Discrete Price Updates Yield Fast Convergence in Ongoing Markets with Finite Warehouses*

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Abstract

This paper shows that in suitable markets, even with out-of-equilibrium trade allowed, a simple price update rule leads to rapid convergence toward the equilibrium. In particular, this paper considers a Fisher market repeated over an unbounded number of time steps, with the addition of finite sized warehouses to enable non-equilibrium trade.

The main result is that suitable tatonnement style price updates lead to convergence in a significant subset of markets satisfying the Weak Gross Substitutes property. Throughout this process the warehouse are always able to store or meet demand imbalances (the needed capacity depends on the initial imbalances).

Our price update rule is robust in a variety of regards:

- The updates for each good depend only on information about that good (its current price, its excess demand since its last update) and occur asynchronously from updates to other prices.
- The process is resilient to error in the excess demand data.
- Likewise, the process is resilient to discreteness, i.e. a limit to divisibility, both of goods and money.

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1 Introduction

This paper investigates when a tatonnement-style price update in a dynamic market setting could lead to fast convergent behavior.

The impetus for this work comes from the following question: why might well-functioning markets be able to stay at or near equilibrium prices? This raises two issues: what are plausible price adjustment mechanisms and in what types of markets are they effective?

This question was considered by Walras in 1874, when he suggested that prices adjust by tatonnement: upward if there is too much demand and downward if too little [51]. Since then, the study of market equilibria, their existence, stability, and their computation has received much attention in Economics, Operations Research, and most recently in Computer Science. McKenzie [33] gives a fairly recent account of the classic perspective in economics. The recent activity in Computer Science has led to a considerable number of polynomial time algorithms for finding approximate and exact equilibria in a variety of markets with divisible goods; we cite a selection of these works [8, 9, 14, 16, 15, 22, 27, 38, 49, 50]. However, these algorithms do not seek to, and do not appear to provide methods that might plausibly explain these markets' behavior.

We argue here for the relevance of this question from a computer science perspective. Much justification for looking at the market problem in computer science stems from the following argument: If economic models and statements about equilibrium and convergence are to make sense as being realizable in economies, then they should be concepts that are computationally tractable. Our viewpoint is that it is not enough to show that the problems are computationally tractable; it is also necessary to show that they are tractable in a model that might capture how a market works. It seems implausible that markets with many interacting players (buyers, sellers, traders) would perform overt global computations, using global information.

Analogous goals have arisen when considering convergence to equilibria in game theory. Perhaps the most successful approach has been to use regret minimizing procedures, which has yielded convergence in several settings [40, 30, 17, 6]. Other works have considered best response [19, 37] and proportional response dynamics [52].

A central concern with the tatonnement model of price updates, and other models, is whether they cause prices to converge to equilibria. Such results were shown for some types of markets; early examples include [1, 2, 36, 47]. However, there is no demonstration that these proposed update models converge reasonably quickly. Indeed, without care in the specific details, they will not.[1]

It has long been recognized that the tatonnement price adjustment model is far from a realistic model: for example, in 1972, Fisher [21] wrote "such a model of price adjustment $\cdots$ describes nobody's actual behavior." This has led to to work on other price dynamics ranging from the Hahn process [21] to non-equilibrium dynamics in the trading post model [45, 4], and others [12]. At the same time, there has been a continued interest in the plausibility of tatonnement, and indeed its predictive accuracy in a non-equilibrium trade setting has been shown in some experiments [26].

Plausibly, in many consumer markets buyers are myopic: based on the current prices, goods are assessed on a take it or leave it basis. It seems natural that this would lead to out of equilibrium trade. This is the type of setting in which we wish to consider our main question: under what conditions can tatonnement style price updates lead to convergence?

Accordingly, in this paper we propose a simple market model in which the market extends over

[1] Of the referenced papers, only one formalization [47] is a discrete algorithm, and, as we make more specific later, it may not converge quickly.

time and trading occurs out of equilibrium (as well as at equilibrium). We call this the *Ongoing Market*. Here, the market repeats from one time unit to the next; we call the basic unit a *day*. The link from one day to the next is that goods unsold one day are available the next day, in addition to the new supply, which for simplicity, we take as being the same from day to day. This appears to provide a simple and natural way of allowing out-of-equilibrium trade. The algorithmic task is to converge to equilibrium prices while clearing unsold stocks. We use the version of tatonnment in which price updates are proportional both to the current price and to the excess demand (modulo some minor details to prevent very large price changes). We show that it results in rapid convergence toward equilibrium prices in this market.

Our analysis of the algorithm for the Ongoing Market relies in part on understanding which we develop by analyzing convergence in the traditional market problem (used in the auctioneer model for example). In this paper, we call this the *One-Time Market*. The algorithmic technique of iteratively computing prices for the One-Time Market can be seen as a plausible approximation to the Ongoing Market (but with no carry over of unsold goods). Our work can be seen as a formal justification for this approach. Further, in our opinion, the intuitive understanding that markets are usually similar from one time period to the next has been a factor in the previous appeal of iterative price update algorithms, including, in the Computer Science literature, the recent tatonnement algorithm of Codenotti et al. [8] and the auction algorithms of Garg et al. [22].

Our proposed price update protocols capture some of the characteristics of trading as proposed in the trading post model [4], features that are lacking from previous algorithms subject to asymptotic analysis. Namely, our algorithm consists of price updates satisfying the following criteria: the price update for a good depends only on the price, demand, and supply for that good, and on no other information about the market; in addition, the price update for each good occurs distributively and asynchronously (i.e. at independent times). Further desirable features are that: the algorithm can start with an arbitrary set of prices; the algorithm tolerates inaccuracy in demand data; finally, it can tolerate discreteness: the fact that neither goods nor money are infinitely divisible. We show that our update protocols converge quickly in many markets that satisfy the weak gross substitutes property. In the process, we identify several natural parameters characterizing these markets, parameters which govern the rate of convergence.

1.1 The Market Problems

The One-Time Fisher Market[2] A market comprises a set of goods G, with $|G| = n$, and two sets of agents, buyers B, with $|B| = m$, and sellers S. The sellers bring the goods G to market and the buyers bring money with which to buy them. The trade is driven by a collection of prices p_i for good i, $1 \le i \le n$. For simplicity, we assume that there is a distinct seller for each good; further it suffices to have one seller per good. The seller of good i brings a supply w_i of this good to market. Each seller seeks to sell its goods for money at the prices p_i.

Each buyer $b_j \in B$ comes to market with money v_j; buyer b_j has a utility function $u_j(x_{1j}, \cdots, x_{nj})$ expressing its preferences: if b_j prefers a basket with x_{ij} units (possibly a real number) of good i, to the basket with y_{ij} units, for $1 \le i \le n$, then $u_l(x_{1j}, \cdots, x_{nj}) > u_j(y_{1j}, \cdots, y_{nj})$. Each buyer b_j intends to buy goods so as to achieve a personal optimal combination (basket) of goods given their available money.

[2]The market we describe here is often referred to as the Fisher market. We use a different term because we consider this problem in a new Ongoing Market model as described below.

Prices $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ are said to provide an *equilibrium* if, in addition, the demand for each good is bounded by the supply: $\sum_{j=1}^{m} x_{ij} \le w_i$, and the total cost of the goods is bounded by the available money: $\sum_i p_i w_i \le \sum_j v_j$. The market problem is to find equilibrium prices.[3]

The Fisher market is a special case of the more general *Exchange* market or *Arrow-Debreu* market.

While we define the market in terms of a set of buyers B, all that matters for our algorithms is the aggregate demand these buyers generate, so we will tend to focus on properties of the aggregate demand rather than properties of individual buyers' demands. This will prove significant when we consider the case of (somewhat) indivisible goods.

Standard notation $x_i = \sum_l x_{il}$ is the demand for good i, and $z_i = x_i - w_i$ is the excess demand for good i (which can be positive or negative). Note that while w is part of the specification of the market, x and z are functions of the vector of prices as determined by individual buyers maximizing their utility functions given their available money. We will assume that x_i is a function of the prices p, that is a set of prices induce unique demands for each good.

The Ongoing Fisher Market In order to have non-equilibrium trade, we need a way to allocate excess supply and demand. To this end, we suppose that for each good there is a *warehouse* which can store excess demand and meet excess supply. The seller has a warehouse of finite capacity to enable it to cope with fluctuations in demand. It will change prices as needed to ensure its warehouse neither overfills nor runs out of goods.

The market consists of a set G of n goods and a set B of m buyers. The market repeats over an unbounded number of time intervals called days. Each day, the seller of good i (called seller i) receives w_i new units of good i, and buyer ℓ is given v_ℓ money, $1 \le \ell \le m$. As before, each buyer ℓ has a utility function $u_\ell(x_{1\ell}, \cdots, x_{n\ell})$ expressing its preferences. Each day, buyer ℓ selects a maximum utility basket of goods $(x_{1\ell}, \cdots, x_{n\ell})$ of cost at most v_ℓ. Each seller i provides the demanded goods $\sum_{\ell=1}^{m} x_{i\ell}$. The resulting excess demand or surplus, $\sum_{\ell=1}^{m} x_{i\ell} - w_i$, is taken from or added to the warehouse stock. Seller i has a warehouse of capacity c_i.

Given initial prices p_i°, warehouse stocks s_i°, where $0 < s_i^\circ < c_i$, $1 \le i \le n$, and ideal warehouse stocks s_i^*, $0 < s_i^* < c_i$, the task is to repeatedly adjust prices so as to converge to equilibrium prices with the warehouse stocks converging to their ideal values. We let s_i denote the current contents of warehouse i, and $s_i^e = s_i - s_i^*$ denote the *excess warehouse reserves*.

We suppose that it is the sellers that are adjusting the prices of their goods. In order to have progress, we require them to change prices at least once a day. However, for the most part, we impose no upper bound on the frequency of price changes. Indeed, we will see that having more frequent price changes is helpful in the event that very low prices are present. This entails measuring demand on a finer scale than day units. We take a very simple approach: we assume that each buyer spends their money at a uniform rate throughout the day. (Equivalently, this is saying that buyers with collectively identical profiles occur throughout the day, though really similar profiles suffice for our analysis.) Likewise, if one supposes there is a limit to the granularity, this imposes a limit on the frequency of price changes.

Market Properties In an effort to capture the distributed nature of markets and the possibly limited knowledge of individual price setters, we impose several constraints on procedures we wish to consider:

[3] Equilibria exist under quite mild conditions (see [32] §17.C, for example).

1. Limited information: the price setter for good i knows only the price, supply, and excess demand of good i, both current and past history. Thus the price updates can depend on this information only. Notably, this precludes the use not only of other prices or demands, but also of any information about the specific form of utility functions.

2. Simple actions: The price setters' procedures should be simple.

3. Asynchrony: Price updates for different goods are allowed to be asynchronous.

4. Fast Convergence: The price update procedure should converge quickly toward equilibrium prices from any initial price vector.

5. Robustness to Inaccuracy: Even if the demand data is somewhat inaccurate, the procedure should still converge to approximately the equilibrium prices.

We call procedures that satisfy the first three constraints *local*, by contrast with centralized procedures that use more complete (global) information about the market.

Next, we discuss the motivations for these constraints.

Constraint (1) stems from the plausible assertion that not everything about the market will be known to a single price setter. While no doubt some information about several goods is known to a price setter, it is a conservative assumption to assume less is known, for any convergence result carries over to the broader setting. Further, it is far from clear how to model the broader setting.

Constraint (2), simplicity, is in the eye of the beholder. Its presence reflects our view that without further information, this is both generally applicable and plausible.

Constraint (3), asynchrony, is an inherent property of independent price adjustments. Since the price setter of good i reacts only to trade in good i, the price adjustment of good i occurs independently of other price adjustments.

Constraint (4) arises in an effort to recognize the dynamic nature of real markets, which are subject to changing supplies and demands over time. However, surely much of the time, markets are changing gradually, for otherwise there would be no predictability. A natural approximation is to imagine fixed conditions and seek to come close to an equilibrium in the time they prevail — hence the desire for rapid convergence.

Constraint (5) recognizes the reality that in practice data is always is a little dated and inaccurate, and it seems more realistic if procedures tolerate this.

A final issue is that it is an approximation to allow goods and money to be infinitely divisible. It is useful to understand to what extent this limits convergent behavior. This entails a variety of changes to the standard definitions used for continuous (divisible) markets: it turns out it is no longer sensible to constrain demands (or utilities) at the individual level, but only in the aggregate, and the notion of weak gross substitutes needs to be modified (relaxed in fact), but not in the way used in the literature for (discrete) matching markets. We call this modified setting the indivisible goods market, but we are intending a setting in which there are still many copies of each good, rather than a setting in which each good is unique or close to unique.

1.2 Previous Work

To the best of our knowledge, asynchronous price update algorithms have not been considered previously. Further, there has been no complexity analysis of even synchronous tatonnement algorithms with this type of limited information. While Uzawa [47] gave a synchronous algorithm of this type, he only showed convergence, and did not address speed of convergence.

The existence of market equilibria has been a central topic of economics since the problem was formulated by Walras in 1874 [51]. Tatonnement was described more precisely as a differential equation by Samuelson [42]:

$$dp_i/dt = \mu_i z_i. \tag{1}$$

The μ_i are arbitrary positive constants that represent rates of adjustment for the different prices; they need not all be the same. Arrow, Block, and Hurwitz, and Nikaido and Uzawa [1, 2, 36] showed that for markets of gross substitutes the above differential equation will converge to an equilibrium price.

Unfortunately, for general utility functions (i.e. that do not lead to gross substitutability), the equilibrium need not be stable and the differential equation (and thus also discretized versions) need not converge [43]. Partly in response, Smale described a convergent procedure that uses the derivative matrix of excess demands with respect to prices [46]. Following this, Saari and Simon [41] showed that any price update algorithm which uses an update that is a fixed function of excesses and their derivatives with respect to prices needs to use essentially all the derivatives in order to converge in all markets. However, this is viewed as being an excessive amount of information, in general.

There are really two questions here. The first is how to find an equilibrium, and the second is how does the market find an equilibrium. The first question is partially addressed by the work of Arrow et al. and Smale, and addressed further in papers in operations research (notably Scarf [44] gives a (non-polynomial) algorithm for computing equilibrium prices), and theoretical computer science, where there are a series of very nice results demonstrating equilibria as the solutions to convex programs, or describing combinatorial algorithms to compute such equilibria exactly or approximately. (An early example of a polynomial algorithm for computing market equilibria for restricted settings is [16]. An extensive list of references is given in the surveys [10, 48].)

We are interested in the second question. The differential equations provide a start here, but they ignore the discrete nature of markets: prices typically change in discrete increments, not continuously. In 1960, Uzawa showed that there is a choice of λ for which an obvious discrete analog of (1) does converge [47]. However, determining the right λ depends on knowing properties of the matrix of derivatives of demand with respect to price, or in other words, this requires global information.

In relatively recent work, three separate groups have proposed three distinct discrete update algorithms for finding equilibrium prices and showed that their algorithms converge in markets of gross substitutes [29, 12, 8]. However, all of these algorithms use global information. With the exception of [8], none of this work gives (good) bounds on the rate of convergence. The algorithm in Codenotti et al. [8] describes a tatonnement algorithm (albeit not asynchronous); however, it begins by modifying the market by introducing a fictitious player with some convenient properties that capture global information about the market and have a profound effect on market behavior. Even in this transformed setting, the price update step uses a global parameter based on the desired approximation guarantee, and starts with an initial price point that is restricted to lie within a bounded region containing the equilibrium point. Translating their algorithm back into the real market, one can see that it does not meet our definition of simplicity or locality.

There are some auction-style algorithms for finding approximate equilibria which also have a distributed flavor but depend on buyer utilities being separable over the set of goods [22, 23].

However, these algorithms are not seeking to explain market behavior and not surprisingly do not obey natural properties of markets.[4]

The one work in the Computer Science literature considering the indivisible goods setting is by Deng et al. [13]; this paper was also the one to start the study of market equilibria in the CS theory community. They showed that it is APX-Hard to approximate equilibrium prices and allocations for indivisible exchange markets. They also gave an exhaustive algorithm for computing an approximate equilibrium in polynomial time for markets with a constant number of distinct goods.

Markets of indivisible goods have been studied by mathematical economists also. Ausubel, Gul and Stacchetti [3] introduce an "individual substitutes" property so as to ensure that equilibria exist. Milgrom and Strulovici [34] also consider this setting, replacing the individual substitutes constraint with the standard WGS constraint. Both works present exhaustive algorithms to find the equilibrium prices.

The design and analysis of procedures and convergence to equilibria has been a recent topic of study for game theoretic problems, using the technique of regret minimization in particular [40, 30, 17, 6]. Other work has studied convergence in network routing and network design games [7, 20, 24, 35]. In partial contrast, it is known that finding equilibria via local search (e.g., via best response dynamics) is PLS-complete in many contexts [28, 18]. Recently, Hart and Mansour [25] gave communication complexity lower bounds to show that in general games, players with limited information require an exponential (in the number of players) number of steps to reach an equilibrium.

The design and analysis of convergent asynchronous distributed protocols has also arisen in network routing, for example [31], and in high latency parallel computing [5]. These lines of work are perhaps the most similar to ours.

1.3 Our Contribution

As devising a tatonnement-style price update in unsolvable for general markets, our goal has been to devise plausible constraints that enable rapid convergence. Our overall task has been to devise a reasonable model, a price update algorithm, a measure of closeness to equilibrium, and then to analyze the system to demonstrate fast convergence. This also entails identifying appropriate parameterized constraints on the market.

Loosely speaking, our measure computes the cost of resource misallocation (compared to the equilibrium allocation). Roughly, we show that the price update process runs in time $O(\frac{1}{\kappa}\log(\text{initial cost/final cost}))$, where κ is a parameter of the price update protocol, a parameter that depends on the properties of the market demand. Further, in the case of indivisible goods, we give nearly tight bounds on the achievable minimum for the final cost. We also bound the needed warehouse size in terms of the initial (imbalanced) state of the market.

A precursor of this work [11] introduced the Ongoing Market and analyzed it in the context of divisible markets. The analysis given here is quite distinct and conceptually simpler; it also yields improved bounds in terms of the parameter constraints. (It is a little difficult to compare the two works, as the earlier work required more constraints and parameters, and used slightly different rules for the price updates — the rules in the present paper strike us as more realistic.)

[4]These algorithms start their computation at a non-arbitrary set of artificially low prices; global information is used for price initialization; and they work with a global approximation measure — each price update uses the goal approximation guarantee in its update.

An initial analysis of markets of indivisible goods appeared in Rastogi's thesis [39].

The main technical tool for the analysis is the use of a potential function, whose value corresponds to the market resource misallocation. We show the potential decreases continuously at or above a suitable minimum rate, and never increases.

This work identifies three parameters which govern the rate of convergence and the needed warehouse sizes. The first parameter, which we call the *market demand elasticity*, or the *market elasticity* for short, governs the rate of convergence of the one-time markets: it is a lower bound on the fractional rate of change of the demand for any good with respect to its own price: $\min_i \frac{dx_i}{dp_i} / \frac{x_i}{p_i}$.

The second parameter is the standard elasticity of wealth[5] it will be used in the analysis of convergence rates when rapid price rises are used to mitigate the effects of large demands.

The third parameter is somewhat less intuitive; it governs the needed warehouse sizes. We call it the *equilibrium flex*, and it is defined as follows. Let $p_i^{(c)}$ denote the equilibrium prices for supplies cw_i, for $1 \le i \le n$. Then we define the equilibrium flex, $e(c)$, by

$$e(c) = \ln \max_i \left\{ \frac{p_i^*}{p_i^{(c)}}, \frac{p_i^{(1/c)}}{p_i^*} \right\}$$

where $p_i^* = p_i^{(1)}$ are the equilibrium prices for the market at hand. We remark that the carry-forward of demand imbalances in Ongoing Markets will create temporary changes in effective supply levels and hence in the desired equilibrium prices. It is plausible that the size of these changes affects the convergence rate and not surprising that it affects the needed warehouse size also.

Bounds on $e(c)$ follow from bounds on the elasticity of wealth, but it is not clear that this is a tight connection.

For a market of buyers all having CES utilities, the values of all these parameters are modest, as we will see.

2 Results

2.1 The 1-Good Case in a One-Time Market

We begin by considering a tatonnement process in the context of a one-time Fisher market with a single good, in order to determine reasonable constraints on the demand function, namely constraints that enable a fairly rapid convergence of the price toward its equilibrium value.

Let x denote the demand for the good and p its price. We assume that $x(p)$ is a strictly decreasing function. Let w denote the supply of the good. Our goal is to update p repeatedly so as to cause x to converge toward w. We will use the following update rule:

$$p' = p \left(1 + \lambda \min \left\{ 1, \frac{x - w}{w} \right\} \right)$$

where $0 < \lambda \le \frac{1}{2}$ is a suitable fixed parameter.

[5]This is the the fractional rate of change of demand with respect to changes in wealth; it is usually defined for individual agents; we will use an aggregate definition, obtained when all buyers have the same fractional change in wealth, or equivalently all prices change by the same fraction.

We are going to argue that this update function is appropriate for demands x satisfying:

$$\frac{x(p)}{p} \leq -\frac{dx}{dp} \leq E\frac{x(p)}{p}, \quad \text{where } 1 \leq E \leq \frac{1}{2\lambda}. \tag{2}$$

We will also explain why the update function can be viewed as an approximation of Newton's method, but an approximation quite unlike the standard secant method.

Definition 1. *Let x' denote $x(p')$. Price p' is said to be on the* same side *as p if either both $x, x' \geq w$ or both $x, x' \leq w$. We also say that p' is a* same-side update*.*

Let p^* denote the value of p for which $x(p^*) = w$. Note that p' is on the same side as p exactly if both $p, p' \geq p^*$ or if both $p, p' \leq p^*$.

While same side updates are not important in the 1d setting, in the multi-dimensional setting they are needed to ensure progress. And so in the next lemma we investigate how large a step size is possible while having a same side update.

Lemma 1. *Suppose that $-p\frac{dx}{dp} \leq Ex(p)$, for some $E \geq 1$. Then the update $p' = p(1+\frac{1}{2E}\min\{1, \frac{x-w}{w}\})$ is on the* same side *as p, but the update $p' = p(1 + \frac{1}{E}\frac{x-w}{w})$ need not be.*

Next we investigate the rate of convergence. It might seem natural to determine how quickly $|x^* - x(p)|$ decreases, but in higher dimensions, this proves a less effective measure. Instead, we will measure the "distance" from equilibrium by the measure, or potential function, $\phi(x, p) = |x - w|p$. This can be thought of as the cost in money of the current amount of disequilibrium.

Theorem 1. *If $x \leq 2w$ initially, then in one iteration ϕ reduces by at least $w|p' - p| = \lambda\phi$, where p is the price before the update and p' the price after the update. Hence in $O(\frac{1}{\lambda}\log\frac{\phi_I}{\phi_F})$ iterations, ϕ reduces from ϕ_I to at most ϕ_F. If $x \geq 2w$ initially, then in one interation ϕ reduces by at least $w|p' - p| = \lambda\frac{w}{x-w}\phi$. Hence in $O(\frac{1}{\lambda}\frac{x-w}{w}\log\frac{\phi_I}{\phi_F})$ iterations, ϕ reduces from ϕ_I to at most ϕ_F.*

2.2 The Multidimensional Case, Synchronous Updates, One-Time Market

We start by investigating the multidimensional case in the event that all prices are updated simultaneously as the analysis is relatively simple. We will see that the rate of convergence is very similar to that which was obtained in the 1d case.

Recall that there are n goods $g_1, g_2, \cdots, g_n$. Let $p = (p_1, p_2, \cdots, p_n)$ be the n-vector of prices for these goods, and let $x_i(p)$, or x_i for short, denote the demand for the ith good at prices p (that is, it is a function of all the prices, not just the price of the ith good).

Again, we use the update rule

$$p_i' = p_i\left(1 + \lambda\min\left\{1, \frac{x_i - w_i}{w_i}\right\}\right).$$

Again, we assume that the demands satisfy the bounded rate of change property which we term *bounded elasticity* in the current context.

Definition 2. *Good i has* bounded elasticity *E, if for any collection p of prices:*

$$\frac{x_i(p)}{p_i} \leq -\frac{dx_i}{dp_i} \leq E\frac{x_i(p)}{p_i}, \quad \textit{where } 1 \leq E \leq \frac{1}{2\lambda}.$$

The market has (demand) elasticity *E if every good has bounded elasticity E.*

We are thinking of E's value as a constant. To motivate this, we give the value of E for some standard utility functions: if all buyers have Cobb-Douglas utilities [32] (p. 612), then $E = 1$; if all buyers have CES utilities [32] (p. 97) with parameter ρ, $0 \leq \rho \leq 1$, then $E = 1/(1-\rho)$.

We will also need the standard assumption of Weak Gross Substitutes (WGS) to somewhat constrain how one price change affects demands for other goods.

Definition 3. *A market satisfies the* gross substitutes property *(GS) if for any good i, increasing p_i leads to increased demand for all other goods. The market satisfies* weak gross substitutes *if the demand for every other good increases or stays the same.*

We use the potential function $\phi = \sum_i \phi_i$, where $\phi_i = p_i|x_i - w_i|$. So the effect of an update to p_i on ϕ_i is just as in the 1-d case. However the update may also affect the demands x_j, and hence the potentials ϕ_j, for goods $j \neq i$.

Theorem 2. *Suppose that the market obeys WGS and has elasticity E. When all the prices are updated simultaneously, if $\lambda(2E-1) \leq \frac{1}{2}$, $\phi = \sum_i \phi_i$ reduces by at least $\sum_i \lambda \phi_i^I \min\left\{1, \frac{w_i}{|x_i - w_i|}\right\}$. If the demands satisfy $x_i \leq dw_i$ throughout, then ϕ reduces from ϕ_I to at most ϕ_F in $O(\frac{1}{\lambda}(d-1)\log\frac{\phi_I}{\phi_F})$ iterations.*

Bounding d is non-trivial, for an update to x_j, $j \neq i$, may cause x_i to increase. We return to this issue later.

2.3 The Multidimensional Case, Asynchronous Updates, One-Time Market

But we don't want to assume that all the price updates are simultaneous. For this to make sense we have to reinterpret the meaning of the parameters for supply and demand. We now view these as rates.

To make this concrete, let us call the basic time unit a day. w_i is the daily supply of good i, which is assumed to arrive continuously throughout the day. x_i will be the instantaneous rate of demand given the current prices. The demand over some time period $[t_1, t_2]$ is given by $\int_{t_1}^{t_2} x_i(t)dt$. We are interested in scaling this demand by the time period; we call this the average demand, denoted $\overline{x}_i[t_1, t_2]$; i.e., $\overline{x}_i[t_1, t_2] = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x_i(t)dt$. We are particularly interested in the average demand for good i since the time of last update to p_i, at time τ_i say: we define the average demand at the current time by $\overline{x}_i = \overline{x}_i[\tau_i, t] = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x_i(t)dt$. As we will see later when we introduce warehouses, the average demand can be measured in our setting and so this is the value for the demand that will be used in the price update function:

$$p_i' = p_i\left(1 + \lambda \min\left\{1, \frac{\overline{x}_i - w_i}{w_i}\right\}\right).$$

To ensure progress we require that each price updates at least once a day.

We use the following potential function: $\phi = \sum_i \phi_i$, where

$$\phi_i(x_i, \overline{x}_i, w_i) = p_i\left[\mathrm{span}(x_i, \overline{x}_i, w_i) - \alpha_1 \lambda |w_i - \overline{x}_i|(t - \tau_i)\right], \tag{3}$$

with $\alpha_1 > 0$ being a suitable constant and $\mathrm{span}(x, y, z)$ denoting the length of the interval spanned by its arguments, i.e. $\max\{x, y, z\} - \min\{x, y, z\}$.

We now show a bound on the convergence rate which is broadly the same as the one that applies when the updates are synchronous.

