DISCRETE PAINLEVÉ EQUATIONS
FOR RECURRENCE COEFFICIENTS
OF SEMICLASSICAL LAGUERRE POLYNOMIALS

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ABSTRACT. We consider two semiclassical extensions of the Laguerre weight and their associated sets of orthogonal polynomials. These polynomials satisfy a three-term recurrence relation. We show that the coefficients appearing in this relation satisfy discrete Painlevé equations.

1. INTRODUCTION

Discrete Painlevé equations (dP) are second-order, nonlinear difference equations which have a continuous Painlevé equation as a continuous limit. They pass an integrability test called singularity confinement [9]. This integrability detector is the discrete analogue of the Painlevé property for differential equations. The discrete Painlevé equations share many features of their continuous counterparts: some can be fitted into degeneration cascades; for others Lax pairs have been established, hierarchies have been computed, and special solutions can be found, as well as Miura and Bäcklund transformations. At least one interesting difference between discrete and continuous Painlevé is the following: there are a lot more dP's than the six continuous ones (and as we name the dP's after their continuous limit, there are various nonequivalent dP$_\mathrm{I}$'s, etc.). There is a 'standard list' ([7]) consisting of the earliest derived discrete Painlevé equations. A more elaborate list was compiled by Peter Clarkson and can be found in [14]. For a comprehensive overview of discrete Painlevé equations, see [8].

Orthonormal polynomials on the real line obey a three-term recurrence relation of the form

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x)$$

(with $p_{-1} = 0$). The recurrence coefficients of classical orthogonal polynomials can be obtained explicitly. For semiclassical weights, i.e., weights w satisfying a Pearson equation $D(\sigma w) = \tau w$, where σ and τ are polynomials with $\deg \sigma > 2$ or $\deg \tau \neq 1$, the recurrence coefficients obey nonlinear recurrence relations, which in many cases can be identified as discrete Painlevé equations ([4], [5], [11], [12], [14]). Our aim is to find the nonlinear recurrence relations explicitly for other semiclassical

weights and to identify them as discrete Painlevé equations. In this way we hope to establish a connection between semiclassical orthogonal polynomials and discrete Painlevé equations. The fact that one deals with discrete Painlevé equations for the recurrence coefficients indicates that no simple formula will be available for the recurrence coefficients (except for some special cases that can be identified). On the other hand, the *integrability* of the Painlevé equations implies that the complexity of the recurrence coefficients does not grow exponentially [1]. Painlevé equations are on the borderline between linearisability and nonintegrability [8].

In this paper we make the connection between orthogonal polynomials which are semiclassical extensions of the Laguerre polynomials and some discrete Painlevé equations. We prove the following results:

Theorem 1.1. *Consider the orthonormal polynomials with respect to the weight*

$$w(x) = x^\alpha e^{-x^2 + tx}, \qquad x > 0,$$

with $\alpha > -1$ and $t \in \mathbb{R}$. The recurrence coefficients in the three-term recurrence relation

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x)$$

satisfy

(1.1)
$$
\begin{cases}
x_n x_{n-1} &= \dfrac{y_n + z_n}{y_n^2 - \frac{\alpha^2}{4}}, \\[2mm]
y_n + y_{n+1} &= \dfrac{1}{x_n}\left(\dfrac{t}{\sqrt{2}} - \dfrac{1}{x_n}\right),
\end{cases}
$$

where $z_n = n + \alpha/2$, $y_n = 2a_n^2 - n - \alpha/2$ and $x_n = \sqrt{2}/(t - 2b_n)$.

The equations in (1.1) bear similarities with an asymmetric Painlevé equation dP$_{\mathrm{IV}}$ in the list of Clarkson [14],

(1.2)
$$
\begin{cases}
u_n u_{n-1} &= \dfrac{a(v_n + z_n - b)}{v_n^2 - \gamma^2}, \\[2mm]
v_n + v_{n+1} &= \dfrac{c}{u_n} + \dfrac{z_{n+1/2} + d}{u_n - 1}
\end{cases}
$$

with $z_n = \alpha_1 n + \beta_1$. In fact, we can obtain (1.1) from (1.2) by putting $u_n = x_n/\epsilon$, $v_n = \epsilon y_n$, $z_n = \epsilon \alpha n + \epsilon \beta$, $a = 1/\epsilon$, $\gamma = \epsilon \alpha/2$, $c = 1/\epsilon + t/\sqrt{2}$, $d = -1/\epsilon$ and letting $\epsilon \to 0$. It was identified as a Miura transform of an asymmetric dP$_{\mathrm{I}}$ in [13], a result we will reestablish from the point of view of orthogonal polynomials in Section 2.3.

The case $t = 0$ gives a somewhat simpler equation:

Corollary 1.2. *If we consider the orthonormal polynomials with weight*

$$w(x) = x^\alpha e^{-x^2}, \qquad x > 0,$$

then the recurrence coefficients satisfy

(1.3)
$$(y_n + y_{n+1})(y_n + y_{n-1}) = \frac{(y_n^2 - \alpha^2/4)^2}{(y_n + z_n)^2},$$

where $z_n = n + \alpha/2$ and $y_n = 2a_n^2 - n - \alpha/2$. The recurrence coefficients b_n can be obtained from

$$2b_n^2 = -y_n - y_{n+1}.$$

Equation (1.3) is an instance of dP$_{IV}$ which was found in [6]. Theorem 1.1 and Corollary 1.2 will both be proven in Section 2. In Section 3 we prove

