Discrete Pulse Transform of Images

Roumen Anguelov and Inger Fabris-Rotelli*

Department of Mathematics and Applied Mathematics
University of Pretoria
Pretoria 0002, South Africa
roumen.anguelov@up.ac.za,
inger.fabris-rotelli@up.ac.za

Abstract. The Discrete Pulse Transform (DPT) of images is defined by using a new class of LULU operators on multidimensional arrays. This transform generalizes the DPT of sequences and replicates its essential properties, e.g. total variation preservation. Furthermore, the discrete pulses in the transform capture the contrast in the original image on the boundary of their supports. Since images are perceived via the contrast between neighbour pixels, the DPT may be a convenient new tool for image analysis.

Keywords: LULU, discrete pulse transform, total variation preservation.

1 Introduction

One of the powerful ideas resulting from the theory of the LULU operators for sequences, developed by Carl Rohwer during the last two decades or so, is the Nonlinear Multiresolution Analysis based on a Discrete Pulse Transform (DPT), [8]. A DPT maps a bi-infinite sequence $\xi = (..., \xi_{-1}, \xi_0, \xi_1, \xi_2, ...)$ onto an infinite vector

$$DPT(\xi) = (D_1(\xi), D_2(\xi), ...) \tag{1}$$

where $D_n(\xi)$ is a sequence consisting of well separated discrete block pulses with support n. In a recent work by the authors the definition of the LULU operators was extended from sequences to multidimensional arrays, namely, functions on $\mathbb{Z}^d$, $d > 1$. This development opened the opportunity of developing the DPT for functions on $\mathbb{Z}^d$ by using iterative application of the generalized LULU operators, similar to the one-dimensional case. In this paper we consider the DPT of images. Let us denote by $\mathcal{A}(\mathbb{Z}^2)$ the set of all functions defined on $\mathbb{Z}^2$ which have finite support. A grey scale image is a function $f \in \mathcal{A}(\mathbb{Z}^2)$ such that the support of f is a finite rectangular subset Ω of $\mathbb{Z}^2$. The DPT of a function $f \in \mathcal{A}(\mathbb{Z}^2)$ is a vector of the form

$$DPT(f) = (D_1(f), D_2(f), ..., D_N(f)) \tag{2}$$

which is finite due to the finite support of f. Here N is the number of pixels in the image and $D_n(f) = \sum_{s=1}^{\gamma(n)} \phi_{ns}$, the functions ϕ_{ns}, $s = 1, ..., \gamma(n)$, being discrete

pulses with support of size n, $n = 1, ..., N$. In this context a discrete pulse is a function $\phi \in \mathcal{A}(\mathbb{Z}^2)$ which is zero everywhere except on a connected set V, where it is constant. The set V is called the support of the pulse ϕ and we write $V = \mathrm{supp}(\phi)$. The value of ϕ on V is shortly called the value of the pulse. If the value of ϕ is positive then ϕ is an up-pulse, if it is negative, ϕ is a down-pulse. Via the DPT we represent a function $f \in \mathcal{A}(\mathbb{Z}^2)$ as a sum of pulses

$$f = \sum_{n=1}^{N} D_n(f) = \sum_{n=1}^{N} \sum_{s=1}^{\gamma(n)} \phi_{ns} \ , \tag{3}$$

where the supports of the pulses are either disjoint or nested. Furthermore, we show that, similar to the DPT of sequences, the decomposition (3) preserves the total variation of f, that is, we have

$$TV(f) = \sum_{n=1}^{N} TV(D_n(f)) = \sum_{n=1}^{N} \sum_{s=1}^{\gamma(n)} TV(\phi_{ns}) \ . \tag{4}$$

The total variation is an important characteristic of an image and has been successfully used in noise removal procedures, e.g. [3], [13]. The equalities in (4) show that the decomposition (3) does not create artificial variation, i.e. noise. We should further remark that the pulses in (3) have also a more direct meaning, namely, the contrast in the original image at the boundary of the support of any pulse is at least as much as the value of that pulse.

