Discrete Potential Theory and Boundaries

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0. Introduction. In [10], HUNT has studied a general potential theory based on continuous parameter Markov processes. The purpose of this paper is to outline the corresponding theory based on discrete parameter processes, and to apply the theory to obtain the Martin exit and entrance boundaries in the special case of Markov chains (countable state space). The discrete parameter theory is, of course, essentially simpler than that treated by HUNT (who did did not treat boundary problems), and has the advantage that it involves topological considerations only at an advanced stage. We shall not discuss the analogues of all the results of HUNT, but only give enough to ensure understanding of the issues involved, and to cover the application to boundaries.

Let R be a measurable space, that is, a space with a distinguished Borel field of sets, called measurable sets. Let p be a substochastic transition function with state space R, that is, a function of the pair (ξ, A), where ξ is a point of R and A is a measurable subset of R, satisfying the following conditions:

(a) For each point ξ, $p(\xi, \cdot)$ is a completely additive function of measurable sets with

$$0 \leqq p(\xi, A) \leqq p(\xi, R) \leqq 1. \tag{0.1}$$

(b) For each measurable set A, $p(\cdot\ , A)$ is a measurable function on R.

In particular, if $p(\cdot\ , R) = 1$ on R, the function p will be called a stochastic transition function.

Define $p^{(n)}$ recursively by

$$\begin{aligned} p^{(0)}(\xi, A) &= 1 \quad \text{if} \quad \xi \in A \\ &= 0 \quad \text{otherwise}, \\ p^{(n)}(\xi, A) &= \int_R p^{(n-1)}(\eta, A)p(\xi, d\eta), \qquad n \geqq 1. \end{aligned} \tag{0.2}$$

Then $p^{(n)}$ is also a substochastic transition function, stochastic if p is. Define

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g by

$$g(\xi, A) = \sum_{n=0}^{\infty} p^{(n)}(\xi, A). \tag{0.3}$$

The measurable set A will be called transient if $g(\cdot\,, A)$ is finite on R. The sets of interest for potential theory are the transient sets and their countable unions. The function g will be the kernel (Green function) defining potentials on R. By means of this kernel, potentials of point functions and of set functions, and the analogues (and their duals) of harmonic and superharmonic functions will be defined and discussed briefly. The analogue of the Fatou boundary limit theorem will be proved, and, in the particular case when R is countable, the Martin entrance and exit boundaries will be defined. FELLER [9] has also defined a boundary when R is countable, and much of his preliminary work will be found in this paper in a somewhat different form, generalized (usually trivially) to the case of a general state space. The point of view is quite different, however, since in this paper the main stress is on the potential-theoretic significance of the results, and on the analogy with the classical theory of potentials and harmonic functions. These aspects are developed more than in FELLER's paper, which was devoted more to the construction of processes involving absorption or reflection at the boundary. Thus in this paper (when R is countable) the first boundary value problem for regular functions (the analogues of harmonic functions) is solved, boundary limit theorems are obtained which are close analogues (closer than for general R) of FATOU's boundary limit theorem for harmonic functions in a disc, and an integral representation of positive regular functions is derived which is the analogue of MARTIN's representation of positive harmonic functions.

In Section 11, results of BLACKWELL [1] and BREIMAN [2,3] are interpreted in terms of boundary theory.

For a discussion of the significance of the various properties of potential kernels see CHOQUET & DENY [5].

We shall rely heavily on martingale theory. In order to make the relations between this theory and potential theory more perspicuous, what are called semimartingales and lower semimartingales in [6] will be called (as they always should have been) submartingales and supermartingales respectively.

In the stochastic case, we shall denote by $\{x_{\xi n}\,, n \geqq 0\}$ a Markov stochastic process, defined on a measure space Ω (a point of which will be denoted by ω), with $x_{\xi 0} = \xi$ and transition function p. In the substochastic case, we sometines adjoin an absorbing state ρ. The new class of measurable sets consists of the old measurable sets and also their unions with $\{\rho\}$, and we define

$$\begin{aligned} p(\xi, \rho) &= 1 - p(\xi, R) \quad \text{if} \quad \xi \in R, \\ p(\rho, \rho) &= 1, \end{aligned} \tag{0.4}$$

where we write ρ instead of $\{\rho\}$. The extended transition function corresponds to a Markov process $\{x_{\xi n}, n \geqq 0\}$ whose paths remain at ρ if they ever reach that point. We shall always use "R" to denote the unextended space.

The sample paths of an $x_{\xi n}$ process will be called paths from ξ, and $g(\xi, A)$ is thus the expected number of times that a path from ξ lies in A.

In the following, operational notation will be useful, and the operators corresponding to functions will be denoted by the corresponding boldface letters. Thus, if p is a transition function and if u is a function on R, $\mathbf{p}u$ is the function $\int_R u(\eta)p(\cdot, d\eta)$; if μ is a countably additive function of measurable sets, $\boldsymbol{\mu}u$ is the integral of u with respect to μ, $\boldsymbol{\mu}p$ is the set function $\int_R p(\xi, \cdot)\, \mu(d\xi)$; $\boldsymbol{\mu}\mathbf{p}$ is the integration operator taking u into the number

$$\int_R u(\eta)\, d_\eta \int_R p(\xi, A_\eta)\mu(d\xi),$$

and so on.

1. Superregular and regular functions. Let A be a measurable subset of R. Let $p_A(\xi, B)$ be defined as $p(\xi, B)$ if $\xi \in A$ and as $p^{(0)}(\xi, B)$ otherwise. Then p_A is substochastic and we write its iterate $(p_A)^{(n)}$ as $p_A{}^{(n)}$. Let u be a measurable function on R, satisfying the following conditions:

(a) $-\infty < u \leqq \infty$ on A.

(b) For each strictly positive integer n, $\mathbf{p}_A^{(n)}u$ is well defined, finite on A, and $\mathbf{p}_A u \leqq u$, that is $\mathbf{p}u \leqq u$ on A.

Then we shall call u superregular on A, strictly superregular at a point if there is strict inequality in (b) at the point.

A function will be called subregular on A if its negative is superregular there. It will be called regular on A if it is both superregular and subregular on A.

If (b) is weakened by allowing $\mathbf{p}_A^{(n)}u$ to assume the value ∞ on A, the corresponding functions will be called loosely superregular or subregular as the case may be. (In the spirit of this definition, a loosely regular function is regular.)

If u and v are [loosely] superregular on A, $\min[u, v]$ is also. The positive finite constant functions are superregular on R. The function identically $+\infty$ is loosely superregular on R. If p is stochastic, the constant finite functions are regular on R.

If R is replaced by the extended space $R \cup \{\rho\}$, p becomes stochastic. A superregular function on the extended space is necessarily finite at ρ and satisfies the inequality

$$(\mathbf{p}u)(\xi) + u(\rho)p(\xi, \rho) \leqq u(\xi). \tag{1.1}$$

If v is the function $u - u(\rho)$, v vanishes at ρ, is superregular on R, and is regular on R if u was regular on the extended space. Thus there is no essential generalization obtainable by replacing R by the extended space.

If u is a positive (by which we shall always mean nonnegative) regular function on R, which has the property that any positive regular function v on R with $v \leqq u$ is proportional to u, then u is called minimal. The identically vanishing function is trivially minimal.

If u is loosely superregular on R,

$$u \geqq \mathbf{p}u \geqq \mathbf{p}^{(2)}u \geqq \cdots . \tag{1.2}$$

We denote the limit of this monotone sequence by $\mathbf{p}^{(\infty)}u$. It is clear that, if $n < \infty$, $\mathbf{p}^{(n)}u$ is loosely superregular, and is superregular on a measurable set A if $\mathbf{p}^{(m)}u$ is finite-valued on A when $m > n$. Moreover, in view of the monotoneity of convergence, if, on some measurable set A, $\mathbf{p}^{(\infty)}u$ is finite-valued, then $\mathbf{p}^{(n)}\mathbf{p}^{(\infty)}u = \mathbf{p}^{(\infty)}u$ on A. Hence $\mathbf{p}^{(\infty)}u$ is regular on A.

Thus $\mathbf{p}^{(\infty)}u$ is a regular minorant of u, if finite-valued. Conversely, if v is a regular minorant of u on R,

$$v = \mathbf{p}^{(n)}v \leqq \mathbf{p}^{(n)}u, \tag{1.3}$$

so that $v \leqq \mathbf{p}^{(\infty)}u$. Thus, if u has a regular minorant, it has a greatest one, $\mathbf{p}^{(\infty)}u$.

If u is superregular on R, and if u has a regular minorant,

$$u = \mathbf{p}^{(\infty)}u + \sum_0^\infty [\mathbf{p}^{(n)}u - \mathbf{p}^{(n+1)}u] = \lim_{n\to\infty} \sum_{j=0}^n \mathbf{p}^{(j)}(u - \mathbf{p}u). \tag{1.4}$$

It follows that the set on which u is strictly superregular is the union of a sequence of transient sets and that we can write u in the form

$$u = \mathbf{p}^{(\infty)}u + \mathbf{g}(u - \mathbf{p}u). \tag{1.5}$$

Here we adopt the usual convention that the integral of a function on a set on which the function vanishes identically is 0, even if the measure of the set is $+\infty$.