Theorem 1.3. *Consider the monic orthogonal polynomials with respect to the weight*

$$w(x) = \frac{x^\alpha}{(-x^2; q^2)_\infty}, \qquad x > 0,$$

with $\alpha > 0$ and $0 < q < 1$. The recurrence coefficients in the three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$$

satisfy

$$(1.4) \qquad (x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = \frac{q^{-2n-\alpha} \left(x_n - q^{-\alpha/2}\right)^2 \left(x_n - q^{\alpha/2}\right)^2}{\left(x_n - q^{-n-\alpha/2}\right)^2},$$

where $x_n = q^{-n-\alpha/2}(1 - q^{2n+\alpha-1}\beta_n)$. We have $\alpha_0^2 = x_0 x_1 + 1$. The remaining recurrence coefficients α_n for $n \geq 1$ can be obtained from

$$\alpha_n^2 x_n^2 q^{2n+\alpha} = 1 + x_n x_{n+1} + (x_n x_{n-1} - 1)(1 - q^{n+\alpha/2} x_n)^2 + 2x_n(x_n - q^{\alpha/2} - q^{-\alpha/2}).$$

Equation (1.4) is an instance of the q-Painlevé V equation in the list of Clarkson [14]; see also [8, p. 269].

2. dP$_{IV}$ FOR A SEMICLASSICAL LAGUERRE WEIGHT

We consider $w(x) = x^\alpha e^{-x^2 + tx}$ on $\mathbb{R}^+$ with $\alpha > -1$ and $t \in \mathbb{R}$. Denote the orthonormal polynomials with respect to this weight by $\{p_n\}$. The orthonormality condition is

$$\int_0^\infty p_n(x) p_m(x) w(x)\, dx = \delta_{mn}.$$

We will use the following notation for the coefficients of p_n:

$$p_n(x) = \gamma_n x^n + \delta_n x^{n-1} + \cdots,$$

where $\gamma_n > 0$. The three-term recurrence relation then takes the form

$$xp_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x)$$

with

$$a_n = \int_0^\infty xp_n(x) p_{n-1}(x) w(x)\, dx,$$

$$b_n = \int_0^\infty xp_n^2(x) w(x)\, dx.$$

It follows that all b_n are positive. Comparing leading coefficients on both sides of this identity expresses a_n as the ratio of the leading coefficients of the polynomials:

$$a_n = \frac{\gamma_{n-1}}{\gamma_n}$$

for $n > 0$ and $a_0 = 0$. Hence all a_n are positive for $n \geq 1$.

In Sections 2.1 and 2.2 we prove Theorem 1.1 and Corollary 1.2. We look at the relation between the recurrence coefficients of the orthogonal polynomials with respect to w and those of the Freud weight in Section 2.3, recovering a result of [13]. We conclude this section by looking at extensions of w to the real line.

2.1. **Deriving the Lax pair.** We rewrite the three-term recurrence relation in matrix form

$$xP = JP$$

with

$$P = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \end{pmatrix}, \quad J = \begin{pmatrix} b_0 & a_1 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix},$$

where J is the three-diagonal Jacobi matrix. The Pearson equation for w is

$$[xw(x)]' = \left(-2x^2 + tx + \alpha + 1\right) w(x).$$

We consider the Fourier expansion of the polynomial $xp'_n(x) = \sum_{k=0}^n \alpha_k p_{n-k}$ with

$$\alpha_k = \int_0^\infty xp'_n(x)p_{n-k}(x)w(x)\, dx.$$

Integration by parts and the Pearson equation give

$$\alpha_k = -\int_0^\infty p_n(x)xp'_{n-k}(x)w(x)\, dx + \int_0^\infty p_n(x)p_{n-k}(x)(2x^2 - tx - \alpha - 1)w(x)\, dx.$$

These integrals are zero for $k > 2$ due to orthonormality. For $k = 2$ we have

$$\alpha_2 = 2\int_0^\infty p_n(x)x^2 p_{n-2}(x)w(x)\, dx,$$

which is equal to $2a_n a_{n-1}$ by applying the recurrence relation twice on p_{n-2}. Furthermore, by comparing the coefficients of x^n, we see that $\alpha_0 = n$. In matricial form, we have

$$xP' = LP \qquad \text{with } L = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ A_1 & 1 & 0 & 0 & \cdots \\ 2a_2 a_1 & A_2 & 2 & 0 & \cdots \\ 0 & 2a_3 a_2 & A_3 & 3 & \\ \vdots & & & \ddots & \ddots & \ddots \end{pmatrix}$$

with

$$A_n = \int_0^\infty xp'_n(x)p_{n-1}(x)w(x)\, dx.$$

Together with the recurrence relation, this leads to the following Lax pair:

$$\begin{cases} xP &= JP, \\ xP' &= LP. \end{cases}$$

In the next section, we derive the compatibility condition for this Lax pair. From this condition we will be able to recover the recurrence coefficients as solutions of dP_{IV}.

2.2. **Computations.** The compatibility condition of the Lax pair can be found by calculating $x(xP)'$ in two ways and is

$$JL - LJ = J.$$

This gives rise to three nontrivial equations:

$$(2.1) \qquad b_n \;=\; a_{n+1}A_{n+1} - a_nA_n,$$

$$(2.2) \qquad A_n(b_{n-1} - b_n) + 2a_n \;=\; 2a_n(a_{n+1}^2 - a_{n-1}^2),$$

$$(2.3) \qquad a_{n-1}A_n - a_nA_{n-1} \;=\; 2a_na_{n-1}(b_n - b_{n-2}).$$

Dividing (2.3) by a_na_{n-1} and taking a telescopic sum we get

$$\frac{A_n}{a_n} = \frac{A_1}{a_1} + 2(b_n + b_{n-1} - b_1 - b_0).$$

Observe that

$$A_1 = -\gamma_0 \int_0^\infty p_1(x)[-2x^2 + tx + \alpha + 1]w(x)\,dx.$$

As the polynomial occurring in this integral has the Fourier expansion

$$-2x^2 + tx + \alpha + 1 = \frac{-2}{\gamma_2}p_2(x) + \frac{1}{\gamma_1}\left(2\frac{\delta_2}{\gamma_2} + t\right)p_1(x) + \text{ constant},$$

