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Discrete Riemann Surfaces and the Ising Model

Christian Mercat[⋆]

Université Louis Pasteur, Strasbourg, France. E-mail: mercat@math.u-strasbg.fr

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Abstract: We define a new theory of discrete Riemann surfaces and present its basic results. The key idea is to consider not only a cellular decomposition of a surface, but the union with its dual. Discrete holomorphy is defined by a straightforward discretisation of the Cauchy–Riemann equation. A lot of classical results in Riemann theory have a discrete counterpart, Hodge star, harmonicity, Hodge theorem, Weyl's lemma, Cauchy integral formula, existence of holomorphic forms with prescribed holonomies. Giving a geometrical meaning to the construction on a Riemann surface, we define a notion of criticality on which we prove a continuous limit theorem. We investigate its connection with criticality in the Ising model. We set up a Dirac equation on a discrete universal spin structure and we prove that the existence of a Dirac spinor is equivalent to criticality.

Contents

1. Introduction

We present here a new theory of discrete analytic functions, generalising to discrete Riemann surfaces the notion introduced by Lelong–Ferrand [LF].

Although the theory defined here may be applied wherever the usual Riemann Surfaces theory can, it was primarily designed with statistical mechanics, and particularly the Ising model, in mind [McCW, ID]. Most of the results can be understood without

⋆ *Current address:* Department of Math. and Stats, The University of Melbourne, Parkville, Victoria 3052, Australia. E-mail: mercat@ms.unimelb.edu.au

any prior knowledge in statistical mechanics. The other obvious fields of application in two dimensions are electrical networks, elasticity theory, thermodynamics and hydrodynamics, all fields in which continuous Riemann surfaces theory gives wonderful results. The relationship between the Ising model and holomorphy is almost as old as the theory itself. The key connection to the Dirac equation goes back to the work of Kaufman [K] and the results in this paper should come as no surprise for workers in statistical mechanics; they knew or suspected them for a long time, in one form or another. The aim of this paper is therefore, from the statistical mechanics point of view, to define a general theory as close to the continuous theory as possible, in which claims as "the Ising model near criticality converges to a theory of Dirac spinors" are given a precise meaning and a proof, keeping in mind that such meanings and proofs already exist elsewhere in other forms. The main new result in this context is that there exists a discrete Dirac spinor near criticality in the finite size Ising model, **before** the thermodynamic limit is taken. Self-duality, which enabled the first evaluations of the critical temperature [KW, Ons,Wan50], is equivalent to criticality at finite size. It is given a meaning in terms of compatibility with holomorphy.

The first idea in order to discretise surfaces is to consider **cellular decompositions**. Equipping a cellular decomposition of a surface with a discrete metric, that is giving each edge a length, is sufficient if one only wants to do discrete harmonic analysis. However it is not enough if one wants to define discrete analytic geometry. The basic idea of this paper is to consider not just the cellular decomposition but rather what we call its *double*, i.e. the pair consisting of the cellular decomposition together with its Poincaré dual. A discrete conformal structure is then a class of metrics on the double where we retain only the ratio of the lengths of dual edges[1]. In Ising model terms, a discrete conformal structure is nothing more than a set of interaction constants on each edge separating neighbouring spins in an Ising model of a given topology.

A function of the vertices of the double is said to be *discrete holomorphic* if it satisfies the discrete Cauchy–Riemann equation, on two dual edges (x, x') and (y, y'),

$$\frac{f(y') - f(y)}{\ell(y, y')} = i \frac{f(x') - f(x)}{\ell(x, x')}.$$

This definition gives rise to a theory which is analogous to the classical theory of Riemann surfaces. We define discrete differential forms on the double, a Hodge star operator, discrete holomorphic forms, and prove analogues of the Hodge decomposition and Weyl's lemma. We extend to our situation the notion of pole of order one and we prove existence theorems for meromorphic differentials with prescribed poles and holonomies. Similarly, we define a Green potential and a Cauchy integral formula.

Up to this point, the theory is purely combinatorial. In order to relate the discrete and continuous theories on a Riemann surface, we need to impose an extra condition on the discrete conformal structure to give its parameters a geometrical meaning. We call this semi-criticality in Sect. 3. The main result here is that the limit of a pointwise convergent sequence of discrete holomorphic functions, on a refining sequence of semi-critical cellular decompositions of the same Riemann surface, is a genuine holomorphic function on the Riemann surface. If one imposes the stronger condition of criticality on the discrete conformal structure, one can define a wedge product between functions and 1-forms which is compatible with holomorphy.

[1] By definition, a discrete Riemann surface is a discrete surface equipped with a discrete conformal structure in this sense.