2   LULU Theory

The LULU operators on multidimensional arrays are defined in [1] by using the morphological concept of connection, [15,16], which characterizes the connected subsets of the domain. The set of connected sets appropriate for images is usually defined via a neighbor relation on the domain. More precisely, a set $C \subseteq \mathbb{Z}^2$ is called *connected* if for any two pixels $p, q \in C$ there exists a set of pixels $\{p_1, p_2, ..., p_k\} \subseteq C$ such that each pixel is neighbour to the next one, p is neighbour to p_1 and p_k is neighbour to q. It was shown in [1] that if the neighbor relation is reflexive, symmetric, shift invariant and includes the pairs of consecutive points both vertically and horizontally, then the collection $\mathcal{C}$ of all connected sets on $\mathbb{Z}^2$ is a morphological connection suitable for the LULU operators. The neighbour relations typically considered for images, e.g. as in Figure 1, satisfy these basic requirements.

Given a point $x \in \mathbb{Z}^2$ and $n \in \mathbb{N}$ we denote by $\mathcal{N}_n(x)$ the set of all connected sets of size $n + 1$ which contain a point x, that is,

$$\mathcal{N}_n(x) = \{V \in \mathcal{C} : x \in V, \ \mathrm{card}(V) = n + 1\} \ . \tag{5}$$

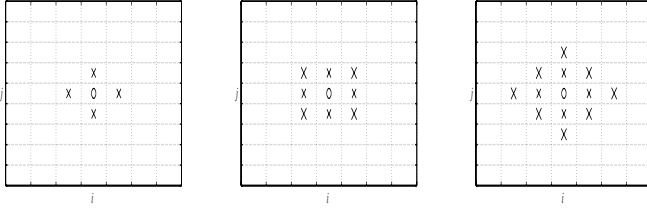Then the basic LULU operators are defined as follows.

Fig. 1. Neighbours of the pixel (i, j)

Definition 1. *Let $f \in \mathcal{A}(\mathbb{Z}^2)$ and $n \in \mathbb{N}$. Then*

$$L_n(f)(x) = \max_{V \in \mathcal{N}_n(x)} \min_{y \in V} f(y), \ x \in \mathbb{Z}^2 \ , \tag{6}$$

$$U_n(f)(x) = \min_{V \in \mathcal{N}_n(x)} \max_{y \in V} f(y), \ x \in \mathbb{Z}^2 \ . \tag{7}$$

The operators in Definition 1 were given in [1] as a generalization of the operators L_n and U_n for sequences. However, it should be noted that these are actually the well known morphological operators of area opening and area closing (section 4.4.2, [17]) considered in a new setting, namely, the LULU theory [8,12]. It was shown in [1] that these operators are smoothers and separators (Chapter 1, [8]). Furthermore, they form, under composition, a four element totally ordered semi-group - the so called LULU semi-group [12,11] - and we have

$$L_n \leq U_n \circ L_n \leq L_n \circ U_n \leq U_n \ . \tag{8}$$

The smoothing effect of the operators, L_n, U_n, on a function $f \in \mathcal{A}(\mathbb{Z}^2)$ is described in terms of the concepts of local minimum and local maximum sets.

Definition 2. *Let $V \in \mathcal{C}$. A point $x \notin V$ is called* **adjacent** *to V if $V \cup \{x\} \in \mathcal{C}$. The set of all points adjacent to V is denoted by $\mathrm{adj}(V)$.*

Definition 3. *A connected subset V of $\mathbb{Z}^2$ is called a* **local maximum set** *of $f \in \mathcal{A}(Z^d)$ if*

$$\max_{y \in \mathrm{adj}(V)} f(y) < \min_{x \in V} f(x) \ .$$

Similarly V is a **local minimum set** *if*

$$\min_{y \in \mathrm{adj}(V)} f(y) > \max_{x \in V} f(x) \ .$$

Then the action of L_n and U_n on $f \in \mathcal{A}(\mathbb{Z}^2)$ can be described in terms of the following properties, [1]:

A1 The application of L_n (U_n) removes local maximum (minimum) sets of size smaller or equal to n.

