\mu},\omega,a^+}\varphi\big)(t) = \sum_{s=a+1}^{t} \big(t-\rho(s)\big)^{\overline{\mu-1}} E_{\overline{\rho},\overline{\mu}}\big(\omega, t-\rho(s)\big)\varphi(s), \quad t \in \mathbb{N}_a.$$

- The discrete (right) generalized fractional integral operator is defined by

$$\big(\mathbf{E}^{1}_{\overline{\rho},\overline{\mu},\omega,b^-}\varphi\big)(t) = \sum_{s=t}^{b-1} \big(s-\rho(t)\big)^{\overline{\mu-1}} E_{\overline{\rho},\overline{\mu}}\big(\omega, s-\rho(t)\big)\varphi(s), \quad t \in {}_b\mathbb{N}.$$

Theorem 7 (Integration by parts for Caputo fractional differences with *ML* kernels) *For functions f and g defined on* $\mathbb{N}_a \cap {}_b\mathbb{N}$, *we have*

$$\sum_{s=a}^{b-1} f(s)\big({}^{ABC}_{a-1}\nabla^{\alpha} g\big)(s) = \sum_{s=a}^{b-1} g(s)\big({}^{ABR}\nabla^{\alpha}_{b-1} f\big)(s) + g\big(\rho(t)\big)\frac{B(\alpha)}{1-\alpha}\big(\mathbf{E}^{1}_{\overline{\alpha},\overline{1},\lambda,b^-} f\big)(t)\Big|_a^b. \tag{49}$$

Similarly,

$$\sum_{s=a+1}^{b} f(s)\big({}^{ABC}\nabla^{\alpha}_{b+1} g\big)(s) = \sum_{s=a+1}^{b} g(s)\big({}^{ABR}_{a+1}\nabla^{\alpha} f\big)(s) - g\big(\sigma(t)\big)\frac{B(\alpha)}{1-\alpha}\big(\mathbf{E}^{1}_{\overline{\alpha},\overline{1},\lambda,a^+} f\big)(t)\Big|_a^b, \tag{50}$$

where $\lambda = \frac{-\alpha}{1-\alpha}$.

Proof By (35) and Theorem 6, we have

$$\begin{aligned}
\sum_{s=a}^{b-1} f(s)\big({}^{ABC}_{a-1}\nabla^{\alpha} g\big)(s) &= \sum_{s=a}^{b-1} f(s)\Big[\big({}^{ABR}_{a-1}\nabla^{\alpha} g\big)(s) - g(a-1)\frac{B(\alpha)}{1-\alpha}E_{\overline{\alpha}}\big(\lambda, s-\rho(a)\big)\Big] \\
&= \sum_{s=a}^{b-1} g(s)\big({}^{ABR}\nabla^{\alpha}_{b-1} f\big)(s) - g(a-1)\frac{B(\alpha)}{1-\alpha}\sum_{s=a}^{b-1} f(s)E_{\overline{\alpha}}\big(\lambda, s-\rho(a)\big) \\
&= \sum_{s=a}^{b-1} g(s)\big({}^{ABR}\nabla^{\alpha}_{b-1} f\big)(s) + g\big(\rho(t)\big)\frac{B(\alpha)}{1-\alpha}\big(\mathbf{E}^{1}_{\overline{\alpha},\overline{1},\lambda,b^-} f\big)(t)\Big|_a^b.
\end{aligned}$$

The second part follows by (42) and the second part of Theorem 6. □

4 Discrete fractional Euler-Lagrange equations

We prove the Euler-Lagrange equations for a Lagrangian containing the left new discrete Caputo derivative.

Theorem 8 *Let $0 < \alpha \leq 1$ be non-integer, $a, b \in \mathbb{R}, a < b, a \equiv b \pmod 1$. Assume that the functional of the form*

$$J(f) = \sum_{t=a}^{b-1} L\big(t, f^\rho(t), {}^{ABC}_{a-1}\nabla^\alpha f(t)\big)$$

has a local extremum in $S = \{y : (\mathbb{N}_{a-1} \cap {}_{b-1}\mathbb{N}) \to \mathbb{R} : y(a-1) = A, y(b-1) = B\}$ at some $f \in S$, where $L : (\mathbb{N}_{a-1} \cap {}_{b-1}\mathbb{N}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Then

$$\big[L_1(s) + {}^{ABR}\nabla^\alpha_{b-1} L_2(s)\big] = 0, \quad \text{for all } s \in (\mathbb{N}_{a-1} \cap {}_{b-1}\mathbb{N}), \tag{51}$$

where $L_1(s) = \frac{\partial L}{\partial f^\rho}(s)$ and $L_2(s) = \frac{\partial L}{\partial {}^{ABC}_{a-1}\nabla^\alpha f}(s)$.