Theorem 3. *Suppose that the market obeys WGS and has elasticity E. If $x_i \le dw_i$, for all i, where $d \ge 2$, $\alpha_1(d-1) \le 1$, $\lambda\alpha_1 + \lambda\left(1 + \frac{2Ed}{1-\lambda E}\right) \le 1$, and each price is updated at least once every day, then ϕ decreases by at least a $1 - \frac{\lambda\alpha_1}{2}$ factor daily. Hence ϕ reduces from ϕ_I to at most ϕ_F in $O(\frac{1}{\lambda\alpha_1}(d-1)\log\frac{\phi_I}{\phi_F})$ days.*

In the current setting, the natural measure of misspending on good i is $|\overline{x}_i - w_i|p_i$, for as we will see when we consider warehouses, the difference $|\overline{x}_i - w_i|$ is what we can measure, and this is used as the indicator of how far from equilibrium p_i currently is, and hence how to update it. However, we also have to take account of the current rate of excess demand, $|x_i - w_i|$. Thus we define the misspending S to be

$$S = \sum_i [|x_i(t') - w_i| + |\overline{x}_i - w_i|]p_i.$$

Clearly, $\phi = \theta(S)$.

We can conclude that, while it is not necessarily decreasing at all times, over time the misspending decreases at the same rate as ϕ.

2.4 The Ongoing Market, or Incorporating Warehouses

For the market model to be self-contained, we need to explain how excess demand is met and what is done with excess supply. The solution is simple: we provide finite capacity warehouses (buffers in computer science terminology) that can store excess supply and meet excess demand. There is one warehouse per good. The price-setter for a good changes prices as needed to ensure the corresponding warehouse neither overfills nor runs out of goods.

Just as the demand is a rate, we imagine the supply to be a rate, which for the purposes of analysis we treat as being a fixed rate. Each Δt long instant, the resulting excess demand or surplus, $(x_i - w_i)\Delta t$, is taken from or added to the warehouse stock.

Let c_i be the capacity of the warehouse for good i. Each warehouse is assumed to have a target ideal content of s_i^* units (perhaps the most natural value is $s_i^* = c_i/2$).

The goal is to repeatedly adjust prices so as to converge to near-equilibrium prices with the warehouse stocks converging to near-ideal values. A further issue is to determine what size warehouse suffices, which we defer to a later section.

The price update rule needs to take account of the current state of the warehouse, namely whether it is relatively full or empty. To this end, let τ_i be the time of the previous update to p_i. let t be the current time, and let s_i denote the current contents of warehouse i. Then the target excess demand, $\overline{z}_i$, is given by $\overline{x}_i[\tau_i, t] - w_i + \kappa_i(s_i - s_i^*)$ where $\kappa_i > 0$ is a suitable (small) parameter. So $\overline{z}_i = \frac{s_i(\tau_i) - s_i(t)}{t - \tau_i} + \kappa_i(s_i - s_i^*)$ and is readily calculated by monitoring warehouse stocks. We let $\widetilde{w}_i$ denote $w_i - \kappa_i(s_i - s_i^*)$, which we call the *target demand.*

We will need the following constraint on κ_i.

Constraint 1. $\kappa_i(s_i - s_i^*) = |\widetilde{w}_i - w_i| \le \frac{1}{3}w_i$.

For simplicity, henceforth we assume that $\kappa = \kappa_i$ for all i.

The price of good i is updated according to the following rule:

$$p_i' \leftarrow p_i\left(1 + \lambda \text{ median}\left\{-1, \frac{\overline{z}_i(p)}{w_i}, 1\right\}\right) \tag{4}$$

This rule ensures that the change to p_i is bounded by $\pm\lambda p_i$.

We redefine the potential ϕ_i to also take account of the imbalance in the warehouse stock as follows:

$$\phi_i = p_i[\text{span}(x_i, \overline{x}_i, \widetilde{w}_i) - \lambda\alpha_1(t - \tau_i)|\overline{x}_i - \widetilde{w}_i| + \alpha_2|\widetilde{w}_i - w_i|]$$

where $1 < \alpha_2 < 2$ is a suitable constant; this is simply Equation 3, with $\widetilde{w}_i$ replacing w_i and with the additional term $\alpha_2 p_i|\widetilde{w}_i - w_i|$.

We redefine the misspending S to take account of the warehouse imbalance:

$$S = \sum_i [|x_i(t') - w_i| + |\overline{x}_i - w_i|]p_i. + |\widetilde{w}_i(t') - w_i|]p_i.$$

Again, $\phi = \theta(S)$.

Theorem 4. *If Constraint 1 holds and $x_i \le d\widetilde{w}_i$, $\alpha_2 = \frac{3}{2}$, $\alpha_1 = \frac{1}{16}$, $\lambda E \le \frac{1}{17}$, $\lambda E d \le \frac{5}{17}$, $\lambda \le \frac{1}{14}$, $\kappa \le \frac{\lambda\alpha_1}{10}$, and each price is updated at least once every day, then ϕ decreases by at least a $1 - \frac{\kappa(\alpha_2 - 1)}{4}$ factor daily.*

In fact, if $\phi \ge 2(1 + 2\alpha_2)\sum_i |\widetilde{w}_i - w_i|p_i$, then ϕ decreases by at least a $1 - \frac{\lambda\alpha_1}{8(1+\alpha_2)}$ factor daily.

Once the potential is largely due to the warehouse imbalances, the daily rate of decrease of ϕ drops from $1 - \theta(\lambda\alpha_1)$ to $1 - \theta(\kappa)$, which seems unavoidable as the improvement to the warehouse contents may be only w_i, which contributes an $\alpha_2\kappa w_i p_i$ reduction to ϕ_i, but ϕ_i could be of size $\theta(w_i p_i)$.

We do not mean to suggest that the above values for the parameters are tight or even nearly tight. Rather the result should be understood as indicating the order of magnitude rate of convergence.

2.5 Bounds on Prices and Demands

Here, we determine an bounds on prices and demands that hold throughout, given initial bounds on demands, and hence what value of d can be used in Theoremlem:war-progr-bdd-dem.

Definition 4. *Prices are f-bounded if p_i always remains in the range $p_i^* e^{\pm f}$ for all i, where p_i^* is the equilibrium value for p_i, and demands are d-bounded if $x_i \le dw_i$ for all i.*

Definition 5. *$p^{(c)}$ denotes the equilibrium prices for supplies cw_i.*

Definition 6. *Prices p are c-*demand bounded *if $p_i \in [p_i^{(c)}, p_i^{(1/c)}]$ for all i. The equilibrium flex, $e(c)$, is defined to be*

$$e(c) = \ln \max_i \left\{ \frac{p_i^*}{p_i^{(c)}}, \frac{p_i^{(1/c)}}{p_i^*} \right\}$$

where $p_i^* = p_i^{(1)}$ are the equilibrium prices for the market at hand. Note that c-demand bounded prices are $e(c)$-bounded.

For CES utilities (with parameter ρ) $e(c) = \ln c$ (it is independent of ρ).

Lemma 2. *If $\frac{\lambda E}{1-\lambda E} \le \frac{1}{6}$, and if initially the prices have all been c-demand bounded for a full day for some $c \ge 2$, they remain c-demand bounded thereafter.*

Definition 7. *Suppose that the prices are always c-demand bounded. Let $f = f(c)$ be the corresponding f-bound on the prices and given the prices are f-bounded, let $d(f)$ be the demand bound (conceivably, $d \gg c$).*

Lemma 3. $d(f) \leq e^{2Ef}$.

It follows that given a c-demand bound for the first day of the update process, there is a value for $f = f(c)$, implied by the uniqueness of equilibria in this setting, which in turn yields a bound on $d(f)$.

Next, we obtain a bound on $e(c)$ given the assumption of 0-homegeneity[6] and a lower bound of 0 on the elasticity of wealth.

Definition 8. *Suppose that the money of each buyer increases by a multiplier γ. We define ξ_i^w, the elasticity of good i with respect to wealth, to be $\xi_i^w = \frac{dx_i}{d\gamma}/x_i$. We say that the market has a bounded wealth elasticity $E^w \geq 0$ if $\xi_i^w \geq -E^w$ for all i.* [7] *When $E^w = 0$, the market is said to have* normal *demands (see [32] page 25).*

For CES utilities, $E^w = -1$ (this is a stronger bound than is used in any of our analyses.)
Notation. Let $\rho = \max_{i,j} \frac{w_i p_i^*}{w_j p_j^*}$.

Lemma 4. *If the demands are all normal, then* $e(c) \leq \ln[c(\rho n)^{(c-1)}]$.

We can still obtain bounds on $e(c)$ for positive E^w, but they are much larger.

We note that even the bound of Lemma 4 is large and implies that a small change in supplies could have a huge effect on the values of some of the equilibrium prices. This seems implausible as a practical matter, and as the bound of Lemma 3 and hence the convergence rate depend on the value of $e(c)$ it suggests that our bounds based on the elasticity of wealth, rather than the equilibrium flex, may be unduly pessimistic.

Prior work [11] made the stronger assumption that $E^w < 0$ (though this was expressed in a different way).

2.6 Faster Updates with Large Demands, Ongoing Market

As we have seen, when demands are large initially, the rate of convergence depends inversely on a bound on a parameter d where for each i, dw_i bounds the demand for good i. As we will see the bound on the warehouse sizes also depends on d.

We now show how to avoid this dependence, by a plausible and modest change to the frequency of the price updates.

It seems reasonable that when demands are large, the seller will observe this quickly (due to stock being drawn from its warehouse) and consequently will quickly adjust its price. Accordingly, we introduce a new rule for the frequency of updates: in addition to the once a day update, whenever w_i units of good i have been sold since the last update, price p_i is updated.

In addition, we will need the E' bound on the elasticity of wealth.

[6] i.e. if the money of each agent and prices all increase by a multiplicative factor $\gamma > 0$ this leaves demands unchanged.

[7] It is usual to define the elasticity of wealth for each individual separately. It is not hard to see that if each individual has wealth elasticity E^w then this implies the E^w bound in our definition.

We significantly modify our basic potential function for this analysis, as will be seen when the analysis is carried out. Here the relationship between S and ϕ is a bit looser: $S = O(\phi) = O(S+M)$, where M is the daily supply of money.

Theorem 5. *If Constraint 1 holds, $d = 5$, $\alpha_2 = \frac{3}{2}$, $\lambda(E+E') \leq \frac{1}{17}$, $\alpha_1 \leq \frac{1}{16}$, $\lambda\alpha_1 + \frac{4}{3}\lambda\left(\frac{7}{4} + \frac{10E}{1-\lambda E}\right) \leq 1$, $\kappa \leq \frac{\lambda\alpha_1}{13}$, and each price is updated at least once every day, then ϕ decreases by at least a $1 - \frac{1}{4}\kappa$ factor daily.*

Again, we could prove a $1 - \theta(\lambda\alpha_1)$ decrease when $\phi = \Omega(\sum_i w_i p_i)$.

2.7 Bounds on Warehouse Sizes

For simplicity, we will assume that c_i/w_i is the same for all i. Also, for simplicity, we suppose that $s_i^* = \frac{1}{2}c_i$, that is the target fullness for each warehouse is half full.

We view each warehouse as having 8 equal sized zones of fullness, with the goal being to bring the warehouse into its central four zones. The role of the outer zones is to provide the buffer to cope with initial price imbalances.

Definition 9. *The four zones above the half way target are called the* high *zones, and the other four are the* low *zones. Going from he center outward, the zones are called the* safe *zone, the* inner buffer*, the* middle buffer*, and the* outer buffer*.*

Theorem 6. *Suppose that the prices are always f-bounded and let $d = d(f)$. Also suppose that each price is updated at least once a day. Suppose further that the warehouses are initially all in their safe or inner buffer zones. Finally, suppose that $\lambda\left(1 + \frac{1}{\alpha_4}\right) \leq \frac{1}{2}$, where $\frac{\alpha_4}{=} \frac{\kappa c_i}{8w_i}$. Then the warehouse stocks never go outside their outer buffers (i.e. they never overflow or run out of stock) if $\frac{\alpha_4}{\kappa} = \frac{c_i}{8w_i} \geq \max\left\{(d-1)D, 2\left(1 + \frac{4}{\alpha_4}\right)f + \frac{8\lambda}{\alpha_4}\right\}$; furthermore, after $D + 2\left(1 + \frac{4}{\alpha_4}\right)f + \frac{8\lambda}{\alpha_4} + \frac{8}{\kappa}$ days the warehouses will be in their safe or inner buffer zones thereafter, where*

$$D = \frac{16(1+\alpha_2)}{\lambda\alpha_1} \log \frac{\phi_{init}}{\frac{1-\lambda\alpha_1}{2} \min_i w_i p_i^*},$$

and ϕ_{init} is the initial value of ϕ.

If the fast updates rule is followed, then it suffices to have $\frac{\alpha_4}{\kappa} = \frac{c_i}{8w_i} \geq 2\left(1 + \frac{4}{\alpha_4}\right)f + \frac{8\lambda}{\alpha_4}$, and then after $\left(1 + \frac{4}{\alpha_4}\right)f + \frac{8\lambda}{\alpha_4}\lambda + \frac{8}{\kappa}$ days the warehouses will be in their safe or inner buffer zones thereafter.

2.8 The Effect of Inaccuracy

Next, we investigate the robustness of the tatonnement process with respect to inaccuracy in the demand data. To mitigate the intricacy of the analysis, we return to the setting without fast updates.

Specifically, we assume that there may be an error of up to ρw_i in the reported values of $s_i(t)$ and $s_i(\tau_i)$, where $\rho > 0$ is a constant parameter. Recall that these are the values which are used to calculate $\overline{z}_i$ $(= \frac{s_i(\tau_i) - s_i(t)}{t - \tau_i} + \kappa_i(s_i(t) - s_i^*))$. Let $\overline{z}_i^c$ denote the correct value for $\overline{z}_i$, and $\overline{z}_i^r$ the reported value.

To enable us to control the effect of erroneous updates, we will place a lower bound on the frequency of updates to a given price. Specifically, successive updates are at most 1 day apart (as before), and at least $1/b$ days apart, where $b \geq 1$ is a parameter.

We consider two scenarios:

(i) The parameter ρ is not known to the price-setters, who then perform updates as before. We show that for $\phi \geq \rho b^2 \frac{\lambda}{\kappa} EM$, ϕ reduces by a $(1 - \Theta(\kappa))$ factor daily.

(ii) The parameter ρ is known to the price-setter for each good i, who performs an update only if the possible error is at most half of the reported value $\overline{z}_i^r$, i.e. if $|\overline{z}_i^r - \overline{z}_i^c| \leq \frac{1}{2}\overline{z}_i^r$. Then, we show that for $\phi \geq \rho b M$, ϕ reduces by a $(1 - \Theta(\kappa))$ factor daily.

It may seem more appealing to allow multiplicative errors of up to $1 \pm \rho$ in the reported s_i. However, this seems a little unreasonable in the case that the actual $s_i(t) - s_i(\tau)$ is relatively small. Also, later we will consider a scenario in which more frequent updates are required when $s_i(t)$ changes rapidly, and then a multiplicative rule for the error in the *change* to the warehouse stock would give an error no larger than the additive rule.

Contrariwise, one might argue that if the warehouse stock is changing only slightly, then the perceived error ought to be small. But once one considers that there is a daily supply of w_i units of the ith good, and that possibly the error reflects fluctuations in the selling of these w_i units, an error of $\pm\rho w_i$ seems reasonable.

Our analysis for Case (i) uses the potential function $\phi = \sum_i \phi_i$, from Section 2.4.

Theorem 7. *If $x_i \leq d\widetilde{w}_i$ for and i and Constraint 1 hold, $\frac{\alpha_2}{2} + \alpha_1 \max\{\frac{3}{2}, (d-1)\} \leq 1$, $\lambda\alpha_1 + \frac{4}{3}\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \leq 1$, $\frac{\kappa(\alpha_2 - 1)}{2} \leq 1$, $4\kappa(1+\alpha_2) \leq \lambda\alpha_1 \leq \frac{1}{2}$, and each price is updated at least once every day, and at most every $1/b$ days, and if $\phi \geq \frac{16\mu M}{\kappa(\alpha_2-1)} \frac{1-\lambda\alpha_1}{1-\lambda\alpha_1-\mu}$ at the start of the day, then ϕ decreases by at least a $1 - \frac{\kappa(\alpha_2-1)}{8}$ factor by the end of the day, where M is the daily supply of money and $\mu = \frac{4}{3}\lambda\rho b(2b+\kappa)\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right)$, supposing that $\kappa(\alpha_2 - 1) \geq 16\mu/[1 - \lambda\alpha_1 - \mu]$.*

Case (ii) yields a less stringent constraint on ρ. We use a slightly different potential for which again $\phi(t) \leq S(t) = O(\phi(t-1)$.

Theorem 8. *If $x_i \leq d\widetilde{w}_i$ for and i and Constraint 1 hold, $\frac{\alpha_2}{2} + \alpha_1 \max\{3, 2(d-1)\} \leq 1$, $\lambda\alpha_1 + 2\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \leq 1$, $4\kappa(1+\alpha_2) \leq \lambda\alpha_1 \leq \frac{1}{2}$ for all i, $\frac{\kappa(\alpha_2-1)}{2} \leq 1$, $\mu\left[\frac{1+\mu/(1-\lambda\alpha_1)}{1-\mu/(1-\lambda\alpha_1)} + \frac{1}{(1-\lambda\alpha_1)}\right] \leq \frac{\kappa(\alpha_2-1)}{2}$, if $\phi \geq \frac{32\mu M}{[1-\frac{\mu}{1-\lambda\alpha_1}][(\kappa(\alpha_2-1)]}$ at the start of the day, where $\mu = 8\kappa(1+\alpha_2)(2b+\kappa)\rho$, and if each price is updated at least once every day, then ϕ decreases by at least a $1 - \frac{\kappa(\alpha_2-1)}{8}$ factor daily.*

Remark. For both update rules in this section, to obtain bounds on the needed warehouse sizes, we need to use the fast update variant; this is left to the interested reader.

2.9 Discrete Goods and Prices

Now we investigate the effect of only allowing integer-valued prices and finitely divisible goods: this is implemented by requiring each w_i to be an integer and goods to be sold in integral quantities.

Demands as a Rate. With limited divisibility, we need to look again at our interpretation of demands as a rate.

$x_i(\mathbf{p})$, the *daily demand* for good i at prices $\mathbf{p}$, is simply the demand were all the prices to remain unchanged over the course of a day. The *ideal demand* for good i over time interval $[t_1, t_2]$ with unchanged prices is defined to be $x_i(p)(t_2 - t_1)$. Ideal demand $x_2^I(t_1, t_2)$ for good i over time interval $[t_1, t_2]$ with possibly varying prices is given by $\int_{t_1}^{t_2} x_i(\mathbf{p})dt$. Note that this is in fact a sum as there are only finitely many price changes.

The *actual demand* $x_i^A(t_1, t_2)$ over the time interval $[t_1, t_2]$ is the supply minus the growth in the warehouse stock: $w_i(t_2 - t_1) - [s_i(t_2) - s_i(t_1)]$.

In order to achieve approximately uniform demand as a function of prices, we require that $|x_i^A(t_1, t_2) - x_i^I(t_1, t_2)| < 1$, for all times t_1, t_2 at which price p_i is considered for an update (i.e. both actual and null updates).

We will also define ideal warehouse contents. The ideal content of warehouse i, s_i^I is simply the contents of the warehouse had x_i^I been the demand throughout. Note that $|s_i^A - s_i^I| = |x_i^A - x_i^I| < 1$. $\widetilde{w}_i^I$ is defined in terms of s_i^I: $\widetilde{w}_i^I = w_i - \kappa(s_i^I - s_i^*)$; $\widetilde{w}_i^A$ can be defined analogously. Note that $|\widetilde{w}_i^I - \widetilde{w}_i^A| < \kappa$. We also define $\overline{x}_i^A = x_i - \widetilde{w}_i^A$ and $\overline{x}_i^I = x_i - \widetilde{w}_i^I$. The computation of price updates uses $\overline{x}_i^A$.

Discrete WGS. We need to redefine WGS and the bounds on the rate of change in demand w.r.t. prices so that we can carry out an analysis similar to that for the divisible case.

In the Fisher market context it is not hard to see that WGS imposes the same constraints on the spending and the demand for each good. This means that in the discrete setting under WGS, if the price p_i increases by one unit, then as the spending does not increase, the demand for good i must drop, so if the demand at price p_i is x_i, at price $p_i + x_i$ it must be zero. This seems unnatural. This impression is reinforced by considering what happens were half units of money to be introduced, with WGS remaining in place. Then at price $p_i + x_i/2$ the demand would have to be 0. This suggests that the property ought to be modified in the discrete setting.

Accordingly, we define a market to satisfy the *Discrete WGS property* if, for any good i, reducing its price p_i to $p_i - \Delta$ only reduces demand for all other non-money goods, and the spending on good i is now at least $p_i x_i(p_i) - [(p_i - \Delta) - 1]$, i.e. the spending, if reduced, is reduced by less than the cost of one item. Note that it need not be that all the money is spent (for there may be left over money which is insufficient to buy one item of any good).

Elasticity of Demand and the Parameter E. We define the following bounded analog for discrete markets. Suppose that the prices of all goods other than good i is set to p_{-i}. Then, for all $l_i \le p_i \le q_i$,

$$\left\lfloor x_i(l_i, p_{-i}) \left(\frac{l_i}{p_i}\right)^E \right\rfloor \le x_i(p) \le \left\lceil x_i(q_i, p_{-i}) \left(\frac{q_i}{p_i}\right)^E \right\rceil.$$

The crucial observation is that there is a fully divisible market with elasticity bound $2E$ that has demands y_i very similar to those for the discrete market: for every price vector p which induces non-zero demand for every good, for all i, $x_i(p) - 1 < y_i(p) \le x_i(p)$. Given this correspondence, the analysis of the discrete case becomes similar to that for inaccurate data.

Indivisibility Parameters. We measure the indivisibility of the market in terms of two parameters, r and s. $r = M/\sum_i w_i$, where M is the daily supply of money; it provides an upper bound on the weighted average price for an item at equilibrium. This can be thought of as the granularity of money at the equilibrium. $s = \min_i w_i$ is the minimum size for the daily supply for any item, and thus indicates the granularity of the least divisible good.

Next, we restate Constraint 1 and Lemma 1, replacing $\widetilde{w}_i$ with $\widetilde{w}_i^I$.

Constraint 2. $|\widetilde{w}_i^I - w_i| \leq \frac{1}{3} w_i$.

Again, if ϕ is large enough that the following theorem guarantees it decreases, then $\phi = O(S)$.

Theorem 9. *If Constraint 2 holds and* $y_i \leq d\widetilde{w}_i$ *for all* i, $\frac{\alpha_2}{2} + \alpha_1 \max\{\frac{9}{2}, 2(d-1)\} \leq 1$, $\lambda\alpha_1 + \frac{8}{3}\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \leq 1$, *each* $w_i \geq 6$, $4\kappa(1+\alpha_2) \leq \lambda\alpha_1 \leq \frac{1}{2}$ *for all* i,
$s \geq \frac{48}{(\alpha_2-1)(1-\lambda\alpha_1)}\left[1 + 6(1+\alpha_2) + (1+\alpha_2)\frac{1-\lambda\alpha_1+\frac{18}{s}\kappa(1+\alpha_2)+\frac{3\kappa}{s}}{1-\lambda\alpha_1-\frac{18}{s}\kappa(1+\alpha_2)}\right]$,
if $\phi(\tau) \geq \frac{48}{(\alpha_2-1)}\left[(1+\alpha_2)\left(\frac{4}{\lambda r} + \frac{24}{s}\right) + \frac{1}{s}\right]\frac{1-\lambda\alpha_1}{1-\lambda\alpha_1-\frac{18}{s}\kappa(1+\alpha_2)} M$ *at the start of the day, and if each price is updated at least once every day, then* ϕ *decreases by at least a* $1 - \frac{\kappa(\alpha_2-1)}{8}$ *factor over the course of the day.*

Remark. Again, to obtain bounds on the needed warehouse sizes, we need to use the fast update rule.

Theorem 10. *In the discrete setting there are markets with* $\Omega(E/r)$ *misspending at any pricing.*

2.10 Extensions

We briefly examine how the analysis can be applied to some divisible markets in which the WGS constraint is relaxed, and also indicate a possible extension to a class of markets that interpolate between the pure Fisher market and an Exchange market.

3 The Analysis

We now prove the above results. For readability, we repeat the statements of lemmas and theorems.

From a technical perspective, the novely in this analysis lies in the approach for coping with asynchrony. The idea is to have a potential which decreases either at or faster than the desired rate of improvement, and whenever an event occurs (a price update), ensure that the potential either stays the same or decreases further. Of course, the "real" decreases are due to the updates, but because of the interplay between the different prices and demands, it proves easier to show the desired rate of progress using our approach.

The most challenging of these analyses is the one used to handle the fast updates. Because the fast updates may introduce some temporarily "bad" events (events that would increase the potential) these are deferred for the purposes of the analysis. Tracking the differences between the real market and the market with deferred events is a delicate matter which has to be done with considerable accuracy. The resulting potential function is fairly elaborate.

We now proceed to prove the results in the order they were introduced in the previous section.

3.1 A Single Good

Let x denote the demand for the good and p its price. We assume that $x(p)$ is a strictly decreasing function. Let w denote the supply of the good. Our goal is to update p repeatedly so as to cause x to converge toward w. We will use the following update rule:

$$p' = p\left(1 + \lambda \min\left\{1, \frac{x-w}{w}\right\}\right)$$

where $0 < \lambda \leq \frac{1}{2}$ is a suitable fixed parameter.

We are going to argue that this update function is appropriate for demands x satisfying:

$$\frac{x(p)}{p} \leq -\frac{dx}{dp} \leq E\frac{x(p)}{p}, \quad \text{where } 1 \leq E \leq \frac{1}{2\lambda}. \tag{5}$$

We will also explain why the update function can be viewed as an approximation of Newton's method, but an approximation quite unlike the standard secant method.

The following fact will be used repeatedly in our analysis.

Fact 1. *(a) If $\delta \geq -1$ and either $a \leq 0$, or $a \geq 1$, then $(1+\delta)^a \geq 1 + a\delta$.*
(b) If $0 > \delta \geq -\frac{1}{2}$ and $0 < a < 1$, then $(1+\delta)^a \geq 1 + 2a\delta$.
(c) If $\delta \geq -1$ and $0 \leq a \leq 1$, then $(1+\delta)^a \leq 1 + a\delta$, and the inequality is strict if $a \neq 0, 1$.
(d) If $-1 < \rho \leq \delta \leq 0$ and $-1 \leq a \leq 0$, or if $a \leq -1$ and $0 \leq a\delta \leq \rho < 1$, then $(1+\delta)^a \leq 1 + a\delta/(1-\rho)$.
(e) If $0 \leq \delta \leq \rho < 1$ and $-1 \leq a \leq 0$, then $(1+\delta)^a \leq 1 + a\delta(1-\rho)$.

Proof. We prove (a)–(c), using a simplified version of Taylor's Theorem:

Theorem 11 (Taylor)**.** *If f is a twice differentiable function in the interval $[0,x]$ (or $[x,0]$, if $x < 0$) then there is a $\xi \in [0,x]$ (or $\xi \in [x,0]$, if $x < 0$) such that*

$$f(x) = f(0) + f'(0) * x + \frac{f''(\xi)}{2}x^2.$$

□

Let $f(x) = (1+x)^a$. Then $f'(x) = a(1+x)^{a-1}$, $f''(x) = a(a-1)(1+x)^{a-2}$, $f(0) = 1$, and $f'(0) = a$. Thus we have that

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2}(1+\xi)^{a-2}x^2.$$

If the last term is nonnegative, then we have that $(1+x)^a \geq 1 + ax$. The last term is nonnegative provided that $x \geq -1$ (implying $\xi \geq -1$) and either $a \geq 1$ or $a \leq 0$. This is (a).

If the last term is nonpositive, then we have that $(1+x)^a \leq 1+ax$. The last term is nonpositive provided that $x \geq -1$ and $0 \leq a \leq 1$, and strictly negative if $0 < a < 1$. This is (c).

(b) holds if the last term is at least ax. This is true since $ax < 0$ and $0 < \frac{(a-1)}{2}(1+\xi)^{a-2}x < 1$ when $0 \geq \xi \geq x \geq -\frac{1}{2}$ and $0 < a < 1$.