If $n > 0$,

$$g = \mathbf{p}^{(n)}g + \sum_0^{n-1} p^{(j)}. \tag{1.6}$$

Let A be any measurable subset of R and let A_f be the set on which $g(\cdot, A)$ is finite. Applying (1.6) with $n = 1$, we find that $g(\cdot, A)$ is loosely superregular, is superregular on A_f, and is regular on $A_f - AA_f$. According to (1.6),

$$\mathbf{p}^{(\infty)}g(\cdot, A) = 0 \quad \text{on} \quad A_f .$$

2. The probability of meeting a set. Let $u_A(\xi)$ be the probability that a path from ξ ever meets a point of the measurable set A, and let $r(\xi, A)$, for ξ in A, be the probability that a path from ξ never returns to A. Then $0 \leqq u_A \leqq 1$, and $u_A(\xi) = 1$ for ξ in A. Moreover,

$$u_A(\xi) = (\mathbf{p}u_A)(\xi) + p^{(0)}(\xi, A)r(\xi, A). \tag{2.1}$$

It follows that u_A is superregular on R, regular on $R - A$. Moreover, operating on this equation with $\mathbf{p}^{(j)}$ and adding the resulting equations for $j \leqq n - 1$, we find that

$$(2.2)\qquad u_A(\xi) = (\mathbf{p}^{(n)}u_A)(\xi) + \sum_{j=0}^{n-1} \int_R p^{(0)}(\eta, A)r(\eta, A)p^{(j)}(\xi, d\eta).$$

When $n \to \infty$, we obtain

$$(2.3)\qquad u_A(\xi) = (\mathbf{p}^{(\infty)}u_A)(\xi) + \int_A r(\eta, A)g(\xi, d\eta).$$

The first term in (2.3) is the probability that a path from ξ meets A infinitely often. The second term is the probability that a path from ξ meets A a finite strictly positive number of times. If $r(\cdot, A) \geqq \delta > 0$ on A, it follows from (2.3) that $g(\cdot, A) \leqq 1/\delta$.

Let $\{x_{\xi n}(A), n \geqq 0\}$ be the $x_{\xi n}$ process stopped when, if ever, a path leaves A. That is, after a path reaches ρ or a point of $R - A$, it remains at that point, which we denote by $x_{\xi\infty}(A)$, throughout all later steps.

If A, B are measurable subsets of R, we define $H_A(\xi, B)$ by

$$(2.4)\qquad H_A(\xi, B) = \mathrm{P}\{x_{\xi\infty}(R - A)(\omega) \in B\}.$$

Then $H_A(\xi, R) = u_A(\xi)$, and, trivially,

$$(2.5)\qquad H_A(\xi, B) = p^{(0)}(\xi, AB) \quad \text{if} \quad \xi \in A.$$

A simple decomposition shows that $H_A(\cdot, B)$ is superregular on AB, regular on $R - A$.

Let u be a function on R, with $-\infty < u \leqq +\infty$, and define $u(\rho) = 0$. The condition that, for every point ξ of A, the process $\{u[x_{\xi n}(A)], n \geqq 1\}$ be a supermartingale [martingale], with the parameter value 0 excluded in the first case if $u(\xi) = \infty$, is precisely the condition that u be superregular [regular] on A. This leads to the following theorem.

Theorem 2.1 *Let u be a function on R, bounded from below on R, and loosely superregular on A. Define $u(\rho) = 0$. Then, if ξ is a point of R, u has a limit (finite if u is superregular), for which we use the notation $u[x_{\xi\infty}(A)]$, on almost every $x_{\xi n}(A)$ path. Moreover,*

$$(2.6)\qquad u(\xi) \geqq \mathrm{E}\{u[x_{\xi\infty}(A)]\}.$$

If $u_{R-A}(\xi) = 1$, then

$$(2.7)\qquad \mathrm{E}\{u[x_{\xi\infty}(A)]\} = (\mathbf{H}_{R-A}u)(\xi).$$

Without this hypothesis on u_{R-A}, but assuming that u is positive,

$$(2.8)\qquad \mathrm{E}\{u[x_{\xi\infty}(A)]\} \geqq (\mathbf{H}_{R-A}u)(\xi).$$

We can and shall suppose in the proof that u is superregular, replacing u by $\min[u, n]$ and letting $n \to \infty$ otherwise. This theorem is trivial for any point ξ not in A. If $\xi \in A$, the existence of the limit is, of course, trivial except for paths that never leave A. If $\xi \in A$, the process $\{u[x_{\xi n}(A)], n \geqq 1\}$ is a supermartingale which is bounded from below. Hence the limit $u[x_{\xi\infty}(A)]$ exists with probability 1 (and the notation for the limit is consistent with our previous convention for paths leaving A) and (2.6) follows from the supermartingale inequality and FATOU's lemma. The remaining assertions of the theorem are trivial.

It is clear that the limit in the theorem exists if the hypothesis that u is bounded from below is replaced by the hypothesis that u has a subregular minorant which is bounded from above.

The fact that the $u[x_{\xi n}(A)]$ process in the theorem is a supermartingale implies that u is finite-valued on almost every path from a point of A as long as it remains in A (except initially, if u is infinite at the initial point of paths). This fact is also readily proved directly from the definition of superregularity.

We shall use the following immediate corollary of Theorem 2.1.

Corollary. *If, in Theorem* 2.1, *u is positive and $u(\eta) \geqq \delta$ whenever $\eta \in R - A$, then*

$$u(\xi) \geqq \delta u_{R-A}(\xi). \tag{2.9}$$

It is clear that if u is even regular on A and if conditions justifying integration to the limit (such as boundedness of u) are satisfied, then there is equality in (2.6).

As an application of the theorem, let A be a subset of R and consider the superregular positive function u_A. We define $u_A(\rho) = 0$ and show that, for almost every path from a point of R, u_A has the limit 1 along the path if the path hits A infinitely often, the limit 0 otherwise. The first part of the assertion is trivial. To prove the second part, let Λ be the set of points ω for which some $x_{\xi n}(\omega)$ lies in A. Then, with probability 1,

$$\begin{aligned} \mathrm{P}\{\Lambda \mid x_{\xi 0}(R - A), \cdots, x_{\xi n}(R - A)\}(\omega) &= 1 \quad \text{if} \quad x_{\xi n}(R - A)(\omega) \in A \\ &= u_A[x_{\xi n}(\omega)] \quad \text{otherwise}. \end{aligned} \tag{2.10}$$

According to LEVY's theorem on successive conditional probabilities, the right side converges with probability 1 to the indicator function of Λ, so that u_A has the limit 0 along almost every path from ξ which never meets A. Hence if k is any positive integer and if we consider the x_n process for $n \geqq k$, we find that u_A has the limit 0 along almost every path from ξ which never meets A for $n \geqq k$, and therefore has the limit 0 along almost every path from ξ which only meets A a finite number of times.

The result of the preceding paragraph implies easily that if $u_\rho(\xi)$ is the probability that a path from ξ ever meets ρ, then u_ρ has the limit 0 along almost every one of those paths from a point of R which do not meet ρ.

It also can easily be deduced from the result on u_A that if $u_A'(\xi)$ is the probability that a path from ξ meets A infinitely often, then u_A' (a regular function) and u_A have the same limits along almost every path from a point of R.

If u is a function on $R \cup \{\rho\}$, vanishing at ρ and bounded from below, and if A is a measurable subset of R, define v by

(2.11) $$v = \mathbf{H}_{R-A}u.$$

Then v and u are identical on $R - A$. Moreover $\mathbf{p}v$ and v are identical on A. Thus v is regular on A, if finite-valued. In particular, suppose that u is superregular and positive. Then $u \geqq v$, according to (2.6) and (2.8). In this case, v is superregular, equal to u on $R - A$, regular and $\leqq u$ on A. In fact, v is the infimum of the positive superregular functions which majorize u on $R - A$. To see this, let w be a function in the stated class. Then

(2.12) $$w \geqq \mathbf{H}_{R-A}w \geqq \mathbf{H}_{R-A}u = v,$$

as was to be proved. Finally, we remark that if $\xi \in A$, the function v has the limit 0 on almost every path from ξ which never leaves A. To see this, define $v(\rho) = 0$ and note that, by definition of v, if $y(\omega)$ is defined as 0 when the $x_{\xi n}(\omega)$ path does not leave A, and 1 otherwise, then

(2.13) $$v(\xi) = \mathrm{E}\{yu[x_{\xi\infty}(A)]\}$$

and

(2.14) $$\mathrm{E}\{yu[x_{\xi\infty}(A)]|\, x_{\xi 0}(A), \cdots, x_{\xi k}(A)\} = v[x_{\xi k}(A)] \quad \text{if} \quad x_{\xi k}(A) \in A.$$

The sequence of conditional expectations converges to $yu[x_{\xi\infty}(A)]$ with probability 1, according to a standard theorem on conditional expectations, and this fact yields the stated result.

