

Discrete Linear Canonical Transform Based on Hyperdifferential Operators

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Abstract—Linear canonical transforms (LCTs) are of importance in many areas of science and engineering with many applications. Therefore, a satisfactory discrete implementation is of considerable interest. Although there are methods that link the samples of the input signal to the samples of the linear canonical transformed output signal, no widely-accepted definition of the discrete LCT has been established. We introduce a new approach to defining the discrete linear canonical transform (DLCT) by employing operator theory. Operators are abstract entities that can have both continuous and discrete concrete manifestations. Generating the continuous and discrete manifestations of LCTs from the same abstract operator framework allows us to define the continuous and discrete transforms in a structurally analogous manner. By utilizing hyperdifferential operators, we obtain a DLCT matrix, which is totally compatible with the theory of the discrete Fourier transform (DFT) and its dual and circulant structure, which makes further analytical manipulations and progress possible. The proposed DLCT is to the continuous LCT, what the DFT is to the continuous Fourier transform. The DLCT of the signal is obtained simply by multiplying the vector holding the samples of the input signal by the DLCT matrix.

Index Terms—Linear canonical transform (LCT), fractional Fourier transform (FRT), operator theory, discrete transforms, hyperdifferential operators.

I. INTRODUCTION

LINEAR canonical transforms (LCTs) are a family of linear integral transforms with three parameters, [1]–[4]. The family of LCTs is a generalization of many important transforms such as the fractional Fourier transform (FRT), chirp multiplication (CM), chirp convolution (CC), and scaling operations. For certain values of the three parameters, the LCT reduces to these transforms or their combinations. LCTs have several applications in signal processing [3] and computational and applied mathematics [5], [6], including fast and efficient optimal filtering [7], radar signal processing [8], [9], speech processing [10], image representation [11], and image encryption and watermarking [12]–[14], to mention a small sample of published

works. LCTs have also been extensively studied for their applications in optics [2], [15]–[20], electromagnetics, and classical and quantum mechanics [1], [3], [21], [22].

In optical contexts, LCTs are commonly referred to as quadratic-phase integrals or quadratic-phase systems [17], [23]. The so-called *ABCD* systems widely used in optics [24] are also represented by linear canonical transforms. They have also been referred to by other names: generalized Huygens integrals [15], generalized Fresnel transforms [25], [26], special affine Fourier transforms [27], [28], extended fractional Fourier transforms [29], and Moshinsky-Quesne transforms [1].

Two-dimensional (2D) LCTs and complex-parametered LCTs (CLCTs) have also been discussed in the literature, [30]–[33]. Bilateral Laplace transforms, Bargmann transforms, Gauss-Weierstrass transforms, [1], [34], [35], fractional Laplace transforms, [36], [37], and complex-ordered FRTs [38]–[41] are all special cases of CLCTs.

The establishment of a discrete framework is essential to the deployment of LCTs in applications. There is considerable work on discrete or finite forms of fractional Fourier transforms, and, to a lesser degree, discrete or finite linear canonical transforms. Being one of the most important special cases of LCTs, discretization and discrete versions of fractional Fourier transforms have been well studied and established [42]–[54].

As for the discretization or digital computation of LCTs, there are many approaches present in the literature, [23], [55]–[70]. Some of these [23], [56]–[59], [61], [62], [66], [67] numerically compute the continuous integral and establish a direct mapping between the samples of the continuous input function and the samples of the LCT-transformed continuous output function. The methods in [55]–[58], [66] directly convert the LCT integral to a summation and [23], [59]–[62], [67] make use of decompositions into elementary building blocks. Moreover, some approaches focus on defining a discrete LCT (DLCT), which can then be used to numerically approximate continuous LCTs, in the same way that the discrete Fourier transform (DFT) is used to approximate continuous Fourier transforms [55], [57], [60], [63]–[65], [68]–[70]. Algorithms in [55], [57], [60] also numerically approximate the continuous LCTs in the same way the DFT approximates the continuous FT. Based on the DLCT definition proposed in [55], Refs. [66] and [67] propose efficient numerical computation algorithms. Ref. [55] also includes a comparison of the properties satisfied by definitions of DLCTs proposed up to that date.

Despite these works, no single definition has been widely established as the definition of the DLCT. In this paper, we

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present a different approach based on hyperdifferential operator theory [1], [2], [71]–[73], to obtain a definition of the DLCT. Why do we propose to use operator theory? Most approaches to discretization are naturally based on sampling of the continuous entities. However, sampling often does not lead to a clean, discrete transform definition that satisfies operational formulas and exhibits desirable analytical properties such as unitarity and preservation of the group structure. So if our purpose is not to merely numerically compute a continuous transform, but to obtain a self-consistent discrete transform definition, it often turns out to be insufficient. A purely numerical method can compute the continuous transform accurately, but it does not provide us with a definition on which further manipulation can be done, and theoretical progress can build upon. We want a discrete definition that is as analogous to the continuous definition as possible. (This is satisfied by the discrete Fourier transform (DFT) and that is why the DFT is so established.)

How does operator theory help? Operators are abstract entities that can have both continuous and discrete concrete manifestations. Thus if we begin from a continuous entity and can appropriately deduce the abstract operator underlying that entity, then, that can form a basis for defining its discrete version. Since both the continuous and discrete versions are based on the same abstract operator, they can be expected to exhibit similar structural characteristics and operational properties to the extent possible. The structure of relationships between different entities can also be preserved and can be expected to mirror the relationships between the abstract operators. Thus we can obtain discrete entities that are not merely numerical approximations, but which exhibit desirable analytical and operational properties. This is the rationale of the present paper.

Our definition of the discrete LCT will be presented in the form of a matrix of size $N \times N$ which, upon multiplication, produces the DLCT of a discrete and finite signal of length N , expressed as a column vector. The main difference from earlier approaches is that the definition is based on hyperdifferential forms of the discrete coordinate multiplication and differentiation operators, which we carefully define so that they are strictly Fourier duals related through the DFT matrix. Our definition provides a self-consistent, pure, and elegant definition of the DLCT which is fully compatible with the theory of the discrete Fourier transform and its dual and circulant structure. By self-consistent we mean that the relations between discrete entities should mirror those between continuous entities as much as possible, e.g. if the coordinate multiplication and differentiation operators are dual in the continuous case, they should also be so in the discrete case. The discrete LCT should be built upon these two operators in the same way that the continuous LCT is, and so forth. By duality we mean that a kind of symmetry between the two domains is exactly satisfied (e.g. coordinate multiplication in one domain is differentiation in the other, translation in one domain is phase multiplication in the other, etc.). All the dual properties of the Fourier transform (such as those in parenthesis above) can be derived from the duality of \mathcal{U} and \mathcal{D} [2], so first and foremost, this duality must be maintained. One of the most important features of our approach is that our

definition maintains this structure by treating both domains totally symmetrically.