we find $A_1 = a_1(2b_0 + 2b_1 - t)$. Hence

$$(2.4) \qquad A_n = a_n(2b_n + 2b_{n-1} - t).$$

Using this expression in (2.2) we obtain, after dividing by a_n and taking a telescopic sum,

$$2(a_n^2 + b_n^2 + a_{n+1}^2) - tb_n = 2n + 2(a_0^2 + b_0^2 + a_1^2) - tb_0.$$

One computes

$$\begin{aligned}
2(a_0^2 + b_0^2 + a_1^2) - tb_0 &= \int_0^\infty (2x^2 - tx)p_0^2(x)w(x)\,dx \\
&= -\gamma_0^2 \int_0^\infty [xw(x)]'\,dx + \alpha + 1 = \alpha + 1;
\end{aligned}$$

hence

$$(2.5) \qquad 2(a_n^2 + b_n^2 + a_{n+1}^2) = tb_n + 2n + \alpha + 1.$$

Taking a telescopic sum of (2.1), we get

$$(2.6) \qquad a_nA_n = \sum_{j=0}^{n-1} b_j.$$

Multiplying (2.2) by a_n and taking a telescopic sum yields

$$\begin{aligned}
2a_n^2a_{n+1}^2 - 2\sum_{j=0}^n a_j^2 &= \sum_{j=1}^n a_jA_j(b_{j-1} - b_j) \\
&= \sum_{j=0}^{n-1}(a_{j+1}A_{j+1} - a_jA_j)b_j - a_nA_nb_n \\
&= \sum_{j=0}^{n-1} b_j^2 - a_nA_nb_n,
\end{aligned}$$

where we have used (2.1) in the last line. We get, using (2.5) and (2.6),

$$
\begin{aligned}
2a_n^2 a_{n+1}^2 + a_n A_n b_n &= \sum_{j=0}^{n-1}(a_j^2 + b_j^2 + a_{j+1}^2) + a_n^2 = \frac{1}{2}\sum_{j=0}^{n-1}(tb_j + 2j + \alpha + 1) + a_n^2 \\
&= \frac{t}{2}a_n A_n + \frac{n(n+\alpha)}{2} + a_n^2,
\end{aligned}
$$

or by using (2.4),

$$(2.7) \qquad 4a_n^2 a_{n+1}^2 = n(n+\alpha) + 2a_n^2 - a_n^2(2b_n - t)(2b_n + 2b_{n-1} - t).$$

Now consider the case $t = 0$. We use (2.5) to substitute b_n^2 in (2.7) and get

$$(2.8) \qquad 4a_n^2 b_n b_{n-1} = (2a_n^2 - n)(2a_n^2 - n - \alpha).$$

Squaring this equation and substituting b_n^2 and b_{n-1}^2, using (2.5) we obtain

$$(2a_n^2)^2(2a_n^2 + 2a_{n+1}^2 - 2n - \alpha - 1)(2a_n^2 + 2a_{n-1}^2 - 2n + \alpha + 1) = (2a_n^2 - n)^2(2a_n^2 - (n+\alpha))^2.$$

Using the substitution $y_n = 2a_n^2 - n - \alpha/2$ gives

$$(2.9) \qquad (y_n + y_{n+1})(y_n + y_{n-1}) = \frac{(y_n^2 - \frac{\alpha^2}{4})^2}{(y_n + z_n)^2},$$

where $z_n = n + \alpha/2$, thus proving Corollary 1.2. This is an instance of $\mathrm{dP_{IV}}$, which was found in [6]. The first initial value, $y_0 = -\alpha/2$, follows from the fact that $a_0 = 0$. As

$$a_1^2 = \frac{\mu_2\mu_0 - \mu_1^2}{\mu_0^2},$$

where μ_k is the kth moment of the weight w, we find for the second initial condition, $y_1 = 2\frac{\mu_2\mu_0 - \mu_1^2}{\mu_o^2} - 1 - \frac{\alpha}{2}$. Using integration by parts one finds

$$\mu_k = \frac{2}{\alpha + k + 1}\mu_{k+2},$$

so we have

$$y_1 = \frac{\alpha}{2} - \frac{2\mu_1^2}{\mu_0^2}.$$

The case $t \neq 0$ doesn't allow a reduction of the system

$$
\begin{cases}
2a_n^2 + 2b_n^2 + 2a_{n+1}^2 = tb_n + 2n + \alpha + 1, \\
4a_n^2 a_{n+1}^2 = n(n+\alpha) + 2a_n^2 - a_n^2(2b_n - t)(2b_n + 2b_{n-1} - t)
\end{cases}
$$

to a single equation. Using the substitutions $y_n = 2a_n^2 - n - \alpha/2$ and $x_n = \sqrt{2}\frac{1}{t - 2b_n}$ gives a system of equations,

$$(2.10) \qquad
\begin{cases}
x_n x_{n-1} = \dfrac{y_n + z_n}{y_n^2 - \frac{\alpha^2}{4}}, \\[2ex]
y_n + y_{n+1} = \dfrac{1}{x_n}\left(\dfrac{t}{\sqrt{2}} - \dfrac{1}{x_n}\right)
\end{cases}
$$

with $z_n = n + \alpha/2$. The initial conditions are given by $a_0 = 0, b_0 = \mu_1/\mu_0$ or

$$y_0 = -\frac{\alpha}{2}, \quad x_0 = \frac{\sqrt{2}\mu_0}{t\mu_0 - 2\mu_1}.$$

This proves Theorem 1.1.