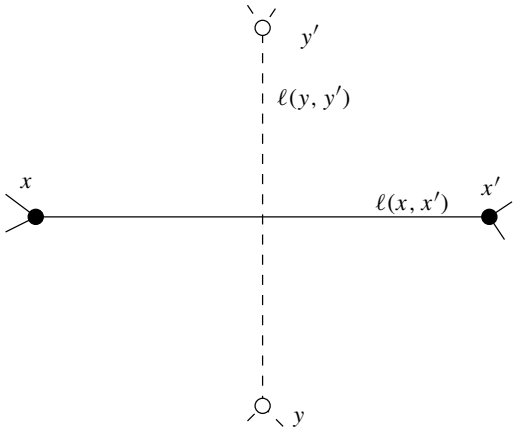


Fig. 1. The discrete Cauchy–Riemann equation

Finally, for applications of this theory to statistical physics, one needs to define a discrete analogue of spinor fields on Riemann surfaces. In Sect. 4 we first define the notion of a discrete spin structure on a discrete surface. It sheds an interesting light onto the continuous notion, allowing us to redefine it in explicit geometrical terms. In the case of a discrete Riemann surface we then define a discrete Dirac equation, generalising an equation appearing in the Ising model, and show that criticality of the discrete conformal structure is equivalent to the existence of a local massless Dirac spinor field.

In Sect. 2, we present definitions and properties of the theory which are purely combinatorial. First, in the empty boundary case, we recall the definitions of dual cellular complexes, notions of deRham cohomology. We define the double Λ, we present the discrete Cauchy–Riemann equation, the discrete Hodge star on Λ, the Laplacian and the Hodge decomposition. In Subsect. 2.2, we prove Dirichlet and Neumann theorems, the basic tools of discrete harmonic analysis. In Subsect. 2.3 we prove existence theorems for 1-forms with prescribed poles and holonomies. In Subsect. 2.4, we deal with the basic difficulty of the theory: The Hodge star is defined on Λ while the wedge product is on another complex, the diamond $\diamond$, obtained from Γ or Γ^* by the procedure of tile centering [GS87]. We prove Weyl's lemma, Green's identity and Cauchy integral formulae.

In Sect. 3, we define semi-criticality and criticality and prove that it agrees with the usual notion for the Ising model on the square and triangular lattices. We present Voronoï and Delaunay semi-critical maps in order to give examples and we prove the continuous limit theorem. We prove that every Riemann surface admits a critical map and give examples. On a critical map, the product between functions and 1-forms is compatible with holomorphy and yields a polynomial ring, integration and derivation of functions. We give an example showing where the problems are.

In Sect. 4, we set up the Dirac equation on discrete spin structures. We motivate the discrete universal spin structure by first showing the same construction in the continuous case. We show discrete holomorphy property for Dirac spinors, we prove that criticality is equivalent to the existence of local Dirac spinors and present a continuous limit theorem for Dirac spinors.

2. Discrete Harmonic and Holomorphic Functions

In this section, we are interested in properties of *combinatorial* geometry. The constructions are considered up to homeomorphisms, that is to say on a combinatorial surface, as opposed to Sect. 3 where criticality implies that the discrete geometry is embedded in a genuine Riemann surface.

2.1. First definitions. Let Σ be an oriented surface without boundary. A cellular decomposition Γ of Σ is a partition of Σ into disjoint connected sets, called cells, of three types: a discrete set of points, the vertices Γ_0; a set of non intersecting paths between vertices, the edges Γ_1; and a set of topological discs bounded by a finite number of edges and vertices, the oriented faces Γ_2. A parametrisation of each cell is chosen, faces are mapped to standard polygons of the euclidean plane, and edges to the segment $(0, 1)$; we recall particularly that for each edge, one of its two possible orientations is chosen arbitrarily. We consider only locally finite decompositions, i.e. any compact set intersects a finite number of cells. In each dimension, we define the space of chains $C_\bullet(\Gamma)$ as the $\mathbb{Z}$-module generated by the cells. The boundary operator $\partial : C_k(\Gamma) \to C_{k-1}(\Gamma)$ partially encodes the incidence relations between cells. It fulfills the boundary condition $\partial\partial = 0$.

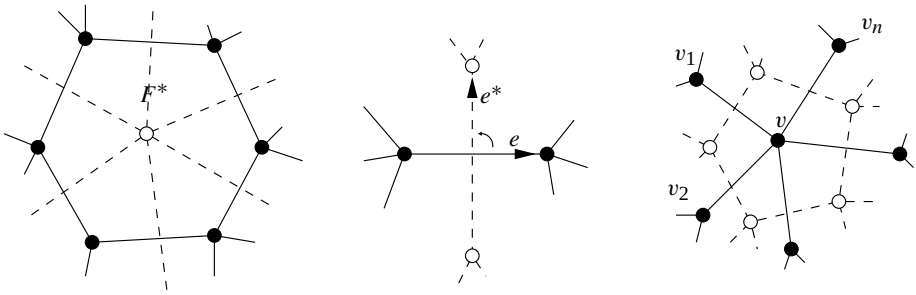
We now describe the **dual cellular decomposition** Γ^* of a cellular decomposition Γ of a surface *without boundary*. We refer to [Veb] for the general definition. Though we formally use the parametrisation of each cell for the definition of the dual, its combinatorics is intrinsically well defined. To each face $F \in \Gamma_2$ we define the vertex $F^* \in \Gamma_0^*$ inside the face F, the preimage of the origin of the euclidean plane by the parametrisation of the face. Each edge $e \in \Gamma_1$, separates two faces, say $F_1, F_2 \in \Gamma_1$ (which may coincide), hence is identified with a segment on the boundary of the standard polygon corresponding to F_1, respectively F_2. We define the dual edge $e^* \in \Gamma_1^*$ as the preimage of the two segments in these polygons, joining the origin to the point of the boundary mapped to the middle of e. It is a simple path lying in the faces F_1 and F_2, drawn between the two vertices F_1^* and F_2^* (which may coincide), cutting no edge but e, once and transversely. As the surface is oriented, to the oriented edge e we can associate the oriented dual edge e^* such that (e, e^*) is direct at their crossing point. To each vertex $v \in \Gamma_0$, with $v_1, \ldots, v_n \in \Gamma_0$ as neighbours, we define the face $v^* \in \Gamma_2^*$ by its boundary $\partial v^* = (v, v_1)^* + \ldots + (v, v_k)^* + \ldots + (v, v_n)^*$.