The paper is organized as follows: Section II reviews the preliminaries and the definition and important properties of LCTs. Section III describes the theory and derivations for the proposed DLCT. Theoretical discussions on defining a discrete LCT and the properties of such a definition that need to exist are given in Section IV. In Section V, numerical examples and comparisons are provided. Lastly, we conclude in Section VI. There is also an Appendix in which we have provided some proofs, necessary fundamental information, justifications and implementation details that are needed for the derivations in Section III.

II. PRELIMINARIES

A. Linear Canonical Transform

LCTs are unitary transforms specified by a 2×2 parameter matrix \mathbf{L} . Because the determinant of \mathbf{L} is required to be equal to 1, an LCT can also be uniquely specified by three independent parameters, often denoted by α, β, γ . The elements A, B, C, D of the 2×2 matrix and α, β, γ are related by:

$$\mathbf{L} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{\gamma}{\beta} & \frac{1}{\beta} \\ -\beta + \frac{\alpha\gamma}{\beta} & \frac{\alpha}{\beta} \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{\beta} & \frac{-1}{\beta} \\ \beta - \frac{\alpha\gamma}{\beta} & \frac{\gamma}{\beta} \end{bmatrix}^{-1}. \quad (1)$$

We can define an LCT through either the parameter set (A, B, C, D) with the condition that $AD - BC = 1$ or the parameter set (α, β, γ) . In this paper, we restrict ourselves to the case where the parameters in both sets are all real. The definition of the LCT as a linear integral transform, using the second set of parameters, can be written as:

$$\mathcal{C}_{\mathbf{L}} f(u) = \sqrt{\beta} e^{-i\pi/4} \int_{-\infty}^{\infty} \exp [i\pi(\alpha u^2 - 2\beta uu' + \gamma u'^2)] f(u') du'. \quad (2)$$

Every triplet (α, β, γ) corresponds to a different LCT. We denote the LCT operator using $\mathcal{C}_{\mathbf{L}}$ where the subscript \mathbf{L} denotes the 2×2 parameter matrix.

B. Important Properties

The utility of the parameter set (A, B, C, D) is best appreciated upon observing the concatenation property: If any two LCTs are concatenated (applied one after the other), the resulting operation is also an LCT whose 2×2 matrix is the product of the 2×2 matrices of the two original LCTs. This can be stated as:

$$\mathcal{C}_{\mathbf{L}} f(u) = \mathcal{C}_{\mathbf{L}_1} \mathcal{C}_{\mathbf{L}_2} f(u), \quad (3)$$

where $\mathbf{L} = \mathbf{L}_1 \mathbf{L}_2$.

An important special case of this property is the reversibility property. It basically states that the 2×2 matrix for the inverse of an LCT is again an LCT whose 2×2 matrix is the matrix inverse of the original LCT:

$$\mathcal{C}_{\mathbf{L}_2} \mathcal{C}_{\mathbf{L}_1} f(u) = f(u), \quad (4)$$

if $\mathbf{L}_2 = \mathbf{L}_1^{-1}$.

C. Special Linear Canonical Transforms

We now give some special transforms and operations, which are all special cases of LCTs.

1) *Scaling*: The parameter matrix for the scaling operation is as follows

$$\mathbf{L}_M = \begin{bmatrix} M & 0 \\ 0 & \frac{1}{M} \end{bmatrix} = \begin{bmatrix} \frac{1}{M} & 0 \\ 0 & M \end{bmatrix}^{-1}. \quad (5)$$

Functionally it can be defined in the following way:

$$\mathcal{C}_{\mathbf{L}_M} f(u) = \mathcal{M}_M f(u) = \sqrt{\frac{1}{M}} f\left(\frac{u}{M}\right). \quad (6)$$

2) *Fractional Fourier Transform*: The Fractional Fourier transform (FRT) is the generalized version of the Fourier transform (FT). It has the following parameter matrix:

$$\mathbf{L}_{\mathcal{F}_{lc}^a} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^{-1}, \quad (7)$$

where $\theta = \pi a/2$ and a is the fractional order. When $a = 1$, the FRT reduces to the FT. (It should be noted that there is a slight difference between the FRT thus defined (\mathcal{F}_{lc}^a) and the more commonly used definition of the FRT (\mathcal{F}^a), [2].)

The a th order fractional Fourier transform \mathcal{F}^a of the function $f(u)$ may be defined as [2]:

$$\begin{aligned} \mathcal{F}^a f(u) &= \int_{-\infty}^{\infty} K_a(u, u') f(u') du', \\ K_a(u, u') &= A_\theta \exp [i\pi(u^2 \cot \theta - 2uu' \csc \theta + u'^2 \cot \theta)], \\ A_\theta &= \frac{\exp(-i\pi \operatorname{sgn}(\sin \theta)/4 + i\theta/2)}{|\sin \theta|^{1/2}} \end{aligned} \quad (8)$$

3) *Chirp Multiplication*: The parameter matrix for the chirp multiplication operation is

$$\mathbf{L}_{\mathcal{Q}_q} = \begin{bmatrix} 1 & 0 \\ -q & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ q & 1 \end{bmatrix}^{-1}. \quad (9)$$

The chirp multiplication operation can be expressed as

$$\mathcal{C}_{\mathcal{Q}_q} f(u) = \mathcal{Q}_q f(u) = \exp(-i\pi q u^2) f(u). \quad (10)$$

Corresponding formulas for chirp convolution may be found in [2].

III. DISCRETE LINEAR CANONICAL TRANSFORMS

We now present our development of the DLCT based on hyperdifferential operator theory. Our approach is based on decomposing the LCT into simpler parts, finding the discrete versions of these parts by using operator theory, and then multiplying those to obtain the final DLCT matrix.

Although there are several ways to decompose the LCT [59], here we choose the Iwasawa decomposition since it includes a greater number of special LCTs than other decompositions, providing the opportunity to discuss their hyperdifferential forms. The method of using hyperdifferential operators outlined here can also be applied to other decompositions.

A. The Iwasawa Decomposition

The linear canonical transform (LCT) operator $\mathcal{C}_{\mathbf{L}}$ can be expressed as combinations of other simpler operators in many ways. Using scaling \mathcal{M}_M , chirp multiplication \mathcal{Q}_q and fractional Fourier \mathcal{F}^a operators, it is possible to construct any linear canonical transform. The Iwasawa decomposition we will employ, breaks down an arbitrary LCT into a fractional Fourier transform followed by scaling followed by chirp multiplication, and can be written in operator notation as follows [3]:

$$\mathcal{C}_{\mathbf{L}} = \mathcal{Q}_q \mathcal{M}_M \mathcal{F}_{lc}^a, \quad (11)$$

When each operator is characterized by their 2×2 LCT parameter matrix, the decomposition looks like

$$\begin{aligned} \mathbf{L} &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{\gamma}{\beta} & \frac{1}{\beta} \\ -\beta + \frac{\alpha\gamma}{\beta} & \frac{\alpha}{\beta} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -q & 1 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & 1/M \end{bmatrix} \begin{bmatrix} \cos a\pi/2 & \sin a\pi/2 \\ -\sin a\pi/2 & \cos a\pi/2 \end{bmatrix} \end{aligned} \quad (12)$$

where a, q, M must be chosen as:

$$M = \begin{cases} \sqrt{1 + \gamma^2}/\beta, & \gamma \geq 0, \\ -\sqrt{1 + \gamma^2}/\beta, & \gamma < 0, \end{cases} \quad (13)$$

$$q = \frac{\gamma\beta^2}{1 + \gamma^2} - \alpha, \quad (14)$$

$$a = \frac{2}{\pi} \operatorname{arccot} \gamma. \quad (15)$$

This decomposition can break down any arbitrary linear canonical transform into a cascade of elementary operations. Our approach will be to find the $N \times N$ discrete transform matrix for each of these three operations and multiply them to obtain the discrete LCT matrix.