2.3. Relation with dP$_\mathrm{I}$.

The recurrence coefficients of orthogonal polynomials for the Freud weight

$$w_\alpha(x) = |x|^{2\alpha+1}\exp(-x^4 + tx^2), \qquad x \in \mathbb{R}, \alpha > -1,$$

were studied by Freud [5] (see also [11], [14]). The set $\{q_n\}$ of orthonormal polynomials with respect to w_α satisfies the recurrence relation

$$xq_n(x) = A_{n+1}q_{n+1}(x) + A_n q_{n-1}(x),$$

and the recurrence coefficients satisfy

(2.11) $$4A_n^2(A_{n-1}^2 + A_n^2 + A_{n+1}^2 - t/2) = n + (2\alpha + 1)\Delta_n,$$

where $\Delta_n = 0$ when n is even and $\Delta_n = 1$ when n is odd. This equation is an instance of the discrete Painlevé equation dP$_\mathrm{I}$.

Now consider the semiclassical Laguerre weight from Theorem 1.1,

$$v_\alpha(x) = x^\alpha \exp(-x^2 + tx), \qquad x \in \mathbb{R}^+, \alpha > -1,$$

and the set $\{p_n^\alpha\}$ of orthonormal polynomials for this weight. We will denote the recurrence relation for this weight by

$$xp_n^\alpha(x) = a_{n+1}^\alpha p_{n+1}^\alpha(x) + b_n^\alpha p_n^\alpha(x) + a_n^\alpha p_{n-1}^\alpha(x).$$

As was shown in [3], the polynomials

$$q_{2n}(x) = p_n^\alpha(x^2), \quad q_{2n+1}(x) = xp_n^{\alpha+1}(x^2)$$

are orthonormal with respect to the weight w_α and the following relations hold between the recurrence coefficients of both weights:

(2.12) $$\begin{cases} a_n^\alpha = A_{2n}A_{2n-1}, \\ b_n^\alpha = A_{2n}^2 + A_{2n+1}^2, \end{cases} \qquad \begin{cases} a_n^{\alpha+1} = A_{2n}A_{2n+1}, \\ b_n^{\alpha+1} = A_{2n+2}^2 + A_{2n+1}^2. \end{cases}$$

Adding the equations

$$\begin{cases} 4A_{2n}^2\left(A_{2n-1}^2 + A_{2n}^2 + A_{2n+1}^2\right) = 2n + 2tA_{2n}^2, \\ 4A_{2n+1}^2\left(A_{2n}^2 + A_{2n+1}^2 + A_{2n+2}^2\right) = 2n + 2\alpha + 2 + 2tA_{2n+1}^2 \end{cases}$$

gives

$$4(a_n^\alpha)^2 + 4(a_{n+1}^\alpha)^2 + 4(b_n^\alpha)^2 = 4n + 2\alpha + 2 + 2tb_n^\alpha,$$

which is exactly the second equation of (2.10). To obtain the first equation of (2.10) one multiplies the equations

$$\begin{cases} 4A_{2n}^2\left(A_{2n}^2 + A_{2n+1}^2 - t/2\right) = 2n - 4A_{2n}^2 A_{2n-1}^2, \\ 4A_{2n-1}^2\left(A_{2n-1}^2 + A_{2n-2}^2 - t/2\right) = 2n + 2\alpha - 4A_{2n}^2 A_{2n-1}^2. \end{cases}$$

In this way the transformations in (2.12) can be considered as a Miura transformation relating dP$_\mathrm{I}$ (2.11) and the asymmetric dP$_\mathrm{IV}$ (1.1) or (2.10), as described in [13].

2.4. **Orthogonality on the real line.** So far we have considered the weight $w(x) = x^{\alpha}e^{-x^2+tx}$ on the positive real axis $\mathbb{R}^+$. On the full real axis we may consider the weight

$$\tilde{w}(x) = |x|^{\alpha}e^{-x^2+tx}, \qquad x \in \mathbb{R}.$$

Observe that it satisfies the same Pearson equation

$$[x\tilde{w}(x)]' = (-2x^2 + tx + \alpha + 1)\tilde{w}(x).$$

All the computations remain valid, and the conclusion is that the recurrence coefficients for the orthogonal polynomials satisfy the same discrete Painlevé equation (1.1) but with a different initial value for b_0:

$$a_0 = 0, \qquad b_0 = \frac{\int_{-\infty}^{\infty} x|x|^{\alpha}e^{-x^2+tx}\,dx}{\int_{-\infty}^{\infty} |x|^{\alpha}e^{-x^2+tx}\,dx}.$$

The b_n need no longer all be positive. The special case $t = 0$ gives a symmetric weight on the real line, and hence $b_n = 0$ for all $n \geq 0$. If we use this information in (1.1), then $1/x_n = 0$ for all $n \geq 0$ and (1.1) reduces to

$$\begin{cases} y_n^2 = \alpha^2/4, \\ y_n + y_{n+1} = 0. \end{cases}$$

With the initial value $y_0 = -\alpha/2$, we then find that $y_n = (-1)^{n+1}\alpha/2$, so that $2a_n^2 = n + \alpha\Delta_n$, where $\Delta_n = 0$ when n is even and $\Delta_n = 1$ when n is odd. These are indeed the recurrence coefficients for the generalized Hermite polynomials, given in Chihara [3, Ch. V, Eq. (2.46)].