Remark 1. Γ^* is a cellular decomposition of Σ [Veb]. If we choose a parametrisation of the cells of Γ^*, we can consider its dual Γ^{**}; it is a cellular decomposition homeomorphic to Γ but the orientation of the edges is reversed. The bidual of $e \in \Gamma_1$ is the reversed edge $e^{**} = -e$ (see Fig. 2).

The **double** Λ of a cellular decomposition is the union of these two dual cellular decompositions. We will speak of a k-cell of Λ as a k-cell of either Γ or Γ^*.

A discrete **metric** ℓ is an assignment of a positive number $\ell(e)$ to each edge $e \in \Lambda_1$, its **length**. For convenience the edge with reversed orientation, $-e$, will be assigned the same length: $\ell(-e) := \ell(e)$. Two metrics $\ell, \ell' : \Lambda_1 \to (0, +\infty)$ belong to the same discrete **conformal structure** if the *ratio* of the lengths $\rho(e) := \frac{\ell(e^*)}{\ell(e)} = \frac{\ell'(e^*)}{\ell'(e)}$, on each pair of dual edges $e \in \Gamma_1, e^* \in \Gamma_1^*$ are equal.

A function f on Λ is a function defined on the vertices of Γ and of Γ^*. Such a function is said to be **holomorphic** if, for every pair of dual edges $(x, x') \in \Gamma_1$ and



0. The vertex dual to a face 1. Dual edges 2. The face dual to a vertex

Fig. 2. Duality

$(y, y') = (x, x')^* \in \Gamma_1^*$, it fulfills

$$\frac{f(y') - f(y)}{\ell(y, y')} = i\frac{f(x') - f(x)}{\ell(x, x')}.$$

It is the naive discretisation of the Cauchy–Riemann equation for a function f, which is, in local orthonormal coordinates (x, y):

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}.$$

Here, we understand two dual edges as being orthogonal.

This equation, though simple, was never considered in such a generality. It was introduced by Lelong–Ferrand [LF] for the decomposition of the plane by the standard square lattice $\mathbb{Z}^2$. It is also called monodiffric functions; for background on this topic, see [Duf]. Polynomials of degree two, restricted to the square lattice, give examples of monodiffric functions. See also the works of Kenyon [Ken] and Schramm and Benjamini [BS96] who considered more than lattices.

The usual notions of deRham cohomology are useful in this setup. We said that k-chains are elements of the $\mathbb{Z}$-module $C_k(\Lambda)$, generated by the k-cells, its dual space $C^k(\Lambda) := \mathrm{Hom}\,(C_k(\Lambda), \mathbb{C})$ is the space of k-cochains. We will denote the coupling by the usual integral and functional notations: $f(x)$ for a function $f \in C^0(\Lambda)$ on a vertex $x \in \Lambda_0$; $\int_e \alpha$ for a 1-form $\alpha \in C^1(\Lambda)$ on an edge $e \in \Lambda_1$; and $\iint_F \omega$ for a 2-form $\omega \in C^2(\Lambda)$ on a face $F \in \Lambda_2$.

The **coboundary** $d : C^k(\Lambda) \to C^{k+1}(\Lambda)$ is defined by the Stokes formula (with the same notations as before):

$$\int_{(x,x')} df := f\left(\partial(x, x')\right) = f(x') - f(x) \qquad \iint_F d\alpha := \oint_{\partial F} \alpha.$$

As the boundary operator splits onto the two dual complexes Γ and Γ^*, the coboundary d also respects the direct sum $C^k(\Lambda) = C^k(\Gamma) \oplus C^k(\Gamma^*)$.

The Cauchy–Riemann equation can be written in the usual form $*df = -idf$ for the following **Hodge star** $* : C^k(\Lambda) \to C^{2-k}(\Lambda)$ defined by:

$$\int_e *\alpha := -\rho(e^*) \int_{e^*} \alpha.$$

We extend it to functions and 2-forms by:

$$\iint_F *f := f(F^*), \qquad *\omega(x) := \iint_{x^*} \omega.$$

As, by definition, for each edge $e \in \Lambda_1$, $\rho(e)\rho(e^*) = \frac{\ell(e^*)}{\ell(e)} \frac{\ell(e)}{\ell(e^*)} = 1$, the Hodge star fulfills on k-forms, $*^2 = (-1)^{k(2-k)} \mathrm{Id}_{C^k(\Lambda)}$.