B. The Hyperdifferential Forms

The term hyperdifferential refers to having differential operators in an exponent. In the LCT context, we only have second order coordinate multiplication and differentiation operators in the exponent. Operators representing an arbitrary LCT or all of its special cases can be generated by exponentiating these second order operators and these constitute the hyperdifferential forms of these transforms. There is correspondence among the integral transforms, hyperdifferential operators and the 2×2 parameter matrices that are given in the preliminaries section. An LCT can be represented by any one of these mathematical objects. More details can be found in [1].

It is well established that the chirp multiplication operator \mathcal{Q}_q , the scaling operator \mathcal{M}_M , and the fractional Fourier transform operator \mathcal{F}_{lc}^a can all be written in hyperdifferential forms as

follows: [1], [2]:

$$\mathbf{Q}_q = \exp\left(-i2\pi q \frac{\mathbf{U}^2}{2}\right), \quad (16)$$

$$\mathbf{M}_M = \exp\left(-i2\pi \ln(M) \frac{\mathbf{U}\mathbf{D} + \mathbf{D}\mathbf{U}}{2}\right), \quad (17)$$

$$\mathbf{F}_{\text{lc}}^a = \exp\left(-ia\pi^2 \frac{\mathbf{U}^2 + \mathbf{D}^2}{2}\right), \quad (18)$$

where \mathcal{U} and \mathcal{D} are the coordinate multiplication and differentiation operators, respectively. We see that all three of the operators we are working with can be expressed in terms of these two building blocks, whose continuous manifestations are:

$$\mathcal{U}f(u) = uf(u) \quad (19)$$

$$\mathcal{D}f(u) = \frac{1}{i2\pi} \frac{df(u)}{du}, \quad (20)$$

where the $(i2\pi)^{-1}$ is included so that \mathcal{U} and \mathcal{D} are precisely Fourier duals (the effect of either in one domain is its dual in the Fourier domain). This duality can be expressed as follows:

$$\mathcal{U} = \mathcal{F}\mathcal{D}\mathcal{F}^{-1}. \quad (21)$$

C. The Discrete Linear Canonical Transform

Our approach is based on requiring that, to the extent possible, all the discrete entities we define observe the same structural relationships as they do in abstract operator form. We want a discrete definition that is as analogous to the continuous definition as possible. To ensure this, we define the discrete LCT and its special cases as the discrete manifestations of Eq. (11), Eq. (16), Eq. (17) and Eq. (18), with the abstract operators being replaced by matrix operators. This can be written as follows:

$$\mathbf{C}_L = \mathbf{Q}_q \mathbf{M}_M \mathbf{F}_{\text{lc}}^a. \quad (22)$$

$$\mathbf{Q}_q = \exp\left(-i2\pi q \frac{\mathbf{U}^2}{2}\right). \quad (23)$$

$$\mathbf{M}_M = \exp\left(-i2\pi \ln(M) \frac{\mathbf{U}\mathbf{D} + \mathbf{D}\mathbf{U}}{2}\right). \quad (24)$$

$$\mathbf{F}_{\text{lc}}^a = \exp\left(-ia\pi^2 \frac{\mathbf{U}^2 + \mathbf{D}^2}{2}\right). \quad (25)$$

Note that $\exp()$ in the above equations are matrix exponentials and how they are computed is discussed in Appendix C. Thus the discrete LCT matrix is given by

$$\begin{aligned} \mathbf{C}_L &= \exp\left(-i2\pi q \frac{\mathbf{U}^2}{2}\right) \times \\ &\exp\left(-i2\pi \ln(M) \frac{\mathbf{U}\mathbf{D} + \mathbf{D}\mathbf{U}}{2}\right) \exp\left(-ia\pi^2 \frac{\mathbf{U}^2 + \mathbf{D}^2}{2}\right). \end{aligned} \quad (26)$$

The discrete LCT matrix is defined as the product of the FRT, scaling, and chirp multiplication matrices, all of which are defined in terms of the \mathbf{U} and \mathbf{D} matrices. To get the DLCT of a function of a discrete variable, we just need to write it as a column vector and multiply it with the DLCT matrix \mathbf{C}_L .

Thus it is seen that all rests on the differentiation and coordinate multiplication matrices \mathbf{D} and \mathbf{U} and computation of the matrix exponentials in Eq. (26). Thus, we move on to how to obtain the \mathbf{U} and \mathbf{D} matrices.

For signals of discrete variables, the closest thing to differentiation is finite differencing. Consider the following definition:

$$\tilde{\mathcal{D}}_h f(u) = \frac{1}{i2\pi} \frac{f(u+h/2) - f(u-h/2)}{h}. \quad (27)$$

If $h \rightarrow 0$, then $\tilde{\mathcal{D}}_h \rightarrow \mathcal{D}$, since in this case the right-hand side approaches $(i2\pi)^{-1} df(u)/du$. Therefore, $\tilde{\mathcal{D}}_h$ can be interpreted as a finite difference operator.

Now, using $f(u+h) = \exp(i2\pi h\mathcal{D})f(u)$, which is another established result in operator theory [1], [2], we express Eq. (27) in hyperdifferential form:

$$\begin{aligned} \tilde{\mathcal{D}}_h &= \frac{1}{i2\pi} \frac{e^{i\pi h\mathcal{D}} - e^{-i\pi h\mathcal{D}}}{h} \\ &= \frac{1}{i2\pi} \frac{2i \sin(\pi h\mathcal{D})}{h} = \text{sinc}(h\mathcal{D}) \mathcal{D}. \end{aligned} \quad (28)$$

Note that if we let $h \rightarrow 0$ in the last equation and take the limit, we can verify that $\tilde{\mathcal{D}}_h \rightarrow \mathcal{D}$ from here as well.

