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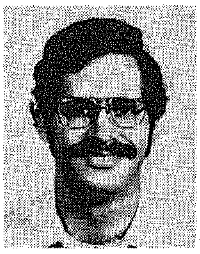
[8] L. R. Nackman, "Three-dimensional shape description using the symmetric axis transform," Ph.D. dissertation, Dep. Comput. Sci., Univ. North Carolina, Chapel Hill, 1982.
[9] H. Blum and R. N. Nagel, "Shape description using weighted symmetric axis features," *Pattern Recognition*, vol. 10, pp. 167-180, 1978.
[10] V. Guillemin and A. Pollack, *Differential Topology*. Englewood Cliffs, NJ: Prentice-Hall, 1974.
[11] W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations*, 2nd ed. New York: Wiley, 1969.
[12] M. Morse and S. S. Cairns, *Critical Point Theory in Global Analysis and Differential Topology*. New York: Academic, 1969.
[13] M. Morse and G. B. Van Schaack, "The critical point theory under general boundary conditions," *Ann. Math.*, vol. 35, pp. 545-571, July 1934.



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# Discrete Representation of Straight Lines

LEO DORST AND ARNOLD W. M. SMEULDERS

*Abstract*—If a continuous straight line segment is digitized on a regular grid, obviously a loss of information occurs. As a result, the discrete representation obtained (e.g., a chaincode string) can be coded more conveniently than the continuous line segment, but measurements of properties (such as line length) performed on the representation have an intrinsic inaccuracy due to the digitization process. In this paper, two fundamental properties of the quantization of straight line segments are treated.

1) It is proved that every "straight" chaincode string can be represented by a set of four unique integer parameters. Definitions of these parameters are given.

2) A mathematical expression is derived for the set of all continuous line segments which could have generated a given chaincode string. The relation with the chord property is briefly discussed.

*Index Terms*—Chaincode string, chord property, coding efficiency, digitized straight lines, quantization error.

## I. Introduction

IN THIS paper, we will derive two fundamental properties of digitized straight line segments. First, it will be shown that the chaincode string of any straight line segment can be one-to-one characterized by a set of four integer parameters. Secondly, we will derive the set of all continuous straight lines which could have generated a given chaincode string.

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The results derived here for the digitization for straight line segments may be used in a first-order approximation to arbitrary curves, the so-called polygon-approximation [1].

The first property derived may be useful for the efficient encoding of digitized line drawings and two-dimensional contours. Examples are the analysis and storage of line patterns in computer-aided design (CAD-CAM) and cartography, and the treatment of two-dimensional contours in industrial inspection [2]. The second property shows the fundamental loss of accuracy caused by the digitization of continuous straight lines. It also gives an explicit mathematical expression for the set of all lines lying *near* a specific chaincode string (*near* understood in the sense of the well-known chord property, as in [3]).

## II. Basic Definitions

For the digitization of two-dimensional binary images two methods are frequently used:

- OBQ (object boundary quantization), in which the outermost points still belonging to an object are digitized by a chaincode string [4] [Fig. 1(a)], which can be coded by the Freeman scheme [5] [Fig. 1(b)];
- GIQ (grid intersection quantization), in which the grid points closest to a curve whenever it intersects a row or a column of the grid are connected by a chaincode string [5] [Fig. 1(c)].

In this paper we will restrict ourselves to straight object boundaries and/or straight line drawings and consequently to "straight strings." By definition, a straight string is a string that could have been generated by the digitization of a straight object boundary and/or straight line drawing. Straight strings

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(a)

(b)

(c)

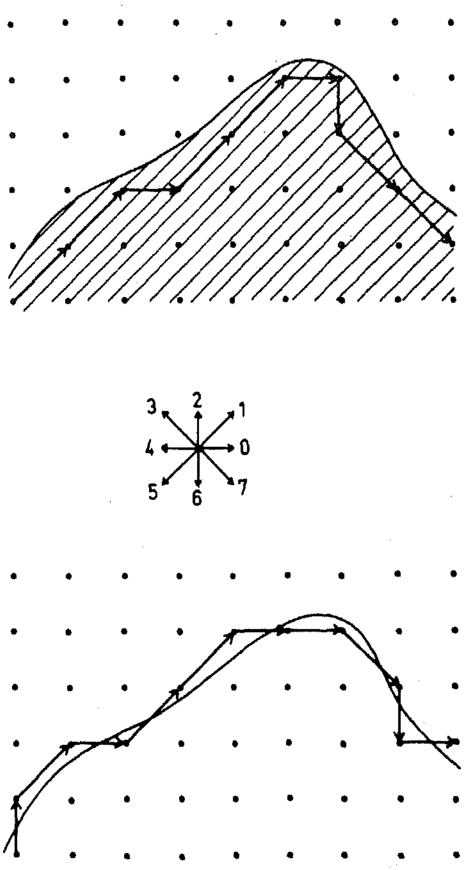



Fig. 1. (a) Object boundary quantization. (b) The Freeman chaincode scheme. (c) Grid intersect quantization.

satisfy the linearity conditions [5] and the chord property [3], while nonstraight strings do not.

We take the digitizing grid to be a square grid, and the chaincode strings to be made up of 8-connected chaincodes. It has been shown that this case can be generalized to other regular grids (e.g., the hexagonal grid) or other connectivities by straightforward computations [6].

For convenience in the mathematics, we will only consider strings consisting of chaincodes 0 and/or 1. This is no restriction: since the straight strings satisfy the linearity conditions, they consist of at most two different chaincode elements, differing 1 (mod 8). By a suitable choice of coordinate axes, one can therefore write a straight string as a string consisting only of the chaincode elements 0 and/or 1, corresponding to the directions $\binom{1}{0}$ and $\binom{1}{1}$ (Fig. 2).

Consider a line in the first octant of a Cartesian coordinate system, given by $y(x) = \alpha x + e$. The OBQ-digitization of this line is, in the first $n + 1$ columns, given by

$$\lfloor y(i) \rfloor = \lfloor \alpha i + e \rfloor \qquad i = 0, 1, 2, \cdots, n. \tag{1}$$

In this paper, $\lfloor x \rfloor$ denotes the floor function, defined as the largest integer not exceeding $x$. We will also need the ceiling-function, $\lceil x \rceil$, defined as the smallest integer not smaller than $x$.

We will denote a chaincode string by its symbol, followed by a description of the $i$th element $c_i$. The chaincode string $C$ corresponding to (1) is

$$C\colon\ c_i = \lfloor y(i) \rfloor - \lfloor y(i-1) \rfloor \qquad i = 1, 2, \cdots, n. \tag{2}$$

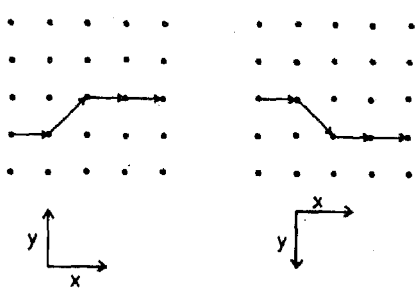



Fig. 2. All straight strings can be considered to consist of codes 0 and 1, by a suitable choice of the coordinate axes. (The nondepicted cases follow by rotation of the figure.)

The GIQ-quantization of a line $y(x) = \alpha x + e'$ is given by

$$y(i) = \lfloor \alpha i + e' - \tfrac{1}{2} \rfloor \qquad i = 0, 1, 2, \cdots, n \tag{3}$$

and the corresponding string again by applying (2). So, in the case of straight lines, GIQ can be obtained from OBQ by the substitution $e \to e' - 1/2$. (This is not true in general!)

In this paper we are solely concerned with straight lines, and therefore we will only treat OBQ. In Section III we will prove that any straight string $C$ can be characterized by a set of four integer parameters. In Section IV we introduce the "domain" of a given chaincode string as the set of all continuous lines that would, after digitization, result in that string. In Section V these domains are related to the chaincode strings, and a mathematical expression for the domain of an arbitrary string is given.

## III. The Quadruple $(n, q, p, s)$

In this section it will be proved that any straight chaincode string $C$ can be uniquely characterized by a quadruple of basic parameters, which we will write as $(n, q, p, s)$.

As a preparation, we need three theorems from the theory of numbers.

*Theorem 1:* Let $P$, $Q$, $K$, and $L$ be integers. If $P/Q$ is an irreducible fraction, then the equation

$$KP = L \pmod{Q}$$

has, for any given $L$, precisely one solution $K$ in the range $0 \leq K < Q$.

*Theorem 2:* Let $P/Q$ be an irreducible fraction, and let $i$ assume $Q$ consecutive values $i = k + 0, i = k + 1, \cdots, i = k + Q - 1$ for some $k \in \mathbf{Z}$. Then $i(P/Q)$ (mod 1) assumes all values $0/Q$, $1/Q, \cdots, (Q-1)/Q$, once and only once (in some order).

*Proof:* A proof of these theorems can be found in most introductory books on number theory; see, e.g., [7].

*Theorem 3:* Let $0 \leq \epsilon < 1$. Then

a) $\lfloor x \rfloor - \lfloor x - \epsilon \rfloor = 0 \Leftrightarrow \lfloor x \rfloor + \epsilon \leq x < \lfloor x \rfloor + 1$

b) $\lfloor x \rfloor - \lfloor x - \epsilon \rfloor = 1 \Leftrightarrow \lfloor x \rfloor \leq x < \lfloor x \rfloor + \epsilon$

c) $\lfloor x + \epsilon \rfloor - \lfloor x \rfloor = 0 \Leftrightarrow \lfloor x \rfloor \leq x < 1 + \lfloor x \rfloor - \epsilon$

d) $\lfloor x + \epsilon \rfloor - \lfloor x \rfloor = 1 \Leftrightarrow 1 + \lfloor x \rfloor - \epsilon \leq x < \lfloor x \rfloor + 1$.

*Proof:* We will prove a) only. The other cases are similar. Let $X = \lfloor x \rfloor$. We have, by the definition of the floor function,

$$\lfloor x \rfloor = X \Leftrightarrow X \leq x < X + 1 \quad \text{and}$$

$$\lfloor x - \epsilon \rfloor = X \Leftrightarrow X + \epsilon \leq x < X + \epsilon + 1.$$

So both equations are satisfied if and only if

$$\max(X, X+\epsilon) \leqslant x < \min(X+1, X+\epsilon+1)$$

and the theorem follows. Q.E.D.

We now show that the string of any straight line can be represented by a set of four integers $N, Q, P, S$.

*Theorem 4:* Any straight string $C$ can be written in the form

$$C\colon\ c_i = \left\lfloor \frac{P}{Q}(i-S) \right\rfloor - \left\lfloor \frac{P}{Q}(i-S-1) \right\rfloor; \qquad i = 1, 2, \cdots, N \tag{4}$$

where $P$, $Q$, $S$, and $N$ are integers, $P/Q$ is an irreducible fraction, and $0 \leqslant S < Q$.