Parts (d) and (e) are obtained by bounding the limit of the infinite Taylor series for $(1+\delta)^a$. Namely,

$$(1+\delta)^a = \sum_{i \geq 0} \delta^i \frac{a(a-1)(a-2)\cdots(a-i+1)}{i!}.$$

If $-1 \leq a \leq 0$ and $\delta \leq 0$, then, for $i \geq 0$, $(a-i)/(i+1) \geq -1$, and so $\delta(a-i)/(i+1) \leq -\delta$; in this case the sum is bounded by $1 + \sum_{i \geq 1} a\delta(-\delta)^{i-1} \leq 1 + a\delta/(1-\rho)$, as claimed for the first result in (d).
If $a \leq -1$ and $0 \leq a\delta < 1$, then $(a-i)/(i+1) \geq a$, and so $\delta(a-i)/(i+1) \leq a\delta$; in this case the sum is bounded by $1 + \sum_{i \geq 1}(a\delta)^i \leq 1 + a\delta/(1-\rho)$, as claimed for the second result in (d).
If $-1 \leq a \leq 0$ and $0 \leq \delta < 1$, then, for $i \geq 0$, $0 \geq (a-i)/(i+1) \geq -1$; in this case the sum is

a sequence of alternating terms, each successive term being smaller in magnitude by at least a δ factor. Consequently, the sum is bounded by $1 + a\delta - a\delta^2 \leq 1 + a\delta(1-\rho)$ (recall that $a \leq 0$), as claimed for the result in (e).

□

Lemma 5. *For* $\Delta > 0$, $\frac{x(p)}{(1+\Delta)^E} \leq x(p(1+\Delta)) \leq \frac{x(p)}{(1+\Delta)}$ *and* $\frac{x(p)}{(1-\Delta)} \leq x(p(1-\Delta)) \leq \frac{x(p)}{(1-\Delta)^E}$.

Proof. If $\frac{dx}{dp} = -Ex/p$ for all p, then $\frac{d(p^E x)}{dp} = 0$ for all p, and then $p^E \cdot x(p) = p^E (1+\Delta)^E \cdot x(p(1+\Delta))$, for all Δ (positive or negative). Similarly, if $\frac{dx}{dp} = -x/p$ for all p, then $\frac{d(px)}{dp} = 0$ for all p, and then $p \cdot x(p) = p(1+\Delta) \cdot x(p(1+\Delta))$. These cases provide the extreme bounds on the growth of x from which the claimed bounds follow. □

Corollary 1. $f(p) = p \cdot x(p)$ *is a non-increasing function of* p.

Proof. $p(1+\Delta) \cdot x(p(1+\Delta)) \leq p(1+\Delta) \cdot x(p)/(1+\Delta) = p \cdot x(p)$. □

Lemma 1. *Suppose that* $-p\frac{dx}{dp} \leq Ex(p)$, *for some* $E \geq 1$. *Then the update* $p' = p(1+\frac{1}{2E}\min\{1, \frac{x-w}{w}\})$ *is on the* same side *as* p, *but the update* $p' = p(1 + \frac{1}{E}\frac{x-w}{w})$ *need not be.*

Proof. If $p' = p(1+\Delta)$ is not on the same side as p for $\Delta > 0$, then $x(p) > w > x(p(1+\Delta))$. By Lemma 5, this implies that $w > x(p)(1+\Delta)^{-E}$; equivalently, $x(p) < w(1+\Delta)^E \leq w(1+2E\Delta)$ if $E\Delta \leq \frac{1}{2}$ (applying Fact 1(d) with $\rho = \frac{1}{2}$); thus either $E\Delta \geq \frac{1}{2}$ or $w(1+2E\Delta) > x(p)$. It follows that $\Delta \geq \frac{1}{2E}\min\{\frac{x-w}{w}, 1\}$.

Next, we show that replacing the $1/(2E)$ parameter by $1/E$ can result in updates which are not on the same side. Consider the case that $\frac{dx}{dp} = -E\frac{x}{p}$ for all p; then, as noted in the proof of Lemma 5, $p^E \cdot x(p)$ is constant. Let $p(w)$ denote the price at which the demand is w. In this case $p(w) = p \cdot (\frac{x}{w})^{1/E} = p \cdot (1 + \frac{x-w}{w})^{1/E} < p \cdot (1 + \frac{1}{E}\frac{x-w}{w})$ if $1/E < 1$ (applying Fact 1(c)). Now suppose that the update rule being used is $p' = p(1 + \frac{1}{E}\frac{x-w}{w})$. Then $p' > p(w)$ and this would not be a same side update.

Finally, suppose that $p' = p(1-\Delta)$ is not on the same side as p for $\Delta > 0$; then $x(p) < w < x(p(1+\Delta))$. By Lemma 5, this implies that $w < x(p)(1+\Delta)^{-E}$; equivalently, $x(p) > w(1-\Delta)^E \geq w(1-E\Delta)$ as $|\Delta| \leq 1$ (applying Fact 1(a)); it follows that $\Delta > -\frac{1}{E}\frac{x-w}{w}$. □

Thus, if we want same-side updates, and we are using the rule $p' = p(1 + \lambda\frac{x-w}{w})$ when $\frac{x-w}{w} \leq 1$ (i.e. $x \leq 2w$) then we are obliged to have a derivative bound more or less of the form given in Equation 2.

For $x > 2w$, the update $p' = p(1+\lambda)$ is not the largest possible same-side update, but it is simple which is why we use it.

Another way to view the update rule is to see it as an approximate form of Newton's method:

$$p' = p - \frac{x(p) - w}{dx/dp}.$$

Instead of dx/dp (which is not available in our setting) we use a lower bound on its value (recall that dx/dp is negative by assumption), for this only reduces the change to p. The lower bound is $-Ex(p)/p$. This yields

$$p' = p + \frac{p}{E}\frac{x(p) - w}{x(p)}.$$

When $x(p) < w$, replacing $x(p)$ by w in the denominator only decreases the lower bound on dx/dp further, and yields the update rule $p' = p + \frac{p}{E}\frac{x(p)-w}{w}$. Note that our update rule achieves half this step size when $\lambda = 1/(2E)$.
When $x(p) > w$ we have two cases:
$x(p) \leq 2w$: then replacing $x(p)$ by $2w$ in the denominator only decreases the lower bound on dx/dp, while our update rule achieves this step size when $\lambda = 1/(2E)$.
$x(p) \geq 2w$: then replacing $x(p)$ by $2(x(p) - w)$ only decreases the lower bound, and once again our update rule achieves this step size when $\lambda = 1/(2E)$.

Next we investigate the rate of convergence. Recall that we measure the "distance" from equilibrium by the potential function $\phi(x, p) = |x - w|p$.

Lemma 6. *ϕ increases as p diverges from p^* (in either direction).*

Proof. By Corollary 1, $p \cdot x(p)$ is a non-increasing function of p. For $p > p^*$, $\phi = pw - px(p)$; pw is strictly increasing, and $-px(p)$ is non-decreasing; thus in this case ϕ is strictly increasing. For $p < p^*$, $\phi = px(p) - pw$; as a function of $-p$, $px(p)$ is non-decreasing, and $-pw$ is strictly increasing; thus ϕ is a strictly increasing function of $-p$ in this case. Together, these observations show that ϕ is strictly increasing as p diverges from p^*. □

Notation. ϕ_I denotes the initial value of ϕ and ϕ_F a target final upper bound for ϕ.

Theorem 1. *If $x \leq 2w$ initially, then in one interation ϕ reduces by at least $w|p' - p| = \lambda\phi$, where p is the price before the update and p' the price after the update. Hence in $O(\frac{1}{\lambda}\log\frac{\phi_I}{\phi_F})$ iterations, ϕ reduces from ϕ_I to at most ϕ_F. If $x \geq 2w$ initially, then in one interation ϕ reduces by at least $w|p' - p| = \lambda\frac{w}{x-w}\phi$. Hence in $O(\frac{1}{\lambda}\frac{x-w}{w}\log\frac{\phi_I}{\phi_F})$ iterations, ϕ reduces from ϕ_I to at most ϕ_F.*

Proof. Case 1: $x < w$.
Let $p' = p(1 - \Delta)$, where $\Delta = -\lambda\min\{1, \frac{x-w}{w}\} = -\lambda\frac{x-w}{w} > 0$.

Then, as the price update is same-sided, the value of the potential after the price is updated is given by $\phi' = p(1 - \Delta)[w - x(p(1 - \Delta))]$. This is bounded as follows:

$$\begin{aligned} p(1-\Delta)[w - x(p(1-\Delta))] &\leq p(1-\Delta)w - p \cdot x(p) \\ &\qquad \text{as } p \cdot x(p) \leq p(1-\Delta) \cdot x(p(1-\Delta)) \text{ by Corollary 1} \\ &\leq p(w - x) - \Delta wp. \end{aligned}$$

Thus $\phi - \phi' \geq \Delta wp = w \cdot \lambda p\frac{w-x}{w} = \lambda\phi$. Also note that $\Delta wp = (p - p')w = w|p - p'|$.
Case 2: $x > w$.
Now, let $p' = p(1 + \Delta)$, where $\Delta = \lambda\min\{1, \frac{x-w}{w}\} > 0$.

Again, as the price update is same-sided, the value of the potential after the price is updated is given by $\phi' = p(1 + \Delta)[x(p(1 + \Delta)) - w]$. This is bounded as follows:

$$\begin{aligned} p(1+\Delta)[x(p(1+\Delta)) - w] &\leq p \cdot x(p) - p(1+\Delta)w \\ &\qquad \text{as } p \cdot x(p) \geq p(1+\Delta) \cdot x(p(1+\Delta)) \text{ by Corollary 1} \\ &\leq p(x - w) - p\Delta w. \end{aligned}$$

Thus $\phi - \phi' \geq \Delta wp = w(p' - p) = w|p - p'|$.
Case 2.1: $x \leq 2w$.

Here $w\Delta p = w \cdot \lambda p \frac{x-w}{w} = \lambda\phi$.

Case 2.2: $x \geq 2w$.

Here $w\Delta p = w \cdot \lambda p = \frac{w}{x-w}\lambda p(x-w) = \lambda \frac{w}{x-w}\phi$.

□

Remark. Changes of variable (and possibly consequent changes to the update rule) can allow various other potential functions to be modified so as to have the form $|x-w|p$.
e.g. 1. $\phi' = |\tilde{x}^\alpha - \tilde{w}^\alpha|\tilde{p}^\beta$, with $\alpha, \beta > 0$; setting $x = \tilde{x}^\alpha$, $w = \tilde{w}^\alpha$, $p = \tilde{p}^\beta$, yields $\phi' = \phi = |x-w|p$.
2. $\phi' = \max\{\frac{\tilde{x}}{\tilde{w}}, \frac{\tilde{w}}{\tilde{x}}\}^{\tilde{p}}$; setting $p = \tilde{p}$, $x = \log \tilde{x}$, $w = \log \tilde{w}$, yields $\log \phi' = \phi = |x-w|p$. The formulation using ϕ seems more natural, since $\phi(w,w,p) = 0$ while $\phi'(w,w,p) = 1$.
3. $\phi' = (\tilde{x}-\tilde{w})^2\tilde{p}$; setting $\phi'' = \sqrt{\phi'} = |\tilde{x}-\tilde{w}|\sqrt{\tilde{p}}$ yields an instance of example 1.

3.2 Multiple Goods

Notation.
1. Let $\Delta_i p_i = p'_i - p_i$ denote the change to p_i due to its update.
2. Let $\Delta_i x_j = x'_j - x_j$ denote the change to x_j due to the update to p_i. (Recall that x'_j denotes $x_j(p_{-i}, p'_i)$.)
3. Let I_i denote $w_i|\Delta_i p_i|$.
4. Let (p_{-i}, p'_i) denote the price vector p with the ith price replaced by p'_i.
5. For all j, let $\Gamma_i\phi_j = \phi_j(p_{-i}, p_i) - \phi_j(p_{-i}, p'_i)$ denote the reduction in ϕ_j when p_i is updated. This may be negative for $j \neq i$. In addition, let $\Gamma_i\phi = \sum_j \Gamma_i\phi_j$; this is the reduction to ϕ when p_i is updated.

By Lemma 1, $\Gamma_i\phi_i$ is at least $I_i = w_i|\Delta_i p_i| = \lambda p_i w_i \min\{\frac{|x_i - w_i|}{w_i}, 1\} = \lambda\phi_i \cdot \min\{1, \frac{w_i}{|x_i-w_i|}\}$.

To achieve a similar rate of progress as in the 1-d case, as we will see, it will suffice to assume that Property 1 below holds.

Definition 10. *The update p'_i is said to be* toward w_i *if* $|x_i(p_{-i}, p'_i) - w_i| < |x_i(p_{-i}, p_i) - w_i|$.

Property 1. *When p'_i is a same-side update toward w_i, $\sum_{j\neq i}|\Gamma_i\phi_j| + \alpha I_i \leq \Gamma_i\phi_i$, for some α, $0 < \alpha \leq 1$.*

The term on the RHS is the reduction in ϕ_i. The sum on the LHS is the the sum of the changes to the ϕ_j, $j \neq i$, which are assumed, in a worst-case way, to all be increases (which is possible with certain w_j, namely $w_j > x_j > x'_j$ or $w_j < x_j < x'_j$). The second term on the LHS is the desired reduction in ϕ_i. In the 1-d case the reduction was by at least I_i; here we relax this to provide more flexibility.

Lemma 7. *If Property 1 holds, then when p'_i is given by a same-side toward w_i update, $\Gamma_i\phi \geq \alpha I_i$.*

Proof. $\Gamma_i\phi = \sum_j \Gamma_i\phi_j \geq \Gamma_i\phi_i - \sum_{j\neq i}|\Gamma_i\phi_j| \geq \alpha I_i$. □

Next, we show that the standard assumption of weak gross substitutes (WGS) in Fisher markets implies Property 1 with $\alpha = 1$. This claim is shown in Lemma 9, below.

Definition 11. *Let price p_i change to $p_i + \Delta_i p_i$. The corresponding* spending neutral *change to demand for good i, $\Delta_i^N x_i$, is given by:*

$$p_i x_i = (p_i + \Delta_i p_i)(x_i + \Delta_i^N x_i). \quad (6)$$

In the following lemma we show that given a same side update toward w_i, the corresponding spending neutral change to x_i decreases ϕ_i by $w_i|\Delta_i p_i| = I_i$.

Lemma 8. *1. If $x_i > x_i + \Delta_i^N x_i \geq w_i$, then $\Delta_i p_i > 0$ and*

$$p_i(x_i - w_i) - (p_i + \Delta_i p_i)(x_i + \Delta_i^N x_i - w_i) = w_i \Delta_i p_i = I_i.$$

2. If $x_i < x_i + \Delta_i^N x_i \leq w_i$, then $\Delta_i p_i < 0$ and

$$p_i(w_i - x_i) - (p_i + \Delta_i p_i)(w_i - x_i - \Delta_i^N x_i) = -w_i \Delta_i p_i = w_i|\Delta_i p_i| = I_i.$$

Proof. We show the first claim. By (6),

$$p_i(x_i - w_i) - (p_i + \Delta_i p_i)(x_i + \Delta_i^N x_i - w_i) = -p_i w_i + (p_i + \Delta_i p_i) w_i = w_i \Delta_i p_i.$$

Note that $\Delta_i p_i > 0$ as $\Delta_i^N x_i < 0$.

The proof of the second claim is similar. □

Lemma 9. *If all demands obey WGS, then Property 1 holds with $\alpha = 1$.*

Proof. We begin by showing that $\sum_{j \neq i} \Gamma_i \phi_j = \sum_{j \neq i} p_j \Delta_i x_j = p_i x_i - (p_i + \Delta p_i)(x_i + \Delta_i x_i)$. This follows from $\sum_j p_j x_j = (p_i + \Delta p_i)(x_i + \Delta_i x_i) + \Sigma_{j \neq i} p_j (x_j + \Delta_i x_j)$, which holds because when p_i is updated, the combined total spending on all the goods is unchanged.

Next we show that $\Gamma_i \phi_i = I_i + \sum_{j \neq i} p_j |\Delta_i x_j|$.

For the case $x_i > w_i$, we have:

$$\begin{aligned} \Gamma_i \phi_i &= \phi_i(p_{-i}, p_i) - \phi_i(p_{-i}, p_i') \\ &= p_i(x_i - w_i) - (p_i + \Delta_i p_i)(x_i + \Delta_i x_i - w_i) \\ &\qquad \text{(recall that a same-side update is being applied to } p_i\text{)} \\ &= w_i \Delta_i p_i + p_i x_i - (p_i + \Delta p_i)(x_i + \Delta_i x_i) \\ &= I_i + \sum_{j \neq i} p_j \Delta_i x_j \quad \text{ as } \Delta_i p_i > 0. \end{aligned}$$

And $\Delta_i x_j \geq 0$ for $j \neq i$, as there is a price increase applied to p_i, which by WGS only increases the demand for good j, $j \neq i$. So $\Gamma_i \phi_i = I_i + \sum_{j \neq i} p_j |\Delta_i x_j|$ in this case.

For the case $x_i < w_i$, $\phi_i(p_{-i}, p_i) = p_i(w_i - x_i)$ and $\phi_i(p_{-i}, p_i') = (p_i + \Delta_i p_i)(w_i - x_i - \Delta_i)$. Running through the same algebra yields $\Gamma_i \phi_i = -w_i \Delta_i p_i - \sum_{j \neq i} p_j \Delta_i x_j = I_i + \sum_{j \neq i} p_j |\Delta_i x_j|$, for in this case there is a price decrease applied to p_i, which implies that $\Delta_i x_j \leq 0$ for $j \neq i$.

Finally, for $j \neq i$, we observe that:

$$\begin{aligned} |p_j \Delta_i x_j| &= |p_j(x_j + \Delta_i x_j - w_j) - p_j(x_j - w_j)| \\ &= \begin{cases} |\phi_j(p_{-i}, p_i') - \phi_j(p_{-i}, p_i)| & \text{if } \mathrm{sign}(x_j - w_j) = \mathrm{sign}(x_j + \Delta_i x_j - w_j) \\ |\phi_j(p_{-i}, p_i') + \phi_j(p_{-i}, p_i)| & \text{otherwise} \end{cases} \\ &\geq |\phi_j(p_{-i}, p_i') - \phi_j(p_{-i}, p_i)| = |\Gamma_i \phi_j|. \end{aligned}$$

Hence $\Gamma_i \phi_i \geq \sum_{j \neq i} |\Gamma_i \phi_j| + I_i$.

□

This will not suffice when we seek to analyze multiple asynchronous updates to different prices. The difficulty can be understood by considering the special case where all prices are updated synchronously. One way to analyze this would be to apply the updates one by one in turn. The difficulty we face is that it seems unreasonable to assume that each successive update is based on x_j values updated to take account of the previous price updates.

Instead, we will show that if we apply the updates one at a time using the price updates given by the original x_i values, then there is an ordering of the goods so that when the update to p_i occurs, $\text{sign}(x_i - w_i)$ has not yet changed, nor does the update change it (though, possibly, it makes $x_i = w_i$). Consequently, by Lemmas 7 and 9, the reduction to the potential ϕ due to the update to p_i (based on the value of x_i prior to any of the updates) is at least I_i. Hence the reduction to ϕ is at least $\sum_i I_i$.

To achieve this we will need $\lambda(2E-1) \leq \frac{1}{2}$. Note that this condition implies $\lambda E \leq \frac{1}{2}$ as $E \geq 1$.

The discussion is simplified by introducing the following function ψ_i.

Definition 12. *$\psi_i = p_i(w_i - x_i)$, if $x_i < w_i$ prior to the price updates; while if $x_i > w_i$ prior to the price updates, $\psi_i = p_i(x_i - w_i)$. We also define $\Gamma_i\psi_i$, analogously to $\Gamma_i\phi_i$; it denotes the decrease to ψ_i due to the update to p_i. More precisely, WLOG, suppose that the price updates are applied in the order $p_1, p_2, \cdots, p_n$. $\Gamma_i\psi_i = \psi_i(p'_1, \cdots, p'_{i-1}, p_i, \cdots, p_n) - \psi_i(p'_1, \cdots, p'_i, p_{i+1}, \cdots, p_n)$.*

Lemma 10. *Suppose that the price updates are performed sequentially in some order. Consider a point when the update to p_i is about to occur. Let ϕ_i^I be the value of ϕ_i prior to any update, and let ψ'_i be the current value of ψ_i. Suppose further that despite price updates to the other goods, $\psi'_i \geq \phi_i^I/2$. Then, assuming that $\lambda(2E-1) \leq \frac{1}{2}$, p_i's update (based on x_i's original value) is toward w_i and same-sided, and it reduces $\psi_i = \phi_i$ as follows: if $\psi'_i \leq \phi_i^I$, then $\Gamma_i\psi_i \leq \frac{1}{2}\phi_i^I$, and if $\psi'_i > \phi_i^I$, then $\Gamma_i\psi_i \leq (\psi'_i - \phi_i^I) + \frac{1}{2}\phi_i^I$. In either case, $\psi_i \geq 0$ right after the update. Further, $\Gamma_i\phi \geq I_i = w_i|\Delta_i p_i|$.*

Proof. **Case 1**. $x_i < w_i$.
If $\Gamma_i\psi_i \leq \psi'_i$, then $\psi_i(p_{-i}, p'_i) \geq 0$ and consequently $x_i(p_{-i}, p'_i) \leq w_i$, in which case the update is same-sided; consequently $\psi_i \geq 0$ right after the update. As $\psi_i > 0$ right before the update also, $\psi_i = \phi_i$, both right before and right after the update, so $\Gamma_i\psi_i = \Gamma_i\phi_i$. Further, by Lemmas 9 and 7, $\Gamma_i\phi \geq I_i$. Thus it suffices to show that $\Gamma_i\psi_i \leq \phi_i^I/2$, as we assume that $\psi'_i \geq \phi_i^I/2$.

$$
\begin{aligned}
\Gamma_i\psi_i &\leq \psi'_i - \left[p_i\left(1 + \lambda\frac{x_i - w_i}{w_i}\right)\right]\left[w_i - \left(w_i - \frac{\psi'_i}{p_i}\right)\left(1 + \lambda\frac{x_i - w_i}{w_i}\right)^{-E}\right] \\
&\leq \psi'_i - p_i w_i + \lambda p_i(w_i - x_i) + (p_i w_i - \psi'_i)\left(1 - \lambda\frac{w_i - x_i}{w_i}\right)^{-(E-1)} \\
&\leq \psi'_i - p_i w_i + \lambda\phi_i^I + (p_i w_i - \psi'_i)\left(1 + 2\lambda(E-1)\frac{w_i - x_i}{w_i}\right)
\end{aligned}
$$

applying Fact 1d, with $a = -(E-1)$, $\delta = -\lambda\frac{w_i - x_i}{w_i}$, $\rho = \frac{1}{2}$ when $E \geq 2$, and $\rho = -\frac{1}{2}$ when $1 \leq E < 2$; for when $E \geq 2$, $\lambda(E-1)\frac{w_i - x_i}{w_i} \leq \frac{1}{2}$ and when $1 \leq E < 2$, $\lambda\frac{w_i - x_i}{w_i} \leq \frac{1}{2}$; further, in both cases, $-(E-1) \leq 0$.

$$
\begin{aligned}
&\leq \lambda\phi_i^I + 2\lambda(E-1)p_i(w_i - x_i) \\
&\leq (2E-1)\lambda\phi_i^I \leq \frac{\phi_i^I}{2} \quad \text{as } \lambda(2E-1) \leq \tfrac{1}{2}.
\end{aligned}
$$

Case 2. $x_i > w_i$.
Again, if $\Gamma_i\psi_i \le \psi_i'$, we can conclude that the update is same-sided, $\psi_i \ge 0$ right after the update, and $\Gamma_i\phi \ge I_i$. Again, this holds either if $\Gamma_i\psi_i \le \phi_i^I/2$ (which holds when $\psi_i' \le \phi_i^I$) or if $\Gamma_i\psi_i \le \phi_i^I/2 + (\psi_i' - \phi_i^I)$ (which holds when $\psi_i' > \phi_i^I$).
Case 2.1. $w_i < x_i \le 2w_i$.

$$\begin{aligned}
\Gamma_i\psi_i &\le \psi_i' - \left[p_i\left(1+\lambda\frac{x_i-w_i}{w_i}\right)\right]\left[\left(w_i+\frac{\psi_i'}{p_i}\right)\left(1+\lambda\frac{x_i-w_i}{w_i}\right)^{-E} - w_i\right]\\
&\le \psi_i' + p_iw_i + \lambda\phi_i^I - (p_iw_i+\psi_i')\left(1+\lambda\frac{x_i-w_i}{w_i}\right)^{-(E-1)}\\
&\le \lambda\phi_i^I + \lambda(E-1)\frac{x_i-w_i}{w_i}(p_iw_i+\psi_i')\\
&\qquad \text{applying Fact 1a, with } a=-(E-1) \text{ and } \delta=\lambda\tfrac{x_i-w_i}{w_i}, \text{ as } \lambda\tfrac{x_i-w_i}{w_i}\le 1 \text{ and } -(E-1)\le 0.\\
&\le \lambda\phi_i^I + \lambda(E-1)[(x_i-w_i)p_i+\psi_i']\\
&\le \begin{cases} \lambda(2E-1)\phi_i^I & \text{if } \psi_i'\le\phi_i^I\\ \lambda(2E-1)\phi_i^I + \lambda(E-1)(\psi_i'-\phi_i^I) & \text{if } \psi_i'>\phi_i^I\end{cases}\\
&\le \begin{cases} \phi_i^I/2 & \text{if } \psi_i'\le\phi_i^I\\ \phi_i^I/2 + (\psi_i'-\phi_i^I) & \text{if } \psi_i'>\phi_i^I\end{cases} \quad \text{as } \lambda(2E-1)\le\tfrac{1}{2}
\end{aligned}$$

Case 2.2. $2w_i < x_i$.

$$\begin{aligned}
\Gamma_i\psi_i &\le \psi_i' - [p_i(1+\lambda)]\left[\left(w_i+\frac{\psi_i'}{p_i}\right)(1+\lambda)^{-E} - w_i\right]\\
&\le \psi_i' + p_iw_i(1+\lambda) - (p_iw_i+\psi_i')(1+\lambda)^{-(E-1)}\\
&\le \lambda p_iw_i + \lambda(E-1)(p_iw_i+\psi_i')\\
&\qquad \text{applying Fact 1a, with } a=-(E-1) \text{ and } \delta=\lambda, \text{ as } \lambda\le 1 \text{ and } -(E-1)\le 0.\\
&\le \lambda E\phi_i^I + \lambda(E-1)(2\phi_i^I+\psi_i'-\phi_i^I) \quad \text{as } p_iw_i\le p_i(x_i-w_i)=\phi_i^I\\
&\le \lambda(2E-1)\phi_i^I + \lambda(E-1)(\psi_i'-\phi_i^I)\\
&\le \phi_i^I/2 + (\psi_i'-\phi_i^I) \quad \text{as } \lambda(2E-1)\le\tfrac{1}{2}.
\end{aligned}$$

□

Lemma 11. *When all the updates are applied, for at least one good, WLOG g_a, either sign$(x_a - w_a)$ is unchanged or $x_a = w_a$ now.*

Proof. WLOG let $g_1, \cdots, g_i$ be the goods for which $x_h \ge w_h$, $h \le i$, $g_{i+1}, \cdots, g_j$ those for which $x_h = w_h$, $i < h \le j$, and $g_{j+1}, \cdots, g_n$, the remaining goods, be those for which $x_h < w_h$, $j < h \le n$. We show by induction that there is an ordering of the updates so that, for all i, just before an update is applied to p_i, $\psi_i' \ge \phi_i^I/2$; then, by Lemma 10, following the update to p_i, $\psi_i \ge 0$.
Case 1 $\phi_1^I + \cdots + \phi_i^I \le \phi_{j+1}^I + \cdots + \phi_n^I$.