Now, we turn our attention to the task of defining $\tilde{\mathcal{U}}_h$. It is tempting to define the discrete version of the coordinate multiplication matrix by simply forming a diagonal matrix with the diagonal entries being equal to the coordinate values. However, upon closer inspection we have decided that this could not be taken for granted. In order to obtain the most self-consistent formulation possible, we must be sure to maintain the structural symmetry between \mathcal{U} and \mathcal{D} in all their manifestations. Therefore, we choose to define $\tilde{\mathcal{U}}_h$ such that it is related to \mathcal{U} , in exactly the same way as $\tilde{\mathcal{D}}_h$ is related to \mathcal{D} :

$$\tilde{\mathcal{U}}_h = \text{sinc}(h\mathcal{U}) \mathcal{U}, \quad (29)$$

from which we can observe that as $h \rightarrow 0$, we have $\tilde{\mathcal{U}}_h \rightarrow \mathcal{U}$, as should be. However, beyond that, it is also possible to show that, $\tilde{\mathcal{U}}_h$, when defined like this, satisfies the same duality expression Eq. (21) satisfied by \mathcal{U} and \mathcal{D} :

$$\tilde{\mathcal{U}}_h = \mathcal{F}\tilde{\mathcal{D}}_h\mathcal{F}^{-1}. \quad (30)$$

To see this, substitute $\tilde{\mathcal{D}}_h$ in this equation:

$$\begin{aligned} \tilde{\mathcal{U}}_h &= \mathcal{F} \left(\frac{1}{i2\pi} \frac{2i \sin(\pi h\mathcal{D})}{h} \right) \mathcal{F}^{-1} \\ &= \frac{1}{i2\pi} \frac{2i \sin(\pi h\mathcal{U})}{h} = \text{sinc}(h\mathcal{U})\mathcal{U}. \end{aligned} \quad (31)$$

When acting on a continuous signal $f(u)$, the operator \mathcal{U} becomes

$$\tilde{\mathcal{U}}_h f(u) = \frac{1}{\pi} \frac{\sin(\pi hu)}{h} f(u). \quad (32)$$

We observe that the effect is not merely multiplying with the coordinate variable. Had we defined $\tilde{\mathcal{U}}_h$ such that it corresponds to multiplication with the coordinate variable, we would have destroyed the symmetry and duality between \mathcal{U} and \mathcal{D} in passing to the discrete world.

Now, by sampling Eq. (32), we can obtain the matrix operator to act on finite discrete signals. The sample points will be taken as $u = nh$ to finally yield the \mathbf{U} matrix defined as:

$$U_{mn} = \begin{cases} \frac{\sqrt{N}}{\pi} \sin\left(\frac{\pi}{N}n\right), & \text{for } m = n \\ 0, & \text{for } m \neq n \end{cases} \quad (33)$$

As always, the value of N should be determined based on the time/space and frequency extent of the signal, along with the required accuracy [59], [74]–[76]. Further detail is provided in Section III-E.

The matrix \mathbf{D} , on the other hand, can be calculated in terms of \mathbf{U} by using the discrete version of the duality relation given in Eq. (21):

$$\mathbf{D} = \mathbf{F}^{-1}\mathbf{U}\mathbf{F}, \quad (34)$$

in which \mathbf{F} is the matrix representing the unitary discrete Fourier transform (DFT) matrix. The elements F_{mn} of the N -point unitary DFT matrix \mathbf{F} can be written in terms of $W_N = \exp(-j2\pi/N)$ as follows:

$$F_{mn} = \frac{1}{\sqrt{N}} W_N^{mn}.$$

When all is put together, the LCT of a signal $x[n]$ of length N , represented by the column vector \mathbf{x} , is then computed by $\mathbf{C}_L \mathbf{x}$, yielding an $N \times 1$ output. Further details of the development of the \mathbf{U} and \mathbf{D} matrices and their applications may be found in [77], which together with the present work, not only establish a formulation of these operators that is fully consistent with the theory of the DFT and its circulant structure, but also pave the way for the utilization of operator theory in deriving other more sophisticated discrete operations. We believe these works are the first to apply operator theory in defining discrete transforms.

D. Unitarity of the Discrete Linear Canonical Transform

One of the most essential properties of the kind of discrete transforms we are working with is unitarity. This leads to Parseval type relationships and manifests itself as energy or power conservation in physical applications.

Here we prove that the proposed DLCT definition is unitary by showing that the matrix \mathbf{C}_L given in Eq. (22) and more explicitly in Eq. (26) is unitary.

Theorem 1: The discrete LCT defined in Eq. (26) is unitary, with M, q, a chosen according to Eqs. (13), (14), (15), and \mathbf{U} and \mathbf{D} defined according to Eqs. (33) and (34).

Before proceeding with the proof, we first recall some fundamental definitions: A matrix \mathbf{A} is said to be Hermitian when $\mathbf{A} = \mathbf{A}^H$ holds, where \mathbf{A}^H denotes the conjugate transpose of \mathbf{A} , and is said to be unitary when $\mathbf{A}^{-1} = \mathbf{A}^H$. Since \mathbf{C}_L is defined as the product of three matrices, showing that each of them is unitary will suffice to show that \mathbf{C}_L is unitary. \mathbf{U} and \mathbf{D} are the fundamental matrices that give rise to those three components. We will first show that these matrices are Hermitian. From that it will follow that the three multiplied matrices are all unitary.

Theorem 2: The matrices \mathbf{U} and \mathbf{D} are Hermitian and the matrices defined in Eqs. (23), (24), (25) are unitary.

Theorem 2 is proved in the Appendix A from which Theorem 1 follows.

E. Discretization, Sampling and Indexing

We introduce discretization by replacing the continuous derivative with a finite difference, such that, as the finite interval goes to zero, it approaches the continuous derivative. Remembering that exponentiation etc. can be expressed as power series, the full LCT development is then based on the following operations on this finite difference operation: inversion, fractional and ordinary Fourier transformation, repeated application, multiplication with a scalar and addition. Now, as the finite difference goes to a derivative, similar will hold for its repeated applications, as well as scalar multiplied and added versions. Likewise, we know that the DFT approximates the continuous Fourier transform more and more closely as the sampling interval is reduced, so if this operation is in succession with finite differencing, the resulting limit will be the succession of Fourier transformation and continuous differentiation. Similar applies to fractional Fourier transformation, of which inversion is a special case.

In this paper we deal with finite-length signals of a discrete (integer) variable. (We could equivalently think of them as being defined on a circulant domain, which would not make a difference in our arguments.) The length of our signal vectors will be denoted by N . When N is even, they will be defined on the interval of integers $[-\frac{N}{2}, \frac{N}{2} - 1]$, and when N is odd, they will be defined on the interval of integers $[-\frac{N-1}{2}, \frac{N-1}{2}]$. We will also consider an alternative, less-common approach based on the device of using “half integers.” In this approach, the domain is defined as the interval of unit-spaced half integers $[-\frac{N}{2} + 0.5, \frac{N}{2} - 1 + 0.5]$ for even N and $[-\frac{N-1}{2} - 0.5, \frac{N-1}{2} - 0.5]$ for odd N . Although not very usual, there is nothing unnatural about this way of indexing signals of a discrete variable; it is merely a particular way of bookkeeping. Note that the indices are still spaced by unity, and there is merely a shift by 0.5 with the purpose of making the interval symmetrical around the origin when N is even (with the consequence that symmetry is lost when N is odd). A few examples of works considering this way of indexing are [51], [78]–[80]. Consistent with this literature, we will refer to the former approach as the *ordinary* DFT and refer to the latter one, in which we use “half integers”, as the *centered* DFT. The DLCT derivation procedure we presented has been carefully written in a manner that it is consistent with both approaches. Readers interested in further details on this issue may refer to [77].