*Proof:* Let the line to be digitized be given by $y = \alpha x + e$ and consider its digitization in $N+1$ columns of the grid, leading to a string of $N$ elements.

We choose two integers $P$ and $Q$ satisfying two constraints.

1) $P/Q$ is an irreducible fraction.

2) In the $N+1$ columns considered, the digitization of the line $y = \alpha x + e$ is identical to the digitization of $y = (P/Q)x + e$.

(These conditions mean that $P/Q$ is a "very good" rational approximation of $\alpha$. Since the set of rationals is dense in the set of reals, pairs of $P$ and $Q$ exist; in fact, one can always find an infinity of values satisfying the constraints.)

For the intercept $\lfloor y(i) \rfloor$ of the column $x = i$ by the digitized line we thus have

$$\begin{aligned} \lfloor y(i) \rfloor &= \lfloor \alpha i + e \rfloor = \left\lfloor \frac{P}{Q} i + e \right\rfloor \\ &= \left\lfloor \frac{Pi + \lfloor eQ \rfloor}{Q} + \frac{eQ - \lfloor eQ \rfloor}{Q} \right\rfloor \\ &= \left\lfloor \frac{Pi + \lfloor eQ \rfloor}{Q} \right\rfloor \end{aligned} \tag{5}$$

where the last transition is allowed since the first term between the brackets in (5) is a fraction with integer numerator and denominator $Q$, and for the second term we have: $0 \leqslant (eQ - \lfloor eQ \rfloor)/Q < 1/Q$. This equation can be rewritten:

$$\lfloor y(i) \rfloor = \left\lfloor \frac{Pi + \lfloor eQ \rfloor}{Q} \right\rfloor = \left\lfloor \frac{P}{Q}(i - \lfloor eQ \rfloor\, l') + \lfloor eQ \rfloor \cdot \frac{l'P + 1}{Q} \right\rfloor$$

for any value of $l'$. In particular, we can take $l'$ to be an integer $L$ in the range $0 \leqslant L < Q$ such that $LP = Q - 1 \pmod{Q}$. Theorem 1 guarantees the existence and uniqueness of $L$, given $P$ and $Q$. It follows that $LP + 1 = 0 \pmod{Q}$, so $(LP+1)/Q$ is an integer, and we have

$$\lfloor y(i) \rfloor = \left\lfloor \frac{P}{Q}(i - \lfloor eQ \rfloor L) \right\rfloor + \lfloor eQ \rfloor \frac{LP+1}{Q}$$

so, using formula (2),

$$c_i = \left\lfloor \frac{P}{Q}(i - \lfloor eQ \rfloor L) \right\rfloor - \left\lfloor \frac{P}{Q}(i - \lfloor eQ \rfloor L - 1) \right\rfloor, \qquad i = 1, 2, \cdots, N$$

which can be rewritten as

$$c_i = \left\lfloor \frac{P}{Q}(i-S) \right\rfloor - \left\lfloor \frac{P}{Q}(i-S-1) \right\rfloor, \qquad i = 1, 2, \cdots, N \tag{6}$$

where $S = \lfloor eQ \rfloor L +$ (any multiple of $Q$). We will choose $S = \lfloor eQ \rfloor L - \lfloor \lfloor eQ \rfloor L/Q \rfloor Q$, implying that $0 \leqslant S < Q$. This proves Theorem 4. Q.E.D.

Theorem 4 states that any string $C$ can be characterized completely by a quadruple of parameters: $N, Q, P, S$. This quadruple was derived from a line whose digitization is $C$, and is not uniquely determined: many different quadruples can represent the same string $C$. For calculations later in this paper we need a "standard" quadruple of defining parameters $(n, q, p, s)$ which *is* uniquely determined and which can be calculated from the chaincode string $C$ itself. This standard quadruple $(n, q, p, s)$ will now be defined. This is done by showing (Lemmas 1-3) how $N, Q, P$, and $S$ can be found in terms of a chaincode string, and then taking a specific, uniquely determined, quadruple as definition for the "standard" representation of the discrete straight line (Definitions 1-4).

First, from (4) it is seen that $N$ is the number of elements of $C$. Therefore, the uniquely determined standard value $n$ of $N$ can be defined simply by the following.

*Definition 1:* $n$ is the number of elements of $C$.

For the determination of the standard value of $Q, P$, and $S$ we have to introduce a string $C_\infty$ defined by

$$C_\infty\colon\ c_{\infty i} = \left\lfloor \frac{P}{Q}(i-S) \right\rfloor - \left\lfloor \frac{P}{Q}(i-S-1) \right\rfloor, \qquad i \in \mathbb{Z} \tag{7}$$

Note that $C$ is the part of $C_\infty$ in the interval $i = 1, 2, \cdots, N$.

The parameter $Q$ has the following property.

*Lemma 1:* $Q$ is the smallest periodicity of $C_\infty$.

*Proof:* By substitution in (7) it is obvious that $c_{i+Q} = c_i$, and therefore that $C_\infty$ has a periodicity $Q$. Suppose $C_\infty$ has a shorter periodicity $K$, with $0 < K < Q$. If $Q = 1$ this is impossible. If $Q \neq 1$, we can always find a value of $j$ such that $c_j = 0$ and $c_{j+K} = 1$. This will now be shown.

We demand the following:

$$\left\lfloor \frac{(j-s)P}{Q} \right\rfloor - \left\lfloor \frac{(j-s)P}{Q} - \frac{P}{Q} \right\rfloor = 0$$
$$\wedge \left\lfloor \frac{(j-s+K)P}{Q} \right\rfloor - \left\lfloor \frac{(j-s+K)P}{Q} - \frac{P}{Q} \right\rfloor = 1.$$

Using Theorem 3a, the first condition is equivalent to

$$\frac{P}{Q} \leqslant (j-s)\frac{P}{Q} - \left\lfloor (j-s)\frac{P}{Q} \right\rfloor < 1, \quad \text{or} \quad \frac{P}{Q} \leqslant \frac{JP}{Q} \bmod 1 < 1 \tag{8}$$

(where we introduced $J = j - s$), and the second condition is equivalent to (Theorem 3b):

$$0 \leqslant (j-s+K)\frac{P}{Q} - \left\lfloor (j-s+K)\frac{P}{Q} \right\rfloor < \frac{P}{Q},$$
$$\text{or} \quad 0 \leqslant (J+K)\frac{P}{Q} \bmod 1 < \frac{P}{Q}. \tag{9}$$

To determine a value of $j$ such that the conditions (8) and (9) are not contradictory, we examine two cases separately.

a) $P/Q \leqslant 1 - (KP/Q \bmod 1)$. In this case we choose $J$ such that $(JP/Q \bmod 1) = 1 - (KP/Q \bmod 1)$. Since the right-hand side of this equality is one of the fractions $0/Q, 1/Q, \cdots, (Q-1)/Q$, it follows from Theorem 2 that $J$ exists. Equation (8) is now satisfied and since

$$(J+K)\frac{P}{Q} \bmod 1 = \left\{\left(J\frac{P}{Q} \bmod 1\right) + \left(K\frac{P}{Q} \bmod 1\right)\right\} \bmod 1$$

$$= 1 \bmod 1 = 0 < \frac{P}{Q}, \qquad \text{(9) is also satisfied.}$$

b) $P/Q > 1 - (KP/Q \bmod 1)$. In this case we choose $J$ such that $(JP/Q \bmod 1) = P/Q$. Equation (8) is satisfied, and since

$$(J+K)\frac{P}{Q} \bmod 1 = \left\{\left(J\frac{P}{Q} \bmod 1\right) + \left(K\frac{P}{Q} \bmod 1\right)\right\} \bmod 1$$

$$= \left\{\frac{P}{Q} + \left(K\frac{P}{Q} \bmod 1\right)\right\} \bmod 1$$

$$= \frac{P}{Q} + \left(K\frac{P}{Q} \bmod 1\right) - 1 < \frac{P}{Q},$$

(9) is also satisfied.

In both cases we have the contradiction $c_{s+J} \neq c_{s+J+K}$ which implies that the string $C_\infty$ as no periodicity $K$ smaller than $Q$. Hence, $Q$ is the smallest periodicity. Q.E.D.

Now, it is obvious that the smallest period of a string $C_\infty$ of the form (7), which is identical to $C$ on the finite interval $i = 1, 2, \cdots, n$, is at most $n$ (we define the periodicity to be $n$ if the string is completely aperiodic on the interval considered). We will take this smallest periodicity as the standard value for $Q$.

So, if we define $q$ to be the smallest periodicity of $C$:

*Definition 2:*

$$q = \min_k \{k \in \{1, 2, \cdots, n\} \mid k = n \vee \forall i \in \{1, 2, \cdots, n-k\}: c_i = c_{i+k}\}$$

then $q$ is uniquely determined if $C$ is a straight string.

In $C_\infty$ defined by (7), the parameter $P$ has the property:

*Lemma 2:* $P = \Sigma_{i=1}^{Q} c_{\infty i}$.

*Proof:* Since $c_{\infty i} = \lfloor P/Q(i-S) \rfloor - \lfloor P/Q(i-S-1) \rfloor$ we have, within one period $Q$:

$$c_{\infty i} = 1 \quad \text{iff} \quad \frac{P}{Q}(i-S) \bmod 1 \in \left\{\frac{0}{Q}, \frac{1}{Q}, \cdots, \frac{P-1}{Q}\right\}$$

$$c_{\infty i} = 0 \quad \text{iff} \quad \frac{P}{Q}(i-S) \bmod 1 \in \left\{\frac{P}{Q}, \frac{P+1}{Q}, \cdots, \frac{Q-1}{Q}\right\}.$$

Since $P/Q$ is irreducible, Theorem 2 yields that every value in the two sets occurs once and only once if $i$ assumes $Q$ consecutive values. Hence $c_{\infty i} = 1$ occurs $P$ times, and $c_{\infty i} = 0$ occurs $Q - P$ times, and we have $P = \Sigma_{i=1}^{Q} c_{\infty i}$. Q.E.D.

Since $q$ is only a special choice for $Q$, defined in the finite string $C$ instead of $C_\infty$, we must define the standard value $p$ of $P$ corresponding to this choice as follows.

*Definition 3:* $p = \Sigma_{i=1}^{q} c_i$.

Therefore $p$ is also uniquely defined.

In $C_\infty$, the parameter $S$ has the following property:

*Lemma 3:* $S$ is the unique integer in the range $0 \leqslant S < Q$ satisfying:

$$\forall i \in \mathbb{Z}: \; c_{\infty i} = \left\lfloor \frac{P}{Q}(i-S) \right\rfloor - \left\lfloor \frac{P}{Q}(i-S-1) \right\rfloor.$$

*Proof:* It is obvious from (7) that $S$ satisfies this condition. It remains to be shown that $S$ is the unique solution. We do this by a reductio ad absurdum.