Consider applying all the price updates for goods $g_1, \cdots, g_i$. We argue that the update to price p_h leaves $\psi_h \ge \phi_h^I/2$. For as the ultimate change to ψ_h is the same regardless of the order in which the updates to $p_1, \cdots, p_i$ are applied, we will analyze the case in which p_h is updated first: then by Lemma 10, its update reduces ψ_h by at most $\phi_h^I/2$; the updates to the other $p_{h'}$, $h' \le i$, price increases, by WGS only increase x_h and hence ψ_h. We conclude that after the updates to $p_1, \cdots, p_i$, for every $h \le i$, $\psi_h \ge \phi_h^I/2$.

Because the total spending is fixed, any reductions to ψ_l, $l > j$, result from matching reductions to the ψ_h, $h \le i$. The total available reduction is at most $\frac{1}{2}\sum_{h\le i}\phi_h^I \le \frac{1}{2}\sum_{l>j}\phi_l^I$. Hence there is some good g_l, such that, after the updates to all of $p_1, \cdots, p_i$, $\psi_l \ge \phi_l^I/2$. WLOG let $l = n$.

Applying the updates to $p_{j+1}, \cdots, p_{n-1}$ (price decreases) only decreases x_n and hence only increases ψ_n.

So the update to p_n can be applied last and just before it is applied, $\psi_n \ge \frac{1}{2}\phi_n^I$. Thus we can choose $a = n$.

Case 2 $\phi_1^I + \cdots + \phi_i^I \ge \phi_{j+1}^I + \cdots + \phi_n^I$.

A symmetric argument applies here.

By induction, the claim applies to the remaining $n-1$ goods. □

Theorem 2. *Suppose that the market obeys WGS and elasticity E. When all the prices are updated simultaneously, if $\lambda(2E-1) \le \frac{1}{2}$, $\phi = \sum_i \phi_i$ reduces by at least $\sum_i \lambda\phi_i^I \min\left\{1, \frac{w_i}{|x_i - w_i|}\right\}$. If the demands satisfy $x_i \le dw_i$ throughout, then ϕ reduces from ϕ_I to at most ϕ_F in $O(\frac{1}{\lambda}(d-1)\log\frac{\phi_I}{\phi_F})$ iterations.*

Proof. Simply apply the updates in the order given by Lemma 11. □

3.3 Asynchronous Price Updates with Bounded Demands

Recall that we use the following potential function: $\phi = \sum_i \phi_i$, where

$$\phi_i(x_i, \overline{x}_i, w_i) = p_i\left[\mathrm{span}(x_i, \overline{x}_i, w_i) - \alpha_1\lambda|w_i - \overline{x}_i|(t - \tau_i)\right], \tag{7}$$

with $\alpha_1 > 0$ being a suitable constant and $\mathrm{span}(x, y, z)$ denoting the length of the interval spanned by its arguments, i.e. $\max\{x, y, z\} - \min\{x, y, z\}$. At the end of this section we will relate ϕ to our definition of misspending.

ϕ_i will decrease continuously: in Lemma 12, we will show that $\frac{d\phi_i}{dt} \le -\lambda\phi_i$. In Corollary 2, we will also show that when the price update occurs, $\phi = \sum_i \phi_i$ only decreases. Together, these imply a daily decrease in potential by at least a $1 - \theta(\lambda)$ factor.

In fact, we will need a more elaborate potential to cope with the following scenario: suppose that $\overline{x}_i < w_i$ and yet owing to last-minute price reductions to other goods $x_i \gg w_i$. Then applying the update to p_i may increase ϕ_i well beyond the available "savings" of $p_i|w_i - x_i|$. So for the moment we assume that all demands are bounded at all times:

Assumption 1. *$x_i \le dw_i$ for all i, where $d \ge 2$ is a suitable constant.*

Later, we will show how to drop this assumption.

Lemma 12. *If $\lambda\alpha_1 \le \frac{1}{2}$, then $\frac{d\phi_i}{dt} \le -\lambda\alpha_1 p_i\,\mathrm{span}(\overline{x}_i, x_i, w_i) \le -\lambda\alpha_1\phi_i$ at any time when no price update is occurring (to any p_j); this bound also holds for the one-sided derivatives when a price update occurs.*

Proof. We begin by showing $\frac{d\overline{x}_i}{dt} = \frac{1}{t-\tau_i}(x_i - \overline{x}_i)$. For $\overline{x}_i = \frac{1}{t-\tau_i}\int_{\tau_i}^t x_i dt$; so $\overline{x}_i + (t - \tau_i)\frac{d\overline{x}_i}{dt} = x_i$.

Now we bound $\frac{d\phi_i}{dt}$.

Case 1: $\overline{x}_i \geq x_i \geq w_i$ (or symmetrically, $\overline{x}_i \leq x_i \leq w_i$). Then $\phi = p_i[(\overline{x}_i - w_i)(1 - \lambda\alpha_1(t - \tau_i))]$.

$$\frac{d\phi_i}{dt} = p_i\left[\frac{-1}{t-\tau_i}(\overline{x}_i - x_i)\left(1 - \lambda\alpha_1(t-\tau_i)\right) - \lambda\alpha_1(\overline{x}_i - w_i)\right] \leq -\lambda\alpha_1 p_i(\overline{x}_i - w_i) \leq -\lambda\alpha_1\phi_i \text{ as } t - \tau_i \leq 1.$$

Case 2: $x_i > \overline{x}_i \geq w_i$ (or symmetrically, $x_i < \overline{x}_i \leq w_i$). Then $\phi_i = p_i[(x_i - w_i) - \lambda\alpha_1(t-\tau_i)(\overline{x}_i - w_i)]$.

$$\frac{d\phi_i}{dt} = -\lambda\alpha_1 p_i\left[(\overline{x}_i - w_i) + (x_i - \overline{x}_i)\right] = -\lambda\alpha_1 p_i(x_i - w_i) \leq -\lambda\alpha_1\phi_i.$$

Case 3: $\overline{x}_i > w_i \geq x_i$ (or symmetrically, $\overline{x}_i < w_i \leq x_i$). Then $\phi_i = p_i[(\overline{x}_i - x_i) - \lambda\alpha_1(t-\tau_i)(\overline{x}_i - w_i)]$.

$$\frac{d\phi_i}{dt} = p_i\left[\frac{-(\overline{x}_i - x_i)}{t-\tau_i} - \lambda\alpha_1((\overline{x}_i - w_i) + (x_i - \overline{x}_i))\right] \leq p_i\left[-(\overline{x}_i - x_i) - \lambda\alpha_1(x_i - w_i)\right]$$

$$\leq -p_i\left(1 - \lambda\alpha_1\right)(\overline{x}_i - x_i) \leq -\left(1 - \lambda\alpha_1\right)\phi_i \leq -\lambda\alpha_1\phi_i, \qquad \text{as } \lambda\alpha_1 \leq \frac{1}{2}.$$

□

Lemma 13.

1. If $\Delta_i p_i > 0$,

$$-\frac{\Delta_i x_i}{x_i} \leq \frac{E\Delta_i p_i}{p_i} = \lambda E \min\left\{1, \frac{\overline{x}_i - w_i}{w_i}\right\}.$$

2. If $\Delta_i p_i < 0$,

$$\frac{\Delta_i x_i}{x_i} \leq \frac{E(-\Delta_i p_i)}{(1-\lambda E)p_i} = -\frac{\lambda E}{1-\lambda E}\frac{(\overline{x}_i - w_i)}{w_i}, \text{ if } \frac{\lambda E}{1-\lambda E} \leq 1.$$

Proof. We begin with (1). By Lemma 5,

$$\frac{x_i - (-\Delta_i x_i)}{x_i} \geq \left(\frac{p_i}{p_i + \Delta_i p_i}\right)^E = \left(1 + \frac{\Delta_i p_i}{p_i}\right)^{-E} \geq 1 - \frac{E\Delta_i p_i}{p_i},$$

using Fact 1a, with $a = E$ and $\delta = \frac{\Delta_i p_i}{p_i}$, as $\frac{E\Delta_i p_i}{p_i} \geq -1$.

(1) now follows readily.

For (2), by Lemma 5,

$$\frac{x_i + \Delta_i x_i}{x_i} \leq \left(\frac{p_i}{p_i - (-\Delta_i p_i)}\right)^E = \left(\frac{p_i - (-\Delta_i p_i)}{p_i}\right)^{-E} \leq 1 + \frac{E}{1-\lambda E}\frac{(-\Delta_i p_i)}{p_i},$$

using Fact 1d, with $a = -E$, $\delta = \frac{\Delta_i p_i}{p_i}$, and $\rho = \lambda E$, as $E(\frac{(-\Delta_i p_i}{p_i}) \leq E\lambda$.

(2) now follows readily. □

Note that any non-same-side update can be split into two same-side updates: the first causes $x_i = w_i$ and the second changes x_i to its final value. Consequently, we will analyze only same-side updates henceforth. We say that the update of p_i to p'_i is *toward* w_i if either $x_i > x'_i \geq w_i$ or $x_i < x'_i \leq w_i$, and it is *away from* w_i if either $x'_i > x_i \geq w_i$ or $x'_i < x_i \leq w_i$, where $x'_i = x_i(p'_i)$.

Let $\psi_i = \phi_i - \text{span}(x_i, x_i^u, w_i) - \alpha_1\lambda|w_i - x_i^u|(t-\tau_i)$, and let $\Delta_i\psi_i$ denote the increase in ψ_i when p_i is updated (this definition is useful in later sections where the definition of ϕ_i is changed with the effect that $\psi_i \neq 0$). x_i^u is the values of x_i used in computing the price update; in this section, $x_i^u = \overline{x}_i$.

Lemma 14. *If Assumption 1 holds ($x_i \le dw_i$), when p_i is updated, ϕ increases by at most the following amount:*
Case 1. the update is toward w_i:

$$\lambda\alpha_1|x_i^u - w_i|p_i + \Delta_i\psi_i - w_i|\Delta_i p_i|. \tag{8}$$

Case 2. The update is away from w_i:

$$\left(1 + \frac{2Ed}{1-\lambda E}\right) w_i|\Delta_i p_i| + \Delta_i\psi_i - (1-\lambda\alpha_1)p_i|x_i^u - w_i|. \tag{9}$$

Proof. Case 1: First, we increase ψ_i by $\Delta_i\psi_i$.

Next, we reduce the term $p_i \operatorname{span}(x_i, x_i^u, w_i)$ in ϕ_i to $p_i|x_i - w_i|$. Following this, we update x_i to $x_i + \Delta_i^N x_i$ and reduce $p_i|x_i - w_i|$ to $(p_i + \Delta_i p_i)|x_i + \Delta_i^N x_i - w_i|$. By Lemma 8, this reduces ϕ_i by $w_i|\Delta_i p_i|$. We also remove the term $\lambda\alpha_1(t-\tau_i)|x_i^u - w_i|p_i$ from ϕ_i. The increase to ϕ_i following these changes is at most

$$\lambda\alpha_1(t-\tau_i)|x_i^u - w_i|p_i + \Delta_i\psi_i - w_i|\Delta_i p_i|.$$

Note that $t - \tau_i \le 1$, as there is a price update at least once a day; this yields the bound in (8) in the statement of the Lemma.

Finally, we change x_i by a further $\Delta_i x_i - \Delta_i^N x_i$. This reduces ϕ_i by $(p_i + \Delta_i p_i)(\Delta_i x_i - \Delta_i^N x_i)$ and may increase other ϕ_j by up to this amount (due to a transfer of this amount of spending from the span term in ϕ_i to span terms in ϕ_j, $j \ne i$). At worst, this leaves the potential ϕ unchanged.
Case 2: First, we increase ψ_i by $\Delta_i\psi_i$.

Next, we reduce $p_i \operatorname{span}(x_i, x_i^u, w_i)$ to $p_i|x_i - w_i|$. This yields a saving of $p_i|x_i^u - w_i|$.

Again, we remove the term $\lambda\alpha_1 p_i(t-\tau_i)|x_i^u - w_i|$, with this cost. Next, we update p_i to $p_i + \Delta_i p_i$ and change x_i by $\Delta_i^N x_i$. By Lemma 8, this increases ϕ_i by $w_i|\Delta_i p_i|$. Then, we update x_i by a further $\Delta_i x_i - \Delta_i^N x_i$. This increases ϕ_i by $(p_i + \Delta_i p_i)|\Delta_i x_i - \Delta_i^N x_i|$. It may also cause up to an equal increase in other ϕ_j due to matching spending transfers between good i and goods $j \ne i$. The net increase in potential is bounded by $w_i|\Delta_i p_i| + 2(p_i + \Delta_i p_i)|\Delta x_i| + \lambda\alpha_1 p_i|x_i^u - w_i| + \Delta_i\psi_i - p_i|x_i^u - w_i|$.
Case 2.1: $x_i \le w_i < x_i^u$.
Here $0 \le \Delta_i p_i \le \lambda p_i$. By Lemma 13, as $\Delta_i p_i \ge 0$, $|\Delta_i x_i| \le x_i E \Delta_i p_i / p_i \le w_i E(\Delta_i p_i)/p_i$. Also $\Delta_i p_i \le \lambda p_i$. So the increase to ϕ is at most $w_i|\Delta_i p_i| + 2p_i(1+\lambda)w_i E(\Delta_i p_i)/p_i + \lambda\alpha_1 p_i|x_i^u - w_i| + \Delta_i\psi_i - p_i|x_i^u - w_i|$; on rearranging, this amounts to $w_i|\Delta_i p_i|(1 + 2E(1+\lambda)) + \Delta_i\psi_i - (1-\lambda\alpha_1)p_i|x_i^u - w_i|$. This is bounded by (9).
Case 2.2: $x_i \ge w_i > x_i^u$.
Here $\Delta_i p_i \le 0$; so $p_i + \Delta_i p_i \le p_i$. By Lemma 13, $|\Delta_i x_i| \le \frac{E}{1-\lambda E} x_i \frac{|\Delta_i p_i|}{p_i} \le \frac{E}{1-\lambda E} dw_i \frac{|\Delta_i p_i|}{p_i}$. So the increase to ϕ is at most $w_i|\Delta_i p_i| + 2p_i \cdot \frac{E}{1-\lambda E} dw_i \frac{|\Delta_i p_i|}{p_i} + \lambda\alpha_1 p_i|x_i^u - w_i| + \Delta_i\psi_i - p_i|x_i^u - w_i|$; on rearranging, this amounts to (9). □

Corollary 2. *If Assumption 1 holds ($x_i \le dw_i$), $\alpha_1(d-1) \le 1$, and $\lambda\alpha_1 + \lambda(\frac{2Ed}{1-\lambda E} + 1) \le 1$ then when p_i is updated, ϕ only decreases.*

Proof. Recall that $\Delta_i p_i = \lambda \min\{1, \frac{\overline{x}_i - w_i}{w_i}\}$. Also note that $\psi_i = 0$ and hence $\Delta_i\psi_i = 0$ too. Finally, recall that $x_i^u = \overline{x}_i$.

Case 1: A toward w_i update.

The increase in ϕ is bounded by (8).

If $|\overline{x}_i - w_i| \le w_i$, $\Delta_i p_i = \lambda p_i(\overline{x}_i - w_i)/w_i$; for the increase to be non-positive, it suffices that $\alpha_1 \le 1$.

If $|\overline{x}_i - w_i| > w_i$, then $\Delta_i p_i = \lambda p_i$. Also $|\overline{x}_i - w_i| \le (d-1)w_i$. Again, for the increase to be non-positive, it suffices that $\alpha_1(d-1) \le 1$.

Case 2: An away from w_i update.
The increase in ϕ is bounded by (9). Again, $\Delta_i \psi_i = 0$. Also, $|\Delta_i p_i| \le p_i \lambda |\overline{x}_i - w_i|/w_i$. For the increase to be non-positive, it suffices that $\lambda\alpha_1 + \lambda\left(1 + \frac{2Ed}{1-\lambda E}\right) \le 1$. ☐

Theorem 3. *Suppose that the market obeys WGS and has elasticity E. If $x_i \le dw_i$, for all i, where $d \ge 2$, $\alpha_1(d-1) \le 1$, $\lambda\alpha_1 + \lambda\left(1 + \frac{2Ed}{1-\lambda E}\right) \le 1$, and each price is updated at least once every day, then ϕ decreases by at least a $1 - \frac{\lambda\alpha_1}{2}$ factor daily. Hence ϕ reduces from ϕ_I to at most ϕ_F in $O(\frac{1}{\lambda\alpha_1}(d-1)\log\frac{\phi_I}{\phi_F})$ days.*

Proof. By Corollary 3, ϕ only decreases whenever a price update occurs. By Lemma 12, $d\phi/dt \le \lambda\alpha_1\phi$, which implies $\phi(t+1) \le e^{-\lambda\alpha_1}\phi(t) \le (1 - \frac{\lambda\alpha_1}{2})\phi(t)$ as $\lambda\alpha_1 \le 1$. ☐

Remark. One could imagine having a distinct elasticity bound for each good, E_i for good i, $1 \le i \le n$, say. Then we would use a distinct parameter λ_i for good i. To ensure uniform progress in reducing the various ϕ_i, a price update for good i would need to occur every $\Theta(\lambda_i f)$ time units, where f is an upper bound on the time between updates for goods with elasticity 1. As this does not alter the character of our results, we leave the details to the interested reader.

3.4 The Ongoing Market, or Incorporating Warehouses

We begin by showing that Constraint1 allows us to relate $\widetilde{w}_i$ and w_i.

Lemma 15. *If Constraint 1 holds, then $\frac{3}{4}\widetilde{w}_i \le w_i \le \frac{3}{2}\widetilde{w}_i$ and $|\widetilde{w}_i - w_i| \le \frac{1}{2}\widetilde{w}_i$. In addition, Constraint 1 holds for all possible warehouse contents if $\kappa \max\{|c_i - s_i^*|, s_i^*\} \le \frac{1}{3}w_i$, and if in addition $s_i^* = c_i/2$, the condition becomes $\kappa \le \frac{2}{3}\frac{w_i}{c_i}$.*

Proof. Constraint 1 implies $\frac{2}{3}w_i \le \widetilde{w}_i \le \frac{4}{3}w_i$. Thus $\frac{3}{4}\widetilde{w}_i \le w_i \le \frac{3}{2}\widetilde{w}_i$, which gives the first claim. The second pair of claims is immediate from the definition of $\widetilde{w}_i$, as $|s_i - s_i^*|$ is maximized either when $s_i = c_i$ or when $s_i = 0$. ☐

We prove results analogous to Lemma 12 and Corollary 2 to demonstrate progress as before. To enable us to apply the following lemma in later sections, we define $\chi_i = \frac{d\widetilde{w}_i}{dt} + \kappa(x_i - w_i)$ (in the current section, $\chi_i = 0$).

Lemma 16. *Suppose that $4\kappa(1+\alpha_2) \le \lambda\alpha_1 \le \frac{1}{2}$. If $|\widetilde{w}_i - w_i| \le 2\,\mathrm{span}(\overline{x}_i, x_i, \widetilde{w}_i)$, then $\frac{d\phi_i}{dt} \le -\frac{\lambda\alpha_1}{4(1+\alpha_2)}\phi_i - p_i\chi_i$, and otherwise $\frac{d\phi_i}{dt} \le -\frac{\kappa(\alpha_2-1)}{2}\phi_i - p_i\chi_i$, at any time when no price update is occurring (to any p_j); this bound also holds for the one-sided derivatives when a price update occurs.*

Proof. The analysis builds on the proof for Lemma 12. Taking account of the two changes to the previous form of the potential, we can conclude

$$\begin{aligned}\frac{d\phi_i}{dt} &\leq -\lambda\alpha_1 p_i \operatorname{span}(\overline{x}_i, x_i, \widetilde{w}_i) + [\kappa(x_i - w_i) - \chi_i] p_i \max\{(1 - \lambda\alpha_1(t - \tau_i)), \lambda\alpha_1(t - \tau_i)\} \\ &\quad - \kappa(x_i - w_i)\alpha_2 p_i \operatorname{sign}(\widetilde{w}_i - w_i).\end{aligned}$$

Case 1: $|\widetilde{w}_i - w_i| \leq 2 \operatorname{span}(\overline{x}_i, x_i, \widetilde{w}_i)$.
Then $|x_i - w_i| \leq 3 \operatorname{span}(\overline{x}_i, x_i, \widetilde{w}_i)$. And

$$\begin{aligned}\frac{d\phi_i}{dt} &\leq -\lambda\alpha_1 p_i \operatorname{span}(\overline{x}_i, x_i, \widetilde{w}_i) + \kappa p_i (1 + \alpha_2)|x_i - w_i| - p_i\chi_i \\ &\leq -p_i(\lambda\alpha_1 - 3\kappa(1 + \alpha_2)) \operatorname{span}(\overline{x}_i, x_i, \widetilde{w}_i) - p_i\chi_i \\ &\leq -p_i \frac{(\lambda\alpha_1 - 3\kappa(1 + \alpha_2))}{1 + \alpha_2} [\operatorname{span}(\overline{x}_i, x_i, \widetilde{w}_i) + \alpha_2|\widetilde{w}_i - w_i|] - p_i\chi_i \\ &\leq -\frac{(\lambda\alpha_1 - 3\kappa(1 + \alpha_2))}{1 + \alpha_2}\phi_i - p_i\chi_i \leq -\frac{\lambda\alpha_1}{4(1 + \alpha_2)}\phi_i - p_i\chi_i.\end{aligned}$$

Case 2: $|\widetilde{w}_i - w_i| > 2 \operatorname{span}(\overline{x}_i, x_i, \widetilde{w}_i)$.
Then $|\widetilde{w}_i - w_i| \leq 2|x_i - w_i|$. And

$$\begin{aligned}\frac{d\phi_i}{dt} &\leq -\lambda\alpha_1 p_i \operatorname{span}(\overline{x}_i, x_i, \widetilde{w}_i) - \kappa p_i(\alpha_2 - 1)|x_i - w_i| - p_i\chi_i \\ &\leq -p_i\lambda\alpha_1 \operatorname{span}(\overline{x}_i, x_i, \widetilde{w}_i) - \kappa p_i \frac{(\alpha_2 - 1)}{2}|\widetilde{w}_i - w_i| - p_i\chi_i \\ &\leq -\min\left\{\lambda\alpha_1, \kappa\frac{(\alpha_2 - 1)}{2}\right\}\phi_i - p_i\chi_i \leq -\frac{\kappa(\alpha_2 - 1)}{2}\phi_i - p_i\chi_i.\end{aligned}$$

□

Corollary 3. *Suppose that* $4\kappa(1 + \alpha_2) \leq \lambda\alpha_1 \leq \frac{1}{2}$. *If* $\phi \geq 2(1 + 2\alpha_2)\sum_i |\widetilde{w}_i - w_i|p_i$, *then* $\frac{d\phi}{dt} \leq -\frac{\lambda\alpha_1}{8(1+\alpha_2)}\phi - \sum_i p_i\chi_i$, *and otherwise* $\frac{d\phi}{dt} \leq -\frac{\kappa(\alpha_2-1)}{2}\phi - \sum_i p_i\chi_i$, *at any time when no price update is occurring (to any* p_j*); this bound also holds for the one-sided derivatives when a price update occurs.*

Proof. Let $I = \{i \mid |\widetilde{w}_i - w_i| \leq 2 \operatorname{span}(\overline{x}_i, x_i, \widetilde{w}_i)\}$. If $\phi \geq 2(1 + 2\alpha_2)\sum_i |\widetilde{w}_i - w_i|p_i$, then $\sum_{i\in I}\phi_i \geq \frac{1}{2}\phi$. Thus

$$\begin{aligned}\frac{d\phi}{dt} &\leq \sum_{i\in I}\frac{d\phi_i}{dt} - \sum_{i\notin I} p_i\chi_i \\ &\leq -\frac{\lambda\alpha_1}{4(1 + \alpha_2)}\sum_{i\in I}\phi_i - \sum_i p_i\chi_i \\ &\leq -\frac{\lambda\alpha_1}{8(1 + \alpha_2)}\phi - \sum_i p_i\chi_i.\end{aligned}$$

□

In Lemma 18 below, $\widetilde{w}_i$ takes on the role of w_i in Corollary 2. Accordingly, the bound of $x_i \le dw_i$ from Assumption 1 is replaced by a bound of $x_i \le d\widetilde{w}_i$, which is ensured by the following assumption.

Assumption 2. *$x_i \le \frac{2}{3}dw_i$ for all i, where $d \ge 2$.*

Lemma 17. *If Assumption 2 and Constraint 1 hold, then $x_i \le d\widetilde{w}_i$.*

Proof. This is immediate from Lemma 15. □

Lemma 18. *If Constraint 1 holds and $x_i \le d\widetilde{w}_i$, then when p_i is updated, ϕ increases by at most the following:*
(i) With a toward $\widetilde{w}_i$ update:

$$\lambda\alpha_1|\overline{x}_i - \widetilde{w}_i|p_i + \frac{1}{2}\alpha_2\widetilde{w}_i|\Delta_i p_i| - \widetilde{w}_i|\Delta_i p_i|. \tag{10}$$

(ii) With an away from $\widetilde{w}_i$ update:

$$\left(1 + \frac{2Ed}{1-\lambda E}\right)\widetilde{w}_i|\Delta_i p_i| + \frac{1}{2}\alpha_2\widetilde{w}_i|\Delta_i p_i| - (1-\lambda\alpha_1)p_i|\overline{x}_i - \widetilde{w}_i|. \tag{11}$$

Proof. We apply Lemma 14, with $x_i^u = \overline{x}_i$ and $\widetilde{w}_i$ replacing w_i, as here ϕ_i is defined in terms of $\widetilde{w}_i$ rather than w_i. Also, now $\psi_i = \alpha_2 p_i|\widetilde{w}_i - w_i|$. So $\Delta_i\psi_i = \alpha_2\Delta_i p_i|\widetilde{w}_i - w_i|$. By Lemma 15, $|\widetilde{w}_i - w_i| \le \widetilde{w}_i/2$, so $\Delta_i\psi_i \le \alpha_2|\Delta_i p_i|\widetilde{w}_i/2$. □

Corollary 4. *If Constraint 1 holds and $x_i \le d\widetilde{w}_i$, $\frac{\alpha_2}{2} + \max\{\frac{3}{2},(d-1)\}\alpha_1 \le 1$, and $\lambda\alpha_1 + \frac{4}{3}\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \le 1$, then when p_i is updated, ϕ only decreases.*

Proof. Recall that by Lemma 17, $\overline{x}_i \le (d-1)\widetilde{w}_i$.
Case 1: An update toward w_i:
The increase in ϕ is is bounded by (10).
(i) $|\overline{x}_i - \widetilde{w}_i| \le w_i$.
$\Delta_i p_i = \lambda p_i(\overline{x}_i - \widetilde{w}_i)/w_i$, and by Lemma 15 $w_i \le \frac{3}{2}\widetilde{w}_i$. Thus for the increase to be non-positive, $\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 \le 1$ suffices.
(ii) $|\overline{x}_i - \widetilde{w}_i| \ge w_i$.
As $|\overline{x}_i - \widetilde{w}_i| \le (d-1)\widetilde{w}_i$ and $|\Delta_i p_i| = \lambda p_i$, for the increase to be non-positive, $\alpha_1(d-1) + \frac{1}{2}\alpha_2 \le 1$ suffices.
Case 2: An update away from w_i:
The increase in ϕ is is bounded by (11). As $\Delta_i p_i \le \lambda p_i|\overline{x}_i - \widetilde{w}_i|/w_i$, for the increase to be non-positive, the condition $\frac{\widetilde{w}_i}{w_i}\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \le 1 - \lambda\alpha_1$ suffices. By Lemma 15, $\widetilde{w}_i \le \frac{4}{3}w_i$, so the condition of the previous sentence is subsumed by $\lambda\alpha_1 + \frac{4}{3}\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \le 1$. □

Theorem 4. *If Constraint 1 holds and $x_i \le d\widetilde{w}_i$, $\frac{\alpha_2}{2} + \alpha_1\max\{\frac{3}{2},(d-1)\} \le 1$, $\lambda\alpha_1 + \frac{4}{3}\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \le 1$, $4\kappa(1+\alpha_2) \le \lambda\alpha_1 \le \frac{1}{2}$, $\frac{\kappa(\alpha_2-1)}{2} \le 1$, and each price is updated at least once every day, then ϕ decreases by at least a $1 - \frac{\kappa(\alpha_2-1)}{4}$ factor daily.*

In fact, if $\phi \ge 2(1+2\alpha_2)\sum_i|\widetilde{w}_i - w_i|p_i$, then ϕ decreases by at least a $1 - \frac{\lambda\alpha_1}{8(1+\alpha_2)}$ factor daily.