A2 The operator L_n (U_n) does not affect the local minimum (maximum) sets directly in the sense that such sets may be affected only as a result of the removal of local maximum (minimum) sets. However, no new local minimum sets are created where there were none. This does not exclude the possibility that the action of L_n (U_n) may enlarge existing local maximum (minimum) sets or join two or more local maximum (minimum) sets of f into one local maximum (minimum) set of $L_n(f)$ $(U_n(f))$.

A3 $L_n(f) = f$ $(U_n(f) = f)$ if and only if f does not have local maximum (minimum) sets of size n or less.

In analogy with the respective concept for sequences we introduce the concept of n-monotonicity for functions in $\mathcal{A}(\mathbb{Z}^2)$.

Definition 4. *Function $f \in \mathcal{A}(\mathbb{Z}^2)$ is called (locally) n-monotone if the size of all local minimum or local maximum sets of f is larger then n.*

It follows from A1–A3 that for every $f \in \mathcal{A}(\mathbb{Z}^2)$

$$(L_n \circ U_n)(f) \text{ is } n\text{-monotone} \tag{9}$$

$$(U_n \circ L_n)(f) \text{ is } n\text{-monotone} \tag{10}$$

$$(L_n \circ U_n)(f) = (U_n \circ L_n)(f) = f \iff f \text{ is } n\text{-monotone} \tag{11}$$

3 Discrete Pulse Transform

The Discrete Pulse Transform (2) for a function $f \in \mathcal{A}(\mathbb{Z}^2)$ is obtained via iterative application of the operators L_n, U_n with n increasing from 1 to N. For a given n, the sequencing of L_n and U_n does not affect the properties of (2) stated in the Introduction. However, it introduces bias towards up-pulses or down-pulses. This issue will be given particular attention later. For now let P_n denote either the composition $L_n \circ U_n$ or the composition $U_n \circ L_n$ and let $Q_n = P_n \circ P_{n-1} \circ ... \circ P_2 \circ P_1$. In the general theory of Mathematical Morphology, Q_n is known as an alternating sequential filter (see Section 8.3 [17]). However, here are interested in the portions of the image which are filtered out by the application of $P_n, n = 1, 2, ..., N$. Indeed, we ultimately obtain $Q_N(f)$, which is a constant function containing no information about the original image except possibly the general level of illumination. The rest of the information carried by f is in the layers peeled off. More precisely, we have

$$f = (id - P_1)(f) + ((id - P_2) \circ Q_1)(f) + ((id - P_3) \circ Q_2)(f) + ...$$
$$+((id - P_{N-1}) \circ Q_{N-2})(f) + ((id - P_N) \circ Q_{N-1})(f) + Q_N(f) , \tag{12}$$

where id denotes the identity operator. Let us note that a similar iterative application of area opening and area closing operators is used by Acton and Mukherjee for image classification. In [2], filtering is for selected values of n and instead of the layers of peeled off portions the authors keep record of filtered images at every scale (this would be $Q_n(f)$ in the notation adopted here). While the approach

in [2] is adequate for the classification methods derived there, the decomposition (12) contains much more information. It is also easy to apply, not least, due to the fact that it gives a representation of f in terms of discrete pulses. Indeed, we show next that (12) is a DPT decomposition of the form (3) by proving that each term is a sum of pulses of certain size. This result is obtained in the next two theorems.