Suppose $S' \neq S$ also satisfies the condition. If $Q = 1$ this is impossible, since there is only one $S$ in the range $0 \leqslant S < Q$, namely $S = 0$. If $Q \neq 1$ we derive a contradiction by finding a value of $j$ for which

$$c_{\infty j} = \left\lfloor \frac{P}{Q}(j-S) \right\rfloor - \left\lfloor \frac{P}{Q}(j-S-1) \right\rfloor = 0$$

but simultaneously

$$c_{\infty j} = \left\lfloor \frac{P}{Q}(j-S') \right\rfloor - \left\lfloor \frac{P}{Q}(j-S'-1) \right\rfloor = 1.$$

By an argument completely analogous to the proof of Lemma 1 (by putting $S + K = S'$) we can show that a value of $j$ can always be found, and hence a contradiction is inevitable. Q.E.D.

As in the case of $q$ and $p$, $s$ is defined by adopting Lemma 3 to the string $C$.

*Definition 4:* Let $c_i$ be the $i$th element of $C$. Then $s$ is the unique integer in the range $0 \leqslant s < q$ for which

$$\forall i \in \{1, 2, \cdots, q\}: \; c_i = \left\lfloor \frac{p}{q}(i-s) \right\rfloor - \left\lfloor \frac{p}{q}(i-s-1) \right\rfloor.$$

It is a direct consequence of the Lemmas 1-3 and the Definitions 1-4 that the quadruple $(n, q, p, s)$ can be determined uniquely from the string $C$; so, we have the following.

*Theorem 5:* Given a straight string $C$, one can determine the quadruple of parameters $(n, q, p, s)$ uniquely.

Conversely, we have the following.

*Theorem 6:* If the quadruple $(n, q, p, s)$ can be determined from a string $C$, then $C$ is a straight string, uniquely determined by $n$, $q$, $p$, and $s$.

*Proof:* Definitions 1-4 imply that the string $C$ can be written uniquely as

$$C: \; c_i = \left\lfloor \frac{p}{q}(i-s) \right\rfloor - \left\lfloor \frac{p}{q}(i-s-1) \right\rfloor, \qquad i \in \{1, 2, \cdots, n\}.$$

To show that this is a straight string we have to give a line which would have $C$ as its digitization. Such a line is

$$y(x) = \frac{p}{q}x + \left\lceil \frac{p}{q}s \right\rceil - \frac{p}{q}s \tag{10}$$

which can be easily verified by applying formula (2). Q.E.D.

Combining the Definitions 1, 2, 3, 4 and Theorems 5 and 6 we have the following.

*Main Theorem:* There is a one-to-one correspondence between a straight string $C$ and the quadruple $(n, q, p, s)$ defined by

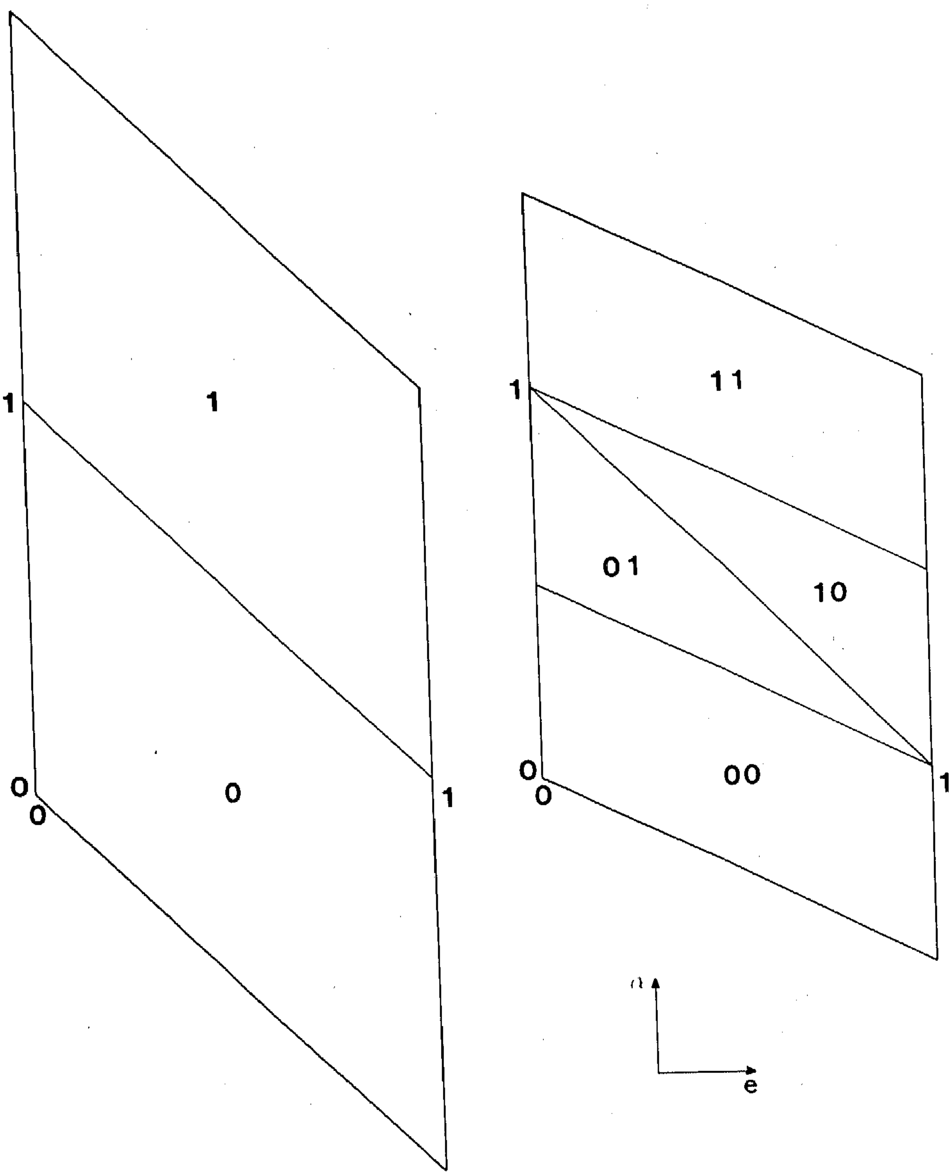



Fig. 3. The $(e, \alpha)$-plane with the domains of all straight strings of up to six elements. The chaincode string is indicated in the corresponding domain.

$$\begin{cases} n \text{ is the number of elements of } C \\ q = \min_k \{k \in \{1, 2, \cdots, n\} \mid \\ \qquad k = n \vee \forall i \in \{1, 2, \cdots, n-k\}: c_{i+k} = c_i\} \\ p = \sum_{i=1}^{q} c_i \\ s: s \in \{0, 1, 2, \cdots, q-1\} \mid \forall i \in \{1, 2, \cdots, q\}: \\ \qquad c_i = \left\lfloor \frac{p}{q}(i-s) \right\rfloor - \left\lfloor \frac{p}{q}(i-s-1) \right\rfloor \end{cases} \quad (11)$$

where $c_i$ is the $i$th element of $C$.

In other words, all information present in the string $C$ is contained in the quadruple $(n, q, p, s)$.

Combination of the Definitions 1-4 therefore leads to a unique representation of $C$ in terms of $n$, $q$, $p$, and $s$:

$$C: c_i = \left\lfloor \frac{p}{q}(i-s) \right\rfloor - \left\lfloor \frac{p}{q}(i-s-1) \right\rfloor; \quad i = 1, 2, \cdots, n \quad (12)$$

which we will sometimes write as

$$C = \text{code}(n, q, p, s).$$

In Section V we will use this description to derive an expression for the domain of $C$.

## IV. The Domain of a Chaincode String

Consider a line $l$: $y = \alpha x + e$, and its OBQ-digitization in $n+1$ columns:

$$\lfloor y(i) \rfloor = \lfloor \alpha i + e \rfloor, \quad i = 0, 1, 2, \cdots, n. \quad (13)$$

This digitized straight line segment does not uniquely determine the line $l$. In fact, there is an infinity of lines that could

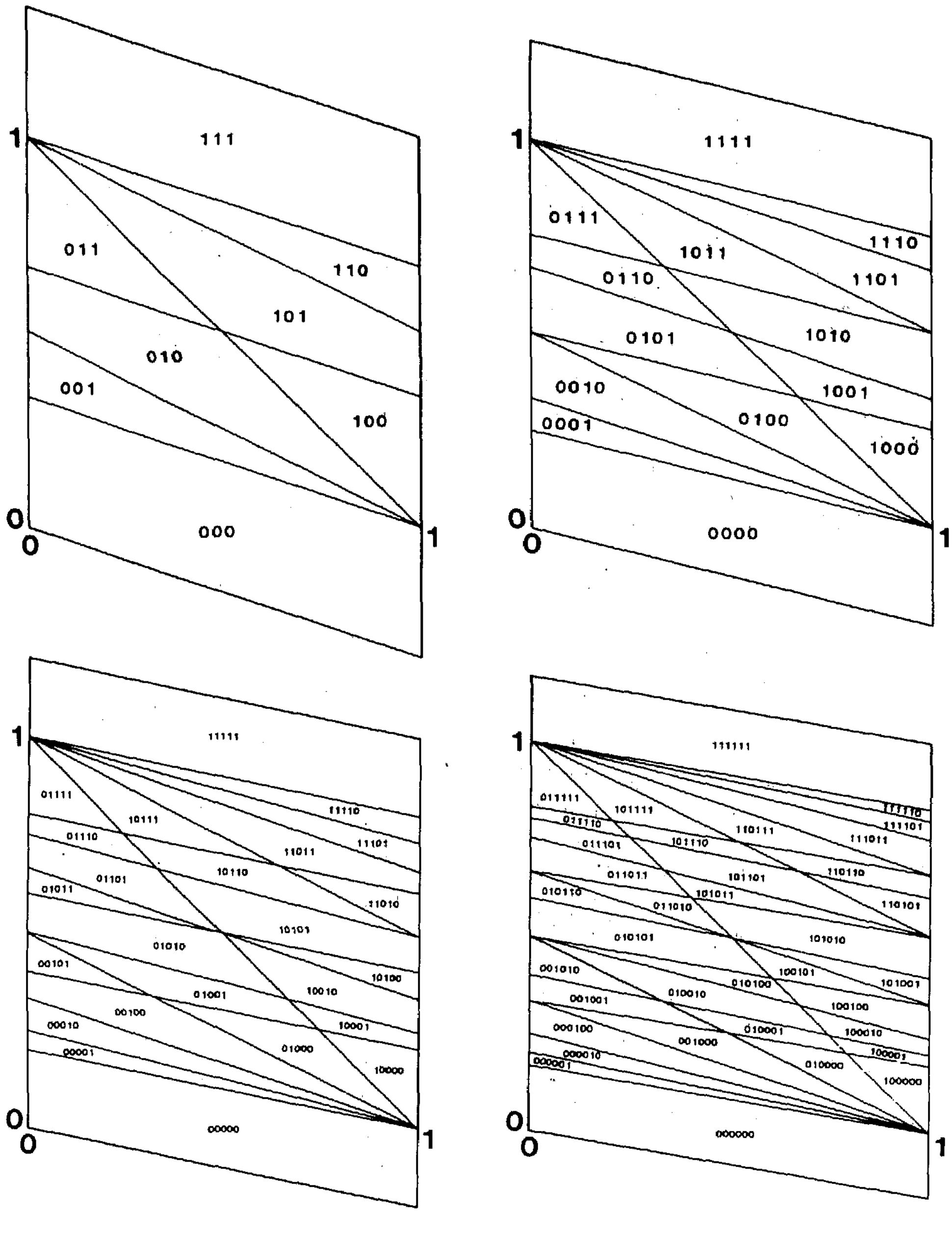



Fig. 3. (*Continued*).

have led to the digitization points $\lfloor y(i) \rfloor$, and the corresponding chaincode string defined by (2).