Even more is true. We may consider the weights

$$\hat{w}(x) = \begin{cases} c_1 x^{\alpha}e^{-x^2+tx}, & x > 0, \\ c_2(-x)^{\alpha}e^{-x^2+tx}, & x < 0, \end{cases}$$

with $c_1, c_2 > 0$. They still satisfy the same Pearson equation, and all the calculus done before remains valid. Hence the recurrence coefficients for the orthogonal polynomials with weight $\hat{w}$ also satisfy (1.1) but with initial conditions

$$a_0 = 0, \quad b_0 = \frac{c_1\int_0^{\infty} x^{\alpha+1}e^{-x^2+tx}\,dx + c_2\int_{-\infty}^0 x|x|^{\alpha}e^{-x^2+tx}\,dx}{c_1\int_0^{\infty} x^{\alpha}e^{-x^2+tx}\,dx + c_2\int_{-\infty}^0 |x|^{\alpha}e^{-x^2+tx}\,dx}.$$

The case $c_2 = 0$ gives the weight w on the positive real line; the case $c_1 = c_2$ gives the weight $\tilde{w}$ on the full real line. If we define $\beta = c_1/c_2$, then this gives a one-parameter family of solutions of (1.1) with $a_0 = 0$ and

$$b_0 = b_0(\beta) = \frac{\beta\int_0^{\infty} x^{\alpha+1}e^{-x^2+tx}\,dx + \int_{-\infty}^0 x|x|^{\alpha}e^{-x^2+tx}\,dx}{\beta\int_0^{\infty} x^{\alpha}e^{-x^2+tx}\,dx + \int_{-\infty}^0 |x|^{\alpha}e^{-x^2+tx}\,dx},$$

where $\beta \in (0, \infty)$. Observe that $b_0(\beta)$ is an increasing function of β. The limiting cases $\beta = 0$ and $\beta = \infty$ correspond to orthogonal polynomials on $\mathbb{R}^-$ and $\mathbb{R}^+$ respectively. Hence for $b_0(0) \leq b_0 \leq b_0(\infty)$ the solution of (1.1) corresponds to recurrence coefficients of orthogonal polynomials, and hence this solution has no singularities.

This concludes our findings in the field of orthogonal polynomials on the real line. In the next section we will look at q-orthogonal polynomials.

3. $q\mathrm{P_V}$ FOR A q-SEMICLASSICAL LAGUERRE WEIGHT

In this section we investigate the weight function

$$(3.1) \qquad w(x) = \frac{x^\alpha}{(-x^2; q^2)_\infty}$$

on $\mathbb{R}^+$ and the recurrence coefficients associated to the set of the monic polynomials orthogonal with respect to this weight. Here, $0 < q < 1$, $\alpha > 0$ and $(z; q)_\infty$ is the q-Pochhammer symbol

$$(z; q)_\infty = \prod_{k=0}^\infty (1 - zq^k).$$

The weight w is a semiclassical variation of a q-analogue of the Laguerre weight

$$w(x) = \frac{x^\alpha}{(-x; q)_\infty}, \qquad x \in \mathbb{R}^+,$$

studied by Chen and Ismail in [2]. They use the theory of q-ladder operators to retrieve explicit expressions for the recurrence coefficients α_n, β_n of the recurrence relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$$

for the monic orthogonal polynomials P_n associated with this weight ($P_{-1} = 0$). In the next subsection we will take a closer look at the q-ladder operators before applying this to the weight (3.1) to find $q\mathrm{P_V}$ as the equation describing the recurrence coefficients of a q-modification of the Laguerre weight in Section 3.2.

3.1. q-Ladder operators.
The theory of ladder operators as introduced in [2] considers monic polynomials orthogonal with respect to the weight w on the positive real axis. Denoting the polynomials by $\{P_n\}$, the orthogonality condition is

$$\int_0^\infty P_n(x) P_m(x) w(x)\, dx = \zeta_n \delta_{m,n}.$$

The potential u is defined as

$$u(x) = -\frac{D_{q^{-1}} w(x)}{w(x)},$$

where D_q is the q-difference operator

$$(D_q f)(x) = \begin{cases} \dfrac{f(x) - f(qx)}{x(1 - q)} & \text{if } x \neq 0, \\ f'(0) & \text{if } x = 0. \end{cases}$$

The main result involves two entities,

$$\begin{aligned}
A_n(x) &= \frac{1}{\zeta_n} \int_0^\infty \frac{u(qx) - u(y)}{qx - y} P_n(y) P_n(y/q) w(y)\, dy, \\
B_n(x) &= \frac{1}{\zeta_{n-1}} \int_0^\infty \frac{u(qx) - u(y)}{qx - y} P_n(y) P_{n-1}(y/q) w(y)\, dy,
\end{aligned}$$

which appear in the lowering relation [2, Theorem 1.1]

$$D_q P_n(x) = \beta_n A_n P_{n-1}(x) - B_n P_n(x).$$

Furthermore, these quantities A_n and B_n satisfy two difference relations,

$$(3.2) \qquad B_{n+1}(x) + B_n(x) = (x - \alpha_n)A_n(x) + x(q-1)\sum_{j=0}^{n} A_j(x) - u(qx),$$

$$(3.3) \qquad 1 + (x - \alpha_n)B_{n+1}(x) - (qx - \alpha_n)B_n(x) = \beta_{n+1}A_{n+1}(x) - \beta_n A_{n-1}(x).$$

These equations will enable us to find expressions for the recurrence coefficients α_n, β_n as solutions of $q\mathrm{P}_\mathrm{V}$.