Theorem 1. *Let $f \in \mathcal{A}(\mathbb{Z}^2)$ be $(n-1)$-monotone. Then*

a) $(id - L_n)(f)$ *is a sum of disjoint up-pulses of size n;*
b) $(id - U_n)(f)$ *is a sum of disjoint down-pulses of size n;*
c) $(id - U_n \circ L_n)(f)$ *is a sum of disjoint pulses of size n;*
d) $(id - L_n \circ U_n)(f)$ *is a sum of disjoint pulses of size n.*

Proof. a) Consider $x \in \mathbb{Z}^2$ such that $(id - L_n)(f)(x) > 0$. By property A1 there exists a local maximum set V of f such that $x \in V$ and $\mathrm{card}(V) \leq n$. Since f is $(n-1)$-monotone it does not have local maximum sets of size smaller then n. Therefore $\mathrm{card}(V) = n$ and f is constant on V. Moreover, we have $(id - L_n)(f)(y) = 0$, $y \in \mathrm{adj}(V)$. Indeed, if $(id - L_n)(f)(y) > 0$ for some $y \in \mathrm{adj}(V)$ then y belongs to a local maximum set W of f and $\mathrm{card}(W) \leq n$. However, any local maximum set containing y must contain V as well. Thus, $\mathrm{card}(W) \geq n+1$. This is a contradiction. So we have that the support of $(id - L_n)(f)$ is a union of disjoint connected sets of size n, i.e.

$$\mathrm{supp}\big((id - L_n)(f)\big) = V_1 \cup V_2 \cup ... \cup V_{\rho(n)} \ ,$$

where $V_s \in \mathcal{C}$ and $\mathrm{card}(V_s) = n$ for $s = 1, 2, ..., \rho(n)$ and $V_{s_1} \cap V_{s_2} = \emptyset$ for $s_1 \neq s_2$. Let $(id - L_n)(f)(x) = \alpha_s$ for $x \in V_s$. Then

$$(id - L_n)(f) = \sum_{s=1}^{\rho(n)} \phi_{ns} \ ,$$

where ϕ_{ns} is pulse with support V_s and value α_s.

The proof of b) is similar. The proof of c) follows from a) and b) by using that $id - L_n \circ U_n = id - L_n + (id - U_n) \circ L_n$. Similar argument holds for d).

Theorem 2. *Let the operators P_n and Q_n be as in (12). Then for every $n = 1, ..., N$ we have that $D_n(f) = ((id - P_n) \circ Q_{n-1})(f)$ is a sum of pulses of the form*

$$D_n(f) = \sum_{s=1}^{\gamma(n)} \phi_{ns} \ ,$$

where $\mathrm{card}(\mathrm{supp}(\phi_{ns})) = n$ and $\mathrm{supp}(\phi_{ns_1}) \cap \mathrm{supp}(\phi_{ns_2}) = \emptyset$ for $s_1 \neq s_2$. Moreover, the supports of all pulses ϕ_{ns}, $s = 1, ..., \gamma(n)$, $n = 1, ..., N$ are either disjoint or nested, that is,

$$(\mathrm{supp}(\phi_{n_1 s_1}) \cap \mathrm{supp}(\phi_{n_2 s_2}) \neq \emptyset, \ n_1 \leq n_2) \Longrightarrow \mathrm{supp}(\phi_{n_1 s_1}) \subseteq \mathrm{supp}(\phi_{n_2 s_2}) \ . \tag{13}$$

Proof. Let $n \in \{2, ..., N\}$. The function $Q_{n-1}(f) = P_{n-1}(Q_{n-2}(f))$ is $(n-1)$-monotone due to (9)–(10). Then it follows from Theorem 1 that $(id - P_n)Q_{n-1}(f)$ is a sum of pulses of size n. For $n = 1$ the function f is trivially 1-monotone and hence $(id - P_1)(f)$ is a sum of pulses of size 1. Note that here we may have $P_n = U_n \circ L_n$ and obtain the result by item c) in Theorem 1, or use item d) if $P_n = L_n \circ U_n$. Due to (9)–(10) the sequencing of L_k and U_k in the operator P_k for $k < n$ also does not matter. Hence it remains to prove (13).