Proof. By Corollary 4, ϕ only decreases whenever a price update occurs. By Lemma 16, $d\phi/dt \leq -\frac{\kappa(\alpha_2-1)}{2}\phi$, as $\chi_i = 0$ for all i, which implies $\phi(t+1) \leq e^{-\frac{\kappa(\alpha_2-1)}{2}}\phi(t) \leq (1 - \frac{\kappa(\alpha_2-1)}{4})\phi(t)$ as $\frac{\kappa(\alpha_2-1)}{2} \leq 1$, arguing as in the proof of Lemma 3.

The second claim is shown in exactly the same way. □

3.5 Bounds on Demands and Prices

Next, we determine an f-bound on prices given that they are c-demand bounded. To obtain this we need to assume a bounded elasticity of wealth $E' = 0$. Recall that when this holds demands are called *normal*.

Definition 13. *For $c \geq 1$, define $r^{(c)} = \max_i p_i^*/p_i^{(c)}$ and $r^{(1/c)} = \max_i p_i^{(1/c)}/p_i^*$.*

Notation. Let $\rho = \max_{i,j} \frac{w_i p_i^*}{w_j p_j^*}$.

Lemma 19. *$r^{(1/c)} \leq cn\rho$.*

Proof. Let $h = \arg\max_i w_i p_i^*$. Then the money M_a spent at equilibrium is at most $nw_h p_h^*$. Let $k = \arg\max_i p_i^{(1/c)}/p_i^*$. The money M_b spent on good k at prices $\mathbf{p}^{(1/c)}$ is $\frac{1}{c}w_k p_k^{(1/c)}$. By WGS, $M_b \leq M_a$. But

$$\frac{1}{c}r^{(1/c)}w_k p_k^* = \frac{1}{c}w_k p_k^{(1/c)} = M_b \leq M_a \leq nw_h p_h^*.$$

$$\text{so} \quad r^{(1/c)} \leq cn\frac{w_h p_h^*}{w_k p_k^*} \leq cn\rho.$$

□

Obtaining a bound on $r^{(c)}$ for $c \geq 1$ entails a more elaborate argument.

Consider the following process for decreasing prices from from $\mathbf{p}^*$ to $\mathbf{p}^{(c)}$, where WLOG $p_1^*/p_1^{(c)} \geq p_2^*/p_2^{(c)} \geq \cdots \geq p_n^*/p_n^{(c)}$.

Begin by decreasing all prices uniformly, until p_n reaches $p_n^{(c)}$. Continue reducing all prices except p_n uniformly until p_{n-1} reaches $p_{n-1}^{(c)}$, and so forth. Call this the *uniform price reduction process* (UPR for short).

Lemma 20. *$r^{(c)}$ is maximized when demand increases as follows, as the UPR is applied. During the reduction of p_n from p_n^* to $p_n^{(c)}$ only x_n changes (by increasing to cw_n). In general, during the reduction of p_i from $p_i^* p_{i+1}^{(c)}/p_{i+1}^*$ to $p_i^{(c)}$, only x_i changes.*

Loosely speaking, the demands increase one time.

Proof. By normality, for those goods whose prices are decreasing during the UPR, demand either stays the same or is increased.

To maximize the price reduction goods $1, \cdots, n-1$ can achieve one needs to minimize the amount of money spent on them. This implies that $p_n^{(c)}$ needs to be as large as possible, which occurs if all the increase in demand goes to good n as the UPR proceeds, in going from p_n to $p_n^{(c)}$.

By induction, all other demands are increased as specified in the lemma. □

Lemma 21. *$r^{(c)} \leq c(\rho n)^{(c-1)}$.*

Proof. By Lemma 20, the price drop is maximized if demands increase one at a time. So consider the initial price drop in which the price of the first good increases. Now imagine dividing this good into two goods, with their demands increasing in sequence. This only increases the maximum price drop. We iterate this process ad infinitum, leading to a continuous process which we express as follows.

By rescaling if needed, we imagine that all goods have the same equilibrium price. Note that this is now a continuum of goods. As the prices are multiplied by a factor $s < 1$ (a decrease), we track a measure $W(s)$, the quantity of goods whose demand has not yet changed. $W(1) - W(s)$ is the quantity of goods whose demand has grown by a factor of c. We can express W as a differential process.

Consider decreasing s to $s - ds$. Suppose this results in dW of the goods having demand grow by a factor of c. Assuming the total spending on goods does not increase (this maximizes the price drop), $Ws = (W - dW)(s - ds) + cdW(s - ds)$. To first order, $Wds = (c-1)sdW$, which yields $\frac{W}{(c-1)s} = \frac{dW}{ds}$ or $W = W(1)s^{1/(c-1)}$.

To get back to the discrete process, the price reduction is stopped when W is reduced to $W(1)/\rho n$, for this is a lower bound on the amount of good k with $k = \arg\min_j w_j$.

Good k has a further price drop of no more than a factor of c to bring its demand to cw_k.

The maximum possible price reduction in reducing W to $W(1)/\rho n$ occurs when $s^{1/(c-1)} = 1/(\rho n)$. Thus $r^{(c)} \le c/s \le c(\rho n)^{(c-1)}$. □

Lemma 4 follows immediately from Lemmas 20 and 21.

3.6 Faster Updates with Large Demands

Our previous analysis in Lemma 18 (see Lemma 14 also) uses the assumption $x_i \le d\widetilde{w}_i$ in Case 2. This in used in the following scenario: x_i increases from less than $\widetilde{w}_i$ to much more than $\widetilde{w}_i$ between two successive updates to p_i. The price update (a decrease) could then increase the overall potential. However, with our new price update rule, as x_i is then large, this price decrease will be followed quickly by several price increases. These more than undo the just mentioned potential increase.

To analyze this, the potential 'pretends' that the troublesome price decrease did not occur yet, by delaying its instantiation in one of two ways: either this price decrease is combined with the next one or two price increases so that the net effect is no change or a price increase, or the potential instantiates the price update when next $x'_i \le (d-1)\widetilde{w}_i$, whichever occurs sooner, where x'_i is the value of x_i resulting from the delayed price updates. All other price changes continue as before by their previous amounts.

We need some significant changes to the potentials. For goods i that do not have a delayed price increase currently, we define

$$\phi_i = \psi_i^r = p_i \left[\operatorname{span}(x'_i, \overline{x}'_i, \widetilde{w}_i) - \lambda\alpha_1(t-\tau_i)|\overline{x}'_i - \widetilde{w}_i| + [1 - \lambda\alpha_1(t-\tau_i)] \int_{\tau_i}^{t} (x'_i - x_i)dt + \alpha_2|\widetilde{w}_i - w_i| \right],$$

where $\overline{x}'_i$ is the average value of x'_i since time τ_i. Note that by WGS, $x'_i \ge x_i$ since there is no delayed update to p_i in this case. (The superscript r on ψ denotes *regular* or non-delayed updates.)

For goods i with a currently delayed update, we define

$$\phi'_i = \psi^d_i \quad = \quad p_i \left[\text{span}(x'_i, d\widetilde{w}_i, \widetilde{w}_i) + (\widetilde{w}_i(\tau^s_i) - \overline{x}_i(\tau^s_i)) \left[1 - \lambda\alpha_1(t - \tau_i)\right] - \lambda\alpha_1 \int_{\tau^s_i}^t (x'_i - \widetilde{w}_i)dt + \alpha_2|\widetilde{w}_i - w_i| \right]$$
$$- p_i \left[\frac{\lambda E}{1 - \lambda E} [\widetilde{w}_i(\tau^s_i) - \overline{x}_i(\tau^s_i)] \int_{\tau^s_i}^t \frac{x'_i}{w_i} dt \right]$$

where τ^s_i is the time at which the delayed price decrease occurred in reality. (The superscript d on ψ denotes *delayed*.)

Next, we bound $x'_i - x_i$.

Lemma 22. *Suppose that the market has bounded wealth elasticity E' and bounded demand elasticity E. Let q and p be price vectors with $q_j = (1 - \lambda)p_j$ for all $j \neq i$, and and $q_i = p_i$. Then,*

$$x_i(q) \geq (1 - \lambda)^{E+E'} x_i(p).$$

Proof. Consider prices $q'(j) = p(j)(1 - \lambda)$ for all j. Then, by the bound on wealth elasticity, $x_i(q') \geq (1 - \lambda)^{E'} x_i(p)$. Increasing the price $q'(i)$ to $p(i)$ decreases x_i by at most $(1 - \lambda)^E$. So $x_i(q) \geq (1 - \lambda)^{E'}(1 + \lambda)^{-E} 1 x_i(p) \geq (1 - \lambda)^{E+E'} x_i(p)$. □

Lemma 23. $x'_i \leq \left(1 + \frac{\lambda(E+E')}{1-\lambda(E+E')}\right) x_i$, *if* $\lambda(E + E') < 1$.

Proof. We apply the bound from Lemma 22. Let p denote the actual prices, q the prices as defined in Lemma 22, and q' the prices where the delayed updates have not been performed. Now, for each j, the delayed update to p_j is a decrease by at most a $1 - \lambda$ factor, so $q_j \leq q'_j \leq p_j$, ofr $j \neq i$. At prices q the demand for good i is at least $x'_i(1 - \lambda)^{E+E'}$, by Lemma 22. Increasing prices q_j to q'_j, for $j \neq i$, only increases the demand for good i. Reducing $q'_i = p_i$ to q_i also only increases the demand for good i. Thus $x_i = x_i(p) \geq (1 - \lambda)^{E+E'} x_i(q) = (1 - \lambda)^{E+E'} x_i(q') = (1 - \lambda)^{E+E'} x'_i$; or, $x'_i \leq (1 - \lambda)^{-(E+E')} x_i \leq \left(1 + \frac{\lambda(E+E')}{1-\lambda(E+E')}\right) x_i$, by Fact 1(d) with $\rho = \lambda(E + E')$. □

Notation. Let E'' denote $E' + E$.

The following lemma bounds how long a price update can be delayed until it is instantiated in the potential function.

Lemma 24. *If $d \geq 5$, and $\lambda E'' \leq \frac{1}{4}$ then the instantiation of a price update to p_i is delayed by at most one day, and it is instantiated by the time of the second subsequent price increase to p_i. (If the net effect of the price increase following the delayed decrease is to leave the price unchanged, this is considered the performing of the price update.)*

Proof. While the instantiation of update is being delayed, by Lemma 23, $x_i \geq (1 - \lambda E'')x'_i > (1 - \lambda E'')(d - 1)\widetilde{w}_i$. Hence, there has been an excess demand of at least $2w_i$ after a time period of at most $\frac{2w_i}{(1-\lambda E'')(d-1)\widetilde{w}_i} \leq \frac{3}{(1-\lambda E'')(d-1)} \leq 1$ day, as $\widetilde{w}_i \geq \frac{2}{3}w_i$ by Lemma 15; thus two price increases to p_i will occur in this time. The net change to p_i is a multiplicative increase of at least $(1 - \lambda)(1 + \lambda)^2 \geq 1$ as $\lambda \leq \lambda E \leq \frac{1}{4}$. Consequently, no later than the second price increase, the net change to the prices is instantiated. □

Lemma 25. *When p_i receives its delayed update, ϕ_i only decreases if $d \geq 5$, $1 \leq \alpha_2 \leq 2$,*

$$\left[1 - 2\lambda\alpha_1 - \frac{3\lambda E}{1-\lambda E}\left(1 + \frac{\lambda E''}{1-\lambda E''}\right)\right] \geq \frac{4}{3}\lambda\left[2(d-1)\frac{E}{1-\lambda E} + 1\right], \tag{12}$$

$$\lambda(\frac{2}{3} - \eta - \eta\lambda)\left(1 - \frac{1}{3}\alpha_2\right) \geq \lambda\alpha_1\left(3\left(\frac{\lambda E''}{1-\lambda E''} + 1\right) - \frac{2}{3}\right), \tag{13}$$

with

$$\eta = \frac{\lambda\alpha_1\left(3\left(\frac{\lambda E''}{1-\lambda E''} + 1\right) - \frac{2}{3}\right)}{\frac{4}{3}\lambda\left[2(d-1)\frac{E}{1-\lambda E} + 1\right]}. \tag{14}$$

This holds if $d = 5$, $\lambda E, \lambda E'' \leq \frac{1}{17}$, $\alpha_1 \leq \frac{1}{5}$, and $\alpha_2 = \frac{3}{2}$.

Proof. By Lemma 24, this update occurs within one day of its actual time, and no later than the second subsequent price increase. So when the update occurs, $\int_{\tau_i^s}^t (x_i - w_i)dt \leq 2w_i$. Hence $\int_{\tau_i^s}^t x_i \leq 3w_i$. Now,

$$\begin{aligned}\int_{\tau_i^s}^t x_i' dt &\leq \int_{\tau_i^s}^t x_i\left(1 + \frac{\lambda E''}{1-\lambda E''}\right)dt \qquad \text{(using Lemma 23)}\\ &\leq 3\left(1 + \frac{\lambda E''}{1-\lambda E''}\right)w_i.\end{aligned}$$

Thus,

$$\begin{aligned}\psi_d^i &\geq p_i\Bigg[\text{span}(x_i', d\widetilde{w}_i, \widetilde{w}_i) + [\widetilde{w}_i(\tau_i^s) - \overline{x}_i(\tau_i^s)]\left[1 - 2\lambda\alpha_1 - \frac{3\lambda E}{1-\lambda E}\left(1 + \frac{\lambda E''}{1-\lambda E''}\right)\right]\\ &\quad -\lambda\alpha_1\left(3\left(\frac{\lambda E''}{1-\lambda E''} + 1\right) - \frac{2}{3}\right)w_i + \alpha_2|\widetilde{w}_i - w_i|\Bigg] \qquad \text{as } (t - \tau_i) \leq 2.\end{aligned}$$

We need to show that this expression is at least as large as the potential following the update. Note that after the update $\phi_i = (p_i + \Delta_i p_i)[(x_i' + \Delta_i x_i' - \widetilde{w}_i) + \alpha_2|\widetilde{w}_i - w_i|]$. Of course, some of the ϕ_j, $j \neq i$, may also change as a result of the update.

Case 1: $x_i' = (d-1)\widetilde{w}_i$.

Then the original price update, a decrease, is applied at this moment. First, the span term is reduced by $\widetilde{w}_i$ to $x_i' - \widetilde{w}_i$, reducing the potential by $p_i\widetilde{w}_i \geq \frac{2}{3}p_i w_i$ (by Lemma 15). The term $-p_i\lambda\alpha_1\left(3\left(\frac{\lambda E''}{1-\lambda E''} + 1\right) - \frac{2}{3}\right)w_i$ is also removed. As $\lambda\alpha_1 \leq \frac{1}{9}$ and $\lambda E'' \leq \frac{1}{2}$, this is a net reduction.

Now, an analysis similar to that of Case 2.1 in Lemma 14 can be applied, as follows. First, we update p_i to $p_i + \Delta_i p_i$, and increase x_i' by a spending neutral change to $x_i' + \Delta_n x_i'$. By Lemma 13 (with $\widetilde{w}_i$ replacing w_i), this adds $\widetilde{w}_i\Delta_i p_i$ to the term $p_i(x_i' - \widetilde{w}_i)$ in ϕ_i. Next, we further increase x_i' to its final value $x_i' + \Delta_i x_i$, which increases the potentials ϕ_i and ϕ_j for $j \neq i$ by at most $2(p_i + \Delta_i p_i)x_i'\left[(1 - \frac{\Delta_i p_i}{p_i})^{-E} - 1\right] \leq 2(d-1)\widetilde{w}_i \cdot \frac{E}{1-\lambda E}\Delta_i p_i$, using Fact 1(d) with $\rho = \frac{E}{1-\lambda E}$ (recall that $\rho < 1$ suffices). The update leaves ϕ no larger if

$$p_i[\widetilde{w}_i(\tau_i^s) - \overline{x}_i(\tau_i^s)]\left[1 - 2\lambda\alpha_1 - \frac{3\lambda E}{1-\lambda E}\left(1 + \frac{\lambda E''}{1-\lambda E''}\right)\right] \geq \lambda\left[2(d-1)\frac{E}{1-\lambda E}\widetilde{w}_i\Delta_i p_i + \widetilde{w}_i\Delta_i p_i\right].$$

(Note that the term involving α_2 only decreases as the update is a price decrease.) But $|\Delta_i p_i| \le \lambda[\widetilde{w}_i(\tau_i^s) - \overline{x}_i(\tau_i^s)]p_i/w_i$. So it suffices that (12) holds (using Lemma 15 to bound $\widetilde{w}_i/w_i$ by $\frac{4}{3}$).

Case 2: $x_i' > (d-1)\widetilde{w}_i$.

The price update in this case is an increase.

Case 2.1: $\widetilde{w}_i - \overline{x}_i(\tau_i^s) \ge \eta w_i$.

If

$$\eta\left[1 - 2\lambda\alpha_1 - \frac{3\lambda E}{1-\lambda E}\left(1 + \frac{\lambda E''}{1-\lambda E''}\right)\right] \ge \lambda\alpha_1\left(3\left(\frac{\lambda E''}{1-\lambda E''}+1\right) - \frac{2}{3}\right), \tag{15}$$

then

$$\psi_i^d \ge p_i[\text{span}(x_i', d\widetilde{w}_i, \widetilde{w}_i) + \alpha_2|\widetilde{w}_i - w_i|]. \tag{16}$$

Note that the definition of η in (14) makes (15) equivalent to (12).

Now, we proceed as in the analysis of Case 1 of Lemma 14. We start by reducing the span term to $x_i' - \widetilde{w}_i$. Next, we update p_i to $p_i + \Delta_i p_i$, and decrease x_i' by a spending neutral change to $x_i' + \Delta_n x_i'$. By Lemma 13, this decreases ϕ_i by $\Delta_i p_i \widetilde{w}_i$. Finally, we further decrease x_i' to its final value $x_i' + \Delta_i x_i$; this only reduces ϕ_i and possibly increases the other ϕ_j by up to an equal amount. Thus the overall change to ϕ is only a decrease if $\Delta_i p_i \widetilde{w}_i \ge \Delta_i p_i \alpha_2 |\widetilde{w}_i - w_i|$. On applying Lemma 15, we see that $\frac{2}{3} \ge \frac{1}{3}\alpha_2$ suffices, i.e. $\alpha_2 \le 2$.

Case 2.2: $\widetilde{w}_i - \overline{x}_i(\tau_i^s) < \eta w_i$.

$$\text{Then,} \quad \psi_i^d \ge p_i\left[\text{span}(x_i', d\widetilde{w}_i, \widetilde{w}_i) - 3\lambda\alpha_1\left(3\left(\frac{\lambda E''}{1-\lambda E''}+1\right) - \frac{2}{3}\right)w_i + \alpha_2|\widetilde{w}_i - w_i|\right]. \tag{17}$$

We proceed as in Case 2.1. As there, the drop in the first term is at least $\widetilde{w}_i \Delta p_i$, and we need

$$\widetilde{w}_i \Delta p_i \ge \lambda\alpha_1 p_i w_i\left(3\left(\frac{\lambda E''}{1-\lambda E''}+1\right) - \frac{2}{3}\right) + \frac{1}{3}\alpha_2 w_i \Delta p_i.$$

Since $\widetilde{w}_i(\tau_i^s) - \overline{x}_i(\tau_i^s) \le \eta w_i$, the delayed price reduction was by at most a factor of $(1-\eta\lambda)$. The next (current) price increase is by a factor of $(1+\lambda)$, yielding a net increase of at least $(1+\lambda)(1-\eta\lambda) \ge 1 + (1-\eta)\lambda - \eta\lambda^2 \ge 1 + \lambda(1-\eta-\eta\lambda)$.

So it suffices that (13) holds. □

Lemma 26. *If* $2\lambda\alpha_1 \le 1$*, at the original time of a price decrease to good* i*,*

$$\phi_i \ge p_i\left[\text{span}(x_i', \overline{x}_i, \widetilde{w}_i) - \lambda\alpha_1(t-\tau_i)|\overline{x}_i - \widetilde{w}_i| + \alpha_2|\widetilde{w}_i - w_i|\right].$$

Proof. At the original time of the update, $\phi_i = \psi_i^r$. And

$$\int_{\tau_i}^{t}(x_i' - x_i)dt = (\overline{x}_i' - \overline{x}_i)(t - \tau_i).$$

Thus,

$$\psi_i^r = p_i\left[\text{span}(x_i', \overline{x}_i', \widetilde{w}_i) - \lambda\alpha_1(t-\tau_i)|\overline{x}_i' - \widetilde{w}_i| + (\overline{x}_i' - \overline{x}_i)\left[1 - \lambda\alpha_1(t-\tau_i)\right] + \alpha_2|\widetilde{w}_i - w_i|\right].$$

Case 1: $\overline{x}_i' \le \widetilde{w}_i$.

Recall that $\overline{x}'_i \geq \overline{x}_i$. Then $\text{span}(x'_i, \overline{x}'_i, \widetilde{w}_i) + (\overline{x}'_i - \overline{x}_i) \geq \text{span}(x'_i, \overline{x}_i, w\widetilde{w}_i)$ and $\lambda\alpha_1|\overline{x}'_i - \widetilde{w}_i| + \lambda\alpha_1(\overline{x}'_i - \overline{x}_i) = \lambda\alpha_1(\widetilde{w}_i - \overline{x}_i)$. So $\psi^r_i \geq p_i[\text{span}(x'_i, \overline{x}_i, \widetilde{w}_i) - \lambda\alpha_1(t - \tau_i)(\widetilde{w}_i - \overline{x}_i)]$. The claim holds in this case.

Case 2: $\overline{x}'_i > \widetilde{w}_i$.

Note that $\overline{x}_i < \widetilde{w}_i$ as there is supposed to be a price decrease at this time. So,

$$\text{span}(x'_i, \overline{x}'_i, \widetilde{w}_i) + (\widetilde{w}_i - \overline{x}_i)[1 - \lambda\alpha_1(t - \tau_i)] \geq \text{span}(x'_i, \overline{x}_i, \widetilde{w}_i)) - \lambda\alpha_1(t - \tau_i)(\widetilde{w}_i - \overline{x}_i).$$

And

$$-[1-\lambda\alpha_1(t-\tau_i)](\widetilde{w}_i-\overline{x}_i)-\lambda\alpha_1(t-\tau_i)(\overline{x}'_i-\widetilde{w}_i)+(\overline{x}'_i-\overline{x}_i)[1-\lambda\alpha_1(t-\tau_i)] = [1-2\lambda\alpha_1(t-\tau_i)](\overline{x}'_i-\widetilde{w}_i) \geq 0.$$

Thus

$$\psi^r_i \geq p_i[\text{span}(x'_i, \overline{x}_i, \widetilde{w}_i) - \lambda\alpha_1(t - \tau_i)(\widetilde{w}_i - \overline{x}_i) + \alpha_2|\widetilde{w}_i - w_i|].$$

The claim holds in this case too. □

Corollary 5. *If $2\lambda\alpha_1 \leq 1$, at the original time of a delayed price increase, $\psi^r_i \geq \psi^d_i$.*

Proof. Note that as the price increase is delayed, $x'_i \geq d\widetilde{w}_i > \widetilde{w}_i$. By Lemma 26,

$$\begin{aligned}\psi^r_i &\geq p_i\left[\text{span}(x'_i, \overline{x}_i, \widetilde{w}_i) - \lambda\alpha_1(t - \tau_i)|\overline{x}_i - \widetilde{w}_i| + \alpha_2|\widetilde{w}_i - w_i|\right] \\ &= p_i\left[|x'_i - \widetilde{w}_i| + |\widetilde{w}_i - \overline{x}_i| - \lambda\alpha_1(t - \tau_i)|\overline{x}_i - \widetilde{w}_i|(t - \tau_i) + \alpha_2|\widetilde{w}_i - w_i|\right].\end{aligned}$$

Note that at this time, $t = \tau^s_i$, and this expression in ψ^d_i. □

We restate Lemma 14 with x_i replaced by x'_i, w_i replaced by $\widetilde{w}_i$, and $x^u_i = \overline{x}_i$. Note that as in Lemma 18, $\psi_i = \alpha_2 p_i|\widetilde{w}_i - w_i|$.

Lemma 27. *If Constraint 1 holds and $x'_i \leq d\widetilde{w}_i$, then when p_i is updated, ϕ increases by at most the following:*
(i) With a toward $\widetilde{w}_i$ update:

$$\lambda\alpha_1|\overline{x}_i - \widetilde{w}_i|p_i + \frac{1}{2}\alpha_2\widetilde{w}_i|\Delta_i p_i| - \widetilde{w}_i|\Delta_i p_i|. \tag{18}$$

(ii) With an away from $\widetilde{w}_i$ update:

$$\left(1 + \frac{2Ed}{1 - \lambda E}\right)\widetilde{w}_i|\Delta_i p_i| + \frac{1}{2}\alpha_2\widetilde{w}_i|\Delta_i p_i| - (1 - \lambda\alpha_1)p_i|\overline{x}_i - \widetilde{w}_i|. \tag{19}$$

Lemma 28. *If Constraint 1 holds and $x'_i \leq d\widetilde{w}_i$, $\frac{\alpha_2}{2} + \max\{\frac{3}{2}, (d-1)\}\alpha_1 \leq 1$, and $\lambda\alpha_1 + \frac{4}{3}\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \leq 1$, and if p_i is updated at the regular time, then ψ^r_i only decreases.*

Proof. The proof is identical to that of Corollary 4. □

Our next goal is to show $\frac{d\phi}{dt} \leq -\Theta(\kappa)\phi$. We begin with two technical claims.

Lemma 29. *If $\frac{\lambda E}{1-\lambda E} < 1$, $x_i - x'_i \leq \frac{\lambda E''}{1-\lambda E''}[\widetilde{w}_i(\tau^s_i) - \overline{x}_i(\tau^s_i)]\frac{x'_i}{w_i}$ and $x_i \leq \left(1 + \frac{\lambda E}{1-\lambda E}\right)x'_i$.*

Proof. We seek to upper bound x_i/x'_i. Starting from the prices with delayed updates yielding demand x'_i, if one updated p_j, $j \neq i$, this would only decrease the demand for good i. So the only price change that increases the demand for good i is the update to p_i, which is by a $1 - \lambda \min\{1, [\widetilde{w}_i(\tau_i^s) - \overline{x}_i(\tau_i^s)]/w_i\}$ factor. Thus, by bounded elasticity, for a good i with a delayed price update:

$$\begin{aligned} \frac{x_i}{x'_i} &\leq \left[\frac{1}{1 - \lambda \min\{1, [\widetilde{w}_i(\tau_i^s) - \overline{x}_i(\tau_i^s)]/w_i)\}}\right]^{-E} \\ &\leq 1 + \frac{\lambda E}{1 - \lambda E} \min\{1, [\widetilde{w}_i(\tau_i^s) - \overline{x}_i(\tau_i^s)]/w_i\} \\ &\qquad \text{applying Fact 1(d) with } \rho = \frac{\lambda E}{1 - \lambda E}. \end{aligned}$$

So, $x_i - x'_i \leq \frac{\lambda E}{1-\lambda E} \min\{x'_i, [\widetilde{w}_i(\tau_i^s) - \overline{x}_i(\tau_i^s)]\frac{x'_i}{w_i}\}$. The claims follow. □

Lemma 30.

$$\sum_{i:\textit{delayed update}} p_i(x_i - x'_i) \geq \sum_{i:\textit{undelayed update}} p_i(x'_i - x_i).$$

Proof. Consider the change in spending that would occur were all the delayed price updates (decreases) to be made. This can only reduce the amount of money being held. Thus:

$$\sum_{i:\text{delayed update}} (p_i + \Delta_i p_i)x_i - p_i x'_i \quad + \sum_{i:\text{undelayed update}} p_i(x_i - x'_i) \quad \geq 0,$$

where $\Delta_i p_i < 0$ for each delayed update. The claim follows. □

Notation: Let $\text{span}_i = \text{span}(x'_i, \overline{x}'_i, \widetilde{w}_i)$.