3.2. Computations. Given the weight

$$w(x) = \frac{x^\alpha}{(-x^2; q^2)_\infty}, \quad x \in \mathbb{R}^+,$$

we have the Pearson equation

$$w\left(\frac{x}{q}\right) = q^{-\alpha+2}\frac{w(x)}{q^2 + x^2}.$$

This leads to the following expression for the potential:

$$u(x) = \frac{q}{1-q}\left(\frac{1 - q^{-\alpha}}{x} + \frac{q^{-\alpha}x}{x^2 + q^2}\right),$$

so that

$$\frac{u(qx) - u(y)}{qx - y} = \frac{1}{1-q}\left(\frac{q^{-\alpha} - 1}{xy} + \frac{q^{-\alpha}(q - xy)}{(x^2 + 1)(y^2 + q^2)}\right).$$

Before being able to give expressions for A_n, B_n we introduce the following notation for the coefficients of P_n :

$$P_n(x) = x^n + p_1(n)x^{n-1} + p_2(n)x^{n-2} + \cdots,$$

where the boundary conditions $p_1(0), p_2(1), p_2(0), \ldots$ are all zero. This enables us to write the first terms of the Fourier series of both $P_n(qx)$ and $xP_n(qx)$:

$$\begin{aligned}
P_n(qx) &= q^n P_n(x) + (1-q)q^{n-1}p_1(n)P_{n-1}(x) \\
&\quad + q^{n-2}(1-q)[p_2(n)(1+q) - qp_1(n)p_1(n-1)]P_{n-2}(x) + \cdots, \\
xP_n(qx) &= q^n P_{n+1}(x) + q^{n-1}[p_1(n) - qp_1(n+1)]P_n(x) \\
&\quad + q^{n-2}[p_2(n) - q^2p_2(n+1) - qp_1(n)^2 \\
&\quad\quad + q^2 p_1(n)p_1(n+1)]P_{n-1}(x) + \cdots.
\end{aligned}$$

Note that we can express the recurrence coefficients in terms of the coefficients of the polynomials as

$$\begin{aligned}
\alpha_n &= p_1(n) - p_1(n+1), \\
\beta_n &= p_2(n) - p_2(n+1) - p_1(n)^2 + p_1(n)p_1(n+1).
\end{aligned}$$

We then get

$$\begin{aligned}
A_n(x) &= \frac{1}{\zeta_n}\frac{q^{-\alpha} - 1}{1 - q}\frac{1}{x}\int_0^\infty P_n(y)P_n(y/q)\frac{w(y)}{y}\,dy \\
&\quad + \frac{1}{\zeta_n}\frac{q^{-\alpha}}{1 - q}\frac{1}{1 + x^2}\int_0^\infty (q - xy)P_n(y)P_n(y/q)\frac{w(y)}{y^2 + q^2}\,dy.
\end{aligned}$$

Denoting

$$R_n = \frac{1}{\zeta_n}\frac{q^{-1} - 1}{1 - q}\int_0^\infty P_n(y)P_n(y/q)\frac{w(y)}{y}\,dy$$

we get

$$A_n(x) = \frac{R_n}{x} + \frac{1}{\zeta_n}\frac{q^{-2}}{1-q}\frac{1}{1+x^2}\int_0^\infty (q-xy)P_n(y)P_n(y/q)w(y/q)\,dy.$$

Using the substitution $v = y/q$ this gives

$$\begin{aligned}
A_n(x) &= \frac{R_n}{x} + \frac{1}{\zeta_n}\frac{1}{1-q}\frac{1}{1+x^2}\int_0^\infty (1-xv)P_n(qv)P_n(v)w(v)\,dv\\
&= \frac{R_n}{x} + \frac{q^n}{1-q}\frac{1}{1+x^2} - \frac{q^{n-1}}{1-q}\frac{x}{1+x^2}[p_1(n) - qp_1(n+1)].
\end{aligned}$$

Introducing $D_n = p_1(n) - qp_1(n+1)$ we finally arrive at

$$A_n(x) = \frac{R_n}{x}\frac{q^n}{1-q}\frac{1}{1+x^2} - \frac{q^{n-1}D_n}{1-q}\frac{x}{1+x^2}.$$

Note that

$$\sum_{j=0}^n q^{j-1}D_j = -q^n p_1(n+1).$$

For B_n we get

$$\begin{aligned}
B_n(x) &= \frac{1}{\zeta_{n-1}}\frac{q^{-\alpha}-1}{1-q}\frac{1}{x}\int_0^\infty P_n(y)P_{n-1}(y/q)\frac{w(y)}{y}\,dy\\
&+ \frac{1}{\zeta_{n-1}}\frac{q^{-\alpha}}{1-q}\frac{1}{1+x^2}\int_0^\infty (q-xy)P_n(y)P_{n-1}(y/q)\frac{w(y)}{q^2+y^2}\,dy.
\end{aligned}$$

Denoting

$$r_n = \frac{1}{\zeta_{n-1}}\frac{q^{-\alpha}-1}{1-q}\int_0^\infty P_n(y)P_{n-1}(y/q)\frac{w(y)}{y}\,dy$$

and using manipulations similar to those in the calculation of A_n, we get

$$\begin{aligned}
B_n(x) = \frac{r_n}{x} &+ \frac{q^{n-1}p_1(n)}{1+x^2}\\
&- \frac{q^{n-2}}{1-q}\frac{x}{1+x^2}[p_2(n) - q^2p_2(n+1) - qp_1(n)^2 + q^2p_1(n)p_1(n+1)].
\end{aligned}$$

Introducing

$$C_n = q^{n-2}\left(p_2(n) - q^2p_2(n+1) - qp_1(n)^2 + q^2p_1(n)p_1(n+1)\right)$$

we arrive at

$$B_n(x) = \frac{r_n}{x} + \frac{q^{n-1}p_1(n)}{1+x^2} - \frac{1}{1-q}\frac{x}{1+x^2}C_n.$$