Let $\operatorname{supp}(\phi_{n_1 s_1}) \cap \operatorname{supp}(\phi_{n_2 s_2}) \neq \emptyset$. It follows from the construction of the pulses in the proof of Theorem 1 that the functions $Q_n(f)$ and $P_{n+1}(Q_n(f))$, $n \geq n_1$, are constants on the set $\operatorname{supp}(\phi_{n_1 s_1})$. Furthermore, the set $\operatorname{supp}(\phi_{n_2 s_2})$ is a local maximum set or a local minimum set of either $Q_{n_2-1}(f)$ or $P_{n_2}(Q_{n_2-1}(f))$. Then by the definition of local maximum and local minimum sets it follows that $\operatorname{supp}(\phi_{n_1 s_1}) \subset \operatorname{supp}(\phi_{n_2 s_2})$.

As mentioned earlier the sequencing of the operators L_n and U_n in P_n results in bias towards up-pulses or down-pulses. The operator F_n, called a *floor operator* given by $F_n = P_n \circ P_{n-1} \circ ... \circ P_2 \circ P_1$ where $P_n = U_n \circ L_n$, is biased towards up-pulse. This means that for any particular size the up-pulses are extracted before the down pulses. Similarly the operator C_n, called a *ceiling operator* given by $C_n = P_n \circ P_{n-1} \circ ... \circ P_2 \circ P_1$ where $P_n = L_n \circ U_n$, is biased towards down-pulses. A DPT using F_n will generally have more up-pulses and fewer down-pulses compared to DPT using C_n. It is important to realize that the bias extremes will differ for different original images due to their size and contrast so interpretation should be relative to the current image.

The alternating bias separators Z_n^+ and Z_n^- (see [4] and references therein) are given by:

$$Z_{n+1}^- = \begin{cases} L_{n+1}U_{n+1}Z_n^- & \text{if } n \text{ is even} \\ U_{n+1}L_{n+1}Z_n^- & \text{if } n \text{ is odd} \end{cases}$$

$$Z_{n+1}^+ = \begin{cases} L_{n+1}U_{n+1}Z_n^+ & \text{if } n \text{ is odd} \\ U_{n+1}L_{n+1}Z_n^+ & \text{if } n \text{ is even} \end{cases}$$

Using the order in the LULU semi-group (8) it is easy to see that

$$F_n \leq Z_n^{\pm} \leq C_n .$$

We should note that if $f \in \mathcal{A}(\mathbb{Z}^2)$ is such that $F_n = C_n$, $n = 1, ..., N$, then f has a unique DPT decomposition of the form (12).

Finally we discuss the total variation preservation property (4). There are several different concepts used for total variation of a function of multidimensional argument in literature. For a function $f \in \mathcal{A}(\mathbb{Z}^2)$ we define the total variation as

$$TV(f) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \left(|f(i+1, j) - f(i, j)| + |f(i, j+1) - f(i, j)| \right) . \tag{14}$$

It was proved in [1] that the operators L_n and U_n are total variation preserving. Furthermore, the composition of total variation preserving operators is also total variation preserving. Therefore the decomposition (3) obtained via (12) preserves the total variation, i.e. (4) holds.

Table 1. Bias Percentages ($\times 10^{-3}$) between F_n, C_n, Z_n^+ and Z_n^-

	C_n	F_n	Z_n^+	Z_n^-
C_n	0	0.2609	0.2269	0.0340
F_n	0.2609	0	0.0340	0.2269
Z_n^+	0.2269	0.0340	0	0.2610
Z_n^-	0.0340	0.2269	0.2610	0

The decomposition of the original image can be done using F_n, C_n, Z_n^+ or Z_n^-. While it is possible that these lead to different results it is seldom significantly observable in practice. For example, the percentage bias between the operators, when applied to the image (a) in Figure 2, is given in Table 1.