Proof Without loss of generality, assume that J has local maximum in S at f. Hence, there exists an $\epsilon > 0$ such that $J(\widehat{f}) - J(f) \leq 0$ for all $\widehat{f} \in S$ with $\|\widehat{f} - f\| = \sup_{t \in \mathbb{N}_a \cap_b \mathbb{N}} |\widehat{f}(t) - f(t)| < \epsilon$. For any $\widehat{f} \in S$ there is an $\eta \in H = \{y : (\mathbb{N}_{a-1} \cap {}_{b-1}\mathbb{N}) \to \mathbb{R} : y(a-1) = y(b-1) = 0\}$ such that $\widehat{f} = f + \epsilon\eta$. Then the ϵ-Taylor's theorem and the assumption implies that the first variation quantity $\delta J(\eta, y) = \sum_{t=a}^{b-1} [\eta^\rho(t) L_1(t) + ({}^{ABC}_{a-1}\nabla^\alpha \eta)(t) L_2(t)]\, dt = 0$, for all $\eta \in H$. To make the parameter η free, we use the integration by parts equation (49) to obtain

$$\delta J(\eta, f) = \sum_{s=a}^{b-1} \eta^\rho(s)\big[L_1(s) + {}^{ABR}\nabla^\alpha_{b-1} L_2(s)\big] + \eta^\rho(t) \frac{B(\alpha)}{1-\alpha} \big(\mathbf{E}^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} L_2\big)(t)\Big|_a^b = 0,$$

for all $\eta \in H$, and hence the result follows by the discrete fundamental lemma of the calculus of variation. □

The term $(\mathbf{E}^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b^-} L_2)(t)|_a^b = 0$ above is called the natural boundary condition.

Similarly, if we allow the Lagrangian to depend on the discrete right Caputo fractional derivative, we can state the following.

Theorem 9 *Let $0 < \alpha \leq 1$ be non-integer, $a, b \in \mathbb{R}, a < b, a \equiv b \pmod 1$. Assume that the functional J of the form*

$$J(f) = \sum_{a+1}^{b} L\big(t, f^\sigma(t), {}^{ABC}\nabla^\alpha_{b+1} f(t)\big)$$

has a local extremum in $S = \{y : (\mathbb{N}_{a+1} \cap_{b+1} \mathbb{N}) \to \mathbb{R} : y(a+1) = A, y(b+1) = B\}$ at some $f \in S$, where $L : (\mathbb{N}_{a+1} \cap_{b+1} \mathbb{N}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Then

$$\big[L_1(s) + {}^{ABR}_{a+1}\nabla^\alpha L_2(s)\big] = 0, \quad \text{for all } s \in (\mathbb{N}_{a+1} \cap_{b+1} \mathbb{N}), \tag{52}$$

where $L_1(s) = \frac{\partial L}{\partial f^\sigma}(s)$ and $L_2(s) = \frac{\partial L}{\partial {}^{ABC}\nabla^\alpha_{b+1} f}(s)$.

Proof The proof is similar to Theorem 8 by applying the second integration by parts equation (50) to get the natural boundary condition of the form $(\mathbf{E}^{1}_{\overline{\alpha,1,\frac{-\alpha}{1-\alpha}},a^+}L_2)(t)|_a^b = 0$. □

Example 2 We here study an interested physical action to support Theorem 8. Namely, let us consider the following fractional discrete action:

$J(y) = \sum_{t=a}^{b-1}[\frac{1}{2}({}^{ABC}_{a-1}\nabla^{\alpha}y(t))^2 - V(y^{\rho}(t))]$, where $0 < \alpha < 1$ and with $y(a-1), y(b-1)$ are assigned or with the natural boundary condition

$$\left(\mathbf{E}^{1}_{\overline{\alpha,1,\frac{-\alpha}{1-\alpha}},b^-}{}^{ABC}_{a-1}\nabla^{\alpha}y(t)\right)(t)\Big|_a^b = 0.$$

Then the Euler-Lagrange equation by applying Theorem 8 is

$$\left({}^{ABR}\nabla^{\alpha}_{b-1} \circ {}^{ABC}_{a-1}\nabla^{\alpha}y\right)(s) - \frac{dV}{dy}(s) = 0 \quad \text{for all } s \in (\mathbb{N}_{a-1} \cap {}_{b-1}\mathbb{N}).$$

Here, we remark that it is of interest to deal with the above Euler-Lagrange equations obtained in the above example, where we have a composition of discrete right and discrete left type fractional derivatives. For the sake of comparisons with the classical discrete fractional Euler-Lagrange equations within nabla we refer to [38]. For classical fractional dynamical systems composed by the left and right fractional operators under the presence of delay we refer to [39].