We now define the following.

*Definition:* The *domain* $D_C$ of a chaincode string $C$ is the set of all continuous lines whose chaincode string is $C$.

In the next section we will calculate the domain $D_C$ for an arbitrary straight string $C$, using the quadruple $(n, q, p, s)$. In the remainder of this section we will try to elucidate the concept of domain in a more intuitive way.

The formula for an arbitrary line

$$y = \alpha x + e \tag{14}$$

is a linear relationship between $x$ and $y$, given two parameters $\alpha$ and $e$. If we introduce an $(e, \alpha)$-plane, we can represent the line by a point $(e, \alpha)$, and consider (14) as a transformation formula from the $(x, y)$-plane to the $(e, \alpha)$-plane [8]. A point in the $(x, y)$-plane transforms by (14) to a linear relationship between $e$ and $\alpha$, i.e., a line in the $(e, \alpha)$-plane.

The domain $D_C$ of a string $C$ is a set of lines in the $(x, y)$-plane, and hence transforms to a set of points in the $(e, \alpha)$-plane. A plot of the $(e, \alpha)$-plane with the domains of all chaincode strings of lengths of up to six elements (consisting of codes 0 and 1) is given in Fig. 3. Similar figures were given in [6].

Since every line in the $(x, y)$-plane leads to a unique chaincode string, every point in the $(e, \alpha)$-plane belongs to a unique domain: there is no overlap, and there are no gaps between the domains.

The domains in the $(e, \alpha)$-plane are bounded by straight lines, corresponding to points in the $(x, y)$-plane. This can be easily understood as follows. If a line $l$ traverses a grid point in the $(x, y)$-plane, its chaincode string changes. But in the $(e, \alpha)$-

plane, this movement of the line $l$ transforms to the traversing of a point by a line, and the change in chaincode string corresponds to a change in domain.

Closer scrutiny reveals that an arbitrary domain is either triangular or quadrangular and hence the digitization of any line is determined by three or four grid points only (this will be proved in Section V).

A geometrical interpretation of this fact is shown in Fig. 4, where an arbitrary chaincode string $C$ has been drawn. As stated above, there are many lines leading to this string. One of these lines will be found to be of special importance (Section V). This is the line $Y$ given by

$$Y\colon\ y = \frac{p}{q}(x - s) + \left\lceil \frac{sp}{q} \right\rceil$$

where $q$, $p$, and $s$ are derived from $C$ by (12). It was shown that this line has indeed $C$ as its digitization (proof of Theorem 6). This line is indicated in Fig. 4(a).

The line $Y$ passes through a grid point in the columns

$$x = s, s + q, s + 2q, \cdots .$$

Let the last column in which it passes through a grid point be $L(s)$. (We will explain this notation in the next section.) Shifting this line vertically, parallel to itself, we encounter other grid points, lying some distance above the line $Y$. The lowest grid points above the line lie at a vertical distance $1/q$. If the one with the smallest $x$-coordinate lies in the column $F(s + l)$, then the other points of this kind lie in the columns

$$x = F(s + l), F(s + l) + q, F(s + l) + 2q, \cdots .$$

(Again, this will be shown in Section V.) Let the last column of this kind be the column $L(s + l)$. In Section V we will show that the four points indicated in the columns $s, L(s), F(s + l)$, and $L(s + l)$ determine the domain $D_C$ completely. More precisely, all lines passing entirely through the shaded area in Fig. 4(b) have as their chaincode string the string $C$, while lines not passing entirely through the shaded area have a different chaincode string. Therefore, the shaded area determines the domain of $C$.

For some strings the columns $s$ and $L(s)$, or $F(s + l)$ and $L(s + l)$ may coincide. In that case the domain is determined by three points, and its representation in the $(e, \alpha)$-plane will be triangular.

## V. The Calculation of the Domain of Code $(n, q, p, s)$

In this section the domain $D_C$ of the string $C = \text{code}\ (n, q, p, s)$ is calculated.

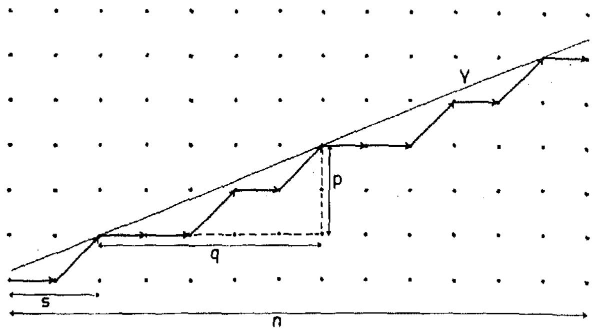
A line whose chaincode string is $C$ is [see (10)]

$$y(x) = (x - s)\frac{p}{q} + \left\lceil \frac{sp}{q} \right\rceil .$$

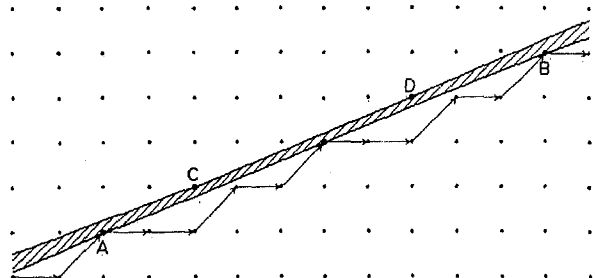
We are looking for all lines $y = \alpha x + e$ whose digitization is $C$, i.e., we have to solve

$$\lfloor \alpha i + e \rfloor = \left\lfloor (i - s)\frac{p}{q} \right\rfloor + \left\lceil s\frac{p}{q} \right\rceil \qquad (i = 0, 1, 2, \cdots, n). \quad (15)$$

This is in fact a set of $n + 1$ conditions on $\alpha$ and $e$.



(a)





(b)

Fig. 4. (a) A chaincode string and a line which could have generated it. (b) A chaincode string and the region which contains the set of all lines that could have generated it.

The solution of this equation is rather elaborate. To simplify the notation and for reasons of symmetry a few additional variables and functions are introduced first.

We introduce an integer $l$ by the implicit definition:

$$0 \leqslant l < q \quad \text{and} \quad 1 + \left\lfloor \frac{lp}{q} \right\rfloor - \frac{lp}{q} = \frac{1}{q} . \quad (16a)$$

The second condition can also be read as $lp = q - 1 \pmod q$, and hence Theorem 2 guarantees existence and uniqueness of $l$. Note that the definition of $l$ implies that

$$\frac{lp + 1}{q} \text{ is an integer,} \quad (16b)$$

a fact we will use in the proofs of the following lemmas and theorems. Note that $l = 0$ if and only if $q = 1$.

We will need two functions $L_{n,q}(x)$ and $F_q(x)$ (pronounced "last $x$" and "first $x$," respectively) defined as

$$L_{n,q}(x) = x + \left\lfloor \frac{n - x}{q} \right\rfloor q$$

and

$$F_q(x) = x - \left\lfloor \frac{x}{q} \right\rfloor q.$$

Note that $n - q < L_{n,q}(x) \leqslant n$ and $0 \leqslant F_q(x) < q$. In this paper $n$ and $q$ will always indicate the $n$ and $q$ of the string $C$ considered. Therefore we will drop the subscripts $n$ and $q$ and write $L(x)$ and $F(x)$ for $L_{n,q}(x)$ and $F_q(x)$, respectively.

It should be noted that $F(s) = s$, since $0 \leqslant s < q$. In the remainder of this section, we will often use $F(s)$ instead of $s$, since the symmetry of some of the formulas is then apparent.

Fig. 4(b) represents a graphic illustration of the "first" and the "last" function.

It can be shown that (see the Appendix)

$$\begin{cases} L(s+l) - F(s) = n > 0 & \text{if } l = 0 \\ L(s+l) - F(s) \geqslant l > 0 & \text{if } l \neq 0 \end{cases} \tag{17a}$$

and

$$\begin{cases} L(s) - F(s+l) = n > 0 & \text{if } l = 0 \\ L(s) - F(s+l) \geqslant q - l > 0 & \text{if } l \neq 0. \end{cases} \tag{17b}$$

For convenience, let us define the following quantities:

$$q_+ = L(s+l) - F(s)$$

$$q_- = L(s) - F(s+l)$$

$$\alpha_+ = \frac{p}{q} + \frac{1}{qq_+}$$

$$\alpha_- = \frac{p}{q} - \frac{1}{qq_-}$$

$$Y_i = \frac{p}{q}(i - F(s)) + \left\lceil F(s)\frac{p}{q} \right\rceil \quad \left( = \frac{p}{q}(i - s) + \left\lceil \frac{sp}{q} \right\rceil \right). \tag{18}$$

It follows from (17a) and (17b) that $\alpha_- < p/q < \alpha_+$.

In Fig. 5 the shaded area of Fig. 4(b) is schematically drawn. The interpretation of the variables defined above can be inferred from this figure.

After these preliminaries let us proceed with the calculation of the solutions $(e, \alpha)$ of (15).

*Lemma 4:* A solution $(e, \alpha)$ of the set of equations:

$$\forall i \in \{0, 1, \cdots, n\}: \ \lfloor \alpha i + e \rfloor = \lfloor Y_i \rfloor \tag{19}$$

exists if and only if $\alpha_- < \alpha < \alpha_+$.

*Proof:* We will prove this lemma by specifying a solution to the $n + 1$ conditions given in (19) in the range $\alpha_- < \alpha < \alpha_+$, and proving the impossibility of a solution outside this range.

Unfortunately, the proof consists of several cases and subcases. Some of these cases are very similar; we will then only treat one representative in detail.