4 Illustrative Examples

4.1 A Partial Reconstruction

Consider image (a) in Figure 2. We wish to extract the boat as a feature and remove the background water. We consider any feature not part of the boat as

(a)

(b)

(c)

(d)

Fig. 2. (a) Original image; (b) Pulses of size up to 80; (c) Pulses of size from 81 to 4000; (d) Pulses of size larger than 4000

noise. The images (b) - (d) in Figure 2 are partial reconstructions, see [1], of the original. We can consider (b) as the microfeatures e.g. noise, (c) as the medium-sized features and (d) as the large features. We see that pulses of size up to 80 represent the water, pulses from 81 to 4000 the boat and pulses larger than 4000 represent larger noise patterns.

4.2   Identifying Pulses of Interest

In Figure 3 the pulses of size 100 to 300 with relative luminosity greater than 1 represent the solar glint on the waves (see image (b) in Figure 3) and they are emphasized in image (c) by giving them all black borders. So we are able to extract the small solar glint patches from the image and can subsequently reconstruct the image without them. The pulses of size 1501 to 2500 are the pulses corresponding to the boat (as seen in image (d) of Figure 3).
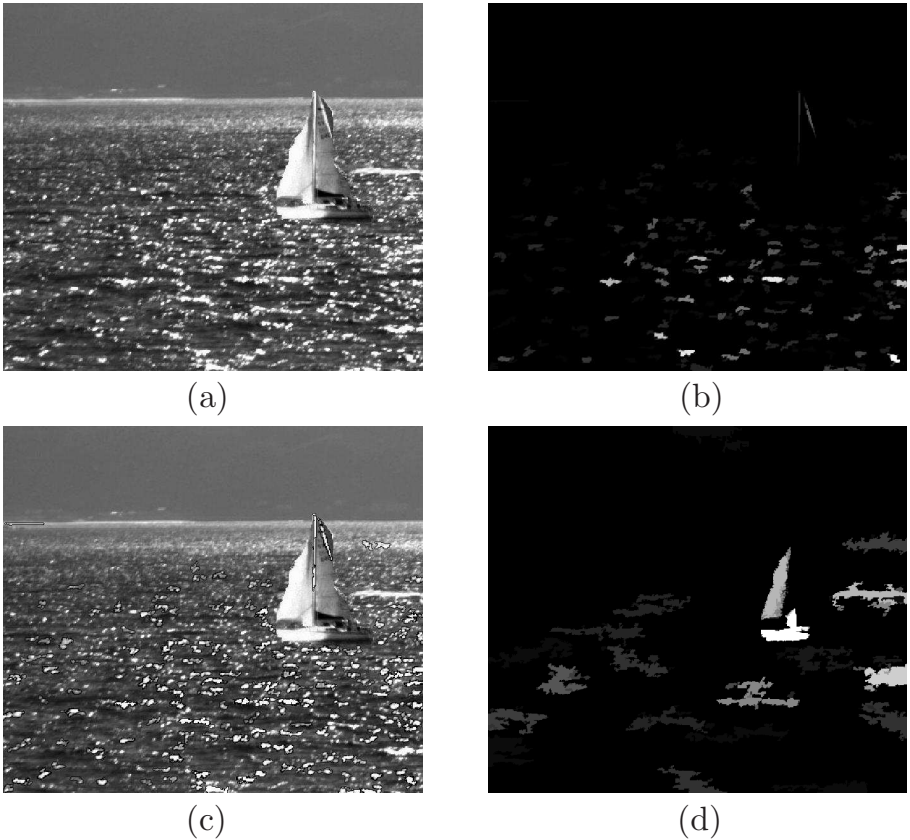

(a)

(b)

(c)

(d)

Fig. 3. (a) Original image; (b) Pulses of size 100 to 300 with relative luminosity greater than 1; (c) Borders of the pulses of size 100 to 300 with relative luminosity greater than 1 superimposed on the original image; (d) Pulses of size larger than 1501 to 2500

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