What we have described here is the analogue of the sweeping out process of classical potential theory. If we had defined v as $\mathrm{E}\{u[x_{\xi\infty}(A)]\}$, v would again have been dominated by u, superregular on R, regular on A, but a minor modification of the proof just given would have shown that u and v would then have had the same limit along almost every path, with initial point in A, which never leaves A.

3. Potentials of functions. If v is a function on R, $\mathbf{g}v$ (not necessarily defined everywhere on R) will be called its potential.

If v is a positive function, its potential is defined everywhere on R, and

(3.1) $$\mathbf{g}v = \mathbf{p}^{(n)}\mathbf{g}v + \sum_0^{n-1} \mathbf{p}^{(i)}v, \qquad n > 0.$$

Then $\mathbf{g}v$ is loosely superregular on R, is superregular on the set F of finiteness of $\mathbf{pg}v$, strictly so at any point ξ of F if and only if $v(\xi) > 0$, and is therefore regular on the intersection of F with the set of zeros of v.

The potential of a function does not necessarily determine the function. For example, let R contain only one point and let $p = p^{(0)}$. Then $g(\xi,A) = \infty$ if A is not the null set, so that the potential of every strictly positive function is identically ∞. It is trivial from (3.1) with $n = 1$, however, that the potential of a positive function v determines v uniquely on the set of finiteness of $\mathbf{pg}v$. More generally, if v is any measurable function, (3.1) is applicable, when $n = 1$, at any point at which $\mathbf{g}v$ is well defined and $\mathbf{pg}v$ is well defined and finite, to show that v is uniquely determined at that point by its potential.

Theorem 3.1 *Let u be the potential of the positive function v, and define $u(\rho) = 0$. Then $\mathbf{p}^{(\infty)}u$ only has the values 0, ∞. Let A be the set of zeros of $\mathbf{p}^{(\infty)}u$. Then u has the limit 0 along almost every path from a point of A, and this limit assertion is also true in the L_1 sense: $\lim_{n\to\infty} \mathrm{E}\{u(x_{\xi n})\} = 0$ if $\xi \in A$.*

Since $\mathbf{p}^{(\infty)}u$ is loosely superregular, $p(\cdot, R_0) = 1$ on R_0, if R_0 is the set of finiteness of $\mathbf{p}^{(\infty)}u$. Hence it is no restriction in proving the theorem to assume that $R_0 = R$, and we shall do so. Let B_m be the set of points ξ with $v(\xi) \geqq 1/m$. Then

$$\mathbf{p}^{(\infty)}u \geqq \mathbf{p}^{(\infty)}\left[\int_{B_m} v(\eta)g(\cdot, d\eta)\right] \geqq \frac{1}{m}\mathbf{p}^{(\infty)}g(\cdot, B_m). \tag{3.2}$$

Hence $\mathbf{p}^{(\infty)}g(\cdot, B_m)$ is finite. When n is sufficiently large, depending on ξ, the left side of the equation

$$\mathbf{p}^{(n)}g(\xi, B_m) + \sum_0^{n-1} p^{(i)}(\xi, B_m) = g(\xi, B_m) \tag{3.3}$$

is therefore finite. Thus B_m is transient. When $n \to \infty$ in (3.3), we find that $\mathbf{p}^{(\infty)}g(\cdot, B_m) = 0$. Now let C_m be the set of points ξ with either $v(\xi) < 1/m$ or $v(\xi) > m$. Then, using what we have already proved,

$$\begin{aligned}\mathbf{p}^{(\infty)}u = \mathbf{p}^{(\infty)}\left[\int_{C_m} v(\eta)g(\cdot, d\eta)\right] &= \lim_{n\to\infty} \mathbf{p}^{(n)}\left[\int_{C_m} v(\eta)g(\cdot, d\eta)\right] \\ &= \lim_{n\to\infty}\int_{C_m} v(\eta)(\mathbf{p}^{(n)}g)(\cdot, d\eta) = \int_{C_m} v(\eta)(\mathbf{p}^{(\infty)}g)(\cdot, d\eta).\end{aligned} \tag{3.4}$$

Since $\mathbf{p}^{(\infty)}u$ is finite, the integral on the right is finite, and, when $m \to \infty$, we find that $\mathbf{p}^{(\infty)}u = 0$. Thus $A = R$, and only the limit assertions of the theorem remain to be proved. According to Theorem 2.1, u has a limit along almost every path from ξ in A. Since

$$\mathrm{E}\{u(x_{\xi n})\} = (\mathbf{p}^{(n)}u)(\xi) \to 0, \tag{3.5}$$

it follows that the limit is 0 in the L_1 sense, and hence also with probability 1. If the initial point ξ is given any distribution on A, this result implies that u still has the limit 0 in the L_1 sense along almost all probability paths.

As an example, let B be a measurable set and choose v as the indicator function of B. Then $u = g(\cdot, B)$, and we have proved that $g(\cdot, B)$ has the limit 0 on almost all paths from a point of R at which $g(\cdot, B)$ is finite.

Theorem 3.1 has the following converse.

Theorem 3.2 *If u is a positive loosely superregular function and if $\mathbf{p}^{(\infty)}u$ has only the two values 0, ∞, then u is the potential of a positive function.*

In fact, if we define v by

$$\begin{aligned} v(\xi) &= u(\xi) - (\mathbf{p}u)(\xi) \quad \text{if} \quad u(\xi) < \infty \\ &= \infty \qquad\qquad\qquad\quad \text{otherwise,} \end{aligned}$$

then u is the potential of v. To see this, we need only remark that if $u(\xi) < \infty$, (1.5) is applicable, and that if $u(\xi) = \infty$, then

$$(\mathbf{g}v)(\xi) \geqq v(\xi) = \infty .$$

If u is a positive superregular function, we conclude, more specifically, that u is the potential of a function v if and only if $\mathbf{p}^{(\infty)}u = 0$. According to (3.5) the condition $\mathbf{p}^{(\infty)}u = 0$ implies that, with the convention that $u(\rho) = 0$, u has the limit 0 on almost every path from each point of R, and this limit assertion is also valid in the L_1 sense. Conversely, suppose that u is a positive superregular function with limit 0 in these two senses. Then, if ρ must be adjoined, $u(\rho) = 0$, and, using (3.5), $\mathbf{p}^{(\infty)}u = 0$.

According to a theorem of F. Riesz, a positive superharmonic function can be expressed as the sum of a positive harmonic function and the potential of a positive mass distribution. The corresponding result in the present study is (1.5), which states that a positive superregular function can be expressed as the sum of a positive regular function and the potential of a positive function. Note that since a positive superregular function u is a potential if and only if $\mathbf{p}^{(\infty)}u = 0$, any positive superregular function dominated by one which is a potential is itself a potential.

It is not necessarily true, without further hypotheses, that *every positive superregular function is the limit of a monotone increasing sequence of potentials of positive functions.* In fact, we need only remark that the positive finite constant functions are always superregular, but that we have already given a trivial example in which $g(\xi, A) = \infty$ unless A is the null set, and in this example the potential of a positive function is either the constant function 0 or the constant function ∞.

The italicized assertion is of fundamental importance in defining the Martin boundary, however, so it is important to have available conditions ensuring its truth. The following result is useful in this connection. *If A is the union of a sequence of transient sets and if v is positive and superregular on R, vanishing on $R - A$, then v is the limit of a monotone increasing sequence of potentials of positive functions.* To prove this, suppose that $A = \bigcup_1^\infty A_n$, where A_n is transient, and define

$$v_k = \min\left[v, kg\left(\cdot, \bigcup_1^k A_n\right)\right]. \tag{3.6}$$

Then v_k is the potential of a positive function, and $v_k \uparrow v$.

4. Relative concepts. Let h be a positive loosely superregular function. Following Feller [9], define a new transition function by

$$(4.1)\qquad q(\xi, A) = \frac{\int_A h(\eta)p(\xi, d\eta)}{h(\xi)} \quad \text{if} \quad 0 < h(\xi) < \infty,$$

$$= 0 \qquad\qquad \text{otherwise}.$$

Probability paths corresponding to this transition function will be called h-paths. The h-path whose initial point is a zero or an infinity of h has no interest. Almost no h-path from a point where h is strictly positive and finite reaches a zero or an infinity of h. Thus the set of zeros and infinities of h can be ignored in discussing h-paths, that is, this set can be excluded from the state space of q. The probability that an h-path from ξ, where $0 < h(\xi) < \infty$, does not reach ρ during the first n steps is

$$(4.2)\qquad q^{(n)}(\xi, R) = \frac{(\mathbf{p}^{(n)}h)(\xi)}{h(\xi)}.$$

Hence $(\mathbf{p}^{(\infty)}h)\ (\xi)/h(\xi)$ is the probability that an h-path from ξ never leaves R. In particular, if h is the potential of a positive function, almost all paths from ξ leave R. At the other extreme, if h is regular, almost no h-path from ξ leaves R.