First of all, the range of $\alpha$ is divided into two parts: $p/q \leqslant \alpha < \alpha_+$ and $\alpha_- < \alpha < p/q$, of which we will only treat the former in detail. Introducing a small but positive $\delta$ $(0 < \delta < 1/qq_+)$, we claim that a solution of (19) is $(e', \alpha')$, with

$$\begin{cases} e' = -(\alpha_+ - \delta) F(s) + \left\lceil F(s)\dfrac{p}{q} \right\rceil \\ \alpha' = \alpha_+ - \delta. \end{cases} \tag{20}$$

(This represents the line $y = \alpha' x + e'$, which passes through a grid point in column $F(s)$, and has a slope slightly smaller than $\alpha_+$. See Fig. 5.) To prove that $(e', \alpha')$ indeed satisfies the $n + 1$ conditions given in (19), the range of $i$ is split into three parts. These are i) the middle range $F(s) \leqslant i < L(s + l)$, ii) the end $L(s + l) \leqslant i \leqslant n$, and iii) the beginning $0 \leqslant i < F(s)$. These parts are schematically indicated in Fig. 5.

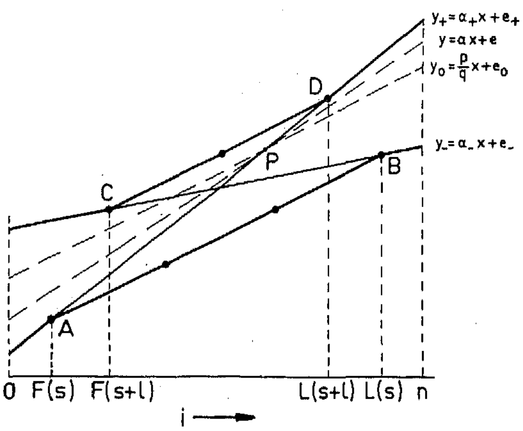



Fig. 5. Schematical representation of Fig. 4(b), as an illustration for the proofs of Lemma 4 and the domain theorem.

i) For $F(s) \leqslant i < L(s + l)$ we have

$$\begin{aligned} &\lfloor \alpha' i + e' \rfloor - \lfloor Y_i \rfloor \\ &\quad = \lfloor (\alpha_+ - \delta)\{i - F(s)\} \rfloor - \left\lfloor \frac{p}{q}\{i - F(s)\} \right\rfloor \\ &\quad = \left\lfloor \frac{p}{q}\{i - F(s)\} + \left( \frac{1}{q\{L(s+l) - F(s)\}} - \delta \right)\{i - F(s)\} \right\rfloor \\ &\qquad - \left\lfloor \frac{p}{q}\{i - F(s)\} \right\rfloor. \end{aligned}$$

Using the fact that

$$0 \leqslant \left( \frac{1}{q\{L(s+l) - F(s)\}} - \delta \right) \{i - F(s)\} < \frac{1}{q}$$

in the interval of $i$ considered, we have that $\lfloor \alpha' i + e' \rfloor - \lfloor Y_i \rfloor = 0$ in this interval, in agreement with (19).

ii) For the case $L(s + l) \leqslant i \leqslant n$ we put $i = L(s + l) + j$ with $0 \leqslant j \leqslant n - L(s + l) < q$.

Denoting the small and positive entity $\{L(s + l) - F(s) + j\}\,\delta$ by $\Delta$, we have

$$\begin{aligned} &\lfloor \alpha' i + e' \rfloor - \lfloor Y_i \rfloor \\ &\quad = \lfloor (\alpha_+ - \delta)\{j + L(s+l) - F(s)\} \rfloor \\ &\qquad - \left\lfloor \frac{p}{q}\{j + L(s+l) - F(s)\} \right\rfloor \\ &\quad = \left\lfloor \frac{p}{q}\{L(s+l) - F(s)\} + \frac{jp}{q} + \frac{1}{q} + \frac{j}{qq_+} - \Delta \right\rfloor \\ &\qquad - \left\lfloor \frac{p}{q}\{j + L(s+l) - F(s)\} \right\rfloor \\ &\quad = \left\lfloor \frac{p}{q}\left( l + \left\lfloor \frac{n-s-l}{q} \right\rfloor q \right) + \frac{jp}{q} + \frac{1}{q} + \frac{j}{qq_+} - \Delta \right\rfloor \\ &\qquad - \left\lfloor \frac{p}{q}\left( j + l + \left\lfloor \frac{n-s-l}{q} \right\rfloor q \right) \right\rfloor \\ &\quad = \left\lfloor (j+l)\frac{p}{q} + \frac{1}{q} + \frac{j}{qq_+} - \Delta \right\rfloor - \left\lfloor (j+l)\frac{p}{q} \right\rfloor \end{aligned} \tag{21}$$

which, by Theorem 3c, equals 0 if and only if

$$\left\lfloor (j+l)\frac{p}{q} \right\rfloor - (j+l)\frac{p}{q} \leq 0 < 1 + \left\lfloor (j+l)\frac{p}{q} \right\rfloor - (j+l)\frac{p}{q} - \frac{1}{q} - \frac{j}{qq_+} + \Delta. \tag{22}$$

The first inequality follows directly from the definition of the floor function. For the second inequality, two subcases are to be considered.

a) $j$ is a multiple of $l$, say $j = kl$, with $k = 0, 1, \cdots, K$. ($K$ is the maximum value of $k$, determined by $Kl \leq n - L(s+l)$ and $(K+1)l > n - L(s+l)$. Note that $0 \leq kl < q$, so $0 \leq K < q$.) We have

$$\begin{aligned} 1 + \left\lfloor (j+l)\frac{p}{q} \right\rfloor - (j+l)\frac{p}{q} &= 1 + \left\lfloor (k+1)\frac{lp}{q} \right\rfloor - (k+1)\frac{lp}{q} \\ &= 1 + \left\lfloor (k+1)\frac{lp+1}{q} - \frac{k+1}{q} \right\rfloor - (k+1)\frac{lp}{q} \\ &= 1 + (k+1)\frac{lp+1}{q} + \left\lfloor -\frac{k+1}{q} \right\rfloor - (k+1)\frac{lp}{q} \quad \text{[by (16b)]} \\ &= 1 + \left\lfloor -\frac{k+1}{q} \right\rfloor + \frac{k+1}{q} = \frac{k+1}{q} \end{aligned} \tag{23}$$

since $0 < k+1 \leq q$.

Furthermore,

$$\begin{aligned} \frac{1}{q} + \frac{j}{qq_+} - \Delta &\leq \frac{1}{q} + \frac{j}{ql} - \Delta \quad \text{[by (17a)]} \\ &= \frac{k+1}{q} - \Delta. \end{aligned}$$

So, in the case $j = kl$,

$$1 + \left\lfloor (j+l)\frac{p}{q} \right\rfloor - (j+l)\frac{p}{q} - \frac{1}{q} - \frac{j}{qq_+} + \Delta \geq \Delta > 0 \tag{24a}$$

in agreement with (22), and $\lfloor \alpha' i + e' \rfloor$ equals $\lfloor Y_i \rfloor$ for the values of $i$ considered in this subcase.

b) $j$ is not a multiple of $l$. We know that $0 \leq j < q$, and from Theorem 2 it therefore follows that in this interval of $j$ the expression $1 + \lfloor (j+l)p/q \rfloor - (j+l)p/q$ can only assume each of the values $0/q, 1/q, \cdots, (q-1)/q \pmod 1$ at most once. Equation (23) shows that for $j = 0, l, \cdots, Kl$ the expression assumes the values $1/q, 2/q, \cdots, (K+1)/q$.

Therefore we have, if $j \neq kl$,

$$1 + \left\lfloor \frac{(j+l)p}{q} \right\rfloor - (j+l)\frac{p}{q} \geq \frac{K+2}{q}.$$

Furthermore,

$$\begin{aligned} \frac{1}{q} + \frac{j}{qq_+} - \Delta &\leq \frac{1}{q} + \frac{j}{ql} - \Delta \quad \text{[by (17a)]} \\ &< \frac{K+2}{q} - \Delta \quad (\text{since } j < (K+1)l). \end{aligned}$$

Hence,

$$1 + \left\lfloor (j+l)\frac{p}{q} \right\rfloor - (j+l)\frac{p}{q} - \frac{1}{q} - \frac{j}{qq_+} + \Delta > \Delta > 0 \tag{24b}$$

in agreement with (21), and again $\lfloor \alpha' i + e' \rfloor = \lfloor Y_i \rfloor$.

The equations (24a) and (24b), together prove (22) and show that $\lfloor \alpha' i + e' \rfloor = \lfloor Y_i \rfloor$ if $L(s+l) \leq i \leq n$.

iii) For the case $0 \leq i < F(s)$, we put $i = F(s) - j$, with $0 < j \leq F(s)$. We then proceed in a way entirely analogous to ii) to show that

$$\lfloor \alpha' i + e' \rfloor = \lfloor Y_i \rfloor \quad \text{if } 0 \leq i < F(s).$$

Combining i), ii), and iii) we have shown that, for $0 \leq i \leq n$, $\lfloor \alpha' i + e' \rfloor = \lfloor Y_i \rfloor$. This proves that a solution $(e, \alpha)$ can be found such that

$$\forall i \in \{0, 1, \cdots, n\}: \lfloor \alpha i + e \rfloor = \lfloor Y_i \rfloor \quad \text{if } \frac{p}{q} \leq \alpha < \alpha_+. \tag{25}$$

We now prove that $\alpha < \alpha_+$ is the strictest upper bound, by deriving a contradiction if $\alpha \geq \alpha_+$.