It decomposes 1-forms into $-i$, respectively $+i$, eigenspaces, called **type** $(1, 0)$, resp. **type** $(0, 1)$ **forms**:

$$C^1(\Lambda) = C^{(1,0)}(\Lambda) \oplus C^{(0,1)}(\Lambda).$$

The associated projections are denoted:

$$\pi_{(1,0)} = \frac{1}{2}(\mathrm{Id} + i*)\colon C^1(\Lambda) \to C^{(1,0)}(\Lambda),$$
$$\pi_{(0,1)} = \frac{1}{2}(\mathrm{Id} - i*)\colon C^1(\Lambda) \to C^{(0,1)}(\Lambda).$$

A 1-form is **holomorphic** if it is closed and of type $(1, 0)$:

$$\alpha \in \Omega^1(\Lambda) \iff d\alpha = 0 \text{ and } *\alpha = -i\alpha.$$

It is **meromorphic** with a pole at a vertex $x \in \Lambda_0$ if it is of type $(1, 0)$ and not closed on the face x^*. Its **residue** at x is defined by

$$\mathrm{Res}_x(\alpha) := \frac{1}{2i\pi} \oint_{\partial x^*} \alpha.$$

The residue theorem is merely a tautology in this context.

We define d', d'', the composition of the coboundary with the projections on eigenspaces of $*$ as its holomorphic and anti-holomorphic parts:

$$d' := \pi_{(1,0)} \circ d, \qquad d'' := \pi_{(0,1)} \circ d$$

from functions to 1-forms,

$$d' := d \circ \pi_{(0,1)}, \qquad d'' := d \circ \pi_{(1,0)}$$

from 1-forms to 2-forms and $d' = d'' = 0$ on 2-forms. They verify $d'^2 = 0$ and $d''^2 = 0$.

The usual discrete **Laplacian**, which splits onto Γ and Γ^* independently, reads $\Delta := -d*d* - *d*d$ as expected. Its formula for a function $f \in C^0(\Lambda)$ on a vertex $x \in \Lambda_0$, with $x_1, \ldots, x_n$ as neighbours is

$$(\Delta f)(x) = \sum_{k=1}^{n} \rho(x, x_k)\,(f(x) - f(x_k))\,. \tag{2.1}$$

As in the continuous case, it can be written in terms of d' and d'' operators: For functions, $\Delta = i * (d'd'' - d''d')$, in particular holomorphic and anti-holomorphic functions are harmonic. The same result holds for 1-forms.

In the compact case, the operator $d^* = - * d *$ is the adjoint of the coboundary with respect to the usual scalar product, $(f, g) := \sum_{x \in \Lambda_0} f(x)\bar{g}(x)$ on functions, similarly on 2-forms and

$$(\alpha, \beta) := \sum_{e \in \Lambda_1} \rho(e) \left(\int_e \alpha \right) \left(\int_e \bar{\beta} \right) \text{ on 1-forms.}$$

It gives rise to the **Hodge decomposition**,

Proposition 1 (Hodge theorem). *In the compact case, the k-forms are decomposed into orthogonal direct sums of exact, coexact and harmonic forms:*

$$C^k(\Lambda) = \text{Im } d \oplus^{\perp} \text{Im } d^* \oplus^{\perp} \text{Ker } \Delta,$$

harmonic forms are the closed and coclosed ones:

$$\text{Ker } \Delta = \text{Ker } d \cap \text{Ker } d^*.$$

In particular the only harmonic functions are locally constant. Harmonic 1*-forms are also the sum of holomorphic and anti-holomorphic ones:*

$$\text{Ker } \Delta = \text{Ker } d' \oplus^{\perp} \text{Ker } d''.$$

Beware that Λ being disconnected, the space of locally constant functions is 2-dimensional. The function ε which is $+1$ on Γ and -1 on Γ^* is chosen as the second basis vector.

The proof is algebraic and the same as in the continuous case. As the Laplacian decomposes onto the two dual graphs, this result tells also that for any harmonic 1-form on Γ, there exists a unique harmonic 1-form on the dual graph Γ^* such that the couple is a holomorphic 1-form on Λ, it's simply $\alpha_{\Gamma^*} := i * \alpha_\Gamma$. These decompositions don't hold in the non-compact case; there exist non-closed and/or non-co-closed, harmonic 1-forms.

2.2. Dirichlet and Neumann problems.

Proposition 2 (Dirichlet problem). *Consider a finite connected graph Γ, equipped with a function ρ on the edges, and a certain non-empty set of points D marked as its boundary. For any boundary function $f^\partial : (\partial\Gamma)_0 \to \mathbb{C}$, there exists a unique function f, harmonic on $\Gamma_0 \setminus D$ such that $f|_{\partial\Gamma} = f^\partial$.*

We refer to the usual laplacian defined by Eq. (2.1).