5 Some tools and properties for fractional derivatives with nonsingular *ML* kernels and their discrete versions

Theorem 10 [37] *Let* $\rho, \mu, \gamma, \nu, \sigma, \lambda \in \mathbb{C}$ $(\mathrm{Re}(\rho), \mathrm{Re}(\mu), \mathrm{Re}(\nu) > 0)$, *then*

$$\int_0^x (x-t)^{\mu-1}E^{\gamma}_{\rho,\mu}\left(\lambda[x-t]^{\rho}\right)t^{\nu-1}E^{\sigma}_{\rho,\nu}\left(\lambda t^{\rho}\right)dt = x^{\mu+\nu-1}E^{\gamma+\sigma}_{\rho,\mu+\nu}\left(\lambda x^{\rho}\right). \tag{53}$$

In particular, if $\gamma = 1, \mu = 1$ *and* $\rho = \alpha$, *we have*

$$\int_0^x E_{\alpha}\left(\lambda[x-t]^{\alpha}\right)t^{\nu-1}E^{\sigma}_{\alpha,\nu}\left(\lambda t^{\alpha}\right)dt = x^{\nu}E^{1+\sigma}_{\alpha,1+\nu}\left(\lambda x^{\alpha}\right). \tag{54}$$

Remark 4 If we use the modified notation of *ML* functions, then (53) takes the form

$$\int_0^x E^{\gamma}_{\rho,\mu}(\lambda, x-t)E^{\sigma}_{\rho,\nu}(\lambda, t)\,dt = E^{\gamma+\sigma}_{\rho,\mu+\nu}(\lambda, x), \tag{55}$$

and (54) takes the form

$$\int_0^x E_{\alpha}(\lambda, x-t)E^{\sigma}_{\alpha,\nu}(\lambda, t)\,dt = E^{1+\sigma}_{\alpha,1+\nu}(\lambda, x). \tag{56}$$

From [3] we recall also the following differentiation formula, expressed in a modified way, which will be helpful.

For $\alpha, \mu, \gamma, \lambda \in \mathbb{C}$ $(\mathrm{Re}(\alpha) > 0)$ and $n \in \mathbb{N}$ we have

$$\left(\frac{d}{dz}\right)^n \left[E^{\gamma}_{\alpha,\mu}(\lambda, z)\right] = E^{\gamma}_{\alpha,\mu-n}(\lambda, z). \tag{57}$$

Now, with the help of (56) and (57), we have

$$ {}_{0}^{ABR}D^{\alpha}\big[E^{\sigma}_{\alpha,\nu}(\lambda,x)\big] = \frac{B(\alpha)}{1-\alpha}\frac{d}{dx}\big[E^{1+\sigma}_{\alpha,1+\nu}(\lambda,x)\big] = \frac{B(\alpha)}{1-\alpha}E^{1+\sigma}_{\alpha,\nu}(\lambda,x). \tag{58} $$

Similarly, with the help of (57) and (56), we have

$$ \begin{aligned} {}_{0}^{ABC}D^{\alpha}\big[E^{\sigma}_{\alpha,\nu}(\lambda,x)\big] &= \frac{B(\alpha)}{1-\alpha}\int_0^x E_\alpha(\lambda,x-t)\frac{d}{dt}\big[E^{\sigma}_{\alpha,\nu}(\lambda,t)\big]\,dt \\ &= \frac{B(\alpha)}{1-\alpha}E^{1+\sigma}_{\alpha,\nu}(\lambda,x). \end{aligned} \tag{59} $$

Remark 5 Noting that

$$ E^{-1}_{\alpha,\nu}(\lambda,x) = x^{\nu-1}E^{-1}_{\alpha,\nu}\big(\lambda x^{\alpha}\big) = \frac{x^{\nu-1}}{\Gamma(\nu)} - \frac{\lambda x^{\alpha+\nu-1}}{\Gamma(\alpha+\nu)} $$

and

$$ E^{0}_{\alpha,\nu}(\lambda,x) = \frac{x^{\nu-1}}{\Gamma(\nu)} \to 0, \quad \nu \to 0^+, $$

one can conclude from (58) and (59) with $\sigma = -1$ that the function

$$ g(x) = \lim_{\nu\to 0^+}\left[\frac{1-\alpha}{B(\alpha)}E^{-1}_{\alpha,\nu}(\lambda,x)\right] = \frac{\alpha}{B(\alpha)}\frac{x^{\alpha-1}}{\Gamma(\alpha)} $$

is a nonzero function whose fractional *ABR* and *ABC* derivative is zero. Note that the function $g(x)$ tends to the constant function $\frac{\alpha}{B(\alpha)\Gamma(\alpha)}$ as α tends to 1.