The probability that an h-path from ξ leaves R, going to ρ directly from the set A, on which h is finite and strictly positive, is

$$(4.3)\qquad \sum_{n=0}^{\infty} \int_A \left[1 - \frac{(\mathbf{p}h)(\eta)}{h(\eta)}\right] \frac{h(\eta)p^{(n)}(\xi, d\eta)}{h(\xi)} = \int_A \frac{h(\eta) - (\mathbf{p}h)(\eta)}{h(\xi)}\, g(\xi, d\eta).$$

Comparing this evaluation with the Riesz-type decomposition of h,

$$(4.4)\qquad h = \mathbf{p}^{(\infty)}h + \mathbf{g}[h - \mathbf{p}h],$$

we see that this decomposition has a simple probability interpretation: $(\mathbf{p}^{(\infty)}h)(\xi)/h(\xi)$ is the probability that an h-path from ξ never leaves R; the density $(h - \mathbf{p}h)(\eta)$, the mass density of the potential component, is the probability density, relative to $g(\xi, \cdot)/h(\xi)$, that an h-path from ξ leaves R for ρ, stepping from η. Note that almost no h-path steps to ρ from a point of regularity of h.

Concepts relative to the transition function q will be qualified by the prefix "h". Thus it is clear what is meant by an "h-superregular function". The classical analogues of these functions, the h-superharmonic functions, were introduced into potential theory by BRELOT [4].

Theorem 4.1 *Let u, h be positive loosely superregular functions, suppose that h is the potential of a positive function v, and suppose that $u \geqq h$ wherever v is strictly positive and finite. Then $u \geqq h$ wherever h is finite, and if u is superregular, $u \geqq h$ everywhere on R.*

The function u/h (considered only where h is strictly positive and finite) is loosely h-superregular, and almost every h-path reaches ρ, stepping to ρ from a point at which h is strictly positive, finite, and not regular, hence from a point at which v is strictly positive and finite (the last because $h \geqq v$), that is, from a point at which u/h has a value at least 1. The first conclusion of the theorem is then an immediate consequence of Theorem 2.1, as applied in the present context, with loose superregularity replaced by loose h-superregularity. If h is even superregular, the second conclusion of the theorem follows from an application of the first, with h replaced by min $[h, c]$, where c is an arbitrary positive constant.

The theorem has been stated in this way because its analogue in classical potential theory is usually so stated. The most general hypothesis in the spirit of this study would be: Let u, h be positive loosely superregular functions, and if the potential component of h in the Riesz-type decomposition of h is the potential of the function v, then it is supposed (a) that $u \geqq h$ wherever v is strictly positive and finite and (b) that the limits of u/h on h-paths are $\geqq 1$ for almost all h-paths which do not leave R. The proofs of the conclusions of the theorem are as before.

Now let h be superregular and be the potential of the positive function v, and let A be the set on which v is strictly positive and finite. Then h is regular on $R - A$ and is determined there by its values on A. To show this we prove that

$$h = \mathbf{H}_A h. \tag{4.5}$$

According to Theorem 2.1, (4.5) is true if "$=$" is replaced by "$\geqq$". On the other hand (see Section 2), $\mathbf{H}_A h$ is a superregular function on R, equal to h on A, regular on $R - A$. Hence, applying the theorem we have just proved, the reverse inequality is also true, as was to be proved.

5. Superregular and regular set functions. In the following we shall consider signed measures, that is, countably additive, not necessarily finite-valued functions of measurable subsets of R. If μ is such a set function, if f is a measurable, not necessarily finite-valued function on R, and if A is a measurable subset of R, $\int_A f\, d\mu$ is finite only (but not always) if the restriction of μ to the class of measurable subsets of A on which f does not vanish is σ-finite. Thus the standard integration theory of σ-finite set functions is applicable with little change. Note that if p is a transition function, $\mathfrak{y}p$ is well defined for all measurable sets and is countably additive.

Let A be a measurable subset of R and let μ be a countably additive set function, satisfying the following conditions:

(a) $-\infty < \mu \leqq \infty$ on the subsets of A.

(b) For each strictly positive integer n and measurable subset B of R, $(\mathfrak{y}p^{(n)})(B)$ is well defined, is finite if $B \subset A$, and, in that case, $(\mathfrak{y}p)(B) \leqq \mu(B)$.

Then we shall call μ "superregular" on A.

A set function will be called "subregular" on A if its negative is superregular there; it will be called "regular" if it is both superregular and subregular.

If (b) is weakened by allowing $(\mathfrak{y}p^{(n)})(B)$ to assume the value ∞ for $B \subset A$, the corresponding set function will be called "loosely" superregular or subregular, as the case may be. (In the spirit of this definition, a loosely regular function is regular.)

If μ is positive and totally finite and if m, n are positive integers, it is well known that

$$\mathfrak{u}p^{(m+n)} = \mathfrak{u}p^{(m)}p^{(n)}. \tag{5.1}$$

A slight extension yields the truth of the equality for any positive countably additive set function, and the linearity of (5.1) in μ then yields the truth of the equality for arbitrary μ for which the left side is well-defined. If ξ is any point of R, $g(\xi, \cdot)$ is a loosely superregular set function, regular on A if A does not contain ξ and if $(\mathbf{g}p)(\cdot, A)$ is finite at ξ.

If μ is a loosely superregular set function on R,

$$\mu \geqq \mathfrak{u}p \geqq \mathfrak{u}p^{(2)} \geqq \cdots . \tag{5.2}$$

Let ν be the limit of the sequence of set functions. Then the restriction of ν to the subsets of any measurable set A with $\nu(A)$ finite is a countably additive set function. Suppose that there is a sequence $\{A_n, n \geqq 1\}$ of disjunct measurable sets, with union R, such that $\nu(A_n)$ is finite and that either the sum over n of the positive or of the negative components of ν on A_n is finite. Then we define $\mathfrak{u}p^{(\infty)}$ as that (uniquely determined) countably additive function of measurable sets with the same value as ν on any subset of any A_n, or, equivalently, with the same value as ν whenever ν is finite. We shall say that $\mathfrak{u}p^{(\infty)}$ is "defineable" on R under these circumstances. Even without these hypotheses, however, we shall use the notation $(\mathfrak{u}p^{(\infty)})(A)$ for $\nu(A)$ whenever $\nu(A)$ is finite.

Suppose that $\mathfrak{u}$ is loosely superregular on R, that $\mathfrak{u}p^{(\infty)}$ is defineable, and that $(\mathfrak{u}\mathbf{p}^{(\infty)})\ p^{(n)}$ is well defined for all measurable sets and every positive integer n. Since

$$(\mathfrak{u}\mathbf{p}^{(n)})p \leqq \mathfrak{u}p^{(n)} \leqq (\mathfrak{u}\mathbf{p}^{(k)})p \quad \text{if} \quad n > k, \tag{5.3}$$

we find, if first $n \to \infty$ and then $k \to \infty$, that $\mathfrak{u}p^{(\infty)}$ is regular on A, if finite on A. We can write μ, on the subsets of A, in the form

$$\begin{aligned} \mu &= \mathfrak{u}p^{(\infty)} + \sum_0^\infty [\mathfrak{u}p^{(n)} - \mathfrak{u}p^{(n+1)}] \\ &= \mathfrak{u}p^{(\infty)} + \sum_0^\infty [\mathfrak{u} - \mathfrak{u}\mathbf{p}]p^{(n)} = \mathfrak{u}p^{(\infty)} + \mathfrak{u}_1 g, \end{aligned} \tag{5.4}$$

where

$$\mu_1 = \mu - \mathfrak{u}p. \tag{5.5}$$

This is the dual of the Riesz-type decomposition of a superregular point function, as given in Section 1.

6. Potentials of set functions. If v is a countably additive set function, the set function $\mathbf{v}g$ will be called the potential of ν. If well defined for all measurable subsets of R and if never $-\infty$, or if never $+\infty$, this potential will be countably additive.

If ν is positive and if n is a strictly positive integer,

$$\mathbf{v}\mathbf{g}p^{(n)} + \sum_{0}^{n-1} \mathbf{v}p^{(i)} = \mathbf{v}g, \tag{6.1}$$

so that (from the case $n = 1$) $\mathbf{v}g$ is loosely superregular, and even regular, on any set A for which $(\mathbf{v}g)(A)$ is finite and $\nu(A) = 0$. Moreover, when $n \to \infty$ in (6.1) we see that if A is a set for which $(\mathbf{v}g)(A)$ is finite, then $(\mathbf{v}\mathbf{g})p^{(\infty)}(A) = 0$.

Conversely, suppose that μ is a positive loosely superregular set function, and suppose that $(\mathbf{\mu}p^{(\infty)})(A) = 0$ if $\mu(A)$ is finite. Then we see from (5.4) that if σ-finite, μ is the potential of a positive set function.

If μ is a loosely superregular set function for which $\mathbf{\mu}p^{(\infty)}$ is defineable and $(\mathbf{\mu}\mathbf{p}^{(\infty)})p^{(n)}$ is well defined for all measurable sets and all positive integers n, (5.4) shows that μ, if σ-finite, is the sum of a regular set function and the potential of a positive set function.