If $f^\partial = 0$, the solution is the null function. Otherwise, it is the minimum of the strictly convex, positive functional $f \mapsto (df, df)$, proper on the non-empty affine subspace of functions which agree with f^∂ on the boundary.

Definition 1. *Given Γ a cellular decomposition of a compact surface with boundary Σ, define the double $\Sigma^2 := \Sigma \cup \bar{\Sigma}$, union with the opposite oriented surface, along their boundary. The double Γ^2 is a cellular decomposition of the compact surface Σ^2. Consider its dual Γ^{2*} and define $\Gamma^* := \Sigma \cap \Gamma^{2*}$. We don't take into account the faces of Γ^{2*} which are not completely inside Σ but we do consider the half-edges dual to boundary edges of Γ as genuine edges noted $(\partial\Gamma^*)_1$ and define $(\partial\Gamma^*)_0 := \Gamma_1^{2*} \cap \partial\Sigma$ as the set of their boundary vertices.*

A function ρ on the edges of Γ yields an extension to Γ_1^* by defining $\rho(e^*) := \frac{1}{\rho(e)}$.

Remark 2. Γ^* is not a cellular decomposition of the surface; the half-edges dual to boundary edges do not bound any face of Γ^*.

Proposition 3 (Neumann problem). *Consider Γ a cellular decomposition of a disk, equipped with a function ρ on its edges. Choose a boundary vertex $y_0 \in (\partial\Gamma^*)_0$, a value $f_0 \in \mathbb{C}$, and on the set of boundary edges $e \in (\partial\Gamma^*)_1$, not incident to y_0, a 1-form α.*

Then there exists a unique function f, harmonic on $\Gamma^* \setminus (\partial\Gamma^*)_0$ such that $f(y_0) = f_0$ and $\int_e df = \int_e \alpha$ for all $e \in (\partial\Gamma^*)_1$ not incident to y_0.

It is a dual problem. Let $e_0^* \in (\partial\Gamma^*)_1$, be the edge incident to y_0 and $e_0 \in (\partial\Gamma)_1$ its dual. Consider, on the set of boundary edges $e \in (\partial\Gamma)_1$ different from e_0, the 1-form defined by $i * \alpha$. Integrating it along the boundary, we get a function f^∂ on $(\partial\Gamma)_0$, well defined up to an additive constant. The Dirichlet theorem gives us a function f harmonic on $\Gamma_0 \setminus (\partial\Gamma)_0$ corresponding to f^∂. Integrating the closed 1-form $i * df$ on Γ^* yields the desired harmonic function f.

Remark 3. The number of boundary points in Γ is the same as in Γ^*, and as every harmonic function on Γ, when the surface is a disk, defines a harmonic function on Γ^* such that their couple is holomorphic, unique up to an additive constant, the space of holomorphic functions, resp. 1-forms, on the double decomposition with boundary Λ is $|(\partial\Lambda)_0|/2 + 1$, resp. $|(\partial\Lambda)_0|/2 - 1$ dimensional.

The theorem is true for more general surfaces than a disk but the proof is different, see the author's PhD thesis [M]. There are ℓ^2 versions of these theorems too.

2.3. Existence theorems. We have very similar existence theorems to the ones in the continuous case. We begin with the main difference:

Proposition 4. *The space of discrete holomorphic 1-forms on a compact surface without boundary is of dimension twice the genus.*

The Hodge theorem implies an isomorphism between the space of harmonic forms and the cohomology group of Λ. It is the direct sum of the cohomology groups of Γ and of Γ^* and each is isomorphic to the cohomology group of the surface which is $2g$ dimensional on a genus g surface. It splits in two isomorphic parts under the type $(1, 0)$ and type $(0, 1)$ sum. As any holomorphic form is harmonic, the dimension of the space of holomorphic 1-forms is $2g$.

We can give explicit basis to this vector space as in the continuous case [Sie]. To construct them, we begin with meromorphic forms:

Proposition 5. *Let Σ be a compact surface with boundary. For each vertex $x \in \Lambda_0 \setminus \partial\diamond$, and a simple path λ on Λ going from x to the boundary there exists a pair of meromorphic 1-forms α_x, β_x with a single pole at x, with residue $+1$ and which have pure imaginary, respectively real holonomies, along loops which don't have any edge dual to an edge of λ.*

Proposition 6. *Let Σ be a compact surface. For each pair of vertices $x, x' \in \Lambda_0$ with a simple path λ on Λ from x to x', there exists a unique pair of meromorphic 1-forms $\alpha_{x,x'}$, $\beta_{x,x'}$ with only poles at x and x', with residue $+1$ and -1 respectively, and which have pure imaginary, respectively real holonomies, along loops which don't have any edge dual to an edge of λ.*

In both cases, the forms are $(\mathrm{Id} + i*)df$ with f a solution of a Dirichlet problem at x (and x') for α and of a Neumann problem on the surface split open along the path λ for β. The uniqueness in the second proposition is given by the difference: the poles cancel out and yield a holomorphic 1-form with pure imaginary, resp. real holonomies, so its real part, resp. imaginary part, can be integrated into a harmonic, hence constant function. So this part is in fact null. Being a holomorphic 1-form, the other part is null too. We refer to the author's PhD thesis [M] for details.