The proof of the following lemma follows by Lemma 1(i) and the definition of discrete *ML* functions in Definition 7.

Lemma 7 *For* $\gamma,\alpha,\beta,\lambda \in \mathbb{C}$ $(\mathrm{Re}(\beta) > 0)$, *and* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 0$, $|\lambda s^{-\alpha}| < 1$, *we have*

$$ \big(\mathcal{N}E^{\gamma}_{\overline{\alpha,\beta}}(\lambda,t)\big)(s) = s^{-\beta}\big[1-\lambda s^{-\alpha}\big]^{-\gamma}. $$

In particular

$$ \big(\mathcal{N}E_{\overline{\alpha,\beta}}(\lambda,t)\big)(s) = s^{-\beta}\big[1-\lambda s^{-\alpha}\big]^{-1}. $$

The proof of the following lemma just follows by applying the discrete Laplace transform $\mathcal{N}$ and its inverse in final step via the help of Lemma 7 and the discrete convolution theorem.

Lemma 8 *Let* $\rho,\mu,\gamma,\nu,\sigma,\lambda \in \mathbb{C}$ $(\mathrm{Re}(\rho),\mathrm{Re}(\mu),\mathrm{Re}(\nu) > 0)$, *then*

$$ \sum_{s=1}^{t} E^{\gamma}_{\overline{\rho,\mu}}\big(\lambda,t-\rho(s)\big)E^{\sigma}_{\overline{\rho,\nu}}(\lambda,s) = E^{\gamma+\sigma}_{\overline{\rho,\mu+\nu}}(\lambda,t). \tag{60} $$

Now, with the help of Proposition 1(iii) and Lemma 8, we have

$$ {}_{0}^{ABR}\nabla^{\alpha}\bigl[E_{\overline{\alpha,\nu}}^{\sigma}(\lambda,t)\bigr]=\frac{B(\alpha)}{1-\alpha}\nabla_{t}\bigl[E_{\overline{\alpha,1+\nu}}^{1+\sigma}(\lambda,t)\bigr]=\frac{B(\alpha)}{1-\alpha}E_{\overline{\alpha,\nu}}^{1+\sigma}(\lambda,x). \tag{61}$$

Similarly, with the help of Proposition 1(iii) and Lemma 8, we have

$$\begin{aligned} {}_{0}^{ABC}\nabla^{\alpha}\bigl[E_{\overline{\alpha,\nu}}^{\sigma}(\lambda,t)\bigr]&=\frac{B(\alpha)}{1-\alpha}\sum_{s=1}^{t}E_{\alpha}\bigl(\lambda,t-\rho(s)\bigr)\nabla_{t}\bigl[E_{\overline{\alpha,\nu}}^{\sigma}(\lambda,s)\bigr]\\ &=\frac{B(\alpha)}{1-\alpha}E_{\overline{\alpha,\nu}}^{1+\sigma}(\lambda,t). \end{aligned}\tag{62}$$

Remark 6 Noting that

$$E_{\overline{\alpha,\nu}}^{-1}(\lambda,t)=\frac{x^{\overline{\nu-1}}}{\Gamma(\nu)}-\frac{\lambda t^{\overline{\alpha+\nu-1}}}{\Gamma(\alpha+\nu)}$$

and

$$E_{\overline{\alpha,\nu}}^{0}(\lambda,t)=\frac{t^{\overline{\nu-1}}}{\Gamma(\nu)}\to 0,\quad \nu\to 0^{+},$$

one can conclude from (61) and (62) with $\sigma=-1$ that the function

$$h(t)=\lim_{\nu\to 0^{+}}\biggl[\frac{1-\alpha}{B(\alpha)}E_{\overline{\alpha,\nu}}^{-1}(\lambda,t)\biggr]=\frac{\alpha}{B(\alpha)}\frac{t^{\overline{\alpha-1}}}{\Gamma(\alpha)}$$