Let $\alpha'' = \alpha_+ + \delta$ with $\delta \geq 0$. At $i = F(s)$ we have

$$\begin{aligned} \lfloor \alpha'' i + e \rfloor - \lfloor Y_i \rfloor &= \lfloor (\alpha_+ + \delta) F(s) + e \rfloor - \left\lceil F(s)\frac{p}{q} \right\rceil \\ &= \left\lfloor (\alpha_+ + \delta) F(s) - \left\lceil F(s)\frac{p}{q} \right\rceil + e \right\rfloor. \end{aligned}$$

The minimal value of $e$ for which this is 0 is

$$e_{min} = -(\alpha_+ + \delta) F(s) + \left\lceil F(s)\frac{p}{q} \right\rceil.$$

So, only if $e \geq e_{min}$ the digitizations $\lfloor \alpha'' i + e \rfloor$ and $\lfloor Y_i \rfloor$ can be identical for all $i$. However, at the same time we have for $i = L(s+l)$:

$$\begin{aligned} &\lfloor \alpha'' i + e \rfloor - \lfloor Y_i \rfloor \\ &= \lfloor (\alpha_+ + \delta) L(s+l) + e \rfloor - \left\lfloor \frac{p}{q}\{L(s+l) - F(s)\} \right\rfloor - \left\lceil F(s)\frac{p}{q} \right\rceil \\ &= \lfloor e - e_{min} + (\alpha_+ + \delta)\{L(s+l) - F(s)\} \rfloor - \left\lfloor \frac{p}{q}\{L(s+l) - F(s)\} \right\rfloor \\ &\geq \lfloor (\alpha_+ + \delta)\{L(s+l) - F(s)\} \rfloor - \left\lfloor \frac{p}{q}\{L(s+l) - F(s)\} \right\rfloor \\ &= \left\lfloor \left(\frac{p}{q} + \delta\right)\{L(s+l) - F(s)\} + \frac{1}{q} \right\rfloor - \left\lfloor \frac{p}{q}\{L(s+l) - F(s)\} \right\rfloor \\ &= \left\lfloor \frac{p}{q}\left\{l + \left\lfloor \frac{n-s-l}{q} \right\rfloor q\right\} + \delta\{L(s+l) - F(s)\} + \frac{1}{q} \right\rfloor - \left\lfloor \frac{p}{q}\left\{l + \left\lfloor \frac{n-s-l}{q} \right\rfloor q\right\} \right\rfloor \\ &= \left\lfloor \frac{lp+1}{q} + \{L(s+l) - F(s)\}\delta \right\rfloor - \left\lfloor \frac{lp}{q} \right\rfloor \\ &= 1 + \left\lfloor \frac{lp}{q} \right\rfloor + \lfloor \delta\{L(s+l) - F(s)\} \rfloor - \left\lfloor \frac{lp}{q} \right\rfloor \quad \text{[by (16a)]} \\ &= 1 + \lfloor \delta\{L(s+l) - F(s)\} \rfloor \geq 1. \end{aligned} \tag{26}$$

This indicates a conflict between $\lfloor \alpha'' i + e \rfloor$ and $\lfloor Y_i \rfloor$ at $i = L(s+l)$. Hence $\lfloor \alpha i + e \rfloor$ is not identical to $\lfloor Y_i \rfloor$ for all $i$ if $\alpha \geqslant \alpha_+$. So, if $\alpha \geqslant \alpha_+$, there is no solution to (19). In combination with (25) we have therefore proved the lemma for the range $p/q \leqslant \alpha < \alpha_+$.

The case $\alpha_- < \alpha \leqslant p/q$ is similar. In this case it can be shown that $(e', \alpha')$ with

$$\begin{cases} e' = -(\alpha_- + \delta)\, L(s) + \left\lceil L(s) \dfrac{p}{q} \right\rceil \\ \alpha' = \alpha_- + \delta \end{cases}$$

(with $0 < \delta < 1/qq_-$) is a solution of (19), and that a contradiction occurs if $\alpha \leqslant \alpha_-$. Therefore the lemma is proved.

Q.E.D.

With this preparation, we are ready to prove the following.

*The Domain Theorem:* The only values of $e$ and $\alpha$ satisfying the equations

$$\forall i \in \{0, 1, \cdots, n\}: \ \lfloor \alpha i + e \rfloor = \lfloor Y_i \rfloor \tag{27}$$

are as follows.

1) For $p/q \leqslant \alpha < \alpha_+$:

$$\left\lceil F(s) \frac{p}{q} \right\rceil - \alpha F(s) \leqslant e < 1 + \left\lfloor L(s+l) \frac{p}{q} \right\rfloor - \alpha L(s+l).$$

2) For $\alpha_- < \alpha \leqslant p/q$:

$$\left\lceil L(s) \frac{p}{q} \right\rceil - \alpha L(s) \leqslant e < 1 + \left\lfloor F(s+l) \frac{p}{q} \right\rfloor - \alpha F(s+l).$$

*Proof:* By Lemma 4, only values of $\alpha$ in the range $\alpha_- < \alpha < \alpha_+$ need to be considered. As in Lemma 4, we will only treat the case $p/q \leqslant \alpha < \alpha_+$, since the case $\alpha_- < \alpha < p/q$ is completely analogous.

Consider a line $y_+(x) = \alpha_+ x + e_+$, with $e_+ = -\alpha F(s) + \lfloor F(s) (p/q) \rfloor$. This line passes through the grid point $A$ in column $F(s)$, and the grid point $D$ in column $L(s+l)$ (see Fig. 5). Let this line intersect an arbitrary line $y = \alpha x + e$ in a point $P = (x_p, y_p)$, with $x_p = (e - e_+)/(\alpha_+ - \alpha)$. Let us also define a line with slope $p/q$ through $P$: $y_0(x) = p/qx + e_0$, with $e_0 = -p/qx_p + y_p$. Due to the fact that $p/q \leqslant \alpha < \alpha_+$ we have

$$\text{for } \ i \geqslant x_p: \ \frac{p}{q} i + e_0 \leqslant \alpha i + e < \alpha_+ i + e_+ \tag{28a}$$

and

$$\text{for } \ i < x_p: \ \alpha_+ i + e_+ < \alpha i + e \leqslant \frac{p}{q} i + e_0. \tag{28b}$$

It will be convenient to introduce a line also passing through $P$, but with a slope infinitesimally smaller than $\alpha_+$. If we use $\delta$ as a small but positive number, this line is

$$y'_+(x) = (\alpha_+ - \delta)\{x - F(s)\} + \delta\{x_p - F(s)\} + \left\lceil F(s) \frac{p}{q} \right\rceil$$

as may be easily verified.

The inequalities (28a) and (28b) can now be reformulated as

$$\text{for } \ i \geqslant x_p: \ y_0(i) \leqslant \alpha i + e \leqslant y'_+(i) \tag{29a}$$

and

$$\text{for } \ i < x_p: \ y'_+(i) \leqslant \alpha i + e \leqslant y_0(i). \tag{29b}$$

The theorem will be proved by deriving bounds for $x_p$ such that (27) is satisfied. For this purpose, the range of values of $x_p$ is split in three parts.

i) $F(s) \leqslant x_p < L(s+l)$. In this range we have

$$\begin{aligned} \left\lfloor \frac{p}{q} i + e_0 \right\rfloor &= \left\lfloor \frac{p}{q}(i - x_p) + y_p \right\rfloor \\ &= \left\lfloor \frac{1}{qq_+}\{x_p - F(s)\} + \frac{p}{q}\{i - F(s)\} \right\rfloor + \left\lceil F(s) \frac{p}{q} \right\rceil \\ &\qquad (\text{since } \ y_+ = \alpha_+ x_p + e_+) \\ &= \left\lfloor \frac{p}{q}\{i - F(s)\} \right\rfloor + \left\lceil F(s) \frac{p}{q} \right\rceil \\ &= \lfloor Y_i \rfloor \end{aligned} \tag{30}$$

where we used the fact that

$$0 \leqslant \frac{1}{qq_+}\{x_p - F(s)\} < \frac{1}{q}$$

in the range of $x_p$ considered.

In the proof of Lemma 4 it was shown that

$$\begin{aligned} &\lfloor (\alpha_+ - \delta)\{i - F(s)\} \rfloor + \left\lceil F(s) \frac{p}{q} \right\rceil \\ &\qquad = \lfloor Y_i \rfloor \quad \text{ for } \ i = 0, 1, 2, \cdots, n. \end{aligned}$$

In a similar way one can show that

$$\begin{aligned} \lfloor y'_+(i) \rfloor &= \lfloor (\alpha_+ - \delta)\{i - F(s)\} + \delta\{x_p - F(s)\} \rfloor + \left\lceil F(s) \frac{p}{q} \right\rceil \\ &= \lfloor Y_i \rfloor \quad \text{ for } \ i = 0, 1, 2, \cdots, n. \end{aligned} \tag{31}$$

The extra term in (31) is infinitesimally small and detailed calculation shows that it does not lead to different values for the floor function if $F(s) \leqslant x_p < L(s+l)$. (This can also be understood geometrically from Fig. 5: the line $y'_+(x)$ is the line $y_+(x)$, slightly tilted around the point $P$. Therefore the critical grid points for the line $y_+(x)$, especially point $D$ and $A$, are harmless for $y'_+(x)$.) Equations (30) and (31), combined with (29a) and (29b) imply that $\alpha i + e$ always lies between two numbers $y_0(i)$ and $y'_+(i)$ whose floor function equals $\lfloor Y_i \rfloor$. Using the definition of the floor-function, this implies that the floor of $\alpha i + e$ also equals $\lfloor Y_i \rfloor$, for $i = 1, 2, \cdots, n$. We therefore have the result that a solution of (27) can always be found if $F(s) \leqslant x_p < L(s+l)$.

ii) $x_p < F(s)$. With $x_p$ in this range, no solution of (27) is possible. More specifically, we can show that a contradiction between the values of $\lfloor \alpha i + e \rfloor$ and $\lfloor Y_i \rfloor$ arises for $i = F(s)$. At $i = F(s)$ we have

$$\begin{aligned} &\lfloor \alpha i + e \rfloor - \lfloor Y_i \rfloor \\ &\quad = \lfloor \alpha F(s) + e \rfloor - \left\lceil F(s) \frac{p}{q} \right\rceil \\ &\quad = \lfloor \alpha\{F(s) - x_p\} + e + \alpha x_p \rfloor - \left\lceil F(s) \frac{p}{q} \right\rceil \\ &\quad = \lfloor (\alpha_+ - \alpha)\{x_p - F(s)\} \rfloor \\ &\qquad\qquad (\text{since } \ \alpha x_p + e = y_p = \alpha_+ x_p + e_+) \\ &\quad < 0. \end{aligned}$$

So $\lfloor \alpha i + e \rfloor \neq \lfloor Y_i \rfloor$ at $i = F(s)$. Therefore, (27) is not satisfied for all $i$, if $x_p < F(s)$.

iii) $x_p \geqslant L(s+l)$. In this case a conflict between $\lfloor \alpha i + e \rfloor$ and $\lfloor Y_i \rfloor$ arises for $i = L(s+l)$. At $i = L(s+l)$ we have

$$\begin{aligned}
&\lfloor \alpha i + e \rfloor - \lfloor Y_i \rfloor \\
&\quad = \lfloor \alpha L(s+l) + e \rfloor - \lfloor Y_{L(s+l)} \rfloor \\
&\quad \geqslant \lfloor \alpha_+ L(s+l) + e_+ \rfloor - \lfloor Y_{L(s+l)} \rfloor \qquad \text{[by (28b)]} \\
&\quad = \lfloor \alpha_+ \{L(s+l) - F(s)\} \rfloor + \left\lceil F(s) \frac{p}{q} \right\rceil - \lfloor Y_{L(s+l)} \rfloor \\
&\quad = \left\lfloor \frac{p}{q} \{L(s+l) - F(s)\} + \frac{1}{q} \right\rfloor - \left\lfloor \frac{p}{q} \{L(s+l) - F(s)\} \right\rfloor \\
&\quad = \left\lfloor \frac{lp+1}{q} \right\rfloor - \left\lfloor \frac{lp}{q} \right\rfloor \\
&\quad = 1 \qquad \text{[by (16a)]}.
\end{aligned}$$

So, if $x_p \geqslant L(s+l)$, no solution to (27) is possible for all $i$ due to the conflict at $i = L(s+l)$.