Lemma 31. *Let* $\nu = \min\left\{\frac{\lambda\alpha_1(d-2)}{2(d-1)+\alpha_2}, \frac{\kappa(\alpha_2-1)}{2}\right\}$. *If* $d \geq 5$, $4\kappa(1+\alpha_2) \leq \lambda\alpha_1 \leq \frac{1}{2}$, *and* $\kappa\left(d - 1 + \alpha_2\left(1 + \frac{\lambda E}{1-\lambda E}\right)\frac{d-1}{d-2}\right) \leq \frac{1}{2}\lambda\alpha_1$, *then* $\frac{d}{dt}\phi \leq -\nu\phi$.

Proof. We will show that

$$\begin{aligned} &\text{(i)} \qquad \frac{d\psi_i^r}{dt} \leq -\frac{\kappa(\alpha_2 - 1)}{2}\psi_i^r + (x'_i - x_i)p_i, \\ \text{and} \quad &\text{(ii)} \qquad \frac{d\psi_i^d}{dt} \leq -\frac{\lambda\alpha_1(d-2)}{2(d-1)+\alpha_2}\psi_i^d - (x_i - x'_i)p_i. \end{aligned}$$

Summing over all i, and then applying Lemma 30, we conclude that

$$\begin{aligned} \frac{d\psi}{dt} &\leq -\nu\psi \quad - \sum_{i:\text{delayed update}} p_i(x_i - x'_i) + \sum_{i:\text{undelayed update}} p_i(x'_i - x_i) \\ &\leq -\nu\psi. \end{aligned}$$

Next, we show (i). Let $\widetilde{\psi}_i^r = p_i[\text{span}_i - \lambda\alpha_1(t - \tau_i)|\overline{x}'_i - \widetilde{w}_i| + \alpha_2|\widetilde{w}_i - w_i|]$.

Lemma 16 shows that

$$\frac{d\widetilde{\psi}_i^r}{dt} \leq -\frac{\kappa(\alpha_2 - 1)}{2}\widetilde{\psi}_i^r$$

for $\widetilde{\psi}_i^r$ is obtained from ψ_i^r by replacing x_i with x_i' in the function ϕ_i used in Lemma 16 and having $\chi_i = 0$ there. Now,

$$\frac{d\psi_i^r}{dt} = \frac{d\widetilde{\psi}_i^r}{dt} + p_i \left[-\lambda\alpha_1 \int_{\tau_i^s}^{t} (x_i' - x_i)dt + [1 - \lambda\alpha_1(t - \tau_i)](x_i' - x_i) \right].$$

As $x_i' \geq x_i$, (i) follows.

We show (ii) next.

$$\frac{d\psi_i^d}{dt} \leq p_i \left[(d-1)\kappa|x_i - w_i| - \lambda\alpha_1[\widetilde{w}_i(\tau_i^s) - \overline{x}_i(\tau_i^s)] - \lambda\alpha_1(x_i' - \widetilde{w}_i) + \alpha_2\kappa|x_i - w_i| \right] - p_i \left(1 + \frac{\lambda E}{1 - \lambda E}\right) [\widetilde{w}_i(\tau_i^s) - \overline{x}_i(\tau_i^s)] \frac{x_i'}{w_i}.$$

Next, we bound $(d - 1 + \alpha_2)\kappa|x_i - w_i|$ by $\frac{1}{2}\lambda\alpha_1(x_i' - \widetilde{w}_i)$. By Lemma 29, $x_i \leq \left(1 + \frac{\lambda E}{1-\lambda E}\right) x_i'$. Recall that $x_i' \geq (d-1)\widetilde{w}_i$, and so $x_i \leq \left(1 + \frac{\lambda E}{1-\lambda E}\right) [(x_i' - \widetilde{w}_i) + \widetilde{w}_i] \leq \left(1 + \frac{\lambda E}{1-\lambda E}\right) (1 + \frac{1}{d-2})(x_i' - \widetilde{w}_i) = \left(1 + \frac{\lambda E}{1-\lambda E}\right) \frac{d-1}{d-2}(x_i' - \widetilde{w}_i)$. As $\kappa(d-1+\alpha_2) \left(1 + \frac{\lambda E}{1-\lambda E}\right) \frac{d-1}{d-2} \leq \frac{1}{2}\lambda\alpha_1$, if $x_i \geq w_i$, $(d-1+\alpha_2)\kappa|x_i - w_i| \leq \frac{1}{2}\lambda\alpha_1(x_i' - \widetilde{w}_i)$; otherwise, if $x_i < w_i$, $(d-1+\alpha_2)\kappa|x_i - w_i| \leq (d-1+\alpha_2)\kappa w_i \leq \frac{3}{2}(d-1+\alpha_2)\kappa\widetilde{w}_i \leq \frac{3}{2(d-2)}(d-1+\alpha_2)\kappa(x_i' - \widetilde{w}_i) \leq \frac{1}{2}\lambda\alpha_1(x_i' - \widetilde{w}_i)$. Thus

$$\frac{d\psi_i^d}{dt} \leq -\lambda\alpha_1[\widetilde{w}_i(\tau_i^s) - \overline{x}_i(\tau_i^s)] - \frac{1}{2}\lambda\alpha_1(x_i' - \widetilde{w}_i) - (x_i - x_i').$$

As $x_i' \geq (d-1)\widetilde{w}_i$, $\text{span}(x_i', d\widetilde{w}_i, \widetilde{w}_i) \leq \frac{d-1}{d-2}(x_i' - \widetilde{w}_i)$. Similarly, $\alpha_2|\widetilde{w}_i - w_i| \leq \frac{1}{2}\alpha_2\widetilde{w}_i \leq \frac{1}{2(d-2)}\alpha_2(x_i' - \widetilde{w}_i)$. It follows that $\text{span}(x_i', d\widetilde{w}_i, \widetilde{w}_i) + \alpha_2|\widetilde{w}_i - w_i| \leq \left[\frac{d-1}{d-2} + \frac{\alpha_2}{2(d-2)}\right](x_i' - \widetilde{w}_i)$. Thus

$$\frac{d\psi_i^d}{dt} \leq \frac{(d-2)}{2(d-1)+\alpha_2}\lambda\alpha_1\psi_i^d - (x_i - x_i').$$

□

Theorem 5. *If Constraint 1 holds, $d = 5$, $\alpha_2 = \frac{3}{2}$, $\lambda E'' \leq \frac{1}{17}$, $\alpha_1 \leq \frac{1}{16}$, $\lambda\alpha_1 + \frac{4}{3}\lambda\left(\frac{7}{4} + \frac{10E}{1-\lambda E}\right) \leq 1$, $\kappa \leq \frac{loa}{13}$, and each price is updated at least once every day, then ϕ decreases by at least a $1 - \nu$ factor daily, where $\nu = \min\left\{\frac{\lambda\alpha_1(d-2)}{2(d-1)+\alpha_2}, \frac{\kappa(\alpha_2-1)}{2}\right\}$.*

Proof. By Lemmas 25 and 28, ϕ only decreases whenever a price update occurs. By Lemma 31, $d\phi/dt \leq -\nu\phi$, which implies $\phi(t+1) \leq e^{-\nu}\phi(t) \leq (1 - \frac{\nu}{2})\phi(t)$ as $\nu \leq \frac{1}{2}$. On substitution of the values and bounds for $d, \alpha_2, \lambda E, \lambda E''\alpha_1$, the constraints $\frac{\alpha_2}{2} + \alpha_1 \max\{\frac{3}{2}, (d-1)\} \leq 1$, $\lambda\alpha_1 + \frac{4}{3}\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \leq 1$, $4\kappa(1+\alpha_2) \leq \lambda\alpha_1 \leq \frac{1}{2}$, $\kappa\left(d - 1 + \alpha_2\left(1 + \frac{\lambda E}{1-\lambda E}\right)\frac{d-1}{d-2}\right) \leq \frac{1}{2}\lambda\alpha_1$, from Lemmas 25 , 28 and 31, reduce to the constraints stated in the current lemma. □

Finally, we relate the potential to the misspending.

Notation. Let $S_i = p_i(|x_i - \widetilde{w}_i| + |\overline{x}_i - \widetilde{w}_i|) + p_i|\widetilde{w}_i - w_i|$; this is called be the misspending on the ith good. SO $S = \sum_i S_i$; this is the total misspending.

Lemma 32. *$S = O(\phi) = O(S + M)$, where M is the daily supply of money.*

Proof. We observe that $\psi_i^r = \theta\left(p_i\left[\text{span}(x_i', \overline{x}_i', \widetilde{w}_i) + \int_{\tau_i}^t (x_i' - x_i)dt + \alpha_2|\widetilde{w}_i - w_i|\right]\right)$, and $\psi_i^d = \theta(\text{span}(x_i', d\widetilde{w}_i, \widetilde{w}_i) + \alpha_2|\widetilde{w}_i - w_i|)$. The first bound follows from the fact that $\lambda\alpha_1 \le \frac{1}{2}$. For the second bound, we argue as in the proof of Lemma 25. As no update has been applied, $x_i' > (d-1)\widetilde{w}_i$, so Case 2 applies. In Case 2.1 the claim is immediate by (16). In Case 2.2 the claim follows from (17), as $3\lambda\alpha_1\left(3\left(\frac{\lambda E}{1-\lambda E} + 1\right) - \frac{2}{3}\right) w_i \le \widetilde{w}_i \le \frac{1}{2}\text{span}(x_i', d\widetilde{w}_i, \widetilde{w}_i)$.

Now, we bound S.

The following observation will be used twice: for a good i with a delayed update,

$$x_i - x_i' \le \frac{\lambda E}{1 - \lambda E} x_i' \le \frac{\lambda E}{1 - \lambda E}\frac{d-1}{d-2}(x_i' - \widetilde{w}_i). \tag{20}$$

This follows by using Lemma 29 for the second inequality, and the bound $x_i' > (d-1)\widetilde{w}_i$ for the third.

Note that for a good i with a delayed price update, $S_i = \theta[(x_i' - \widetilde{w}_i)p_i + p_i|\widetilde{w}_i - w_i|] = \theta(\psi_i^d)$, for if $x_i \ge \widetilde{w}_i$, $x_i - \widetilde{w}_i \le (x_i - x_i') + (x_i' - \widetilde{w}_i) \le \frac{\lambda E}{1-\lambda E}\frac{d-1}{d-2}(x_i' - \widetilde{w}_i) + (x_i' - \widetilde{w}_i)$, using(20); while if $x_i < \widetilde{w}_i$, then $\widetilde{w}_i - x_i \le \widetilde{w}_i \le x_i' - \widetilde{w}_i$.

For a good i with an up to date price, define

$$\begin{aligned}
S_i &= O(\text{span}(x_i, \overline{x}_i, \widetilde{w}_i)p_i + \alpha_2 p_i|\widetilde{w}_i - w_i|) \\
&= O(\text{span}(x_i', \overline{x}_i', \widetilde{w}_i)p_i + p_i(x_i' - x_i) + p_i(\overline{x}_i' - \overline{x}_i) + \alpha_2 p_i|\widetilde{w}_i - w_i|) \\
&= O(\text{span}(x_i', \overline{x}_i', \widetilde{w}_i)p_i + \int_{\tau_i}^t p_i(x_i' - x_i)dt + \alpha_2 p_i|\widetilde{w}_i - w_i| + p_i(x_i' - x_i)) \\
&= O(\psi_i^r + p_i(x_i' - x_i)).
\end{aligned}$$

Now, by Lemma 30 for the first inequality, and (20) for the second,

$$\begin{aligned}
\sum_{i \text{ update not delayed}} p_i(x_i' - x_i) &\le \sum_{i \text{ update delayed}} p_i(x_i - x_i') \\
&\le \sum_{i \text{ delayed}} p_i \frac{\lambda E}{1 - \lambda E}\frac{d-1}{d-2}(x_i' - \widetilde{w}_i) \\
&= O(\sum_i \psi_i^d).
\end{aligned}$$

Thus $S = \sum_i S_i = O(\sum_i \phi_i) = O(\phi)$.

Finally, we bound ϕ.

$$\begin{aligned}
\psi_i^d &= O(x_i' - \widetilde{w}_i) + \alpha_2 p_i|\widetilde{w}_i - w_i|) \\
&\le O(p_i(x_i - \widetilde{w}_i) + \alpha_2 p_i|\widetilde{w}_i - w_i| + p_i(x_i' - x_i)) \\
&= O(S_i + p_i x_i) \quad \text{using Lemma 23.}
\end{aligned}$$

So $\sum_i \psi_i^d = O(S + \sum_i p_i x_i) = O(S + M)$.

$$\begin{aligned}\psi_i^r &= O\left(p_i \operatorname{span}(x_i', \overline{x}_i', \widetilde{w}_i) + \alpha_2 p_i|\widetilde{w}_i - w_i|) + p_i \int_{\tau_i}^t (x_i' - x_i)dt\right)\\ &= O(p_i \operatorname{span}(x_i, \overline{x}_i, \widetilde{w}_i) + \alpha_2 p_i|\widetilde{w}_i - w_i|) + p_i(x_i' - x_i) + p_i(\overline{x}_i' - \overline{x}_i))\\ &= O(S_i + p_i x_i' + p_i \overline{x}_i') \quad \text{using Lemma 23.}\end{aligned}$$

So $\sum_i \psi_i^r = O(S + \sum_i p_i x_i' + p_i \overline{x}_i') = O(S + M)$. □

3.7 Bounds on Warehouse Sizes

We begin with two technical lemmas which show that (i) if a warehouse is rather full and the price does not decrease too much henceforth then the warehouse eventually becomes significantly less full, and (ii) an analogous result for the event that a warehouse is rather empty.

Let $\alpha_4 = \min_i \frac{\kappa c_i}{8 w_i}$. Recall that $\widetilde{w}_i - w_i = \kappa(s_i - s_i^*)$. Thus $|\widetilde{w}_i - w_i| \le 4\alpha_4 w_i$.

Lemma 33. *Suppose that $s_i \ge s_i^* + \frac{\alpha_4}{\kappa} w_i$. Consider the next k updates to p_i. Let p_{i_1} be p_i's current value, and $p_{i_2}, \dots, p_{i_{k+1}}$ be its k successive values. Let $\overline{x}_{i_j}$ be the average value of x_i while $p = p_{i_j}$. Suppose that $p_{i_{k+1}} \ge e^{-\lambda f} p_{i_1}$ for some $f \ge 0$. If $k \ge \left(1 + \frac{2}{\alpha_4}\right) f + \frac{2}{\alpha_4} c$, then the warehouse stock will have decreased to less than $s_i^* + \frac{\alpha_4}{\kappa} w_i$ at some point, or by at least $c w_i$, whichever is the lesser decrease.*

Proof. Suppose that $s_i \ge s_i^* + \frac{\alpha_4}{\kappa} w_i$ at the time of each of the k price updates (or the result holds trivially).

Then a multiplicative price decrease by $(1 - \lambda)$ is preceded by an additive increase to the warehouse stock of at most w_i. A smaller price decrease, by $1 + \lambda(\min\{1, \overline{x}_{i_j} - \widetilde{w}_{i_j})/w_i\}$, for $\widetilde{w}_{i_j} - \overline{x}_{i_j} < w_i$, follows a warehouse increase of exactly $w_i - \overline{x}_{i_j} \le \widetilde{w}_{i_j} - \overline{x}_{i_j} - \alpha_4 w_i$.

Note that $1 + x \le e^x$ for $|x| \le 1$.

Suppose that there are $f + a$ price decreases by $1 - \lambda$. Then there are at least a price increases (as the total price decreases is by at most $e^{-\lambda f}$). Suppose that there are $a + a'$ price changes other than the price decreases by $1 - \lambda$. Now, $\sum_{\widetilde{w}_{i_j} - \overline{x}_{i_j} < w_i} \lambda \min\{1, (\overline{x}_{i_j} - \widetilde{w}_{i_j})/w_i\} - (f + a)\lambda \ge -\lambda f$, as this is the exponent in an upper bound on the price decrease. Hence $\sum_{\widetilde{w}_{i_j} - \overline{x}_{i_j} < w_i} \widetilde{w}_{i_j} - \overline{x}_{i_j} \le \sum_{\widetilde{w}_{i_j} - \overline{x}_{i_j} < w_i} \max\{-w_i, (\widetilde{w}_{i_j} - \overline{x}_{i_j})\} \le [f - (f + a)] w_i$. It follows that the warehouse stock increases are bounded by $(f + a) w_i + \sum_{\widetilde{w}_{i_j} - \overline{x}_{i_j} < w_i} (\widetilde{w}_{i_j} - \overline{x}_{i_j} - \alpha_4 w_i) \le [f - \alpha_4(a + a')] w_i$. As $f + 2a + a' = k \ge \left(1 + \frac{2}{\alpha_4}\right) f + \frac{2}{\alpha_4} c$, $a + a' \ge \frac{1}{\alpha_4}(f + c)$, and so the decrease in warehouse stock is at least $c w_i$. □

Lemma 34. *Suppose that $\lambda \left(1 + \frac{1}{\alpha_4}\right) \le \frac{1}{2}$ and $s_i \le s_i^* + \frac{\alpha_4}{\kappa} w_i$. Let $p_{i_1}, \dots, p_{i_{k+1}}$, $\overline{x}_{i_1}, \dots, \overline{x}_{i_k}$ be as in Lemma 33. Suppose that $p_{i_{k+1}} \le e^{\lambda f} p_{i_1}$ for some $f \ge 0$. Further suppose that the excess demand between successive price increases is at most w_i. If*

$$k \ge (1 + \lambda) \left(1 + \frac{4}{\alpha_4}\right) f + \frac{8}{\alpha_4} \lambda + \frac{4}{\alpha_4} c,$$

then the warehouse stock will have increased to more than $s_i^* - \frac{\alpha_4}{\kappa} w_i$ at some point, or by at least $c w_i$, whichever is the lesser increase.

Proof. Similarly to the proof of Lemma 33, suppose that $s_i \leq s_i^* - \frac{\alpha_4}{\kappa} w_i$ throughout this time.

Then a multiplicative price increase by $(1+\lambda)$ is preceded by an additive decrease of at most w_i in the warehouse stock. A smaller price increase, by $1+\lambda(\max\{-1, \overline{x}_{i_j} - \widetilde{w}_{i_j})/w_i\}$ for $\overline{x}_{i_j} - \widetilde{w}_{i_j} < w_i$, follows a warehouse increase of $w_i - \overline{x}_{i_j} \geq \widetilde{w}_{i_j} - \overline{x}_{i_j} + \alpha_4 w_i$.

Note that $1 + \lambda x \geq e^{\lambda x/(1+\lambda)} \geq e^{\lambda x - 2\lambda^2}$ for $0 \leq \lambda, x \leq 1$, and $1 + \lambda x \geq e^{\lambda x/(1-\lambda)} \geq e^{\lambda x - 2\lambda^2}$, for $-1 \leq x \leq 1$ and $0 < \lambda \leq \frac{1}{2}$.

Suppose that there are $(1+\lambda)(f+a)$ price increases by $1+\lambda$. Then there are at least $a(1-\lambda)$ price decreases (for each such change is a drop of at most $1 - \lambda \geq e^{-\lambda/(1-\lambda)}$, and the price increases yielded an increase of least $e^{(a+f)(1+\lambda)\lambda/(1+\lambda)} = e^{\lambda(a+f)}$).

Now,

$$\sum_{\overline{x}_{i_j} - \widetilde{w}_{i_j} < w_i} \left[\lambda(\overline{x}_{i_j} - \widetilde{w}_{i_j})/w_i - 2\lambda^2\right] + (1+\lambda)(f+a)\frac{\lambda}{1+\lambda} \leq \lambda f,$$

as this is the exponent in a lower bound on the price increase. That is

$$\sum_{\overline{x}_{i_j} - \widetilde{w}_{i_j} < w_i} (\widetilde{w}_{i_j} - \overline{x}_{i_j} + 2\lambda) \geq a w_i$$

Suppose that there are $a(1-\lambda) + a'$ price changes other than the increases by $(1+\lambda)$.

The decrease in the warehouse stock is at most:

$$\begin{aligned} & (1+\lambda)(f+a)w_i - \sum_{\overline{x}_{i_j} - \widetilde{w}_{i_j} < w_i} (\widetilde{w}_{i_j} - \overline{x}_{i_j} + \alpha_4 w_i) \\ \leq \quad & (1+\lambda)(f+a)w_i - a w_i + 2\lambda w_i - \alpha_4(a' + (1-\lambda)a)w_i \\ \leq \quad & w_i \left[(1+\lambda)f + 2\lambda - \alpha_4 \left[a' + \left(1 - \left(1 + \frac{1}{\alpha_4}\right)\lambda\right) a\right]\right] \end{aligned}$$

Now $f(1+\lambda) + 2a + a' = k$, so $\left(1 - \left(1 + \frac{1}{\alpha_4}\right)\lambda\right) a + a' \geq \frac{\left(1 - \left(1 + \frac{1}{\alpha_4}\right)\lambda\right)}{2}[k - f(1+\lambda)]$.

If $\alpha_4 \left[a' + \left(1 - \left(1 + \frac{1}{\alpha_4}\right)\lambda\right) a\right] \geq 2\lambda + (1+\lambda)f + c$, the warehouse stock increases by at least $c w_i$. It suffices to have

$$k \geq (1+\lambda)f + \frac{2\alpha_4}{\left(1 - \left(1 + \frac{1}{\alpha_4}\right)\lambda\right)}\left[2\lambda + (1+\lambda)f + c\right].$$

By assumption $\left(1 + \frac{1}{\alpha_4}\right)\lambda \leq \frac{1}{2}$, so $k \geq (1+\lambda)\left(1 + \frac{4}{\alpha_4}\right) f + \frac{8}{\alpha_4}\lambda + \frac{4}{\alpha_4} c$ suffices. □

In order to apply Lemma 34, we need to bound the length of the initial time interval after which all demands satisfy $x_i \leq 2w_i$. We will need several preliminary lemmas. In the next section, we will avoid this difficulty by modifying the price update rule for the case that demands are large.

Lemma 2. *If $\frac{\lambda E}{1-\lambda E} \leq \frac{1}{6}$, and if initially the prices have all been c-demand bounded for a full day for some $c \geq 2$, they remain c-demand bounded thereafter.*

Proof. First we analyze the bounds for low prices.

For any good i, if $p_i \leq p_i^{(c)}(1+\lambda)^2$, then by WGS and bounded elasticity, $x_i \geq c(1+\lambda)^{-2E} w_i \geq c(1 - 2\lambda E) w_i$, as $2\lambda E \leq 1$, and hence $\overline{x}_i \geq c(1 - 2\lambda E) w_i$ also, which means that the next update to p_i will be not be a decrease so long as $(1-2\lambda E)w_i \geq \widetilde{w}_i$. Applying Lemma 15 to give $\widetilde{w}_i \leq \frac{4}{3}w_i$, shows $\lambda E \leq \frac{1}{6}$ suffices.

Otherwise, a price decrease decreases p_i to at most $p_i^{(c)}(1+\lambda)^2(1-\lambda) \geq p_i^{(c)}$; that is, p_i remains above the lower bound.

Similarly, if $p_i \geq p_i^{(1/c)}(1-\lambda)^2$, then $x_i \leq \frac{1}{c}(1-\lambda)^{-2E}w_i \leq \frac{1}{c}(1+2\frac{\lambda E}{1-\lambda E})w_i$, as $2\frac{\lambda E}{1-\lambda E} \leq 1$, and hence $\overline{x}_i \geq \frac{1}{c}(1-2\frac{\lambda E}{1-\lambda E})w_i$ also, which means that the next update to p_i will be not be an increase so long as $\frac{1}{c}(1+2\frac{\lambda E}{1-\lambda E})w_i \leq \widetilde{w}_i$. Here $\frac{\lambda E}{1-\lambda E} \leq \frac{1}{6}$ suffices.

Otherwise, a price increase increases p_i to at most $p_i^{(1/c)}(1-\lambda)^2(1+\lambda) \leq p_i^{(1/c)}$; that is, p_i remains below the upper bound. □

Lemma 35. *$\sum_i w_i p_i \leq 3(\phi + M)$, where M is the daily supply of money.*

Proof. We note that if $x_i \leq \frac{1}{3}w_i$, then by Lemma 15 $x_i \leq \frac{1}{2}\widetilde{w}_i$, and $\widetilde{w}_i - x_i \geq \frac{1}{2}\widetilde{w}_i$. Now

$$\begin{aligned}\sum_i w_i p_i &= \sum_{x_i \leq w_i/3} w_i p_i + \sum_{x_i \geq w_i/3} w_i p_i \\ &\leq \sum_i |\widetilde{w}_i - x_i| p_i + \sum_i 3x_i p_i \leq \phi + M.\end{aligned}$$

□

Lemma 36. *Let $i = \text{argmax}\{\frac{p_i^*}{p_i}\}$. If $p_i < p_i^*$, then the total misspending, $\sum_h |x_h - w_h| p_h$, is at least $w_i(p_i^* - p_i)$. Similarly, let $j = \text{argmax}\{\frac{p_j}{p_j^*}\}$. If $p_j > p_j^*$, then the total misspending is at least $w_i(p_j - p_j^*)$.*

Proof. By WGS, when the price of a good is reduced, the spending on that good only increases. Consider reducing the prices from their equilibrium values, one by one, for each of the goods whose price is below its equilibrium value. Each price reduction can be viewed as occuring in two stages, first a spending neutral change, which leaves the spending on every good unchanged, and second a stage which increases the spending on the good whose price is being reduced. Clearly, there is no reduction of spending in total, on the goods whose prices are reduced. Thus the excess spending is at least that which would be obtained were all the prices spending neutral. Consider good i in this spending neutral scenario. At price p_i, the excess spending on good i would be $p_i(x_i - w_i) = p_i^* w_i - p_i w_i$, as claimed.

The proof of the second claim is analogous. □

Lemma 37. *Let ϕ_{init} be the initial value for ϕ. If $\alpha_2 > 1$ and $|\widetilde{w}_i - w_i| \leq \alpha_3 w_i = 4\alpha_4 w_i$ for all i, where*

$$\alpha_3 \leq \frac{(1-\lambda\alpha_1)(1-6(1+2\alpha_2))}{12\gamma(1+2\alpha_2)}$$

and $\gamma = \max_i \frac{M}{p_i^* w_i}$, then after

$$D = \frac{16(1+\alpha_2)}{\lambda\alpha_1} \log \frac{2\phi_{init}}{(1-\lambda\alpha_1)\min_i p_i^* w_i}$$

days all demands satisfy $x_i \leq 2w_i$ henceforth.

Proof. We want to show that $\phi \geq 2(1+2\alpha_2)\sum_i |\widetilde{w}_i - w_i|p_i$, while some $p_i \notin [\frac{1}{2}p_i^*, \frac{3}{2}p_i^*]$, for then we can apply Corollary 3 to obtain that $\frac{d\phi}{dt} \leq -\frac{\lambda\alpha_1}{8(1+\alpha_2)}\phi$ (as $\chi_i = 0$ for all i in the present setting). The condition on ϕ holds if $\phi \geq 2(1+2\alpha_2)\alpha_3 \sum_i w_i p_i$, and by Lemma 35, this holds if $\phi \geq 6(1+2\alpha_2)\alpha_3(\phi + M)$; in turn, it suffices that

$$\phi \geq \frac{6(1+2\alpha_2)\alpha_3 M}{1 - 6(1+2\alpha_2)\alpha_3}.$$

Now

$$\begin{aligned}
\phi &\geq \sum_i [(1-\lambda\alpha_1)|x_i - \widetilde{w}_i| + \alpha_2|\widetilde{w}_i - w_i|]p_i \\
&\geq \sum_i [(1-\lambda\alpha_1)(|x_i - w_i| - |\widetilde{w}_i - w_i|) + \alpha_2|\widetilde{w}_i - w_i|]p_i \\
&\geq (1-\lambda\alpha_1)\sum_i (|x_i - w_i|p_i \qquad \text{as } \alpha_2 > 1.
\end{aligned}$$

Suppose there is a good h such that $p_h \notin [\frac{1}{2}p_h^*, \frac{3}{2}p_h^*]$; then let $h = \text{argmax}\{\frac{p_i}{p_i^*}, \frac{p_i^*}{p_i}\}$. By Lemma 36, $\sum_i |x_i - w_i|p_i \geq w_h|p_h - p_h^*| > \frac{1}{2}w_h p_h^* \geq \frac{1}{2\gamma}M$. Thus $\phi \geq \frac{1}{2}w_h p_h^* \geq \frac{1}{2\gamma}M$.