How the number of samples N should be chosen will be determined by factors such as the temporal or spatial extent of the signal, the frequency extent of the signal and therefore the time- or space-bandwidth product. It will also depend on the precision with which the results need to be computed in that application. The choice of N is exogenous to our method. Nevertheless, for completeness, let us elaborate on how the number of samples N is chosen. If the temporal or spatial extent is Δx and the double-sided frequency extent is $\Delta \nu$, then we should be sampling with an interval of $1/\Delta \nu$, which means $\Delta x/(1/\Delta \nu) = \Delta x \Delta \nu$

TABLE I
PERCENTAGE MSE ERRORS FOR DIFFERENT FUNCTIONS AND TRANSFORMS (FOR BOTH ORDINARY AND CENTERED SCHEMES)

Input	N	T1 (ord.)	T2 (ord.)	T3 (ord.)	T4 (ord.)	T1 (cent.)	T2 (cent.)	T3 (cent.)	T4 (cent.)
F1	256	9.82×10^{-4}	4.72×10^{-3}	6.78×10^{-4}	3.93×10^{-2}	9.82×10^{-4}	4.71×10^{-3}	6.78×10^{-4}	3.93×10^{-2}
	1024	6.40×10^{-5}	2.76×10^{-4}	4.26×10^{-5}	2.49×10^{-3}	6.40×10^{-5}	2.76×10^{-4}	4.26×10^{-5}	2.49×10^{-3}
F2	256	4.31	10.6	1.95	6.65	4.31	10.6	1.96	6.65
	1024	0.32	0.87	0.13	0.46	0.32	0.87	0.13	0.46
F3	256	2.49	1.55	2.84	2.85	2.02	1.45	2.37	2.66
	1024	1.09	0.75	1.40	1.44	1.10	0.85	1.34	1.50
F4	256	1.34	0.64	2.29	6.77	1.35	0.63	2.30	6.79
	1024	9.43×10^{-2}	4.38×10^{-2}	0.16	0.49	9.44×10^{-2}	4.38×10^{-2}	0.16	0.49

samples. We call this number of samples N , the time- or space-bandwidth product. If appropriate normalization as described in [59] is applied so that the time/space extent and the frequency extent are made equal in a dimensionless space, it follows that we should sample over an extent \sqrt{N} with sampling interval $h = 1/\sqrt{N}$. Thus as we increase N , we will be making h smaller and smaller. Consequently, the finite difference operator in Eq. (27) approaches a continuous derivative and the finite coordinate multiplication operator will approach the continuous coordinate multiplication operator. The matrix in Eq. (33) will approach $U_{mn} = n/\sqrt{N}$, corresponding to samples of continuous coordinate multiplication. Since all our operators, including the LCT, are defined in terms of coordinate multiplication and differentiation through smooth exponential functions, they will all approach their continuous counterparts.

IV. DISCUSSIONS

Continuous unitary LCTs represented by the parameter matrices \mathbf{L} form the real symplectic group $Sp(2, R)$ with three independent parameters [81]. The desirable properties of a discrete LCT mirror those of the continuous LCT: unitarity, preservation of group structure as expressed by the concatenation property (and its special case reversibility), reduction to important special cases and inverses of special cases, and some satisfactory approximation of the continuous transform. However, a theorem from group theory [82], [83] precludes realization of this ideal: It is theoretically impossible to discretize all LCTs with a finite number of samples such that they are both unitary and they preserve the group structure [82], [83]. More on the group-theoretical properties of LCTs can be found in [1], [2], [82].

That said, no unitary DLCT definition can exhibit exact concatenation/reversibility properties. However, if the proposed definition is to have practical use, we can expect that these properties are at least approximately satisfied. In Section III-D, we theoretically proved that our proposed DLCT is unitary, so that it cannot exactly satisfy the concatenation/reversibility property. Therefore, in the next section, we will numerically show that the concatenation and reversibility properties are satisfied with a reasonable accuracy. We will also show that, regardless of

concatenation, the discrete transform provides a reasonable approximation to the continuous LCT. Before moving on, it needs to be noted that our definition, by construction, reduces to the identity, Fourier and fractional Fourier transforms, chirp multiplication, and magnification (scaling). This result can be trivially obtained by substituting the combination of values leading to the special cases for the parameters a , M , and q in Eq. (26).

V. NUMERICAL RESULTS AND COMPARISONS

We will numerically explore three different aspects of the proposed DLCT definition: (i) approximation of the continuous LCT, (ii) concatenation of multiple transforms, and (iii) reversibility. We will carry out numerical tests regarding these aspects of the proposed DLCT definition.

As the example input functions, the discretized versions of the chirped pulse function $\exp(-\pi u^2 - i\pi u^2)$, denoted F1, the trapezoidal function $1.5\text{tri}(u/3) - 0.5\text{tri}(u)$, denoted F2 ($\text{tri}(u) = \text{rect}(u) * \text{rect}(u)$), rectangular pulse function $\text{rect}(u)$, denoted F3, and the damped sine function $\exp(-2|u|) \sin(3\pi u)$, denoted F4, are used. The number of samples N are taken as 256 and 1024 for two sets of numerical simulations. Four transforms, denoted by T1, T2, T3, and T4, are considered, with parameters $(\alpha, \beta, \gamma) = (-3, -2, -1)$, $(-0.8, 3, 1)$, $(-1.8, -1.75, -1.3)$, and $(0.3, -1.6, -0.9)$, respectively. The LCTs T1, T2, T3 and T4 of the functions F1, F2, F3 and F4 have been computed both by the presented DLCT and by a highly inefficient brute force numerical approach which is taken as a reference. Throughout our numerical comparisons we use percentage mean squared error (MSE) as the performance metric. It is defined as the energy of the difference normalized by the energy of the reference, expressed as a percentage.

A. Approximation of the Continuous LCT

In this subsection, we focus on how well our method approximates the continuous LCT. The “true” continuous LCT of the original function is obtained by highly inefficient brute force numerical integration of the continuous LCT. The resulting percentage MSE scores, for both *ordinary* and *centered* sampling schemes, turn out to be giving very similar results, are tabulated