As in the continuous case, it allows us to construct holomorphic forms with (no poles and) prescribed holonomies:

Corollary 1. *Let $\mathcal{A}, \mathcal{B} \in Z_1(\Lambda)$ be two non-intersecting simple loops such that there exists exactly one edge of $\mathcal{A}$ dual to an edge of $\mathcal{B}$ (dual loops). There exists a unique holomorphic* 1*-form $\Phi_{\mathcal{AB}}$ such that* $\mathrm{Re}(\int_B \Phi_{\mathcal{AB}}) = 1$ *and* $\int_\gamma \Phi_{\mathcal{AB}} \in i\mathbb{R}$ *for every loop γ which doesn't have any edge dual to an edge of $\mathcal{A}$.*

We decompose $\mathcal{A}$ in two paths λ_x^y and λ_y^x. It gives us two 1-forms $\beta_{x,y}$ and $\beta_{y,x}$, then

$$\Phi_{\mathcal{AB}} := \frac{1}{2i\pi}(\beta_{x,y} + \beta_{y,x}) \tag{2.2}$$

fulfills the conditions.

2.4. The diamond $\diamond$ and its wedge product. Following [Whit], we define a wedge product, on another complex, the **diamond** $\diamond$, constructed out of the double Λ: Each pair of dual edges, say $(x, x') \in \Gamma_1$ and $(y, y') = (x, x')^* \in \Gamma_1^*$, defines (up to homeomorphisms) a four-sided polygon (x, y, x', y') and all these constitute the faces of a cellular complex called $\diamond$ (see Fig. 3).

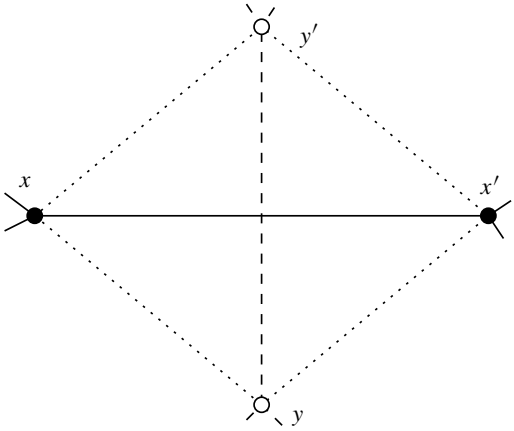


Fig. 3. The diamond $\diamond$

On the other hand, from any cellular decomposition $\diamond$ of a surface by four-sided polygons one can reconstruct the double Λ. A difference is that Λ may not be disconnected in two dual pieces Γ and Γ^*, it is so if each loop in $\diamond$ is of even length; we

will restrict ourselves to this simpler case. This is not very restrictive because from a connected double, refining each quadrilateral in four smaller quadrilaterals, one gets a double disconnected in two dual pieces.

Definition 2. *A **discrete surface with boundary** is defined by a quadrilateral cellular decomposition $\diamond$ of an oriented surface with boundary such that its double complex Λ is disconnected in two dual parts.*

This definition is a generalisation of the more natural previous Definition 1. It allows us to consider any subset of faces of $\diamond$ as a domain yielding a discrete surface with boundary. While any edge of Λ has a dual edge, a vertex of Λ has a dual face if and only if it is an inner vertex. Punctured surfaces can be understood in these terms too: An inner vertex $v \in \Lambda_0$ is a puncture if it is declared as being on the boundary and its dual face v^* removed from Λ_2.

We construct a discrete wedge product, but while the Hodge star lives on the double Λ, the wedge product is defined on the diamond $\diamond$: $\wedge : C^k(\diamond) \times C^l(\diamond) \to C^{k+l}(\diamond)$. It is defined by the following formulae, for $f, g \in C^0(\diamond)$, $\alpha, \beta \in C^1(\diamond)$ and $\omega \in C^2(\diamond)$:

$$
\begin{aligned}
(f \cdot g)(x) &:= f(x) \cdot g(x) \qquad \text{for } x \in \diamond_0, \\
\int_{(x,y)} f \cdot \alpha &:= \frac{f(x)+f(y)}{2} \int (x,y)\alpha \qquad \text{for } (x,y) \in \diamond_1, \\
\iint_{(x_1,x_2,x_3,x_4)} \alpha \wedge \beta &:= \frac{1}{4} \sum_{k=1}^{4} \int_{(x_{k-1},x_k)} \alpha \int_{(x_k,x_{k+1})} \beta - \int_{(x_{k+1},x_k)} \alpha \int_{(x_k,x_{k-1})} \beta \\
\iint_{(x_1,x_2,x_3,x_4)} f \cdot \omega &:= \frac{f(x_1)+f(x_2)+f(x_3)+f(x_4)}{4} \iint_{(x_1,x_2,x_3,x_4)} \omega \\
&\text{for } (x_1, x_2, x_3, x_4) \in \diamond_2.
\end{aligned}
$$