Inserting these expressions for A_n and B_n in (3.2) and (3.3) and multiplying by $x(1+x^2)$ gives two polynomial identities. Comparing coefficients of powers of x

gives, for (3.2), the following identities:

$$(3.4) \qquad r_{n+1} + r_n = -\alpha_n R_n - \frac{1 - q^{-\alpha}}{1 - q},$$

$$(3.5) \qquad q^n p_1(n+1) + q^{n-1} p_1(n) = R_n - \alpha_n \frac{q^n}{1-q} - (1-q) \sum_{j=0}^{n} R_j,$$

(3.6)

$$r_{n+1} + r_n - \frac{C_{n+1}}{1-q} - \frac{C_n}{1-q} = -\alpha_n R_n + \frac{q^n}{1-q} + \alpha_n D_n \frac{q^{n-1}}{1-q} - \frac{1 - q^{n+1}}{1-q} - \frac{1}{1-q},$$

$$(3.7) \qquad 0 = R_n - \frac{q^{n-1}}{1-q} D_n - (1-q) \sum_{j=0}^{n} R_j - q^n p_1(n+1).$$

We can simplify (3.6) by subtracting (3.4), which gives

$$(3.8) \qquad -C_{n+1} - C_n = q^n + \alpha_n q^{n-1} D_n - 1 + q^{n+1} - q^{-\alpha}.$$

From (3.3) we get the identities

$$(3.9) \qquad -\alpha_n r_{n+1} + \alpha_n r_n = \beta_{n+1} R_{n+1} - \beta_n R_{n-1},$$

$$(3.10) \quad 1 + r_{n+1} - \alpha_n q^n p_1(n+1) - q r_n + \alpha_n q^{n-1} p_1(n) = \beta_{n+1} \frac{q^{n+1}}{1-q} - \beta_n \frac{q^{n-1}}{1-q},$$

(3.11)

$$q^n p_1(n+1) + \alpha_n \frac{C_{n+1}}{1-q} - q^n p_1(n) - \alpha_n \frac{C_n}{1-q} = -\beta_{n+1} \frac{q^n}{1-q} D_{n+1} + \beta_n \frac{q^{n-2}}{1-q} D_{n-1},$$

$$(3.12) \qquad 1 + r_{n+1} - \frac{1}{1-q} C_{n+1} - q r_n + \frac{q}{1-q} C_n = 0,$$

where we have already simplified (3.11) by using (3.9). We will now express D_n in terms of R_n. First we rewrite (3.7) as

$$0 = q R_n - \frac{q^{n-1}}{1-q} D_n - (1-q) \sum_{j=0}^{n-1} R_j - q^n p_1(n+1).$$

We can now substitute $\sum R_j$ by using (3.7) for $n-1$. This results in the equation

$$0 = q R_n - R_{n-1} - \frac{q^{n-1}}{1-q} D_n + \frac{q^{n-2}}{1-q} D_{n-1} - q^n p_1(n+1) + q^{n-1} p_1(n),$$

which has integrating factor q^{n-1}. After taking a telescopic sum we have

$$0 = (1-q) q^n R_n - (1-q) R_0 + q^{2n} p_1(n+1) - q^{2n-1} p_1(n) + q^{-1} p_1(0).$$

However, all terms with $R_0, p_1(0)$ and $p_1(1)$ cancel each other when we consider (3.7) for $n = 0$. After dividing by q^n we are left with the identity

$$(3.13) \qquad (1-q) R_n = q^{n-1}(p_1(n) - q p_1(n+1)) = q^{n-1} D_n.$$

Next, we write C_n in terms of r_n. We use (3.11) to obtain

$$\alpha_n q^n + \frac{\alpha_n}{1-q} C_{n+1} - \frac{\alpha_n}{1-q} C_n = \beta_{n+1} R_{n+1} - \beta_n R_{n-1}$$

together with (3.9) to obtain, after dividing by α_n,

$$r_{n+1} - r_n = -q^n + \frac{1}{1-q} C_{n+1} - \frac{1}{1-q} C_n.$$

Taking a telescopic sum and using $r_0 = C_0 = 0$, we find that

$$(1-q)r_n = C_n - 1 + q^n.$$

We now use (3.4) to substitute $\alpha_n R_n$ in (3.10):

$$qr_{n+1} - r_n + q^{-\alpha} = \beta_{n+1}\frac{q^{n+1}}{1-q} - \beta_n\frac{q^{n-1}}{1-q}.$$

This expression has integrating factor q^n. Taking a telescopic sum, we get

$$(3.14) \qquad \beta_n\frac{q^{n-1}}{1-q} = r_n + q^{-\alpha-n}\frac{1-q^n}{1-q}.$$

We have now been able to write β_n, C_n and D_n in terms of r_n and R_n. The same can be done for α_n. Combining (3.5) and (3.7) results in

$$(3.15) \qquad q^{n-1}p_1(n) = R_n - \alpha_n\frac{q^n}{1-q}.$$

We would like to get rid of the $p_1(n)$ in this equation. For that purpose, we rewrite (3.9) using (3.14) and (3.13) to find that

$$\alpha_n(1-q)q^{-\alpha-n-1} = \beta_{n+1}q^n(p_1(n) - qp_1(n+2)) - \beta_n q^{n-2}(p_1(n-1) - qp_1(n+1)).$$

This equation has an integrating factor q^{n+1}, and after taking a telescopic sum we get

$$\begin{aligned}(3.16) \qquad q^{-\alpha}(1-q)p_1(n) &= \beta_n q^{n-1}(qp_1(n+1) - p_1(n-1)) \\ &= \beta_n q^{n-1}(-q\alpha_n - q^{-n+2}(1-q)R_{n-1}).\end{aligned}$$

Combining (3.15) and (3.16) we get

$$\alpha_n(q^n - q^{3n+\alpha-1}\beta_n) = (1-q)R_n + (1-q)R_{n-1}q^{2n+\alpha}\beta_n,$$

which expresses α_n as a function of R_n and r_n.