So the condition holds if

$$\frac{1-\lambda\alpha_1}{2\gamma}M \geq \frac{6(1+2\alpha_2)}{1-6(1+2\alpha_2)\alpha_3}\alpha_3 M.$$

It suffices that

$$\alpha_3 \leq \frac{(1-\lambda\alpha_1)(1-6(1+2\alpha_2))}{12\gamma(1+2\alpha_2)}$$

to ensure that $\frac{d\phi}{dt} \leq -\frac{\lambda\alpha_1}{8(1+\alpha_2)}\phi$. Thus after at most

$$D = \frac{16(1+\alpha_2)}{\lambda\alpha_1}\log\frac{\phi_{\text{init}}}{\frac{1-\lambda\alpha_1}{2}\min_i w_i p_i^*}$$

days, $\phi \leq \frac{1}{2}(1-\lambda\alpha_1)p_i^* w_i$ for all i.

Suppose, for a contradiction, that after D days some $x_h > 2w_h$. Then $p_h(x_h - w_h) > \frac{1}{2}p_h^* w_h$. Now $\phi \geq (1-\lambda\alpha_1)\sum_i |x_i - w_i|p_i \geq (1-\lambda\alpha_1)(x_h - w_h)p_h > \frac{1}{2}(1-\lambda\alpha_1)w_h p_h^* \geq \phi$. □

We now obtain a bound on the depletion of the warehouse stock while the prices are decreasing from their initial values to a value that guarantees 2-demand bound henceforth, which allows Lemmas 33 and 34 to be applied.

Lemma 38. *During the time that ϕ reduces to at most $\frac{1}{2}(1-\lambda\alpha_1)w_h p_h^* \geq \phi$, for each i, the stock of warehouse i reduces by at most*

$$(d-1)Dw_i = (d-1)w_i\frac{16(1+\alpha_2)}{\lambda\alpha_1}\log\frac{\phi_{init}}{\frac{1-\lambda\alpha_1}{2}\min_i w_i p_i^*}.$$

Proof. Each day, the stock can reduce by at most $(d-1)w_i$. By Lemma 37, the stated reduction in potential occurs within D days. □

Now we are ready to bound how large a warehouse suffices, given an upper bound on ϕ_{init}. Recall that we view the warehouse as having 8 equal sized zones of fullness.

Note that if the ith warehouse were completely full or empty, then $|\widetilde{w}_i - w_i| = \frac{1}{2}\kappa c_i$, and as we assumed that $|\widetilde{w}_i - w_i| \le \alpha_3 w_i$, this implies that $\frac{1}{2}\kappa c_i \le \alpha_3 w_i$. Note that Constraint 1 implies $\frac{1}{2}\kappa c_i \le \frac{1}{3} w_i$, which is ensured if $\alpha_3 \le \frac{1}{3}$.

Theorem 6. *Suppose that the prices are always f-bounded and let $d = d(f)$. Also suppose that each price is updated at least once a day. Suppose further that the warehouses are initially all in their safe or inner buffer zones. Finally, suppose that $\lambda\left(1 + \frac{1}{\alpha_4}\right) \le \frac{1}{2}$. Then the warehouse stocks never go outside their outer buffers (i.e. they never overflow or run out of stock) if $\frac{\alpha_4}{\kappa} = \frac{c_i}{8w_i} \ge \max\left\{(d-1)D, 2\left(1 + \frac{4}{\alpha_4}\right)\frac{f}{\lambda} + \frac{8\lambda}{\alpha_4}\right\}$; furthermore, after $D + 2\left(1 + \frac{4}{\alpha_4}\right)\frac{f}{\lambda} + \frac{8\lambda}{\alpha_4} + \frac{8}{\kappa}$ days the warehouses will be in their safe or inner buffer zones thereafter, where*

$$D = \frac{16(1+\alpha_2)}{\lambda\alpha_1} \log \frac{\phi_{init}}{\frac{1-\lambda\alpha_1}{2}\min_i w_i p_i^*},$$

and ϕ_{init} is the initial value of ϕ.

If the fast updates rule is followed, then it suffices to have $\frac{\alpha_4}{\kappa} = \frac{c_i}{8w_i} \ge 2\left(1 + \frac{4}{\alpha_4}\right) f + \frac{8\lambda}{\alpha_4}$, and then after $\left(1 + \frac{4}{\alpha_4}\right) f + \frac{8\lambda}{\alpha_4}\lambda + \frac{8}{\kappa}$ days the warehouses will be in their safe or inner buffer zones thereafter.

Again, substituting our bounds on $\lambda, \alpha_1, \alpha_4$ suggests D is rather large. The result should be viewed as indicating that there is a bound on the needed warehouse size, but not that this is a tight bound (in terms of constants).

Proof. We will consider warehouse i. After an initial D days (as defined in Lemma 37), the demands are henceforth 2-bounded. In this time, by Lemma 38, the warehouse stocks can decrease by at most $(d-1)Dw_i \le \frac{1}{8}c_i$. As a result the warehouses are all in their middle buffers or nearer the center at this point.

We show that henceforth the tendency is to improve, i.e. move toward the safe zone, but there can be fluctuations of up to one zone width. The result is that every warehouse remains within its outer buffer, and after a suitable time they will all be in either their inner buffer or safe zone.

As it is f-bounded, price p_i can decrease by at most $e^{-2(f/\lambda)\lambda}$ from its initial value. If $s_i(t)$ is in middle buffer at time t, then by Lemma 33 (taking $c = 0$), within $2\left(1 + \frac{2}{\alpha_4}\right)\frac{f}{\lambda}$ days the value of s_i will have returned to $s_i(t)$ or remained below this value. During this interval a decrease of ω in the warehouse stock takes at least ω days to replenish; consequently the stock can increase by at most $2\left(1 + \frac{2}{\alpha_4}\right)\frac{f}{\lambda}w_i$. It follows that the warehouse never runs out of stock as each portion of the outer buffer has width at least $\frac{1}{8}c_i \ge 2\left(1 + \frac{2}{\alpha_4}\right)\frac{f}{\lambda}w_i$. Again, by Lemma 33, taking $c = \frac{c_i}{4w_i}$, after $2\left(1 + \frac{2}{\alpha_4}\right)\frac{f}{\lambda} + \frac{c_i}{2\alpha_4 w_i}$ days it will have reached the safe zone. The preceding argument shows that it can never go above the inner buffer henceforth.

We apply the same argument to the low zones using Lemma 34. Now, with $c = 0$, we see that within $2\left(1 + \frac{4}{\alpha_4}\right)\frac{f}{\lambda} + \frac{8\lambda}{\alpha_4}\lambda$ days s_i returns to the inner buffer. Likewise after $2\left(1 + \frac{4}{\alpha_4}\right)\frac{f}{\lambda} + \frac{8\lambda}{\alpha_4} + \frac{c_i}{\alpha_4 w_i}$ days it will never again go below the inner buffer.

If the fast update rule is followed then the first phase in which the potential reduces is not needed, and then the warehouse will never go beyond its middle buffer (so in fact the outer buffers are then not needed). The analysis of the second phase is as before, except that the time needed to first reach the inner buffer has a bound that is half as large. □

Comment. We note that the constraints on λ, α_4, c_i and κ can be met in turn. Lemma 37 constrains $\alpha_3 = 4\alpha_4$ (while λ appears in this constraint we already know from Lemma 4 that we want $\lambda\alpha_1 \le \frac{1}{2}$, which bound can safely be used). In turn, the bound $\lambda\left(1+\frac{1}{\alpha_3}\right) \le \frac{1}{2}$ and Theorem 4 constrain λ. Finally, c_i is upper bounded (and hence κ lower bounded) by the conditions in Theorems 4 and 6.

3.8 The Effect of Inaccuracy

We begin by analyzing Case (i), where the parameter ρ is not known to the price-setters.

Lemma 39. *When p_i is updated, if Assumption 2 and Constraint 1 hold, $\frac{\alpha_2}{2}+\alpha_1\max\{\frac{3}{2}, d-1\} \le 1$, and $\lambda\alpha_1+\frac{4}{3}\lambda\left(1+\frac{2Ed}{1-\lambda E}+\frac{1}{2}\alpha_2\right) \le 1$, then ϕ_i increases by at most $\frac{4}{3}\lambda\rho(2b+\kappa)\left(1+\frac{2Ed}{1-\lambda E}+\frac{1}{2}\alpha_2\right)w_ip_i$.*

Proof. We use Lemma 18. Were $\Delta_i p_i$ accurate, Corollary 4 assures us that ϕ only decreases. With a toward $\widetilde{w}_i$ update $|\Delta_i p_i|$ is too small by at most $\lambda\rho(2b+\kappa)p_i$. Thus the increase in ϕ is at most $\widetilde{w}_i\lambda\rho(2b+\kappa)p_i$; using the bound $\widetilde{w}_i \le \frac{4}{3}w_i$ from Lemma 15, yields that this is at most $\frac{4}{3}\lambda\rho(2b+\kappa)w_ip_i$.

With an away from $\widetilde{w}_i$ update $|\Delta_i p_i|$ may be too large by at most $\lambda\rho(2b+\kappa)p_i$. Thus the increase in ϕ is at most $\widetilde{w}_i\lambda\rho p_i\left(1+\frac{2Ed}{1-\lambda E}+\frac{1}{2}\alpha_2\right)(2b+\kappa)$, which is at most $\frac{4}{3}\lambda\rho\left(1+\frac{2Ed}{1-\lambda E}+\frac{1}{2}\alpha_2\right)(2b+\kappa)w_ip_i$. □

The result of Lemma 16 holds unchanged, namely that $d\phi/dt \le -\frac{\kappa(\alpha_2-1)}{2}\phi$ except when a price increase occurs. We will deduce that so long as ϕ is large enough it will decrease by a $1-\frac{\kappa(\alpha_2-1)}{8})$ rate daily.

The following preliminary lemma is helpful.

Lemma 40. *$\sum_i w_ip_i \le \phi/(1-\lambda\alpha_1)+M$, where M is the daily supply of money.*

Proof. Note that $M=\sum_i x_ip_i$ and that $\phi_i \ge |\widetilde{w}_i - x_i|(1-\lambda\alpha_1)+\alpha_2|\widetilde{w}_i-w_i| \ge |\widetilde{w}_i-x_i|(1-\lambda\alpha_1)+|\widetilde{w}_i-w_i|$. Now

$$
\begin{aligned}
\sum_i w_ip_i &\le \sum_i p_i(x_i+|\widetilde{w}_i-x_i|+|w_i-\widetilde{w}_i|) \\
&\le \sum_i x_ip_i + 2\sum_i p_i[|\widetilde{w}_i-x_i|(1-\lambda\alpha_1)+|\widetilde{w}_i-w_i|]/(1-\lambda\alpha_1) \\
&\le M+\phi/(1-\lambda\alpha_1).
\end{aligned}
$$

□

Before obtaining a bound on the rate of decrease of ϕ, we state a more general result which will be used for a series of results of this nature.

Lemma 41. *Let $D = [\tau, \tau+1]$ be a day. Let $\phi^+ = max_{t\in D}\phi(t)$. Suppose that except when a price increase occurs, $\frac{d\phi}{dt} \le -\nu\phi + \gamma$, for parameters $0 < \nu \le 1$, $\gamma \ge 0$. Further suppose that the price updates over the course of day D collectively increment ϕ by at most $\mu(a_1\phi^+ + a_2\phi(\tau) + a_3 M)$, where M is the daily supply of money, and $\mu, a_1, a_2, a_3 \ge 0$ are suitable parameters.*

If $\mu\left[\frac{a_1(1+\mu a_2)}{1-\mu a_1} + a_2\right] \le \frac{\nu}{8}$ and $\phi(\tau) \ge \frac{8\mu}{\nu} a_3 M \frac{1}{1-\mu a_1}$, then $\phi(\tau+1) \le \left(1 - \frac{\nu}{4}\right)\phi(\tau) + \frac{\gamma}{1-\mu a_1}$.

Proof. $\phi(\tau+1) \le e^{-\nu}\phi(\tau) + \gamma + \mu(a_1\phi^+ + a_2\phi(\tau) + a_3 M)$ as $\frac{d\phi}{dt} \le -\nu\phi + \gamma$. So $\phi(\tau+1) \le (1-\nu/2)\phi(\tau) + \gamma + \mu(a_1\phi^+ + a_2\phi(\tau) + a_3 M)$, if $\nu \le 1$ (using the power series expansion for e^{-x}).

Now $\phi^+ \le \phi(\tau) + \mu(a_1\phi^+ + a_2\phi(\tau) + a_3 M + \gamma)$; hence

$$\phi^+ \le \left[\frac{\phi(\tau)(1+\mu a_2) + \mu a_3 M + \gamma}{1-\mu a_1}\right].$$

Thus

$$\begin{aligned}
\phi(\tau+1) &\le \phi(\tau)(1-\nu/2) + \mu\phi(\tau)\left[\frac{a_1(1+\mu a_2)}{1-\mu a_1} + a_2\right] + \mu a_3 M\left[\frac{\mu a_1}{1-\mu a_1} + 1\right] + \frac{\gamma}{1-\mu a_1} \\
&\le \phi(\tau)(1-\nu/4) + \frac{\gamma}{1-\mu a_1} \\
&\quad \text{if } \mu\phi(\tau)\left[\frac{a_1(1+\mu a_2)}{1-\mu a_1} + a_2\right] \le \frac{\nu}{8}\phi(\tau) \ \text{ i.e. } \ \mu\left[\frac{a_1(1+\mu a_2)}{1-\mu a_1} + a_2\right] \le \frac{\nu}{8} \\
&\quad \text{and } \mu a_3 M\left[\frac{\mu a_1}{1-\mu a_1} + 1\right] \le \frac{\nu\phi(\tau)}{8} \ \text{ i.e. } \ \phi(\tau) \ge \frac{8\mu}{\nu} a_3 M \frac{1}{1-\mu a_1}.
\end{aligned}$$

□

Theorem 7. *If Assumption 2 and Constraint 1 hold, $\frac{\alpha_2}{2} + \alpha_1 \max\{\frac{3}{2}, (d-1)\} \le 1$, $\lambda\alpha_1 + \frac{4}{3}\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \le 1$, $\frac{\kappa(\alpha_2-1)}{2} \le 1$, $4\kappa(1+\alpha_2) \le \lambda\alpha_1 \le \frac{1}{2}$, and each price is updated at least once every day, and at most every $1/b$ days, and if $\phi \ge \frac{16\mu M}{\kappa(\alpha_2-1)}\frac{1-\lambda\alpha_1}{1-\lambda\alpha_1-\mu}$ at the start of the day, then ϕ decreases by at least a $1 - \frac{\kappa(\alpha_2-1)}{8}$ factor by the end of the day, where M is the daily supply of money and $\mu = \frac{4}{3}\lambda\rho b(2b+\kappa)\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right)$, supposing that $\kappa(\alpha_2 - 1) \ge 16\mu/[1 - \lambda\alpha_1 - \mu]$.*

Proof. We will apply Lemma 41. By Lemma 16, $d\phi/dt \le -\frac{\kappa(\alpha_2-1)\phi}{2}$, except when there is a price increase, so $\nu = \frac{\kappa(\alpha_2-1)}{2}$ and $\gamma = 0$.

We now determine the other parameters. Consider a time interval of length $1/b$ days. There is at most one price increase per good during this interval. By Lemma 39, they increase the potential by at most $\sum_i \frac{4}{3}\lambda\rho(2b+\kappa)\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) w_i p_i = \frac{4}{3}\lambda\rho(2b+\kappa)\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right)\left[\frac{\phi^+}{(1-\lambda\alpha_1)} + M\right]$ (on using Lemma 40 to bound $\sum_i w_i p_i$). Thus $\mu = \frac{4}{3}\lambda\rho b(2b+\kappa)\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right)$, $a_1 = 1/(1-\lambda\alpha_1)$, $a_2 = 0$, and $a_3 = 1$. □

To analyze Case (ii) we will use the potential

$$\phi_i = p_i[\text{span}(x_i, \overline{x}_i, \widetilde{w}_i) - 4\kappa(1+\alpha_2)(t-\tau_i)|\overline{x}_i - \widetilde{w}_i| + \alpha_2|\widetilde{w}_i - w_i|].$$

By Lemma 16, $\frac{d\phi_i}{dt} \leq -\frac{\kappa(\alpha_2-1)}{2}\phi_i$ at any time when no price update is occurring (this follows by setting $\lambda = 4\kappa(1+\alpha_2)$ in the statement of Lemma 16).

We say that an attempted update of p_i is a *null update* if p_i is unchanged due to $\overline{z}_i^r$ being too small.

Lemma 42. *When p_i undergoes a null update, ϕ_i increases by at most $8\kappa(1+\alpha_2)\rho(2b+\kappa)w_ip_i(t-\tau_i)$, where t is the current time and τ_i the time of the last update or attempted update.*

Proof. The cost for a null update is $4\kappa(1+\alpha_2)(t-\tau_i)|\overline{x}_i - \widetilde{w}_i|p_i$. As this is a null update, $|\overline{x}_i - \widetilde{w}_i| \leq 2(2b+\kappa)\rho w_i$, so we see that the cost is at most $8\kappa(1+\alpha_2)(2b+\kappa)\rho w_ip_i(t-\tau_i)$. □

Lemma 43. *When p_i undergoes an actual update, if Assumption 2 and Constraint 1 hold, $\frac{\alpha_2}{2} + \alpha_1 \max\{3, 2(d-1)\} \leq 1$, $\frac{\kappa(1+\alpha_2)}{2} \leq 1$, and $\lambda\alpha_1 + 2\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \leq 1$, then ϕ_i only decreases.*

Proof. We use Lemma 18.

Since $|\overline{z}_i^c - \overline{z}_i^r| \geq \frac{1}{2}\overline{z}_i^r$, we know that $\frac{1}{2}\overline{z}_i^r \leq \overline{z}_i^c \leq \frac{3}{2}\overline{z}_i^r$. Consequently, the magnitude of the calculated Δ_ip_i, ranges between $\frac{1}{2}$ and $\frac{3}{2}$ times the ideal value, the value that would be obtained were $\overline{z}_i^r = \overline{z}_i^c$.

In the bounds of Lemma 18, we need to replace the ideal value of Δ_ip_i by the worst case for the calculated value. This yields the following constraints to ensure ϕ_i only decreases.

Case 1. A toward $\widetilde{w}_i$ update:
Case 1.1. $|\overline{x}_i - \widetilde{w}_i| \leq w_i$.
Here $|\Delta_ip_i| \geq \frac{1}{2}\lambda|\overline{x}_i - \widetilde{w}_i|p_i/w_i$. So ϕ only decreases if

$$\lambda\alpha_1|\overline{x}_i - \widetilde{w}_i|p_i \leq \frac{1}{2}(1 - \frac{1}{2}\alpha_2)\widetilde{w}_i\lambda|\overline{x}_i - \widetilde{w}_i|p_i/w_i. \tag{21}$$

By Lemma 15, $\widetilde{w}_i/w_i \geq \frac{2}{3}$, so $\frac{3}{2}\alpha_1 + \frac{1}{4}\alpha_2 \leq \frac{1}{2}$ suffices.
Case 1.2. $|\overline{x}_i - \widetilde{w}_i| > w_i$.
Here $|\Delta_ip_i| \geq \frac{1}{2}\lambda p_i$. So ϕ only decreases if

$$\lambda\alpha_1(d-1)\widetilde{w}_ip_i \leq \frac{1}{2}(1 - \frac{1}{2}\alpha_2)\widetilde{w}_i\frac{1}{2}\lambda p_i. \tag{22}$$

The condition $\alpha_1(d-1) + \frac{1}{4}\alpha_2 \leq \frac{1}{2}$ suffices.
Case 2. An away from $\widetilde{w}_i$ update:
Here $|\Delta_ip_i| \leq \frac{3}{2}\lambda p_i|\overline{x}_i - \widetilde{w}_i|/w_i$. So ϕ only decreases if

$$\left[\widetilde{w}_i + \frac{2E}{1-\lambda E}x_i + \frac{1}{2}\alpha_2\widetilde{w}_i\right]\frac{3}{2}\lambda p_i|\overline{x}_i - \widetilde{w}_i|/w_i \leq (1-\lambda\alpha_1)p_i|\overline{x}_i - \widetilde{w}_i|. \tag{23}$$

By Lemma 15, $\widetilde{w}_i \leq \frac{4}{3}w_i$, so this is subsumed by $\lambda\alpha_1 + 2\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \leq 1$. □

Theorem 8. *If Assumption 2 and Constraint 1 hold, $\frac{\alpha_2}{2} + \alpha_1\max\{3, 2(d-1)\} \leq 1$, $\lambda\alpha_1 + 2\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \leq 1$, $4\kappa(1+\alpha_2) \leq \lambda\alpha_1 \leq \frac{1}{2}$ for all i, $\frac{\kappa(\alpha_2-1)}{2} \leq 1$, $\mu\left[\frac{1+\mu/(1-\lambda\alpha_1)}{1-\mu/(1-\lambda\alpha_1)} + \frac{1}{(1-\lambda\alpha_1)}\right] \leq \frac{\kappa(\alpha_2-1)}{2}$, if $\phi \geq \frac{32\mu M}{[1-\frac{\mu}{1-\lambda\alpha_1}][(\kappa(\alpha_2-1)]}$ at the start of the day, where $\mu = 8\kappa(1+\alpha_2)(2b+\kappa)\rho$, and if each price is updated at least once every day, then ϕ decreases by at least a $1 - \frac{\kappa(\alpha_2-1)}{8}$ factor daily.*

Proof. Again, we apply Lemma 41.

By Lemma 16, $d\phi/dt \le -\frac{\kappa(\alpha_2-1)}{2}$, except when there is a price increase, so $\nu = \frac{\kappa(\alpha_2-1)}{2}$ and $\gamma = 0$.

We now determine the other parameters.

To total the increase due to null updates, we spread the potential increase due to each null update uniformly across the time interval between the present null update and the predeceding update to the same price (null or otherwise). By Lemma 42, the potential increase due to a null update to p_i at time t is at most $8\kappa(1+\alpha_2)(2b+\kappa)\rho w_i p_i(t)\Delta = \mu w_i p_i(t)\Delta$, where Δ is the length of the time interval over which the increase is being spread. Thus the total potential increase due to null updates during D is bounded by $\mu \int_D \sum_i w_i p_i dt$ plus the following term which covers the intervals that extend into the day preceding D: $\mu \sum_i w_i p_i(\tau)$.

Then, using Lemma 40 to bound $\sum_i p_i w_i$, shows that the potential increase over D is bounded by $\mu[M + \phi^+/(1-\lambda\alpha_1)] + \mu[M + \phi(\tau)/(1-\lambda\alpha_1)]$. So we can set $a_1 = a_2 = /1/(1-\lambda\alpha_1)$ and $a_3 = 2$. ☐

Remark. To combine the fast update rule with the rule of at most one update every $1/b$ days, we restate them as follows: an update to p_i occurs after a w_i reduction to the corresponding warehouse stock, and if that does not occur, following a period of between $1/b$ and 1 day since the last update to p_i, null or otherwise.

Small changes will be needed in the proof of Theorem 7.

3.9 Discrete Goods and Prices

The analysis of the discrete case is similar to that for inaccurate data. We begin by examining the sources of "error" in the discrete setting.

Sources of Error.

As $|\overline{x}_i^I - \overline{y}_i| \le 1$, and $|\widetilde{w}_i^A - \widetilde{w}_i^I| \le \kappa$ the previous bound of $2(b+\kappa)\rho w_i$ on the error in calculating $\overline{z}_i$ ($= \overline{y}_i - \widetilde{w}_i$ here) is replaced by $1+\kappa$. This amounts to setting $\rho = \min_i (1+\kappa)/[2(b+\kappa)w_i]$. Note that this implies that a price update occurs only if $|\overline{x}_i^A - \widetilde{w}_i| \ge 2(1+\kappa)$. [8]

We also need to take account of the fact that the prices p_i are integral, but the calculated updates need not be. To avoid overlarge updates, we conservatively round down the the magnitude of each update. Note that this reduces the value of the update by at most a factor of 2. This introduces a second source of error, which we need to incorporate in the analysis. In addition, this has the following implication: no price can be less than $1/\lambda$, for a smaller price would never be updated; furthermore, the price $1/\lambda$ may not be reduced even if the update rule so indicated.

Lemma 44. *If Constraint 2 holds, then* $\frac{3}{4}\widetilde{w}_i^I \le w_i \le \frac{3}{2}\widetilde{w}_i^I$ *and* $|\widetilde{w}_i^I - w_i| \le \frac{1}{2}\widetilde{w}_i^I$*. In addition, Constraint 2 holds for all possible warehouse contents if* $\kappa \max\{|c_i - s_i^*|, s_i^*\} \le \frac{1}{3} w_i$*, and if in addition* $s_i^* = c_i/2$*, the condition becomes* $\kappa \le \frac{2}{3}\frac{w_i}{c_i}$.

The analysis will follow scenario (ii) from Section 3.8, since "ρ" is known. First, we note that Lemma 3.8 still holds, as $M \ge \sum_i x_i p_i$.

[8] We could enforce a condition $|\overline{x}_i^A - \overline{y}_i| \le \frac{1}{(1+\kappa)}$ (or any other convenient positive bound), and then price updates would occur so long as $|\overline{x}_i^A - \widetilde{w}_i| \ge 1$; however, the bound on the elasticity for the y_i demands would increase correspondingly.

We use a potential ϕ_i expressed in terms of y_i:

$$\phi_i = p_i \left[\text{span}(y_i, \overline{y}_i, \widetilde{w}_i^I) - 4\kappa(1+\alpha_2)(t-\tau_i)|\overline{y}_i - \widetilde{w}_i^I| + \alpha_2|\widetilde{w}_i^I - w_i|\right].$$

We restate Lemma 18 in terms of the demands y_i in the corresponding divisible market.

Lemma 45. *If Constraint 2 holds and $y_i \le d\widetilde{w}_i^I$ for all i, then when p_i is updated, ϕ increases by at most the following:*
(i) With a toward $\widetilde{w}_i$ update:

$$\lambda\alpha_1|\overline{y}_i - \widetilde{w}_i^I|p_i + \frac{1}{2}\alpha_2\widetilde{w}_i^I|\Delta_i p_i| - \widetilde{w}_i^I|\Delta_i p_i|. \tag{24}$$

(ii) With an away from $\widetilde{w}_i$ update:

$$\left(1 + \frac{2Ed}{1-\lambda E}\right)\widetilde{w}_i^I|\Delta_i p_i| + \frac{1}{2}\alpha_2\widetilde{w}_i^I|\Delta_i p_i| - (1-\lambda\alpha_1)p_i|\overline{y}_i - \widetilde{w}_i^I|. \tag{25}$$

Proof. The only changes were to replace $\overline{x}_i$ by $\overline{y}_i$ and $\widetilde{w}_i$ by $\widetilde{w}_i^I$ which reflect the corresponding changes to the potential function. □

Again, we say that an update is null if it leaves the price unchanged. Its cost is bounded in the following lemma.