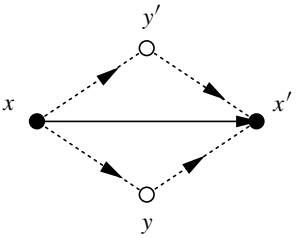
Lemma 1. *$d_\diamond$ is a derivation with respect to this wedge product.*

To take advantage of this property, one has to relate forms on $\diamond$ and forms on Λ where the Hodge star is defined. We construct an **averaging map** A from $C^\bullet(\diamond)$ to $C^\bullet(\Lambda)$. The map is the identity for functions and defined by the following formulae for 1 and 2-forms:

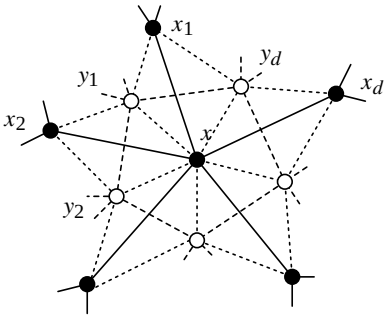
$$
\int_{(x,x')} A(\alpha_\diamond) := \frac{1}{2} \left(\int_{(x,y)} + \int_{(y,x')} + \int_{(x,y')} + \int_{(y',x')} \right) \alpha_\diamond, \tag{2.3}
$$

$$
\iint_{x^*} A(\omega_\diamond) := \frac{1}{2} \sum_{k=1}^{d} \iint_{(x_k,y_k,x,y_{k-1})} \omega_\diamond, \tag{2.4}
$$

where notations are made clear in Fig. 4. With this definition, $d_\Lambda A = A d_\diamond$, but the map A is neither injective nor always surjective, so we can neither define a Hodge star on $\diamond$ nor a wedge product on Λ. An element of the kernel of A is given for example by $d_\diamond \varepsilon$, where ε is $+1$ on Γ and -1 on Γ^*.



(2.3)



(2.4)

Fig. 4. Notations

On the double Λ itself, we have pointwise multiplication between functions, functions and 2-forms, and we construct an *heterogeneous* wedge product for 1-forms: with $\alpha, \beta \in C^1(\Lambda)$, define $\alpha \wedge \beta \in C^1(\diamondsuit)$ by

$$\iint_{(x,y,x',y')} \alpha \wedge \beta := \int_{(x,x')} \alpha \int_{(y,y')} \beta + \int_{(y,y')} \alpha \int_{(x',x)} \beta.$$

It verifies $A(\alpha_\diamondsuit) \wedge A(\beta_\diamondsuit) = \alpha_\diamondsuit \wedge \beta_\diamondsuit$, the first wedge product being between 1-forms on Λ and the second between forms on $\diamondsuit$. Of course, we also have for integrable 2-forms:

$$\iint_{\diamondsuit_2} \omega_\diamondsuit = \iint_{\Gamma_2} A(\omega_\diamondsuit) = \iint_{\Gamma_2^*} A(\omega_\diamondsuit) = \frac{1}{2} \iint_{\Lambda_2} A(\omega_\diamondsuit).$$

And for a function f,

$$\iint_{\diamondsuit_2} f \cdot \omega_\diamondsuit = \frac{1}{2} \iint_{\Lambda_2} A(f \cdot \omega_\diamondsuit) = \frac{1}{2} \iint_{\Lambda_2} f \cdot A(\omega_\diamondsuit)$$

whenever $f \cdot \omega_\diamondsuit$ is integrable.

Explicit calculation shows that for a function $f \in C^0(\Lambda)$, denoting by χ_x the characteristic function of a vertex $x \in \Lambda_0$, $(\Delta f)(x) = -\iint_{\Lambda_2} f \cdot * \Delta \chi_x$. So by linearity one gets **Weyl's lemma**: a function f is harmonic iff for any compactly supported function $g \in C^0(\Lambda)$,

$$\iint_{\Lambda_2} f \cdot \Delta g = 0.$$

One checks also that the usual scalar product on compactly supported forms on Λ reads as expected:

$$(\alpha, \beta) := \sum_{e \in \Lambda_1} \rho(e) \left(\int_e \alpha \right) \left(\int_e \bar{\beta} \right) = \iint_{\diamondsuit_2} \alpha \wedge * \bar{\beta}.$$

In some cases, for example, the decomposition of the plane by lattices, the averaging map A is surjective. We define the inverse map $B : C^1(\Lambda) \to C^1(\diamond)/\mathrm{Ker}\, A$ and $\Delta_\diamond := d_\diamond B * d$ and we then have

Proposition 7 (Green's identity). *For two functions f, g on a compact domain $D \subset \diamond_2$,*

$$\iint_D (f \cdot \Delta_\diamond g - g \cdot \Delta_\diamond f) - \oint_{\partial D} (f \cdot B{*}dg - g \cdot B{*}df) = 0.$$

This means that for any representatives of the classes in $C^1(\diamond)/\mathrm{Ker}\, A$ the equality holds, but each integral separately is not well defined on the classes.