If we look at (3.9), we see that the right hand side has an integrating factor R_n. When multiplying (3.9) with this integrating factor, the expression $\alpha_n R_n$ appears on its left hand side, and we use (3.4) to substitute it in terms of r_n, which gives

$$\beta_{n+1}R_{n+1}R_n - \beta_n R_n R_{n-1} = r_{n+1}^2 - r_n^2 + \frac{1-q^{-\alpha}}{1-q}(r_{n+1} - r_n).$$

Taking a telescopic sum gives

$$(3.17) \qquad \beta_n R_n R_{n-1} = r_n\left(r_n + \frac{1-q^{-\alpha}}{1-q}\right).$$

It is this equation we will use, together with (3.4), to recover $q\mathrm{P_V}$. First, we rewrite (3.4) using the expression found for α_n:

$$q^n\left(r_{n+1} + r_n + \frac{1-q^{-\alpha}}{1-q}\right)(q^{2n+\alpha-1}\beta_n - 1) = (1-q)R_n^2 + (1-q)q^{2n+\alpha}\beta_n R_n R_{n-1}.$$

Using (3.17) in this equation gives an expression for R_n^2:

$$\begin{aligned}(1-q)R_n^2 &= q^n\left(r_{n+1} + r_n + \frac{1-q^{-\alpha}}{1-q}\right)(q^{2n+\alpha-1}\beta_n - 1) \\ &\quad - (1-q)q^{2n+\alpha}r_n\left(r_n + \frac{1-q^{-\alpha}}{1-q}\right).\end{aligned}$$

We insert this in the squared (3.17),

$$\beta_n^2 R_n^2 R_{n-1}^2 = r_n^2 \left(r_n + \frac{1-q^{-\alpha}}{1-q} \right)^2,$$

and replace all β_n using (3.14). This gives a difference equation for r_n:

$$(1-q)^2 r_n^2 (r_n + \frac{1-q^{-\alpha}}{1-q})^2 = q \left((1-q)r_n + (1-q^n q^{-n-\alpha}) \right)^2$$
$$\times \left\{ \left(r_{n+1} + r_n + \frac{1-q^{-\alpha}}{1-q} \right) \left((1-q)q^{n+\alpha} r_n - q^n \right) \right.$$
$$\left. - (1-q)q^{n+\alpha} r_n \left(r_n + \frac{1-q^{-\alpha}}{1-q} \right) \right\}$$
$$\times \left\{ \left(r_{n-1} + r_n + \frac{1-q^{-\alpha}}{1-q} \right) \left((1-q)q^{n+\alpha-1} r_{n-1} - q^{n-1} \right) \right.$$
$$\left. - (1-q)q^{n+\alpha-1} r_{n-1} \left(r_{n-1} + \frac{1-q^{-\alpha}}{1-q} \right) \right\}.$$

We now use the substitution $x_n = q^{-\alpha/2} - q^{\alpha/2}(1-q)r_n = q^{-n-\alpha/2} - q^{n+\alpha/2-1}\beta_n$ and obtain

$$(3.18) \qquad (x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = \frac{q^{-2n-\alpha} \left(x_n - q^{-\alpha/2} \right)^2 \left(x_n - q^{\alpha/2} \right)^2}{\left(x_n - q^{-n-\alpha/2} \right)^2},$$

which is $q\mathrm{P_V}$. Expressed in terms of the kth moments μ_k of the weight w, the initial values are

$$x_0 = q^{-\alpha/2}, \quad x_1 = q^{-1-\alpha/2} - q^{\alpha/2} \frac{\mu_2 \mu_0 - \mu_1^2}{\mu_0^2}.$$

This concludes the proof of Theorem 1.3.

4. Conclusions and outlook

In this paper we have established the connection between various semiclassical variations on the Laguerre weight and discrete Painlevé equations for the recurrence coefficients of the corresponding orthogonal polynomials.

In the case of semiclassical Laguerre orthogonal polynomials (on the positive half-line) we found a connection to the system (1.2) which reduces to $\mathrm{dP_{IV}}$ in a special case. We recovered an earlier result about the quadratic relation between $\mathrm{dP_{IV}}$ and $\mathrm{dP_I}$ from the orthogonal polynomial point of view.

The case of the semiclassical q-Laguerre orthogonal polynomials gives a connection with $q\mathrm{P_V}$. As we know from the coalescence cascades for discrete Painlevé equations, $\mathrm{dP_{IV}}$ is a limiting case of $q\mathrm{P_V}$, just like the semiclassical Laguerre weight on the positive half-line can be found from the semiclassical q-Laguerre weight in an appropriate limit. The symmetrized weight

$$w(x) = \frac{|x|^{2\alpha+1}}{(-x^4; q^4)_\infty},$$

on the full real line was studied by Ismail and Mansour [10]. For $\alpha = -1/2$ they found a q-discrete form of $\mathrm{P_{34}}$ (see [7, Eq. (6.3)]). One would expect $q\mathrm{P_{III}}$ because there is a known quadratic Miura transformation between $q\mathrm{P_V}$ and $q\mathrm{P_{III}}$. We believe that this could be studied in more detail.

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