We have therefore found that the only values of $x_p$ for which a solution to the $(n+1)$ conditions of (27) is possible are given by

$$F(s) \leqslant x_p < L(s+l).$$

Substituting $x_p = (e - e_+)/(\alpha_+ - \alpha)$ and the expressions for $e_+$ and $\alpha_+$, this can be rewritten to a range of $e$:

$$\left\lceil F(s) \frac{p}{q} \right\rceil - \alpha F(s) \leqslant e < \frac{p}{q} \{L(s+l) - F(s)\} + \frac{1}{q} + \left\lceil F(s) \frac{p}{q} \right\rceil - \alpha L(s+l).$$

The right-hand side can be written in a more convenient form, using the identity $\lceil s(p/q) \rceil = 1 + \lfloor (sp-1)/q \rfloor$, which is valid for any integers $s$, $p$, and $q$:

$$\begin{aligned}
&\frac{p}{q}\{L(s+l) - F(s)\} + \frac{1}{q} + \left\lceil F(s) \frac{p}{q} \right\rceil - \alpha L(s+l) \\
&\quad = \frac{lp+1}{q} + \left\lfloor \frac{n-s-l}{q} \right\rfloor p + \left\lfloor \frac{sp-1}{q} \right\rfloor \\
&\qquad + 1 - \alpha L(s+l) \qquad \text{[definition } L(x)] \\
&\quad = \left\lfloor \frac{sp-1}{q} + \frac{lp+1}{q} + \left\lfloor \frac{n-s-l}{q} \right\rfloor p \right\rfloor \\
&\qquad + 1 - \alpha L(s+l) \qquad \text{[by (16b)]} \\
&\quad = 1 + \left\lfloor L(s+l) \frac{p}{q} \right\rfloor - \alpha L(s+l) \qquad \text{[definition of } L(x)].
\end{aligned}$$

Hence, if $p/q \leqslant \alpha < \alpha_+$, the only solutions to (27) are the pairs $(e, \alpha)$ for which

$$\left\lceil F(s) \frac{p}{q} \right\rceil - \alpha F(s) \leqslant e < 1 + \left\lfloor L(s+l) \frac{p}{q} \right\rfloor - \alpha L(s+l).$$

This proves the first part of the theorem. The second part can be proved analogously, by considering the line $y_-(x) = \alpha_- x + e_-$ with $e_- = -\alpha_- L(s) + \lceil L(s)\, p/q \rceil$, which passes through the point $B$ (see Fig. 5), and the lines $y(x) = \alpha x + e$ and $y_0(x) = (p/q) + e_0$. One then arrives at the result that a solution $(e, \alpha)$ with $\alpha_- < \alpha \leqslant p/q$ exists if and only if

$$\left\lceil L(s) \frac{p}{q} \right\rceil - \alpha L(s) \leqslant e < 1 + \left\lfloor F(s+l) \frac{p}{q} \right\rfloor - \alpha F(s+l).$$

This proves the theorem. Q.E.D.

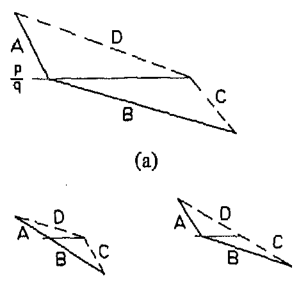



Fig. 6. Schematical representation of the domains in the $(e, \alpha)$-plane. The points of Fig. 5 corresponding to the boundaries have been indicated. (a) General case: quadrilateral. (b) $A$ and $B$ coincide. (c) $C$ and $D$ coincide.

This theorem can be interpreted as indicated in Fig. 6: each chaincode string $C$ is represented by a domain in the $(e, \alpha)$-plane. The domain of $C$ consists of all values $(e, \alpha)$ corresponding to continuous straight lines whose digitization would be $C$.

The domain theorem shows that a domain has in general a quadrilateral shape: it is bounded by four lines in the $(e, \alpha)$-plane. A plot of the domains in the $(e, \alpha)$-plane (see Fig. 3) reveals that sometimes this quadrangle degenerates to a triangle. From the domain theorem we can see that this happens if either $F(s) = L(s)$ or $F(s+l) = L(s+l)$. These conditions can be rewritten to conditions for $n$:

$$F(s) = L(s) \Leftrightarrow s - \left\lfloor \frac{s}{q} \right\rfloor q = s + \left\lfloor \frac{n-s}{q} \right\rfloor q \Leftrightarrow \left\lfloor \frac{n-s}{q} \right\rfloor + \left\lfloor \frac{s}{q} \right\rfloor = 0$$

$$\Leftrightarrow \left\lfloor \frac{n - \left(s - \left\lfloor \frac{s}{q} \right\rfloor q\right)}{q} \right\rfloor = \left\lfloor \frac{n - F(s)}{q} \right\rfloor = 0$$

$$\Leftrightarrow F(s) \leqslant n < F(s) + q$$

corresponding to Fig. 6(b), and similarly

$$F(s+l) = L(s+l) \Leftrightarrow F(s+l) \leqslant n < F(s+l) + q$$

corresponding to Fig. 6(c).

As we have seen in Section IV, the boundaries of the domain correspond to grid points in the $(x, y)$-plane. Therefore, the fact that a domain in the $(e, \alpha)$-plane is, in general, quadrilateral implies that the digitization of a line is determined by at most four grid points. The domain theorem shows that these four grid points lie in the columns $F(s)$, $L(s)$, $F(s+l)$, and $L(s+l)$.

## VI. Conclusion

In this paper we treated two fundamental properties of the quantization of continuous straight line segments. The first property is that the information present in the chaincode string of a continuous line segment can be contained in a set of four integer parameters $(n, q, p, s)$. The second property reveals the set of continuous line segments which could have generated a given chaincode string. This set is called the domain of the string. Formulas for the domain, expressed in $(n, q, p, s)$, are given in the Domain Theorem.

These results were all derived for OBQ-digitization on a square grid, represented by an 8-connected chaincode string in the first octant. It should be noted that they can be easily generalized to other octants by rotation and reflection (Fig. 2), to other regular grids and connectivities by linear transformation [6], and to GIQ-quantization by a simple substitution of variables (Section II of this paper).

The four parameters $(n, q, p, s)$ can be of great practical use since they contain the information present in a chaincode string in a concise form. Fields of application might be the encoding or measurement of digitized objects. In fact, the research reported in this paper was motivated by a paper by Vossepoel and Smeulders [6] on the accuracy of length measurements of digitized straight line segments (see also [9]). By characterizing a chaincode string by three parameters the authors were able to derive an accurate estimator for the line length to be associated with a chaincode string. They remarked, though, that these three parameters were insufficient to characterize a string completely, and that therefore their estimators could still be improved. In a sequel paper, we will use $(n, q, p, s)$ to derive best linear unbiased estimators (BLUE) for the measurement of an arbitrary property of a digitized straight line segment.

At first sight, the choice of $(n, q, p, s)$ as characterizing parameters of a straight string may seem arbitrary, and their definition complex. Actually, it was shown that these parameters summarize basic properties of a straight string: $n$ is its length, $q$ is its smallest periodicity, $p/q$ is the simplest slope in agreement with the string, and $s$ is a phase shift. Moreover, it was shown that the string can be obtained from $(n, q, p, s)$ by the simple formula

$$C: \; c_i = \left\lfloor \frac{p}{q}(i-s) \right\rfloor - \left\lfloor \frac{p}{q}(i-s-1) \right\rfloor \qquad i = 1, 2, \cdots, n.$$

Regarding the complexity of computation, we have developed an algorithm of order $N$ for the decomposition of an arbitrary chaincode string of $N$ elements into straight substrings, which simultaneously computes the parameters $(n, q, p, s)$ of these substrings [10].

Using the parameters $(n, q, p, s)$ we were able to derive mathematical expressions for the domain of an arbitrary chaincode string in the "domain theorem." This theorem summarizes the set of all continuous lines which yield a specific chaincode string after digitization and thus comprises a kind of "inversion" of the digitization process.

There is a close relation between our domain theorem and the well-known chord property. Rosenfeld [3] proved that a chaincode string is straight if and only if it satisfies the chord property. A string has the chord property if there exists a continuous line lying *near* the grid points the string connects. A continuous line lies *near* a set $S$ of grid points if for any point $(x, y)$ of the line, there exist a point of $S$ such that $\max\{|i-x|, |j-y|\} < 1$. In grid intersection quantization, the set of all lines lying *near* a chaincode string $C$ is precisely the set of all lines whose digitization is $C$, and this we called the domain of $C$. Therefore, the analog of the domain theorem for GIQ (instead of OBQ) gives a mathematical expression for the set of all lines lying near a specific string, in the sense of the chord property. This example illustrates that the results derived in this paper relate to very fundamental properties of digitized straight lines.

## Appendix

In this Appendix we will prove

$$L(s+l) - F(s) = n > 0 \qquad \text{if } l = 0$$
$$L(s+l) - F(s) \geqslant l > 0 \qquad \text{if } l \neq 0.$$

We first need three auxiliary lemma's.

Consider two lines

$$y(x) = \frac{p}{q}(x-s) + \left\lceil \frac{sp}{q} \right\rceil$$

and

$$y_l(x) = \frac{\left\lfloor \frac{lp}{q} \right\rfloor + 1}{l}(x-s) + \left\lceil \frac{sp}{q} \right\rceil$$

and the digitizations $\lfloor y(i) \rfloor$ and $\lfloor y_l(i) \rfloor$, for $i = 0, 1, \cdots, n$.

*Lemma A1:*

$$\lfloor y(i) \rfloor = \lfloor y_l(i) \rfloor \qquad \text{for } i = 0, 1, \cdots, s+l-1,$$
$$\text{and } \lfloor y(s+l) \rfloor \neq \lfloor y_l(s+l) \rfloor.$$

In words, the lowest positive $i$ for which $\lfloor y(i) \rfloor$ and $\lfloor y_l(i) \rfloor$ are not identical is $i = s + l$.