is a nonzero function whose discrete fractional *ABR* and *ABC* derivative is zero. Note that the function $h(t)$ tends to the constant function 1 as α tends to 1. As a result of this, if the potential function V in Example 2, with $a=1$, is 0, then the solution takes the form

$$y(t)=\frac{\alpha}{B(\alpha)}\frac{t^{\overline{\alpha-1}}}{\Gamma(\alpha)}.$$

Using the following a-version of equation (14) in [7]:

$$\bigl({}_{a}^{ABC}D^{\alpha}f\bigr)(t)=\bigl({}_{a}^{ABR}D^{\alpha}f\bigr)(t)-\frac{B(\alpha)}{1-\alpha}f(a)E_{\alpha}\bigl(\lambda(t-a)^{\alpha}\bigr),\qquad \lambda=\frac{-\alpha}{1-\alpha}, \tag{63}$$

and the identity (see [3], p.78, for example)

$$\bigl({}_{a}I^{\alpha}(t-a)^{\beta-1}E_{\mu,\beta}\bigl[\lambda(t-a)^{\mu}\bigr]\bigr)(x)=(x-a)^{\alpha+\beta-1}E_{\mu,\alpha+\beta}\bigl[\lambda(x-a)^{\mu}\bigr], \tag{64}$$

we can state the following result which is very useful tool to solve fractional dynamical systems with a Caputo fractional derivative with *ML* kernels.

Proposition 3 *For* $0<\alpha<1$, *we have*

$$\begin{aligned} \bigl({}_{a}^{AB}I^{\alpha}\,{}_{a}^{ABC}D^{\alpha}f\bigr)(x)&=f(x)-f(a)E_{\alpha}\bigl(\lambda(x-a)^{\alpha}\bigr)-\frac{\alpha}{1-\alpha}f(a)x^{\alpha}E_{\alpha,\alpha+1}\bigl(\lambda(x-a)^{\alpha}\bigr)\\ &=f(x)-f(a). \end{aligned}$$

Similarly,

$$\left({}^{AB}I_b^{\alpha\,ABC}D_b^{\alpha}f\right)(x) = f(x) - f(b). \tag{65}$$

Similarly, in the discrete case by recalling that (see [32], Proposition 3.9 or [35])

$$\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1}(t-a)^{\overline{\mu+\alpha}},$$

we can state the following important result in classical discrete fractional calculus.

Theorem 11 *Let* $a \in [0,\infty)$, *and let* $\alpha, \rho, \mu, \gamma, \lambda \in \mathbb{C}$ $(\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\mu) > 0, \mathrm{Re}(\rho) > 0)$. *Then for* $t > a$ *the relations hold*:

- $\nabla_a^{-\alpha} E_{\overline{\rho,\mu}}^{\gamma}(\lambda, t-a) = E_{\overline{\rho,\mu+\alpha}}^{\gamma}(\lambda, t-a)$.
- $\nabla_a^{\alpha} E_{\overline{\rho,\mu}}^{\gamma}(\lambda, t-a) = E_{\overline{\rho,\mu-\alpha}}^{\gamma}(\lambda, t-a)$.

Then we can state the following.

Proposition 4 *For* $0 < \alpha < 1$, *we have*

$$\begin{aligned}\left({}_a^{AB}\nabla^{-\alpha}\,{}_a^{ABC}\nabla^{\alpha}f\right)(t) &= f(t) - f(a)E_{\overline{\alpha}}(\lambda, t-a) - \frac{\alpha}{1-\alpha}f(a)E_{\overline{\alpha,\alpha+1}}(\lambda, t-a)\\ &= f(t) - f(a).\end{aligned}$$

Similarly, by the first part and the action of the Q-operator

$$\left({}^{AB}\nabla_b^{-\alpha\,ABC}\nabla_b^{\alpha}f\right)(t) = f(t) - f(b). \tag{66}$$

6 Conclusions

The modified versions of Mittag-Leffler functions enable us to treat easily the fractional type derivatives with *ML* kernels and enable us to obtain successfully their discrete versions. The *Q*-operator and its discrete version always provide an effective tool to confirm dual definitions and relations when passing from left to right or *vice versa*. There exist nonconstant functions whose usual or discrete *ABC* fractional derivatives are zeros. Hence a zero potential function in a usual or discrete variational problem does not imply only constant solution. The results obtained tend to the ordinary case when α tends to 1. The discrete versions for *AB* type fractional derivatives have been defined and their discrete fractional integrals given with the help of the discrete Laplace transform.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in this article. They read and approved the final manuscript.

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