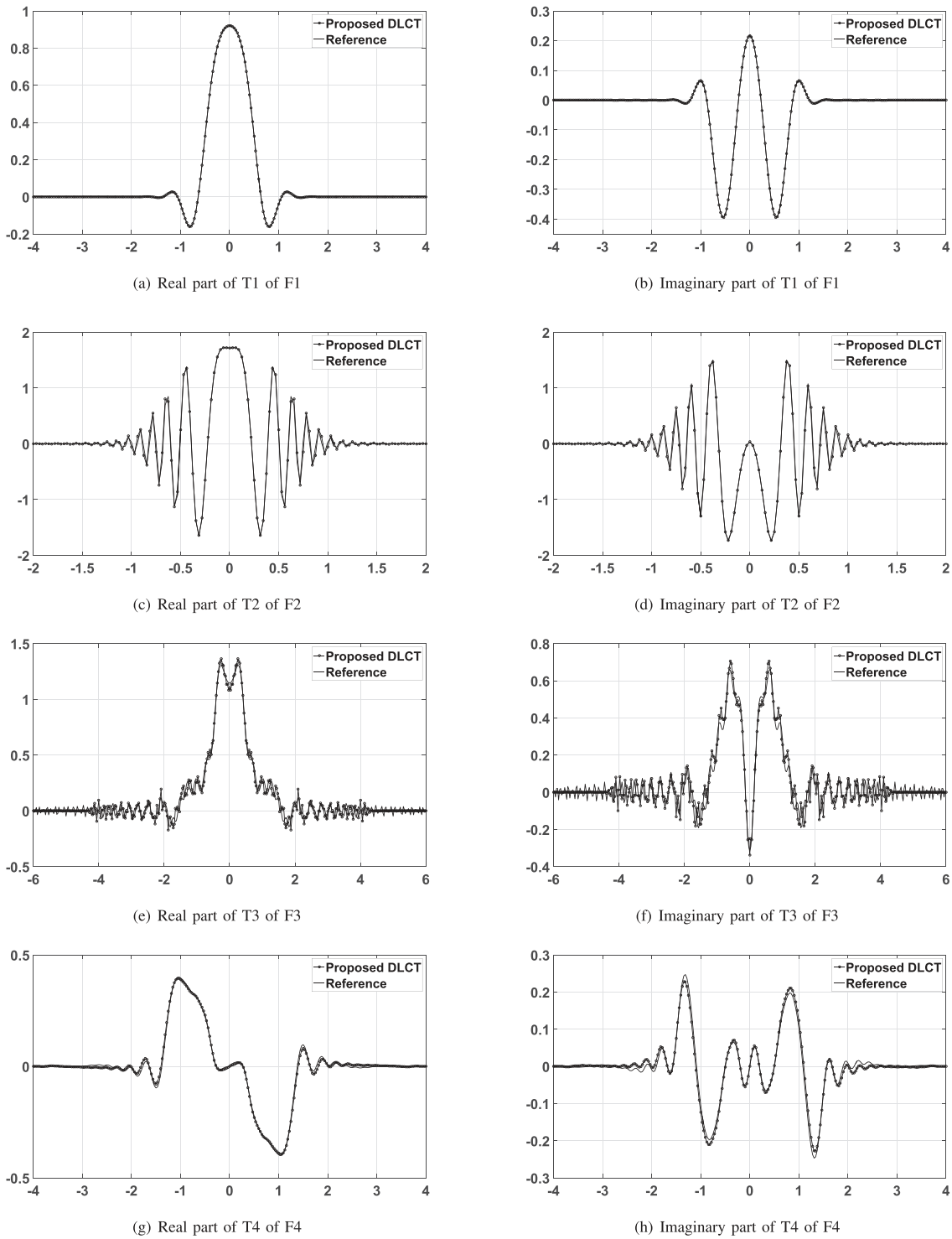


Fig. 1. Comparison of the proposed DLCT of functions with the reference.

in Table I. Plots for some examples for the resulting DLCTs (T1 of F1, T2 of F2, T3 of F3 and T4 of F4) and the corresponding references obtained by the brute force numerical method have been presented for both real and imaginary parts of the signals in Fig. 1.

Although we use the same two values of N for all the signals we consider for fair comparison, normally the value of N should be chosen according to the extent of the signals in both

the time/space and frequency domains. The error is primarily determined by how much of the signal falls outside of the extents implied by the chosen value of N . For example, for F1, which has a very rapidly decaying Gaussian envelope, very little falls outside so the errors are much smaller than for the others. In those cases where the results are not sufficiently accurate for the application at hand, it is possible to obtain higher accuracy by increasing N .

TABLE II
PERCENTAGE MSE ERRORS FOR DIFFERENT CONCATENATIONS AND INVERSES

Input	N	T1-T2	T3-T4	T3-T1	T3-T2	T1-T1 ⁻¹	T3-T3 ⁻¹
F1	256	1.32×10^{-2}	2.78×10^{-3}	1.55×10^{-3}	4.10×10^{-3}	5.85×10^{-3}	9.64×10^{-4}
	1024	6.82×10^{-4}	1.71×10^{-4}	9.58×10^{-5}	2.79×10^{-4}	3.85×10^{-4}	6.29×10^{-5}
F2	256	17.7	0.34	0.35	2.99	1.77	0.49
	1024	1.64	2.47×10^{-2}	2.43×10^{-2}	0.23	0.11	3.48×10^{-2}
F3	256	1.47	1.32	0.99	1.26	6.22	5.31
	1024	1.14	1.05	1.01	1.26	5.67	4.16
F4	256	6.73	1.77	1.03	2.15	18.37	1.83
	1024	0.28	0.14	8.16×10^{-2}	0.17	2.12	0.23

B. Concatenation

In order to test how well the concatenation property is satisfied, we employ the following procedure. Let us consider T1 and T2 as an example: First derive the DLCT matrices C_{L_1} and C_{L_2} for T1 and T2 separately, following the procedure given in Section III. Then, by using Eq. (1), we calculate the 2×2 LCT parameter matrices L_1 and L_2 for T1 and T2. Multiplying these two matrices by using Eq. (3), we obtain the 2×2 parameter matrix of the concatenated system $L_{12} = L_2 L_1$. Then, we obtain $C_{L_{12}}$ from L_{12} , again by using our proposed DLCT procedure. Finally, we compare the result of applying the concatenated transform matrix $C_{L_{12}}$ directly with the result of applying C_{L_1} and C_{L_2} consecutively. More precisely, we compare $C_{L_{12}} \mathbf{x}$ with $C_{L_2} C_{L_1} \mathbf{x}$ where a signal $x[n]$ of length N is represented by the column vector \mathbf{x} . The resulting MSE differences are tabulated in Table II for several such concatenations among T1, T2, T3, and T4. The ordinary sampling scheme is used in these numerical calculations.

C. Reversibility

To test the reversibility property numerically, we follow a similar procedure as in concatenation. This time the second LCTs in the cascade are the inverses of the first ones. For example, we compare \mathbf{x} with $C_{L_1^{-1}} C_{L_1} \mathbf{x}$. Again the ordinary sampling scheme is used in these calculations and the resulting MSE differences are tabulated in Table II.

VI. CONCLUSION

In this paper, a definition of the discrete linear canonical transform (DLCT) based on hyperdifferential operator theory is proposed. For finite-length signals of a discrete variable, a unitary DLCT matrix is obtained so that the LCT-transformed version of the input signal can be obtained by direct matrix multiplication. Given a vector holding the samples of a continuous-time signal, this DLCT matrix multiplies the vector to obtain the approximate samples of the continuous-time LCT-transformed signal, similar to the DFT being used to approximate the continuous-time Fourier transform.