Lemma 46. *When p_i undergoes a null update, ϕ_i increases by at most $6(1+\kappa)\kappa(1+\alpha_2)(t-\tau_i)w_i/\lambda$ if $|\overline{x}_i^A - \widetilde{w}_i^A| \ge 2(1+\kappa)$ and by at most $12(1+\kappa)\kappa(1+\alpha_2)(t-\tau_i)p_i$ if $|\overline{x}_i^A - \widetilde{w}_i^A| < 2(1+\kappa)$, where t is the current time and τ_i the time of the last update or attempted update.*

Proof. The cost for a null update is $4\kappa(1+\alpha_2)(t-\tau_i)|\overline{y}_i - \widetilde{w}_i^I|p_i$. As this is a null update, either $\lambda|\overline{x}_i^A - \widetilde{w}_i^A|p_i/w_i < 1$ or $|\overline{x}_i^A - \widetilde{w}_i^A| < 2(1+\kappa)$. Now $|\overline{y}_i - \widetilde{w}_i^I| \le |\overline{x}_i^I - \widetilde{w}_i^I| + 1 = |\overline{x}_i^A - \widetilde{w}_i^A| + 1$. If $|\overline{x}_i^A - \widetilde{w}_i^A| \ge 2(1+\kappa)$, $|\overline{y}_i - \widetilde{w}_i^I| \le \frac{3}{2}(1+\kappa)|\overline{x}_i^A - \widetilde{w}_i^A|$. So we see that the cost is at most either $6(1+\kappa)\kappa(1+\alpha_2)(w_i/\lambda)(t-\tau_i)$ if $|\overline{x}_i^A - \widetilde{w}_i^A| \ge 2(1+\kappa)$ and $12(1+\kappa)\kappa(1+\alpha_2)(t-\tau_i)p_i$ otherwise. □

Next, we bound $\sum_i w_i$ and $\sum_i p_i$.

Lemma 47. *If $w_i \ge 6$ for all i, $\sum_i w_i = M/r$, and $\sum_i p_i \le \frac{3}{s}\left(\frac{\phi}{(1-\lambda\alpha_1)} + M\right)$.*

Proof. The first claim simply restates the definition of r. For the second claim, we argue as follows.

$$\sum_i p_i = \sum_{y_i^I \le w_i/3} p_i + \sum_{y_i^I > w_i/3} p_i \le \sum_{y_i^I \le w_i/3} \frac{|\widetilde{w}_i^I - y_i^I|p_i}{w_i/3} + \frac{3}{s}\sum_{y_i^I > w_i/3} y_i^I p_i$$

The first term in the inequality following from Lemma 15, as $\widetilde{w}_i^I \ge \frac{2}{3}w_i$. The above total is bounded by $\frac{3}{s}\left(\frac{\phi_i}{(1-\lambda\alpha_1)} + M\right)$. □

Lemma 48. *When p_i undergoes an actual update, if Constraint 2 holds and $y_i \le d\widetilde{w}_i^I$ for all i, $\frac{\alpha_2}{2} + \alpha_1 \max\{\frac{9}{2}, 2(d-1))\} \le 1$, $\lambda\alpha_1 + \frac{8}{3}\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \le 1$, and each $w_i \ge 2$, then ϕ_i only decreases.*

Proof. In the same way that the proof of Lemma 39 is based on Lemma 18, we use Lemma 45.

Since the potential function is based on y_i and $\overline{y}_i$ values, we need to bound $\Delta_i p_i$ in terms of $\overline{y}_i$. In fact, it is calculated in terms of $\overline{x}_i$.

We note that $|\overline{x}_i^A - \widetilde{w}_i^A| - 1 - \kappa \le |\overline{y}_i - \widetilde{w}_i^I| \le |\overline{x}_i^A - \widetilde{w}_i^A| + 1 + \kappa$.

Case 1. A toward $\widetilde{w}_i^I$ update.

Case 1.1. $|\overline{x}_i^A - \widetilde{w}_i^A| \le w_i$.

Here $|\Delta_i p_i| \ge \frac{1}{2}\lambda|\overline{x}_i^A - \widetilde{w}_i^A|p_i/w_i$, as by assumption $\lambda|\overline{x}_i^A - \widetilde{w}_i^A|p_i/w_i \ge 2$ for an actual update. So ϕ is non-increasing as a result of the update if

$$\lambda\alpha_1|\overline{y}_i - \widetilde{w}_i^I|p_i \le (1 - \frac{1}{2}\alpha_2)\widetilde{w}_i^I \cdot \frac{1}{2}\lambda|\overline{x}_i^A - \widetilde{w}_i^A|\frac{p_i}{w_i}. \tag{26}$$

By Lemma 15, $\widetilde{w}_i^I/w_i \ge \frac{2}{3}$, and as already noted $|\overline{y}_i - \widetilde{w}_i^I| \le |\overline{x}_i^A - \widetilde{w}_i^A| + 1 + \kappa$, so $\lambda\alpha_1(|\overline{x}_i^A - \widetilde{w}_i^A| + 1 + \kappa)p_i \le (1 - \frac{1}{2}\alpha_2) \cdot \frac{1}{3}\lambda|\overline{x}_i^A - \widetilde{w}_i^A|p_i$ suffices. As $|\overline{x}_i^A - \widetilde{w}_i^A| \ge 2(1+\kappa)$ for an actual update, $|\overline{x}_i^A - \widetilde{w}_i^A| + 1 + \kappa \le \frac{3}{2}|\overline{x}_i^A - \widetilde{w}_i^A|$, and $\frac{9}{2}\alpha_1 + \frac{1}{2}\alpha_2 \le 1$ suffices.

Case 1.2. $|\overline{x}_i^A - \widetilde{w}_i^A| > w_i$.

Here $|\Delta_i p_i| \ge \frac{1}{2}\lambda p_i$. So ϕ is non-increasing due to the update if $\lambda\alpha_1(d-1)\widetilde{w}_i^I p_i \le (1 - \frac{1}{2}\alpha_2)\widetilde{w}_i^I \cdot \frac{1}{2}\lambda p_i$. In turn, $2\alpha_1(d-1) + \frac{1}{2}\alpha_2 \le 1$ suffices.

Case 2. An away from $\widetilde{w}_i^I$ update.

Here $|\Delta_i p_i| \le \lambda p_i|\overline{x}_i^A - \widetilde{w}_i^A|/w_i$. Recall that an update occurs only if $|\overline{x}_i^A - \widetilde{w}_i^A| \ge 2(1+\kappa)$, and hence $|\overline{y}_i - \widetilde{w}_i^I| \ge \frac{1}{2}(\overline{x}_i^A - \widetilde{w}_i^A)$. Thus $|\Delta_i p_i| \le 2\lambda p_i|\overline{y}_i - \widetilde{w}_i^I|/w_i$. So ϕ is non-increasing due to the update if

$$2\lambda p_i|\overline{y}_i - \widetilde{w}_i^I|\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right)\frac{\widetilde{w}_i^I}{w_i} \le (1 - \lambda\alpha_1)p_i|\overline{y}_i - \widetilde{w}_i^I|. \tag{27}$$

As $\widetilde{w}_i^I/w_i \le \frac{4}{3}$ by Lemma 44, this is subsumed by $\lambda\alpha_1 + \frac{8}{3}\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \le 1$. □

Next, we apply Lemma 16 to get a bound on $d\phi/dt$.

Corollary 6. *If $4\kappa(1+\alpha_2) \le \lambda\alpha_1 \le \frac{1}{2}$, $\frac{d\phi_i}{dt} \le -\frac{\kappa(\alpha_2-1)}{2}\phi_i + \kappa p_i$ at any time when no price update is occurring (to any p_j); this bound also holds for the one-sided derivatives when a price update occurs.*

Proof. This is Lemma 16, with x_i replaced by y_i and $\widetilde{w}_i$ by $\widetilde{w}_i^I$. So $\chi_i = \frac{d\widetilde{w}_i^I}{dt} + \kappa(y_i - w_i) = -\kappa(x_i^I - w_i) + \kappa(y_i - w_i) = -\kappa(x_i - y_i)$, and hence $0 \le -p_i\chi_i \le -\kappa p_i$. □

Theorem 9. *If Constraint 2 holds and $y_i \le d\widetilde{w}_i$ for all i, $\frac{\alpha_2}{2} + \alpha_1 \max\{\frac{9}{2}, 2(d-1)\} \le 1$, $\lambda\alpha_1 + \frac{8}{3}\lambda\left(1 + \frac{2Ed}{1-\lambda E} + \frac{1}{2}\alpha_2\right) \le 1$, each $w_i \ge 6$, $4\kappa(1+\alpha_2) \le \lambda\alpha_1 \le \frac{1}{2}$ for all i,*
$s \ge \frac{48}{(\alpha_2-1)(1-\lambda\alpha_1)}\left[1 + 6(1+\alpha_2) + (1+\alpha_2)\frac{1-\lambda\alpha_1+\frac{18}{s}\kappa(1+\alpha_2)+\frac{3\kappa}{s}}{1-\lambda\alpha_1-\frac{18}{s}\kappa(1+\alpha_2)}\right]$,
if $\phi(\tau) \ge \frac{48}{(\alpha_2-1)}\left[(1+\alpha_2)\left(\frac{4}{\lambda r} + \frac{24}{s}\right) + \frac{1}{s}\right]\frac{1-\lambda\alpha_1}{1-\lambda\alpha_1-\frac{18}{s}\kappa(1+\alpha_2)}M$ at the start of the day, and if each price is updated at least once every day, then ϕ decreases by at least a $1 - \frac{\kappa(\alpha_2-1)}{8}$ factor over the course of the day.

Proof. We prove the result for a day D starting at time τ, using Lemma 41, as in the proof of Lemma 4.

First, we note that by applying Lemma 16, we can set $\nu = \kappa(\alpha_2 - 1)/2$ and $\gamma = \sum_i p_i$.

To total the increase due to null updates during D, we spread the potential increase due to each null update uniformly across the time interval between the present null update and the preceding update to the same price (null or otherwise). By Lemma 46, the potential increase due to a null update to p_i at time t is at most $6\kappa(1+\alpha_2)\frac{w_i}{\lambda}$ if $|\overline{x}_i - \widetilde{w}_i| \geq 2$, and $12\kappa(1+\alpha_2)p_i\Delta$ otherwise, where Δ is the length of the time interval over which the increase is being spread. Thus the total potential increase due to null updates during D is bounded by $6\kappa(1+\alpha_2)\int_D[\sum_i \frac{w_i}{\lambda} + \sum_{|\overline{x}_i - \widetilde{w}_i|<2} 2p_i(t)]dt$ plus the following term which covers the intervals that extend into the day preceding D: $6\kappa(1+\alpha_2)\sum_i \frac{w_i}{\lambda} + \sum_{|\overline{x}_i - \widetilde{w}_i|<2} 2p_i(\tau)$.

The potential increase during D due to the "γ" term is at most $\kappa \sum_i p_i$.

As shown in Lemma 47, $\sum_i w_i \leq M/r$ and $\sum_i p_i \leq 3\phi/[s(1-\lambda\alpha_1)] + 3M/s$. It follows that the potential increase over D is bounded by $6\kappa(1+\alpha_2)\{M/(\lambda r) + 6M/s + 3\phi^+/[s(1-\lambda\alpha_1)] + M/(\lambda r) + 6M/s + 3\phi(\tau)/[s(1-\lambda\alpha_1)]\} + \kappa[3\phi/[s(1-\lambda\alpha_1)] + 3M/s]$. So we can apply Lemma 41 with $\mu = 3\kappa$, $a_1 = 6(1+\alpha_2)/[s(1-\lambda\alpha_1)]$, $a_2 = [6(1+\alpha_2)+1]/[s(1-\lambda\alpha_1)]$ and $a_3 = (1+\alpha_2)[4/(\lambda r) + 24/s] + 1/s$. The condition $\phi(\tau) \geq \frac{8\mu}{\nu} a_3 M \frac{1}{1-\mu a_1}$ in Lemma 47 becomes

$$\phi(\tau) \geq \frac{48}{(\alpha_2 - 1)} \left[(1+\alpha_2)\left(\frac{4}{\lambda r} + \frac{24}{s}\right) + \frac{1}{s}\right] \frac{1-\lambda\alpha_1}{1 - \lambda\alpha_1 - \frac{18}{s}\kappa(1+\alpha_2)} M.$$

Similarly, the condition $\mu\left[\frac{a_1(1+\mu a_2)}{1-\mu a_1)} + a_2\right] \leq \frac{\nu}{8}$ becomes

$$\frac{3(1+\alpha_2)}{s(1-\lambda\alpha_1)}\left[\frac{1-\lambda\alpha_1 + \frac{18}{s}\kappa(1+\alpha_2) + \frac{3\kappa}{s}}{1-\lambda\alpha_1 - \frac{18}{s}\kappa(1+\alpha_2)}\right] + \frac{18(1+\alpha_2)}{s(1-\lambda\alpha_1)} + \frac{3}{s(1-\lambda\alpha_1)} \leq \frac{\alpha_2 - 1}{16};$$

equivalently,

$$s \geq \frac{48}{(\alpha_2 - 1)(1-\lambda\alpha_1)}\left[1 + 6(1+\alpha_2) + (1+\alpha_2)\frac{1-\lambda\alpha_1 + \frac{18}{s}\kappa(1+\alpha_2) + \frac{3\kappa}{s}}{1-\lambda\alpha_1 - \frac{18}{s}\kappa(1+\alpha_2)}\right].$$

□

Theorem 10. *In the discrete setting there are markets with $\Omega(E/r)$ misspending at any pricing.*

Proof. The construction uses one good plus money. Let p_g by the price of the good, x_g the demand for the good and $n_\$$ the demand for money. We define a continuous CES utility $[x_g(r+1/2)]^{1-1/E} + n_\$^{1-1/E}$ for the aggregate demand (it can be due to one buyer), with a supply that at $p_g = r + 1/2$ spends $M/2$ money on $\frac{M}{2r+1}$ units of good g. It is not hard to check that at $p_g = r$ and $r+1$ there is already a change of $\theta(En_g/r)$ in the demand for good g. □

3.9.1 Construction of Virtual Demands y

Our goal is to construct virtual demands y_i, with $x_i - 1 < y_i \leq x_i$, defined for all p for which $x_i \geq 1$. Further the spending on y_i will be a non-increasing function of p_i, will obey WGS at integer price points (which is where it is defined), and will have elasticity parameter $2E$.

y_i is constructed as follows.

For each collection of prices p_{-i}, let $m_i(j)$ be the spending on good i when $p_i = j$; we also write it as m_j for short. Consider the sequence $m_1, m_2, m_3, \cdots$ and let $m_{l_1} = m_1, m_{l_2}, m_{l_3}, \cdots$ be the maximal subsequence of non-increasing nonzero spending. We create a new sequence $m'_1, m'_2, \cdots$ of virtual spending, such that $m'_j \geq m'_{j+1}$ for all j. First, we set $m'_{l_a} = m_{l_a}$ for all a. If $m_{l_a} = m_{l_{a+1}}$, then we set $m'_j = m_{l_a}$ for all $j \in (l_a, l_{a+1})$. Also, if $x_i(l_{a+1} - 1) \geq x_i(l_{a+1}) + 2$, then we set $m'_j = m_{l_a}$ for all $j \in (l_a, l_{a+1})$. Likewise, if $x_i(j) < x_i(j-1)$ for some $j \in (l_a, l_{a+1})$, we set $m'_{l_a+1}, m'_{l_a+2}, \cdots, m'_j = m_{l_a}$. If m_{l_b} is the last term in the non-increasing sequence, we set $m'_j = m_{l_b}$ for all $j > l_b$ with $m_j > 0$. The only other alternative is that there is an $h \geq l_a$ with $m'_{l_a} = m'_h$, and $x_i(h) = x_i(h+1) = \cdots = x_i(l_{a+1} - 1) = x_i(l_{a+1}) + 1$. Then, for $j \in (h, l_{a+1})$, $y'_i(j)$ is defined to interpolate between $y_i(h)$ and $y_i(l_{a+1})$ as follows: $y'(j) = y_i(h)\left(\frac{h}{j}\right)^c$, where c is given by $y_i(h)\left(\frac{h}{l_{a+1}}\right)^c = y_i(l_{a+1})$. Then $y_i(p) = y_i(p_{-i}, j)$ is given by $y_i(p_{-i}, j) = \max_{q_{-i} \leq p_{-i}}\{y'_i(p_{-i}, j), y_i(q_{-i}, j)\}$. Note that $jy'(j) = hy_i(h)\left(\frac{h}{j}\right)^{c-1}$, for $h \leq j \leq l_{a+1}$. As we know that $hy_i(h) = m'_{l_a} > m_{l_{a+1}} = l_{a+1}y_i(l_{a+1})$, in this last case, on setting $j = l_{a+1}$, we can conclude that $c - 1 > 0$, i.e. that $c > 1$.

Lemma 49. *$y_i(j) > x_i(j) - 1$ for all j.*

Proof. **Case 1** $m'_j = m_{l_a}$ for some $l_a < j$.

By Discrete WGS, $m_{l_a} \geq m_j - (l_a - 1)$. Thus

$$y_i(j) = \frac{m'_j}{j} = \frac{m_{l_a}}{j} \geq \frac{m_j}{j} - \frac{l_a - 1}{j} > x_i(j) - 1.$$

Case 2 $m'_j < m_j$ but $m_j \neq m_{l_a}$ for any a.

Then there is a maximal sequence $m_h, m_{h+1}, \cdots, m_k$, with $j \in (h, k]$ and $k = l_{a+1} - 1$ for some a, for which $x_i(h) = x_i(h+1) = \cdots = x_i(k) = x_i(k+1) + 1$. Case 1 shows that $y_i(h) > x_i(h) - 1$. By construction, $y_i(k) > x_i(k+1) = x_i(k) - 1$, and as the values $y'_h, y'_{h+1}, \cdots, y'_k$ form a decreasing sequence by construction, it follows that for j with $h < j \leq k$, $y_i(j) \geq y_i(k) > x_i(k) - 1 = x_i(j) - 1$. ☐

Lemma 50. *$m_i(j) \geq m_i(j+1)$ for all i and j.*

Proof. The only case we need to check is when $y'_i(j+1)$ is defined by interpolation for $j \in (h, l_{a+1})$ say. By construction, $p_i y'_i$ is a decreasing function of p_i on the interpolating interval. Now the result follows by induction on p_{-i}. The claim holds for $p_{-i} = 1$, as then $y'_i(p) = y_i(p)$. For larger p_{-i}, y_i as a function of p_i is simply a max of noncreasing functions and hence is itself non-increasing. ☐

Lemma 51. *At discrete prices for which $x_i > 0$, the demands y_i obey WGS.*

Proof. First, we note that as p_i increases the virtual spending m'_i on good i only decreases, and hence y_i only decreases.

Next, we consider the change to y_i as p_j increases, for $j \neq i$. Let $p_i = h$. Consider prices p_j and $p_j + 1$ for good j. Suppose that $y_i(h, p_j + 1)$ is set by a rule making $m'_h(p_j + 1) = m_{l_a}(p_j + 1)$, where $h \geq l_a$. By Discrete WGS, $m_{l_a}(p_j + 1) \geq m_{l_a}(p_j)$. By construction, $m_{l_a}(p_j) \geq m'_{l_a}(p_j) \geq m'_h(p_j)$. So $m'_h(p_j + 1) \geq m'_h(p_j)$ and hence $y_i(h, p_j + 1) \geq y_i(h, p_j)$ in this case.

Otherwise, by construction, $y_h(p_j + 1) \geq y_h(p_j)$. ☐

Lemma 52. *The demands y_i obey bounded elasticity:*

$$y_i(p_{-i}, p_i) \le y_i(p_{-i}, p_i + d)\left(1 + \frac{d}{p_i}\right)^{2E} \quad \textit{if } y_i(p_{-i}, p_i + d) > 0.$$

Proof. Note that it suffices to prove the bound for $d = 1$, as

$$\left(1 + \frac{a}{p_i}\right)\left(1 + \frac{b}{a + p_i}\right) = \frac{p_i + a + b}{p_i} = \left(1 + \frac{a + b}{p_i}\right).$$

Case 1 $m'_{p_i} = m'_{p_i+1}$.

Then $y_i(p_i) = \frac{m'_{p_i}}{p_i}$, $y_i(p_i + 1) = \frac{m'_{p_i+1}}{p_i+1}$. So, $p_i y_i(p_i) = (p_i + 1)y_i(p_i + 1)$, or $y_i(p_i) \le y_i(p_i + 1)\left(1 + \frac{1}{p_i}\right)$.

Case 2 $m'_h(p_l) = m'_h(p_l - 1)$.

We note that $m'_{h+1}(p_l) \ge m'_{h+1}(p_l - 1)$, so $\frac{m'_h(p_l)}{m'_{h+1}(p_l)} \le \frac{m'_h(p_l-1)}{m'_{h+1}(p_l-1)}$, and $\frac{y'(h,p_l)}{y'(h+1,p_l)} \le \frac{y'(h,p_l-1)}{y'(h+1,p_l-1)}$. Hence the claim follows by induction on p_{-i} since Case 2 does not apply at the base cases when $p_{-i} = 1$.

Case 3 $x_i(h) \ge x(l_{a+1}) + 2$, where $l_a \le h < l_{a+1}$.

This is an intermediate bound, used in Case 4, except when $h + 1 = l_{a+1}$.

By Discrete WGS, $x_i(h) \le \left\lceil x_i(l_{a+1})\left(1 + \frac{l_{a+1}-h}{h}\right)^E \right\rceil$. Hence $x_i(h) - 1 < x_i(l_{a+1})\left(1 + \frac{l_{a+1}-h}{h}\right)^E$, or

$$\frac{x_i(l_{a+1}) + [x_i(h) - x_i(l_{a+1}) - 1]}{x_i(l_{a+1})} \le \left(1 + \frac{l_{a+1} - h}{h}\right)^E.$$

As $x_i(h) - x_i(l_{a+1}) \ge 2$, $x_i(h) - x_i(l_{a+1}) - 1 \ge 1$. Let Δ denote $x_i(h) - x_i(l_{a+1}) - 1$. So $1 + \frac{\Delta}{x_i(l_{a+1})} \le \left(1 + \frac{l_{a+1}-h}{h}\right)^E$. Hence $1 + \frac{2\Delta}{x_i(l_{a+1})} \le \left(1 + \frac{l_{a+1}-h}{h}\right)^{2E}$, or

$$\frac{x_i(h) + [x_i(h) - x_i(l_{a+1}) - 2]}{x_i(l_{a+1})} \le \left(1 + \frac{l_{a+1} - h}{h}\right)^{2E}.$$

Hence $y_i(h) \le x_i(h) \le x_i(l_{a+1})\left(1 + \frac{l_{a+1}-h}{h}\right)^{2E}$.

Case 4 y_j is defined by interpolation.

Then there are an h and an l_a with $j \in (h, l_{a+1})$, such that $x_i(h - 1) > x_i(h) = x_i(h + 1) = \cdots = x_i(k) > x_i(k + 1)$, with $k + 1 = l_{a+1}$, where $y'_{h+1}, y'_{h+2}, \cdots, y'_k$ are obtained by interpolation between y'_h and $y'_{k+1} = x_{k+1}$.

By Case 3, $y_i(h) \le y_i(l_{a+1})\left(1 + \frac{l_{a+1}-h}{h}\right)^{2E}$. Let $c > 1$ satisfy $y_i(h) = y_i(l_{a+1})\left(1 + \frac{l_{a+1}-h}{h}\right)^c$; so $0 < c \le 2E$ (in fact, $1 < c \le 2E$). By construction, for $h \le j < l_{a+1}$, $y_i(j) = y_i(l_{a+1})\left(1 + \frac{l_{a+1}-j}{j}\right)^c$; thus $y_i(j) = y_i(j + 1)\left(1 + \frac{1}{j}\right)^c \le y_i(j + 1)\left(1 + \frac{1}{j}\right)^{2E}$. □

3.10 Extensions

Property 1, while implicit, underpins our analysis, but so far we have limited ourselves to the case $\alpha = 1$. Essentially, reducing α will slow the daily reduction in potential by an α factor. Now we consider two extensions of the ongoing Fisher market with WGS and bounded elasticity which will have an α in the range $(0, 1)$.

3.10.1 Beyond WGS

Instead of having demands for goods $j \neq i$ only increase when p_i increases by $\Delta_i p_i > 0$, we could allow some of the demands to decrease, but with the constraint that the total of all reductions be bounded as follows:

$$\sum_{x'_j < x_j} |x_j - x'_j| p_j \leq (1 - \alpha) w_i \Delta_i p_i, \tag{28}$$

which we call the *α-bounded complements property.* A symmetric condition applies when $\Delta_i p_i < 0$.

Lemma 53. *If demands obey the α-bounded complements property then Property 1 holds.*

Proof. The one change to the proof of Lemma 9 is that the total additional detrimental change in spending when p_i is updated is given by (28), and it increases the potential by at most $(1 - \alpha) w_i |\Delta_i p_i|$.

The prior offsetting reduction in the potential by $I_i = w_i |\Delta_i p_i|$ continues to hold, so there is a net reduction by at least αI_i. □

The resulting change to the overall analysis is that the reductions to the potential by $\alpha_1 w_i |\Delta_i p_i|$, which drive the prior analysis, now become $\alpha \alpha_1 w_i |\Delta_i p_i|$, resulting in a $1/\alpha$ slowdown in the convergence rates.

3.10.2 Ongoing α-Exchange Markets

At this point we review the standard definition of (One-Time) Exchange Markets.

The One-Time Exchange Market A market comprises two sets, goods G, with $|G| = n$, and traders T, with $|T| = m$. The goods are assumed to be infinitely divisible. Each trader l starts with an allocation w_{il} of good i. Each trader j has a utility function $u_j(x_{1j}, \cdots, x_{nj})$ expressing its preferences: if j prefers a basket with x_{ij} units (possibly a real number) of good i, to the basket with y_{ij} units, for $1 \leq i \leq n$, then $u_j(x_{1j}, \cdots, x_{nj}) > u_j(y_{1j}, \cdots, y_{nj})$. Each trader j intends to trade goods so as to achieve a personal optimal combination (basket) of goods given the constraints imposed by their initial allocation. The trade is driven by a collection of prices p_i for good i, $1 \leq i \leq n$. Agent j chooses x_{ij}, $1 \leq i \leq n$, so as to maximize u_j, subject to the basket being affordable, that is: $\sum_{i=1}^{n} x_{ij} p_i \leq \sum_{i=1}^{n} w_{ij} p_i$. Prices $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ are said to provide an *equilibrium* if, in addition, the demand for each good is bounded by the supply: $\sum_{j=1}^{m} x_{ij} \leq \sum_{j=1}^{m} w_{ij}$. The market problem is to find equilibrium prices.

We consider Exchange markets which repeat on a daily basis in the same way as the ongoing Fisher markets. For each good, there will be an agent with a warehouse for that good and this trader is the one who meets demand fluctuations from the warehouse stock.

More specifically, each day there is a collection of identical traders who come to market each with the same basket of goods, including money, to trade. Furthermore, for each good, there is a trader with a warehouse, who is the price-setter for that good and whose goal is to bring the warehouse to being half full.

We further assume that the price-setter for good i brings at least αw_i of good i to the market and is only seeking money for this portion of the supply.

We call this an α-Exchange market.

Then in the same way as for Lemma 53, one can show:

Lemma 54. *In an α-Exchange market Property 1 holds.*

Again, we obtain convergence slowed down by an α factor.

In [11] a related constraint on the buyer side was used. It specified that at equilibrium, traders coming to market with money (buyers), would be buying at least αw_i of the ith good, for each i. To ensure adequate demand among buyers for goods at other prices, they introduced an additional assumption, termed the *wealth effect*. This assumption was a bound on the elasticity of wealth, namely that when all prices dropped by a factor f, all demands increased by a factor f^{β} for some β, $0 < \beta < 1$.

It would be interesting to consider a version of the ongoing exchange market in which the price-setter only meets bounded fluctuations in demand, and demands beyond this range, along with unspent money, are carried forward to the next day. The challenge, with respect to our approach for the analysis, is to ensure a relatively small change in the computed demands for the preponderance of the goods (in terms of their contribution to a suitable potential) for otherwise it is not clear that the price updates will reduce the potential. By computed demands, we have in mind something analogous to the calculations used in the ongoing Fisher market with warehouses: this gives a low weight (a factor $\alpha_2 \kappa$) to the demand contribution used to account for the warehouse excesses.

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