If B is a measurable subset of R, $g(\cdot, B)$ is the potential of a function, in fact, of the indicator function of B. Hence if A is any measurable subset of R, $\mathbf{H}_A g(\cdot, B)$ is a loosely superregular point function, dominated by $g(\cdot, B)$,

$$(\mathbf{H}_A g)(\cdot, B) \leqq g(\cdot, B), \tag{6.2}$$

and there is equality if $B \subset A$. Since the left side of (6.2) is dominated by the right, the left side is the potential of a function, if B is transient.

Let $\mathbf{v}g$ be the potential of the positive set function ν. Then $(\mathbf{v}\mathbf{H}_A)g$ is the potential of the set function $\mathbf{v}H_A$. Applying the remark in the preceding paragraph, $\mathbf{v}\mathbf{H}_A g \leqq \mathbf{v}g$ and there is equality on the subsets of A. On the other hand, $(\mathbf{v}H_A)(R - A) = 0$. The set function $(\mathbf{v}\mathbf{H}_A)g$ is regular on the subsets of $R - A$. if finite-valued on them. The transition from $\mathbf{v}g$ to $(\mathbf{v}\mathbf{H}_A)g$ corresponds to the classical sweeping out process.

Suppose again that $\mathbf{v}g$ is the potential of the positive set function ν, and suppose now that A is a set such that $\nu(R - A) = 0$. We shall obtain a useful expression for $(\mathbf{v}g)(B)$, where B is a transient set. We have remarked above that $(\mathbf{H}_A g)(\cdot, B)$ is the potential of a function, say u, and we can suppose that u vanishes on $R - A$, since $(\mathbf{H}_A g)(\cdot, B)$ is regular there. Then

$$(\mathbf{v}g)(B) = (\mathbf{v}\mathbf{H}_A g)(B) = \mathbf{v}\mathbf{g}u. \tag{6.3}$$

This representation shows that $\mathbf{v}g$ is determined on the transient sets, and so on their countable unions, by its values on the subsets of A.

If $\mathbf{\mu}g$ and $\mathbf{v}g$ are the potentials of the indicated positive set functions and if A is a measurable subset of R, then $\mathbf{\mu}g \leqq \mathbf{v}g$ on the measurable subsets of A implies that $\mathbf{\mu}\mathbf{H}_A g \leqq \mathbf{v}\mathbf{H}_A g$ on the class of transient sets (and therefore on the class of countable unions of transient sets). In fact, using the notation of the preceding paragraph, if B is transient,

$$(\mathbf{\mu}\mathbf{H}_A g)(B) = \mathbf{\mu}\mathbf{g}u \leqq \mathbf{v}\mathbf{g}u = (\mathbf{v}\mathbf{H}_A g)(B). \tag{6.4}$$

7. The Markov chain case. Throughout the remainder of this paper, R will be the set of strictly positive integers (or some other countable set, if more convenient) or, rarely, a finite set, and all subsets of R will be measurable. It will be seen that the boundary obtained is quite different from that defined by FELLER [9], being essentially smaller.

If A is the set containing only the one point j, we shall write $p(i, j)$, $g(i, j)$, $u_j(i)$, $\mu(j)$ for $p(i, A)$, $g(i, A)$, $u_A(i)$, $\mu(A)$ respectively. To avoid trivialities, we suppose that, for every i, $\sum_j p(i, j) > 0$. If either i or j is persistent (recurrent), $g(i, j)$ must be one of the numbers 0, ∞; if j is transient, $g(i, j)$ must be finite. We shall say that a state j is covered by a state i if $g(i, j) > 0$.

If (2.9) is applied to a positive loosely superregular function u, with $A = R - \{i\}$, $\xi = k$, it becomes

$$u(k) \geqq u(i)u_i(k). \tag{7.1}$$

This inequality will play a fundamental role below.

If u is a loosely superregular point function which is bounded from below and if B is an (irreducible closed) persistent class, u is constant on B. To see this, we can suppose without loss of generality that u is positive, replacing u by $u - \inf u$ if the indicated infimum is negative. We now conclude from (7.1) that $u(k) \geqq u(i)$ for every pair i, k in B, so u is constant on B. If we suppose further that u is the potential of a positive function, then the constant value on B must be 0 or ∞, because $g(i, j) = \infty$ if i and j are persistent. Finally, if u is a superregular potential, the constant value on B must be 0.

In the discussion of boundary theory, the states of interest are the transient states, although we do not assume that all states are transient. In some of the discussion, however, it will be convenient to consider all the states of a persistent class as forming a single point of the state space. The remarks of the preceding paragraph furnish a preliminary indication of the advisability of this point of view.

If μ is a loosely superregular set function, such that $(\mathfrak{u}p)(j)$ is finite for all j, then $\mu(i)$ is finite for all i, since $\mu(i) = \infty$ implies that $p(i, j) = 0$ for all j, and we have excluded this case from consideration.

Let μ be a positive countably additive set function which is finite-valued and regular on every finite subset of R, that is, $\mu(i)$ is to be finite for all i and

$$\sum_i \mu(i)p(i, j) = \mu(j). \tag{7.2}$$

Then μ is loosely superregular (but not strictly so) on R. Under these hypotheses on μ, there is a "stationary stochastic process" $\{x_n, -\infty < n < \infty\}$ which is Markovian, has p as transition function, and for which

$$\mathrm{P}\{x_n(\omega) = i\} = \mu(i). \tag{7.3}$$

The quotation marks are used here (but will be dropped below) as tribute to the fact that the measure space on which the random variables are defined may not have measure 1. Only trivial changes in the usual discussion are needed because of this generalization, however.

If, more generally, μ is a positive loosely superregular set function which is finite-valued on every finite subset of R, we introduce a set $\{\rho_{ij}, i, j = 1, 2, \cdots\}$ of additional states and the corresponding transition probabilities,

$$p(\rho_{1j}, j) = 1, \quad p(i, \rho_{kj}) = 0, \quad p(\rho_{k+1j}, \rho_{kj}) = 1, \quad i, j, k = 1, 2, \cdots, \tag{7.4}$$

and we define

$$\mu(\rho_{kj}) = \mu(j) - (\mathbf{u}p)(j). \tag{7.5}$$

Then, as so extended, μ is a positive set function, regular and finite-valued on every finite set of the enlarged space.

If μ is as in the preceding paragraph, no state i with $\mu(i) > 0$ covers a state j with $\mu(j) = 0$, so that the restriction of the transition function to the class of states for which μ is strictly positive amounts probability-wise to the omission of the remaining states as initial states. With this in mind, we define

$$q(i, j) = \frac{\mu(j)p(j, i)}{\mu(i)}, \tag{7.6}$$

for pairs (i, j) for which $\mu(i)$ and $\mu(j)$ are strictly positive. The transition function (matrix) q is substochastic, with state space the class of all j with $\mu(j) > 0$, and

$$q^{(n)}(i, j) = \frac{\mu(j)p^{(n)}(j, i)}{\mu(i)}. \tag{7.7}$$

If the transition function p is used, with μ, to define a stochastic process as described above, the transition function q is that of the process reversed in time and becomes stochastic if we extend it by defining

$$\begin{aligned} q(i, \rho_{1j}) &= 0 \qquad \text{if} \quad j \neq i \\ &= \frac{\mu(\rho_{1i})}{\mu(i)} \quad \text{if} \quad j = i \\ q(\rho_{kj}, \rho_{k+1j}) &= 1. \end{aligned} \tag{7.8}$$

The reverse process is also a Markov process, and for it the states ρ_{kj} can all be amalgamated into a single state, if we consider it for, say, $n \leqq 0$ and under the condition $x_0(\omega) = i$.

If u is a positive and superregular function relative to q, the defining condition for u,

$$\sum_{\mu(j)>0} u(j)\mu(j)p(j, i) \leqq u(i)\mu(i) \quad \text{if} \quad \mu(i) > 0, \tag{7.9}$$

is simply the condition that $u\mu$ be a positive superregular set function, finite for finite subsets of R if u is finite, vanishing for a state if μ does. The last condition is, of course, simply absolute continuity relative to μ. The duals of results already obtained can be derived applying this remark. For example, let A be a set of states for which μ is strictly positive, and let u be the least positive loosely superregular point function relative to q which majorizes 1 on A (see Section 2). Then $u\mu$ is the least positive loosely superregular set function relative to p which majorizes μ on every subset of A. It is regular on every finite subset of $R - A$.

8. The Martin exit boundary. In this section we shall restrict our attention to the class S of states covered by some specified state, which we shall take for convenience to be the state 1. Then every state covered by a state of S lies in S. The set of transient states in S will be denoted by S_t. A (point) function positive and loosely superregular on R vanishes identically on S if it vanishes at 1, is everywhere finite on S if it is finite at 1. We shall study the positive functions u defined on S, with $0 < u(1) < \infty$, and loosely superregular relative to S taken as the state space. Such a function is finite-valued and superregular relative to S.