The advantage of a discrete transform is that it provides a basis for numerical computation. However, our expectations were more than that. The main goal of this work was to obtain a formulation of the discrete LCT based on self-consistent definitions of the discrete coordinate multiplication and differentiation operators, that mirror the structure of their continuous counterparts. Care was taken to ensure that the discrete coordinate multiplication and differentiation operators were strictly duals of each other, related through the DFT. The resulting DLCT matrix is totally compatible with the theory of the discrete Fourier transform (DFT) and its dual and circulant structure. Desirable properties of a discrete LCT definition such as unitarity, preservation of group structure, reversibility and approximation of the continuous LCT were discussed both theoretically and numerically. One immediate possibility for future work is to explore the application of the method to alternative decompositions, such as those discussed in [59], [64], [64].

We showed in [59], that we could digitally compute the continuous LCT to an accuracy limited by the uncertainty relationship, with a fast algorithm. However, this numerical computation method did not exhibit properties we desire from a discrete definition. On the other hand, without a fast algorithm, application of the definition proposed in the present paper involves a matrix multiplication and thus has complexity $O(N^2)$. The best of both worlds would be to find a fast algorithm for the definition proposed in the present paper. This would be analogous to first defining the DFT and then deriving the FFT algorithm for its fast computation. However, such an algorithm is presently not available and will require future work. In the meantime, fast computational methods as in [59], [63], [64], [66] can be used in practical applications when speed is important. The computational complexity of taking the DLCT of signals, which is a matrix multiplication with $O(N^2)$ complexity, should not be confused with the complexity of constructing the proposed DLCT matrix, which has to be done once for a particular LCT. The latter is discussed in Appendix D.

In the present paper our emphasis was to define the DLCT in a manner that preserves structural similarity with the continuous DLCT. The structure in question is how the LCT is defined in terms of coordinate multiplication and differentiation in terms

of hyperdifferential operators, which we followed closely. Since everything rests on these two operators, their accuracy is what defines the accuracy of the method. We chose the conceptually simplest first-order approximations for these. Accuracy can be increased either by increasing N , or by replacing these building blocks with higher-order approximations. Thus, the hyperdifferential formulation provided here constitutes not only a theoretically pure approach to defining the DLCT, it serves as a framework for high accuracy numerical computations.

In conclusion, we have applied hyperdifferential operator theory to the task of defining the discrete LCT in a manner that is fully consistent with the dual and circulant structure of the DFT. Although several definitions for the DLCT have been proposed, a comprehensive evaluation of their relationships remains an important subject for future work. We believe our proposed analytical approach can lead to further possible research directions in the theory of discrete transforms in general.

APPENDIX A PROOF OF UNITARITY

We start with \mathbf{U} given in Eq. (33). \mathbf{U} is a real diagonal matrix, which implies it is Hermitian. The next step is to show \mathbf{D} is also Hermitian. Starting from Eq. (34), we can write

$$\mathbf{D}^{\mathbf{H}} = (\mathbf{F}^{-1}\mathbf{U}\mathbf{F})^{\mathbf{H}} = \mathbf{F}^{\mathbf{H}}\mathbf{U}^{\mathbf{H}}(\mathbf{F}^{\mathbf{H}})^{\mathbf{H}} = \mathbf{F}^{-1}\mathbf{U}\mathbf{F} = \mathbf{D}$$

implying that \mathbf{D} is also Hermitian. Now, we move on to show that \mathbf{Q}_q , \mathbf{M}_M , and \mathbf{F}_{lc}^a are unitary given \mathbf{U} and \mathbf{D} are Hermitian, by showing that their inverses and their Hermitians are equal. The inverse of \mathbf{Q}_q is

$$\mathbf{Q}_q^{-1} = \mathbf{Q}_{-q} = \exp\left(i2\pi q \frac{\mathbf{U}^2}{2}\right) \quad (35)$$

while the Hermitian of \mathbf{Q}_q is

$$\mathbf{Q}_q^{\mathbf{H}} = \exp\left(i2\pi q \frac{(\mathbf{U}^{\mathbf{H}})^2}{2}\right) = \exp\left(i2\pi q \frac{\mathbf{U}^2}{2}\right), \quad (36)$$

which are equal to each other. Similarly, one can follow the same procedure for \mathbf{M}_M as follows:

$$\begin{aligned} \mathbf{M}_M^{-1} &= \mathbf{M}_{1/M} = \exp\left(-i2\pi \ln(1/M) \frac{\mathbf{U}\mathbf{D} + \mathbf{D}\mathbf{U}}{2}\right) \\ &= \exp\left(i2\pi \ln(M) \frac{\mathbf{U}\mathbf{D} + \mathbf{D}\mathbf{U}}{2}\right) \end{aligned} \quad (37)$$

and

$$\begin{aligned} \mathbf{M}_M^{\mathbf{H}} &= \exp\left(i2\pi \ln(M) \frac{(\mathbf{U}\mathbf{D} + \mathbf{D}\mathbf{U})^{\mathbf{H}}}{2}\right) \\ &= \exp\left(i2\pi \ln(M) \frac{\mathbf{D}\mathbf{U} + \mathbf{U}\mathbf{D}}{2}\right) = \mathbf{M}_M^{-1}. \end{aligned} \quad (38)$$

And, finally for \mathbf{F}_{lc}^a we can write:

$$(\mathbf{F}_{\text{lc}}^a)^{-1} = \mathbf{F}_{\text{lc}}^{-a} = \exp\left(ia\pi^2 \frac{\mathbf{U}^2 + \mathbf{D}^2}{2}\right) \quad (39)$$

and

$$(\mathbf{F}_{\text{lc}}^a)^{\mathbf{H}} = \exp\left(ia\pi^2 \frac{(\mathbf{U}^2 + \mathbf{D}^2)^{\mathbf{H}}}{2}\right) = (\mathbf{F}_{\text{lc}}^a)^{-1}. \quad (40)$$

The first equalities in Eqs. (36), (38), and (40) can be shown by considering power expansion formula (Appendix B). Thus we have proven Theorem 2 and therefore Theorem 1. Justifications for the intermediate steps above will be given in the Appendix B.

APPENDIX B SOME FUNDAMENTALS OF OPERATOR THEORY

Here we provide further details regarding the derivations that appear in Section III and Appendix A. These derivations are mostly based on the following elementary definitions or results: (i) The integer power of an operator is defined as its repeated application, e.g. $\mathcal{A}^3 = \mathcal{A}\mathcal{A}\mathcal{A}$. (ii) Therefore, any power of \mathcal{A} commutes with itself, i.e. $\mathcal{A}^n\mathcal{A} = \mathcal{A}\mathcal{A}^n$. (iii) This leads to the fact that any polynomial $p(\mathcal{A})$ of \mathcal{A} commutes with \mathcal{A} , i.e. $p(\mathcal{A})\mathcal{A} = \mathcal{A}p(\mathcal{A})$. (iv) Functions such as $\exp(\mathcal{A})$ and $\sin(\mathcal{A})$ can be defined through power series of $\exp(\cdot)$ and $\sin(\cdot)$, which are essentially like polynomials, therefore these functions of \mathcal{A} also commute with \mathcal{A} . (v) Carrying this one step further, two different functions of \mathcal{A} that can be expressed as power series will also commute with each other, again as a consequence of (ii). (vi) The Hermitian of $p(\mathcal{A})$, and thus also $\exp(\mathcal{A})$ and $\sin(\mathcal{A})$ can be obtained by replacing \mathcal{A} with its Hermitian inside the power series. This follows from the fact that $(\mathcal{A}^n)^{\mathbf{H}} = (\mathcal{A}^{\mathbf{H}})^n$.