*Proof:* First, we have

$$\lfloor y_l(i) \rfloor = \left\lfloor \frac{1}{l}\left(\left\lfloor \frac{lp}{q} \right\rfloor + 1\right)(i-s) \right\rfloor + \left\lceil \frac{sp}{q} \right\rceil$$
$$= \left\lfloor \left(\frac{p}{q} + \frac{1}{ql}\right)(i-s) \right\rfloor + \left\lceil \frac{sp}{q} \right\rceil \qquad \text{by (16a).}$$

The range of $i$ is divided into two parts.

i) If $0 \leqslant i < s$ we put $s - i = j$, with $0 < j \leqslant s$,

$$\lfloor y_l(i) \rfloor - \lfloor y(i) \rfloor = \left\lfloor \frac{-jp}{q} + \frac{-j}{ql} \right\rfloor - \left\lfloor \frac{-jp}{q} \right\rfloor.$$

Thus $\lfloor y_l(i) \rfloor$ is identical to $\lfloor y(i) \rfloor$ in this interval if and only if

$$\left\lfloor \frac{-jp}{q} \right\rfloor + \frac{jp}{q} + \frac{j}{ql} \leqslant 0 < 1 + \left\lfloor \frac{-jp}{q} \right\rfloor + \frac{jp}{q} \qquad (*)$$

by Theorem 3a.

The right-hand side of this inequality is satisfied by the definition of the floor function. For the left-hand side two subcases have to be distinguished.

a) $j$ is a multiple of $l$, say $kl$ (with $k = 0, 1, \cdots, K$). $K$, the maximum value of $k$, is determined by the demands $Kl \leqslant s$

and $(K+1)\,l > s$. We have

$$\left\lfloor \frac{-jp}{q} \right\rfloor + \frac{jp}{q} = \left\lfloor -\frac{klp}{q} \right\rfloor + \frac{klp}{q}$$
$$= \left\lfloor -k\frac{lp+1}{q} + \frac{k}{q} \right\rfloor + \frac{klp}{q}$$
$$= -k\,\frac{lp+1}{q} + \left\lfloor \frac{k}{q} \right\rfloor + \frac{klp}{q} \qquad \text{[by (16b)]}$$
$$= \left\lfloor \frac{k}{q} \right\rfloor - \frac{k}{q} = -\frac{k}{q}.$$

So

$$\left\lfloor -\frac{jp}{q} \right\rfloor + \frac{jp}{q} + \frac{j}{ql} = 0 \leqslant 0$$

in agreement with (*).

b) $j$ is not a multiple of $l$. In this case, $\lfloor -jp/q \rfloor + jp/q$ will not be equal to $-1/q, -2/q, \cdots, -K/q$, since these values were already assumed in case a), and according to Theorem 2 they will be assumed only once in the interval $0 < j \leqslant q$. In this subcase, (*) is again satisfied since $j < (K+1)\,l$:

$$\left\lfloor -\frac{jp}{q} \right\rfloor + \frac{jp}{q} + \frac{j}{ql} < \left\lfloor -\frac{jp}{q} \right\rfloor + \frac{jp}{q} + \frac{K+1}{q} \leqslant 0.$$

So $\lfloor y(i) \rfloor$ and $\lfloor y_l(i) \rfloor$ are identical if $0 \leqslant i < s$.

ii) If $s \leqslant i < s + l$ we have

$$\lfloor y_l(i) \rfloor - \lfloor y(i) \rfloor = \left\lfloor \frac{p}{q}(i-s) + \frac{i-s}{ql} \right\rfloor - \left\lfloor \frac{p}{q}(i-s) \right\rfloor = 0$$

since $0 \leqslant (i-s)/ql < 1/q$, in this interval.

Parts i) and ii) show that in the interval $0 \leqslant i < s + l$, $\lfloor y_l(i) \rfloor$ is identical to $\lfloor y(i) \rfloor$, and thus proves the first part of the lemma. At $i = s + l$ we have

$$\lfloor y_l(i) \rfloor - \lfloor y(i) \rfloor = \left\lfloor \left(\frac{p}{q} + \frac{1}{ql}\right) l \right\rfloor - \left\lfloor \frac{lp}{q} \right\rfloor$$
$$= \left\lfloor \frac{lp+1}{q} \right\rfloor - \left\lfloor \frac{lp}{q} \right\rfloor = 1$$

by (16a) in the main paper.

Hence $i = s + l$ is the lowest positive value of $i$ for which $\lfloor y(i) \rfloor$ and $\lfloor y_l(i) \rfloor$ differ. Q.E.D.

We define two strings $C$ and $C_l$ by

$$C\colon\ c_i = \lfloor y(i) \rfloor - \lfloor y(i-1) \rfloor, \qquad i = 1, 2, \cdots, n$$
$$C_l\colon\ c_i' = \lfloor y_l(i) \rfloor - \lfloor y_l(i-1) \rfloor, \qquad i = 1, 2, \cdots, n.$$

It follows from Lemma A1 that $c_i = c_i'$ for $i = 1, 2, \cdots, s + l - 1$, but $c_{s+l} \neq c_{s+l}'$.

We can now prove the following lemma.

*Lemma A2:* $C_l$ has shortest periodicity $l$.

*Proof:* i) First, we show that the fraction $(\lfloor lp/q \rfloor + 1)/l$ is irreducible:

$$\frac{\left\lfloor \frac{lp}{q} \right\rfloor + 1}{q} = \frac{\frac{lp+1}{q}}{l} \qquad \text{[by (16a)]}$$

Suppose that $l = al'$, with $a$ and $l'$ both integers. Then

$$\frac{\frac{lp+1}{q}}{l} = \frac{\frac{al'p+1}{q}}{al'} = \frac{\frac{l'p}{q} + \frac{1}{aq}}{l'}.$$

The original fraction $(\lfloor lp/q \rfloor + 1)/l$ is therefore reducible if and only if we can find a value of $a$, not equal to 1, such that $l'p/q + 1/aq$ is an integer, say $m$.

This implies

$$l'p = mq - \frac{1}{a}.$$

The term on the left and the first term on the right are both integers. Therefore this equation is insoluble if $a \neq 1$. Hence $(\lfloor lp/q \rfloor + 1)/l$ is irreducible.

ii) Putting $\lfloor lp/q \rfloor + 1 = P$, $l = Q$, and $s = S$, the string $C_l$ has the same form as $C$ in (4) in the main paper. Lemma 1 can then be applied to prove that $l(=Q)$ is the shortest periodicity, in the sense of Definition 2. Q.E.D.

*Lemma A3:* For any string $C$ and the corresponding values of $s$, $l$, and $n$ we have

$$s + l \leqslant n.$$

*Proof:* By Lemma A2, the shortest periodicity of $C_l$ is $l$, which is smaller than $q$ by definition. The first $s + l - 1$ elements of $C$ and $C_l$ are identical by Lemma A1, and hence $C$ has periodicity $l$ if it consists of only $s + l - 1$ elements. By definition, however, $C$ has periodicity $q$. To avoid a contradiction, $C$ must have at least $s + l$ elements, so $n \geqslant s + l$. Q.E.D.

We are now ready to prove

*Lemma A4:*

$$L(s+l) - F(s) = n > 0 \quad \text{if } l = 0$$
$$L(s+l) - F(s) \geqslant l > 0 \quad \text{if } l \neq 0.$$

*Proof:* It follows from the definition of $l$ that $l = 0$ if and only if $q = 1$. In this case we have

$$L(s+l) - F(s) = l + \left\lfloor \frac{n-s-l}{q} \right\rfloor q = n - s = n > 0$$

since $s = 0$ if $q = 1$.

ii) In the case $l \neq 0$,

$$L(s+l) - F(s) = l + \left\lfloor \frac{n-s-l}{q} \right\rfloor q$$
$$\geqslant l \quad \text{(by Lemma A3)}$$
$$> 0. \qquad \text{Q.E.D.}$$

By similar procedure one can prove the following.

*Lemma A5:*

$$L(s) - F(s+l) = n > 0 \quad \text{if } l = 0$$
$$L(s) - F(s+l) \geqslant q - l > 0 \quad \text{if } l \neq 0.$$

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## References

[1] Z. Kulpa, "Area and perimeter measurement of blobs in discrete binary pictures," *Comput. Graphics Image Processing*, vol. 6, pp. 434-454, 1977.

[2] R. C. Gonzalez and Safabakhsh, "Computer vision techniques for industrial applications and robot control," *IEEE Computer*, vol. C-15, pp. 17-32, Dec. 1982.

[3] A. Rosenfeld, "Digital straight line segments," *IEEE Trans. Comput.*, vol. C-23, pp. 1264-1269, 1974.

[4] F. C. A. Groen and P. W. Verbeek, "Freeman-code probabilities of object boundary quantized contours," *Comput. Graphics Image Processing*, vol. 7, pp. 391-402, 1978.

[5] H. Freeman, "Boundary encoding and processing," in *Picture Processing and Psychopictorics*, B. S. Lipkin and A. Rosenfeld, Eds. New York: Academic, 1970, pp. 241-266.

[6] A. M. Vossepoel and A. W. M. Smeulders, "Vector code probability and metrication error in the representation of straight lines of finite length," *Comput. Graphics Image Processing*, vol. 20, pp. 347-364, 1982.

[7] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed. Oxford: Clarendon, 1979, ch. V.

[8] A. Rosenfeld and A. C. Kak, *Digital Picture Processing*, vol. 2, 2nd ed. New York: Academic, 1982, pp. 121-126.

[9] L. Dorst and A. W. M. Smeulders, "The estimation of parameters of digital straight line segments," in *Proc. 6th ICPR*, München, 1982.

[10] A. W. M. Smeulders and L. Dorst, TH Delft, Internal Rep. 1983.


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# Multiprocessor Pyramid Architectures for Bottom-Up Image Analysis

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***Abstract*—This paper describes three hierarchical organizations of small processors for bottom-up image analysis: pyramids, interleaved pyramids, and pyramid trees. Progressively lower levels in the hierarchies process image windows of decreasing size. Bottom-up analysis is made feasible by transmitting up the levels quadrant borders and border-related information that captures quadrant interaction of interest for a given computation. The operation of the pyramid is illustrated by examples of standard algorithms for interior-based computations (e.g., area) and border-based computations of local properties (e.g., perimeter). A connected component counting algorithm is outlined that illustrates the role of border-related information in representing quadrant interaction. Interleaved pyramids are obtained by sharing processors among several pyramids. They increase processor utilization and throughput rate at the cost of increased hardware. Trees of shallow interleaved pyramids, called pyramid trees, are introduced to reduce the hardware requirements of large interleaved pyramids at the expense of increased processing time, without sacrificing processor utilization. The three organizations are compared with respect to several performance measures.**

***Index Terms*—Divide-and-conquer, image analysis, image decomposition, interleaving, parallel processing, performance evaluation, pipelining, pyramid architectures.**

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## I. Introduction

THIS paper explores the use of hierarchical organization of processors to perform strictly bottom-up computations. Three architectures are described: pyramids, interleaved pyramids, and pyramid trees. These architectures perform computations that result in a small number of output bits (small compared to the number of bits necessary to represent the entire image). The architectures are thus intended to compute image properties or to perform image analysis. They are not suitable for performing image transformations, such as seg-

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