The restriction to S of a positive loosely superregular point function u on R is loosely superregular on S, relative to S (actually superregular if $u(1)$ is finite), regular on S relative to S if u was regular on S relative to R. Conversely, if u, defined on S, is positive and superregular relative to S, with $0 < u(1) < \infty$, then u can be defined as $\mathbf{H}_S u$ on $R - S$ to obtain a loosely superregular function on R. This definition yields the least positive loosely superregular extension of u. It is not strictly loosely superregular on $R - S$, and therefore is regular on R, if finite-valued, whenever u is regular on S relative to S.

If u, defined on R, is a minimal regular function relative to R, then u is minimal regular on S, relative to S. In fact, in the first place we have already remarked that u is regular on S, relative to S. In the second place, if v is a positive function defined on S, regular relative to S, with $v \leqq u$, extend v to R minimally, as described above. Then $v \leqq u$ on R, so $v = cu$, where c is a constant, as was to be proved. Conversely, if u is defined on S and is minimal regular relative to S, its extension to R as defined above will also be denoted by u. If v is a positive regular function on R, with $v \leqq u$, then $v = cu$ on S, for some constant c, $0 \leqq c \leqq 1$, because u is minimal relative to S. Therefore, because of the minimal property of the extension method, $v \geqq cu$ on $R - S$. Applying this reasoning to $u - v$, we find that $u - v \geqq (1 - c)u$ on $R - S$. Combining these inequalities, we see that $v = cu$ on R, so that u is minimal on R, if finite-valued on $R - S$.

Define $K(i, j)$ by

$$K(i, j) = \frac{u_j(i)}{u_j(1)} \quad \text{if} \quad i, j \in S, \tag{8.1}$$

so that

$$K(i, j) = \frac{g(i, j)}{g(1, j)} \quad \text{if} \quad j \in S_t . \tag{8.2}$$

Then $K(1, \cdot) = 1$ on S. The function $K(\cdot, j)$ is positive and superregular on S. It is regular except possibly at j, where it is regular if and only if j is persistent; it is constant on each persistent class. If 1 is persistent, S is the persistent class containing 1, and K is identically 1. All the work to follow is applicable but trivial in this case.

Applying (7.1) with $u = u_j$, we find that

$$u_j(1) \geqq u_j(i)u_i(1). \tag{8.3}$$

Hence

$$K(i, j) \leqq \frac{1}{u_i(1)}. \tag{8.4}$$

We metrize S by defining the distance function d,

$$d(\nu_1, \nu_2) = \sum_i |K(i, \nu_1) - K(i, \nu_2)| \, 2^{-i}u_i(1). \tag{8.5}$$

Obviously $d(\nu_1, \nu_2) = 0$ implies that either $\nu_1 = \nu_2$ or else that ν_1 and ν_2 are in the same persistant class. In accepting this metric we are thus adopting the convention that each such class is a single point of the metrized space.

The metrized space is conditionally compact. Completing it, we obtain a compact metric space in which S is dense. Let S' be the set consisting of the point of metrized S corresponding to persistent classes and of the set of limit points of metrized S in the complete space. If p is substochastic, we do *not* consider the adjoined point ρ as a point of S or S'. The set S' will be called the Martin exit boundary of S (or, more properly, the Martin exit boundary of R accessible from the point 1) in accordance with MARTIN's definition [11] in the study of Newtonian potentials. The points corresponding to persistent classes will be called degenerate boundary points. These are isolated points of $S \cup S'$. If R is finite, there are only degenerate boundary points, if any.

If ξ is a degenerate boundary point, $K(i, \xi)$ was defined above. If ξ is a non-degenerate boundary point, define $K(i, \xi)$ by

$$K(i, \xi) = \lim_{j \to \xi} K(i, j). \tag{8.6}$$

The limit exists, by definition of the boundary. Then if ξ is a non-degenerate boundary point and if i is persistent, $K(i, \xi) = 0$. The function $K(i, \cdot)$ is continuous on the metric space $S \cup S'$, and, for every point ξ of this space, $K(\cdot, \xi)$ is positive and superregular on S, with $K(1, \xi) = 1$. Moreover, different points ξ yield different superregular functions, and conversely. Thus the points of the metric space can be identified with these superregular functions.

Finally, we note that $K(i, \cdot)$ is to be regarded as a point function on $S \cup S'$, not as a set function.

9. The representation theorem. Let v be a finite-valued positive superregular function defined on S. Let $A_1, A_2, \cdots$ be the persistent subclasses of S. We have remarked that v is constant on each A_j. Let a_j be the value of v on A_j and define v' and v_n' by

$$v' = \sum_1^\infty a_j u_{A_j}, \qquad v_n' = \sum_1^n a_j u_{A_j}. \tag{9.1}$$

Since A_j is a persistent class, $u_{A_j}(\xi)$ is the probability that a path from ξ meets A_j infinitely often. Hence, according to the work in Section 2, v_n' is regular, with limit value a_j on almost every path, from a point of S, meeting A_j, $j \leqq n$, limit value 0 on almost every path from the point not meeting $\bigcup_1^n A_j$. Thus $v - v_n'$ is a regular function, bounded from below, with positive limit value on almost every path from a point of S, and it follows that $v - v_n' \geqq 0$. We conclude at once that $v \geqq v'$ and that v' is regular relative to S. If there are no transient states in S, the two functions are equal.

We wish to find a canonical representation for the positive finite-valued functions v defined on S and superregular relative to S. We first note that, using the notation of the preceding paragraph, if there are transient points in S, the function $v - v'$ is superregular relative to S and positive, vanishes at every persistent point, and therefore (see Section 3) is the limit of a monotone increasing sequence $\{v_n, n \geqq 1\}$ of potentials of positive functions. Thus if $v' = 0$, we can write v in the form

$$v(i) = \lim_{n\to\infty} \int_{S\cup S'} K(i, \xi)\mu_n(d\xi), \tag{9.2}$$

where μ_n is a measure of Borel subsets of $S \cup S'$ with $\mu_n(S') = 0$. In fact, the integral is simply a complicated notation for the usual sum defining potential on a countable state space. If $v' \neq 0$, (9.2) is still applicable, since v' can be absorbed into the integral at the expense of allowing μ_n to be strictly positive on the set of degenerate boundary points. Moreover,

$$\mu_n(S \cup S') = v_n(1) + v'(1) \leqq v(1), \tag{9.3}$$

and, for each i, the integrand is a continuous function on the compact metric space of integration. Applying the usual compactness argument, there is a measure μ of Borel subsets of $S \cup S'$ such that

$$v(i) = \int_{S\cup S'} K(i, \xi)\mu(d\xi), \qquad \mu(S \cup S') = v(1). \tag{9.4}$$

This is the desired canonical form, due to MARTIN in the classical case of Newtonian potentials.

If $\mathbf{p}K$ is the point function obtained by operating on the first argument of K, then $\mathbf{p}K \leqq K$. Suppose that v is regular at the transient state i. Then

$$(\mathbf{p}v)(i) = \int_{S\cup S'} (\mathbf{p}K)(i, \xi)\mu(d\xi) = v(i). \tag{9.5}$$

Hence

$$(9.6) \qquad (\mathbf{p}K)(i, \xi) = K(i, \xi)$$

for almost all (μ measure) ξ. Thus $K(\cdot, \xi)$ must be regular at i for almost all (μ measure) ξ, and it follows that $\mu(i) = 0$. In particular, if v is a positive regular function on S, it follows that $\mu(S_\iota) = 0$, and the superregular function $K(\cdot, \xi)$ must be regular for almost all (μ measure) ξ.

If v is positive and superregular, the Riesz-type decomposition of v shows that v has the representation (9.4) even if $v(1) = \infty$.

10. Analogues of the classical results. A boundary point ξ will be called minimal if $K(\cdot, \xi)$ is regular and minimal. It is easy to verify that the degenerate boundary points are minimal. The representation (9.4) shows that if v (defined on S) is regular and minimal, finite and strictly positive at 1, there is a minimal boundary point ξ such that

$$(10.1) \qquad v = K(\cdot, \xi)v(1).$$

This point ξ is necessarily uniquely determined by v, since different points of the boundary correspond to different functions $K(\cdot, \xi)$, and will be described as the boundary point corresponding to v.

The methods used in the study of the classical Martin boundary relevant to Newtonian potentials are applicable with no essential change to prove the following results, the analogues of the classical ones, and the proofs will be omitted. See [4, 8, 12] for the ideas involved and the details of the proofs in the classical case.

(a) If h (defined on S) is positive and superregular on S, finite and strictly positive at 1, then almost every h-path from a point at which h is strictly positive converges to a minimal boundary point or reaches ρ. (The alternative cannot arise if h is regular.)