Eq. (31) follows directly from (iv) above. Eq. (32) follows from the fact that the effect of \mathcal{U} on a continuous signal $f(u)$ is to multiply it with u , and the fact that $\sin(\mathcal{U})$ can be written as a power series of \mathcal{U} .

The steps in Eqs. (35) to (40) in the Appendix A are most clearly established as follows. For the first equality in Eq. (36), it follows from (vi) in the established facts above. With regards to Eq. (35), we observe that Eqs. (9) and (10) show that the inverse of the chirp multiplication operator is again a similar operator but with negative parameter. Similar observations can be made for the other operators by referring to their 2×2 matrices. Regarding Eq. (35), this means that the inverse of a chirp multiplication operator is of the same form but with negative parameter $-q$. So we need to show that $\exp(i2\pi q\mathbf{U}^2/2)\exp(-i2\pi q\mathbf{U}^2/2)$ is equal to the identity. Here we can invoke the Baker-Campbell-Hausdorff formula for matrices, [84], [85], which states that

$$\exp(\mathbf{A})\exp(\mathbf{B}) = \exp(\mathbf{A} + \mathbf{B} + 1/2(\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A})), \quad (41)$$

for two complex matrices \mathbf{A} and \mathbf{B} where both \mathbf{A} and \mathbf{B} commute with their commutator $(\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A})$.

In our case, $\mathbf{A} = -\mathbf{B}$, so that $(\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}) = \mathbf{0}$. Therefore, the Baker-Campbell-Hausdorff formula's condition is met since every matrix commutes with the zero matrix. Finally, we observe that the product on the left-hand side of the above identity becomes equal to the exponential of the zero matrix and therefore the identity operator, proving the claim. Exactly the same argument applies for Eq. (37) and Eq. (39) since, although the exponents are more complicated, in each case a minus sign is introduced to the exponent but otherwise the exponent remains

the same. Therefore the exponent of the original and the inverse are merely negatives of each other and will commute, so that the product of the original and inverse matrices will be the identity.

The $\text{sinc}(x) = \sin(\pi x)/(\pi x)$ function has a power series that is obtained by dividing the power series of $\sin(\pi x)$ by (πx) . From number (iv) of our elementary results, $\text{sinc}(h\mathcal{D})$ commutes with \mathcal{D} , so both forms in Eq. (28) are the same. The same is true for Eq. (31).

APPENDIX C

COMPUTATION OF THE MATRIX EXPONENTIAL

Although it may be viewed as an implementation detail, given that it lies at the heart of the proposed method, it is worth clarifying how to compute the matrix exponential operation in Eq. (26). In practice, it is common to use *MATLAB*'s standard routines to compute matrix exponentials. Mathematically, the way in which matrix exponentials are obtained is through the well-known eigen decomposition

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \quad (42)$$

where \mathbf{D} is a diagonal matrix that holds the eigenvalues of \mathbf{A} and \mathbf{P} is the matrix holding the eigenvectors. Then, $\exp(\mathbf{A}) = \mathbf{P}\exp(\mathbf{D})\mathbf{P}^{-1}$ where the $\exp(\cdot)$ that operates on \mathbf{D} is now simply an element-wise exponentiation operation. When \mathbf{A} has a full set of eigenvalues, this procedure works without any complication. Given Eqs. (33) and (34), and the unitarity of the DFT matrix \mathbf{F} , the matrices \mathbf{U} and \mathbf{D} are ensured to have a full set of eigenvalues and eigenvectors, so there is no mathematical complication in using matrix exponentials.

APPENDIX D

COMPUTATIONAL COST OF CONSTRUCTING THE PROPOSED DLCT MATRIX

Given a specified precision (i.e., number of bits used in computations is fixed), to find the complexity of generating the matrix \mathbf{C}_L as a function of N , we first find the complexity of computing the matrices \mathbf{U} and \mathbf{D} . The matrix \mathbf{U} is generated using Eq. 33. This process requires evaluation of the sine function at N points and N multiplications by the constant \sqrt{N}/π . Since we assume a fixed precision, we can take the evaluation of the sine function at a point to be of complexity $O(1)$. The complexity of computing \mathbf{U} is thus $O(N)$. Secondly, to compute \mathbf{D} using Eq. 34, we need to compute the matrix \mathbf{F} and \mathbf{F}^{-1} , both of which can be written in terms of W_N . In generating \mathbf{F} , we compute W_N only once and compute its (mn) 'th power for the (mn) 'th entry. Computing the (mn) 'th entry for the matrices \mathbf{F} and \mathbf{F}^{-1} requires two multiplications and one exponentiation, which are each taken to be $O(1)$. It follows that computing \mathbf{F} and \mathbf{F}^{-1} each takes $O(N^2)$ computations. Finally, multiplying \mathbf{F}^{-1} with \mathbf{U} is $O(N^2)$ since \mathbf{U} is diagonal whereas multiplying $\mathbf{F}^{-1}\mathbf{U}$ with \mathbf{F} is $O(N^2 \log N)$ (by using fast Fourier transform (FFT) algorithm and by noting that neither matrices are diagonal), resulting in an overall complexity of $O(N^2 \log N)$ for \mathbf{D} .

We can now move on to the complexities of computing the matrices \mathbf{Q}_q , \mathbf{M}_M , \mathbf{F}_{lc}^a based on Eqs. 23, 24, and 25. Note that in

Eqs. 23, 24, and 25, the scalar constants can be taken outside the $\exp(\cdot)$ function, be computed separately and then be multiplied with the resulting matrix exponentials. This does not have an effect on the computational complexity with respect to N .

- Complexity of \mathbf{Q}_q : Taking the square of \mathbf{U} is of complexity $O(N)$ since \mathbf{U} is a diagonal matrix. We can compute the matrix exponential of \mathbf{U}^2 simply by taking the exponential of each diagonal element because \mathbf{U}^2 is also a diagonal matrix. This amounts to an overall computational complexity of $O(N)$.
- Complexity of \mathbf{M}_M : One can compute both $\mathbf{U}\mathbf{D}$ and $\mathbf{D}\mathbf{U}$ in $O(N^2)$ time because \mathbf{U} is a diagonal matrix. However, generating \mathbf{D} increases the time to compute the argument of the $\exp(\cdot)$ to $O(N^2 \log N)$. Furthermore, computing matrix exponentials as described in Appendix C is of complexity $O(N^3)$. As a result, the overall complexity is $O(N^3)$.
- Complexity of \mathbf{F}_{lc}^a : This is the same as the complexity of \mathbf{M}_M since it involves computing the matrix exponential of a non-diagonal matrix.

In conclusion, the overall complexity for computing the matrix \mathbf{C}_L is $O(N^3)$.

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