(b) The set of minimal boundary points is a Borel set. If h in (a) has the representation

$$(10.2) \qquad h(i) = \int_{S \cup S'} K(i, \xi)\mu(d\xi), \qquad h(1) = \mu(S \cup S'),$$

μ can be chosen to have the value 0 for the set of non-minimal boundary points and, with this choice, is uniquely determined by h. We assume from now on that this choice has been made. Then the integral in (10.2) over the boundary gives the regular component of the Riesz-type decomposition of h, and the integral over the rest of the space gives the potential component. If A is a Borel subset of the Martin boundary, the probability that an h-path from i, a point of strict positivity of h, converges to a minimal non-degenerate point of A, or reaches a degenerate point of A, is

$$(10.3) \qquad \int_A \frac{K(i, \xi)\mu(d\xi)}{h(i)}.$$

In particular, if h is regular and minimal, almost all h-paths from i converge to the minimal boundary point corresponding to h.

(c) If h (defined on S) is positive and regular on S and if $h(1) > 0$, the Perron-Wiener-Brelot method can be applied to the first boundary value problem for h-regular functions, given a function f on the Martin exit boundary. The measure of sets A in (10.3) is the analogue of h-harmonic measure in the classical Dirichlet problem, and will be called h-regular measure. If the boundary function f is measurable and summable with respect to this measure, for every i, then f is PWB^h resolutive, that is, the PWB method yields a unique h-regular "solution", and the solution is the function u given by

$$u(i) = \int_{S'} \frac{f(\xi)K(i, \xi)\mu(d\xi)}{h(i)}, \qquad h(i) > 0. \tag{10.4}$$

That is, $u(i)$ is the average of f on S', using as weighting the h-regular measure relative to i. Conversely, if f is a PWB^h resolutive boundary function, f is measurable and summable as just described. Moreover, if $h(i) > 0$, the solution u has as limit, on almost every h-path from i, the value of f at the point of the boundary to which the path converges (or which the path finally reaches, if the boundary point is degenerate).

(d) We define a "fine topology" relative to S at a non-degenerate minimal boundary point as follows. A subset B of S will be called a neighborhood of ξ relative to S if almost every $K(\cdot, \xi)$-path from 1 (and therefore from any point of strict positivity of $K(\cdot, \xi)$) to ξ lies in B sufficiently near ξ. If B is not such a neighborhood, almost no such path has the stated property. A limit in terms of this topology will be called a fine limit. Unless "fine" is mentioned explicitly, all topological concepts will refer to the metric topology defined above. A function v on S has the fine limit b at ξ if and only if v has the limit b on almost all $K(\cdot, \xi)$-paths from 1 (and therefore from any point of strict positivity of $K(\cdot, \xi)$) to ξ. We shall consider a degenerate boundary point as an isolated point in the fine topology. If a function v has a fine limit $f(\xi)$ at almost every (h-regular measure on S' relative to state 1) point ξ of S', f will be called the h-fine boundary function of v. If h is a non-trivially minimal regular function, the h-regular measure relative to state 1 is concentrated at the minimal boundary point corresponding to h. Hence every h-fine boundary function is almost everywhere constant with respect to this measure. Without this specialization of h, every positive loosely h-superregular function v on S, or, more generally, every loosely h-superregular function v which is bounded from below by a negative loosely h-subregular function, has an h-fine boundary function. This boundary function is finite almost everywhere (h-regular measure relative to state 1) if v is h-superregular. In particular, each PWB^h first boundary value problem solution for h-regular functions is so bounded and therefore has an h-fine boundary function, which can be taken as the specified boundary function.

(e) Let h be a positive regular function on S, and let $\{x_{in}, n \geqq 1\}$ be an h-path process from the point i of strict positivity of h. For $n \geqq 1$ let ϕ_n be any function on S, and suppose that $\lim_{n\to\infty} \phi_n(x_{in})$ exists with probability 1 for $i = 1$ and therefore for each i. Then there is a function f on S' such that, for almost all x_{in} paths, the indicated limit is $f(\xi)$ if the path approaches ξ. In particular, if h is (non-trivially) minimal, f can be chosen as a constant function.

(f) Since $K(i, \cdot)$ is a continuous function on $S \cup S'$, it follows that

$$\lim_{n\to\infty} K(i, x_n) \tag{10.5}$$

exists and is finite with probability 1. Here $\{x_n, n \geqq 1\}$ is any Markov chain with the given transition function and with initial states distributed in S. For almost every path of the process, the limit in (10.5) on a path converging to ξ is $K(i, \xi)$.

(g) If ξ is a minimal non-degenerate point of S', $K(\cdot, \xi)$-paths are 1-paths conditioned to converge to ξ. More generally, if h is a positive superregular function given by

$$h(i) = \int_A K(i, \xi)\mu'(d\xi), \tag{10.6}$$

where A is a Borel subset of S' and μ' is the measure of subsets of S' which, if used for μ in (10.2), yields the constant function 1, then h-paths from a point are 1-paths from the point, conditioned to converge to a point of A. The "more generally" is used rather loosely here.

11. The Feller boundary and the work of Blackwell and Breiman. FELLER [9] has also constructed an exit boundary for a Markov chain. His boundary is so large that every positive superregular function on the state space can be extended to be continuous on the completed space. It follows easily from this fact that our assertion (a) of Section 10 (convergence of probability paths) is not true for this boundary. This boundary is thus not well adapted to the first boundary value problem for regular functions, and was, in fact, used by FELLER in a quite different application.

Let S be as in Section 10. Following BLACKWELL [1], the subset C of S will be called "almost closed" if (B1) there is strictly positive probability that paths from 1 enter C infinitely often, and if (B2) almost every path which does so remains outside C for only finitely many steps. In FELLER's terminology, (B1) means that C is a "sojourn" set, and the conjunction of (B1) and (B2) means that C is a "representative sojourn set". Such sets play a fundamental role in FELLER's boundary theory. In our language, (B1) can be shown to be equivalent to the hypothesis that if μ_1 is 1-regular measure on S' relative to state 1, then the set of fine limit points of C on S' has strictly positive μ_1 measure. Hypothesis (B2) supposes that, for almost every (μ_1 measure) fine limit point of C, C is a fine neighborhood of the point, relative to S. If, further, (B3) C is not the union of two disjunct almost closed sets, BLACKWELL calls C "atomic". The set C is almost closed and atomic if and only if there is a Martin boundary point of strictly positive μ_1 measure, such that C is a fine neighborhood of this point and has almost (μ_1 measure) no other fine limit point. BLACKWELL's decomposition of S, roughly into atomic almost closed sets and an almost closed set containing no atomic closed set, amounts to the decomposition of μ_1 into its atomic and non-atomic components. The state space is itself atomic if and only if there is a point of S' of μ_1 measure 1. If this point is non-degenerate, that is, if the states of S are all transient, the condition amounts to saying that 1 is a minimal regular function, or, as remarked by BLACKWELL and by BREIMAN [3] in other language, that all 1-fine boundary functions are almost everywhere (μ_1 measure) constant functions.

BREIMAN [2] calls a transient set C of states "denumerably atomic" if (B1) is satisfied and if, in addition, (B4) whenever A is an infinite subset of C, almost every path from 1 meeting C infinitely often also meets A infinitely often. He shows that a set C of transient states satisfying (B1) is denumerably atomic if and only if every bounded subregular (or equivalently superregular) function has a limit on the set C. It follows readily that C, satisfying (B1), is denumerably atomic if and only if there is a minimal point ξ of S' of strictly positive μ_1 measure such that C converges to ξ, in the sense that each fine neighborhood of ξ contains all but finitely many points of C.

12. Space-time exit boundaries. Let $\hat{R}$ be the direct product of the set of integers and R, that is, $\hat{R}$ is space-time. If $\{x_{in}, n \geqq 0\}$ is a Markov chain with state space R and with initial state i, the process $\{(m+n, x_{in}), n \geqq 0\}$ is a Markov chain with state space $\hat{R}$ and with initial state (m, i). If p is the transition function of the chain with state space R, the space-time transition function is given by

$$p(k, i; l, j) = p(i, j)p^{(0)}(l, k+1). \tag{12.1}$$

The space-time Green function is given by

$$g(k, i; l, j) = p^{(l-k)}(i, j) \quad \text{if} \quad l \geqq k \\ = 0 \qquad\qquad \text{otherwise}. \tag{12.2}$$

A space-time regular function u is one for which $\{u(m+n, x_{in}), n \geqq 0\}$ is a martingale, for all initial pairs (m, i). If we wish to define the space-time exit boundary relative to the point (0, 1), the space-time analogue of S is the set $\hat{S}$ of all pairs (k, i) with $p^{(k)}(1, i) > 0$. The K function becomes

$$\hat{K}(k, i; l, j) = \frac{p^{(l-k)}(i, j)}{p^{(l)}(1, j)}, \tag{12.3}$$

and this function determines the space-time exit boundary $\hat{S}'$.

Let h be positive and regular on S, with $h(1) > 0$, and let $\{x_n^{(h)}, n \geqq 0\}$ be an h-path process with initial point 1. Then if $\hat{h}(k, i)$ is defined as $h(i)$, the process $\{n, x_n^{(h)}), n \geqq 0\}$ is an $\hat{h}$-path process in space-time, with initial point (0, 1).

Hence almost every path converges to a point (depending on the path) of $\hat{S}'$. Now suppose that ξ is a minimal point of S' and that $h = K(\cdot, \xi)$. Then, applying assertion (e) of Section 10 to functions with values in space-time, we find that almost every $\hat{h}$-path must converge to the same point, say $\hat{\xi}$, of $\hat{S}'$. Hence almost every $\hat{h}$-path from any point of $\hat{S}$ must converge to $\hat{\xi}$, and we see that $\hat{\xi}$ is a minimal boundary point, corresponding to the minimal function $\hat{h}$. That is, h is not only minimal on S but, considered as a function on $\hat{S}$, is also minimal there. There are, however, in general, many minimal space-time regular functions that cannot be obtained in this way. Dropping the hypothesis that h is necessarily minimal, we observe that

$$(12.4) \qquad \lim_{n\to\infty} K(k, i; n, x_n^{(h)}) = \lim_{n\to\infty} \frac{p^{(n-k)}(i, x_n^{(h)})}{p^{(n)}(1, x_n^{(h)})}$$

exists and is finite with probability 1. In fact, for a path approaching the space-time minimal boundary point $\hat{\xi}$, the limit is $\hat{K}(k, i; \hat{\xi})$. In particular, if the constant functions are minimal relative to space, and so to space-time, the limit value in (12.4) is 1, for almost all paths, if h is a constant function.

13. The Martin entrance boundary. In treating the Martin exit boundary we chose a state, 1, and confined our attention to the set S of states covered by 1. We now dualize this study. Let S^* be the set of states covering 1. Any state covering a state in S^* is itself in S^*. It follows from this fact that if i and j are in S^*, $p^{(n)}(i, j)$ is the probability of a transition from i to j in n steps with all intermediate steps in S^*. The computation of $g(i, j)$ thus involves only states in S^*. Throughout this discussion we could simply drop all states not in S^*, even though these states may be covered by states in S^*. Rather than do so, however, we do what is equivalent, but perhaps clearer, and consider functions on subsets of S^* with various properties relative to S^*.

In the following, it will be convenient to call a set function μ "nearly superregular" if it is loosely superregular and finite on finite sets, "nearly regular" if it is nearly superregular but not strictly so. That is, $\mu(i)$ is to be finite in both cases; in the nearly superregular case,

$$(13.1) \qquad \sum_i \mu(i)p(i, j) \leqq \mu(j),$$

and in the regular case there is to be equality.

If state 1 is persistent, the states in S^* are either transient or in the same persistent class as 1. It is easily seen that any positive nearly superregular set function on S^* vanishes on the transient states and is determined uniquely, up to a constant factor, on the class of persistent states in S^*. Since this situation is of no interest to us, we assume from now on that 1 is transient.

Since 1 is transient, all states in S^* are transient. For each state i in S^*, choose a strictly positive constant c_i, and define ν' by

$$(13.2) \qquad \nu' = \sum_i c_i g(i, \cdot),$$

choosing c_i in such a way that $\nu'(1) < \infty$. Then ν' is the potential of a positive set function and is finite and strictly positive at each state of S^*. If ν, defined on the subsets of S^*, is a positive nearly superregular set function relative to S^*, define ν_n as the completely additive function of subsets of S^* determined by

$$\nu_n(j) = \min [\nu(j), n\nu'(j)]. \tag{13.3}$$

If n is a positive integer, ν_n is the potential of a positive set function (relative to S^*) because ν' is. Thus ν is the limit of an increasing sequence of such potentials.

Define the countably additive set function $K^*(i, \cdot)$ by

$$K^*(i, j) = \frac{g(i, j)}{g(i, 1)} \tag{13.4}$$

for i and j in S^*. Since, for all n,

$$g(i, j)p^{(n)}(j, 1) \leqq g(i, 1), \tag{13.5}$$

it follows that

$$K^*(i, j) \leqq 1/\sup_n p^{(n)}(j, 1). \tag{13.6}$$

Each point i of S^* is identified with the corresponding set function $K^*(i, \cdot)$. The set S^* is metrized by defining the distance function d^*,

$$d^*(i_1, i_2) = \sum_j |K(i_1, j) - K(i_2, j)| \, 2^{-j}/\delta_j , \tag{13.7}$$

where δ_j is the right side of (13.6). The space is then completed, to obtain a compact metric space. The set $S^{*\prime}$ adjoined in the completion will be called the "Martin entrance boundary relative to state 1". The function $K(\cdot, j)$ can be defined on $S^{*\prime}$ to be continuous on the completed space, and we shall assume that this has been done. If $\xi \in S^{*\prime}$, the point function $K^*(\xi, \cdot)$ we have now defined on S^* determines a countably additive function of subsets of S^*, and this notation will be used to refer to the set function from now on. This set function is positive and nearly superregular relative to S^*, with $K^*(\xi, 1) = 1$.

Let ν be a positive nearly superregular function of subsets of S^*, and let $\{\nu_n, n \geqq 1\}$ be a monotone increasing sequence of potentials of positive set functions with limit ν on each finite subset of S^*. Then we can write $\nu_n(j)$ in the form

$$\nu_n(j) = \int_{S^*} K^*(\xi, j)\mu_n(d\xi), \qquad \mu_n(S^*) = \nu_n(1) \leqq \nu(1), \tag{13.8}$$

where μ_n is a positive completely additive set function. When $n \to \infty$ we find that there is a measure μ of Borel subsets of $S^* \cup S^{*\prime}$ for which

$$\nu(j) = \int_{S^* \cup S^{*\prime}} K^*(\xi, j)\mu(d\xi). \tag{13.9}$$

Just as in the dual case, if ν is regular at j, $K^*(\xi, \cdot)$ is also regular at j, for almost all ξ (μ measure), so that, in particular, if ν is nearly regular relative to S^*, $K^*(\xi, \cdot)$ is also, for almost all ξ (μ measure), and $\mu(S^*) = 0$.

If we define a nearly regular positive set function relative to S^* as minimal if every such function it dominates is proportional to it, every minimal set function (not trivially minimal, that is, not vanishing for all sets) corresponds to a unique $\xi \in S^{*\prime}$ in the sense that the set function is proportional to $K^*(\xi, \cdot)$.

Now let μ be a positive nearly superregular set function relative to S^*, with $\mu(1) > 0$. We have seen in Section 7 that μ, together with the transition function p, determines a stochastic process $\{x_{\mu n}, -\infty < n < \infty\}$. It is the character of the sample paths of the process when $n \to -\infty$ that is now of interest. A simple technique for reducing some of the problems to those already solved is the following, used by FELLER [9] to define his entrance boundary. Define q, as in Section 7, by (7.6) and (7.8), so that q is the transition function for the $x_{\mu n}$ process reversed in time. The Green function and K function for q are given by

$$\bar{g}(j, i) = \mu(i)g(i, j)/\mu(j), \qquad (13.10)$$
$$\bar{K}(j, i) = K^*(i, j)\mu(1)/\mu(j).$$

Thus the procedure for defining the exit boundary for q corresponds to the procedure for defining the entrance boundary for p. Note, however, that the boundary involved here depends on μ.

The analogues of the exit boundary theorems, that is, the duals of assertions (a)–(g) in Section 10 can be obtained by this device. Since the transition from an assertion to its dual is straightforward, only a few dual assertions will be stated.

In the following, ν is always a positive nearly superregular set function relative to S^*; the paths of an $x_{\nu n}$ process will be called ν-paths.

(a*) Almost every ν-path converges (back) to a minimal entrance boundary point or else reaches an adjoined state.

(b*) The statement of the dual of (b) should be obvious. We stress that if ν in (a*) is minimal, almost every ν-path converges back to the entrance boundary point corresponding to ν.

The dual of one of our versions of the Fatou boundary limit theorem in the previous context is the following. Let ν be a positive nearly regular set function relative to S^*, and let ν_1 be a positive nearly superregular set function, absolutely continuous with respect to ν. If $i \in S^*$ and if $\nu(i) > 0$, define $u(i) = \nu_1(i)/\nu(i)$. Then $u(x_{\nu n})$ defines a supermartingale, if the parameter sense is reversed, so that u has a finite limit on almost all ν-paths back to the entrance boundary.

The dual of (c) in Section 10 is obtained by studying superregular and regular functions relative to the reverse transition function and needs no further comment. The duals of (d), (e), (f) are clear, remembering that the specifying of a superregular point function h in Section 10 corresponds to the specifying of a nearly superregular set function here. The dual of (g) may not be quite obvious. The dual of the first assertion is simply that if ν is positive and nearly regular relative to S^* and if ν_1 is minimal nearly regular relative to S^*, corresponding to the entrance boundary point ξ^*, and absolutely continuous with respect to ν, then ν_1-paths are ν-paths conditioned to converge to ξ^*. The dual of the second assertion is now clear. The dual of (12.4) is that the limit

$$\lim_{n\to-\infty} \frac{p^{(l-n)}(x_{\nu n}, j)}{p^{(-n)}(x_{\nu n}, 1)} \tag{13.11}$$

exists and is finite for almost all ν-paths passing through state 1 at time 0.

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