2.5. Cauchy integral formula.

Proposition 8. *Let Λ a double map and D a compact region of $\diamond_2$ homeomorphic to a disc. Consider an interior edge $(x, y) \in D$; there exists a meromorphic 1-form $\nu_{x,y} \in C^1(D \setminus (x, y))$ such that the holonomy $\oint_\gamma \nu_{x,y}$ along a cycle γ in D only depends on its homology class in $D \setminus (x, y)$, and $\oint_{\partial D} \nu_{x,y} = 2i\pi$.*

Consider the meromorphic 1-form $\mu_{x,y} = \alpha_x + \alpha_y \in C^1(\Lambda \cap D)$ defined by existence Theorem 5 on D. It is uniquely defined up to a global holomorphic form on D. Its only poles are x and y of residue $+1$ so it verifies a similar holonomy property, but on $\Lambda \cap D \setminus (x, y)$. We define a 1-form $\nu_{x,y}$ on $\diamond \cap D \setminus R$, such that $\mu_{x,y} = A\nu_{x,y}$ in the following way: Let $\int_{(x,a)} \nu_{x,y} := \lambda$, a fixed value, and for an edge $(x', y') \in D_1$, with $x' \in \Gamma_0$, $y' \in \Gamma_0^*$, given two paths in D, $\lambda_{x'}^x \in C_1(\Gamma)$ and $\lambda_y^{y'} \in C_1(\Gamma^*)$ respectively from x' to x and from y to y',

$$\int_{(x',y')} \nu_{x,y} := \int_{\lambda_{x'}^x} \mu_{x,y} + \int_{(x,A)} \nu_{x,y} + \int_{\lambda_y^{y'}} \mu_{x,y} - \oint_{[\gamma]} \mu_{x,y},$$

where $[\gamma]$ is the homology class of $\lambda_{x'}^x + (x, y) + \lambda_y^{y'} + (y', x')$ on the punctured domain.

$\nu_{x,y}$ is the discrete analogue of $\frac{dz}{z-z_0}$ with $z_0 = (x, y)$. It is closed on every face of $D \setminus R$. By definition, the average of $\nu_{x,y}$ on the double map is the meromorphic form $A\nu_{x,y} = \mu_{x,y}$.

It allows us to state

Proposition 9 (Cauchy integral formula). *Let D be a compact connected subset of $\diamond_2$ and $(x, y) \in D_1$ two interior neighbours of D with a non-empty boundary. For each function $f \in C^0(\Lambda)$,*

$$\oint_{\partial D} f \cdot \nu_{x,y} = \iint_D d'' f \wedge \mu_{x,y} + 2i\pi \frac{f(x) + f(y)}{2}.$$

The proof is straightforward: The edge (x, y) bounds two faces in D, let $R = (abcd)$ the rectangle made of these faces (see Fig. 5).

On $D \setminus R$,

$$d_\diamond(f \cdot \nu_{x,y}) = d_\diamond f \wedge \nu_{x,y} + f \cdot d_\diamond \nu_{x,y}.$$

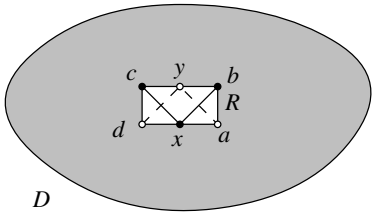


Fig. 5. The rectangle R in a domain D defined by an edge $(x, y) \in \diamond_1$

The $(1, 0)$ part of df disappears in the wedge product against the holomorphic form $\mu_{x,y}$, so we can substitute

$$d_\diamond f \wedge \nu_{x,y} = d_\Lambda f \wedge A\nu_{x,y} = d'' f \wedge \mu_{x,y}.$$

Integrating over D, as $\nu_{x,y}$ is closed on $D \setminus R$, we get:

$$\oint_{\partial D} f \cdot \nu_{x,y} = \iint_{D\setminus R} d'' f \wedge \mu_{x,y} + \oint_{\partial R} f \cdot \nu_{x,y}.$$

Explicit calculus shows that $\oint_{\partial R} f \cdot \nu_{x,y} = \iint_R d'' f \wedge \mu_{x,y} + 2i\pi \frac{f(x)+f(y)}{2}$. □

Remark 4. Since for all $\alpha \in C^1(\diamond)$, the locally constant function ε defined by $\varepsilon(\Gamma) = +1$, $\varepsilon(\Gamma^*) = -1$, verifies $\varepsilon \cdot \alpha = 0$, an integral formula will give the same result for a function f and $f + \lambda\varepsilon$. Therefore such a formula can not give access to the value of the function at one point but only to its average value at an edge of $\diamond$.

Corollary 2. *For $f \in \Omega(\Lambda)$ a holomorphic function, the Cauchy integral formula reads, with the same notations,*

$$\frac{f(x) + f(y)}{2} = \frac{1}{2i\pi} \oint_{\partial D} f \cdot \nu_{x,y}.$$

The Green function on the lattices (rectangular, triangular, hexagonal, Kagomé, square/octogon) is exactly known in terms of hyperelliptic functions ([Hug] and references in Appendix 3). As the potential is real, it means that the discrete Dirichlet problem on these lattices can be exactly solved this way, once the boundary values on the graph and its dual are given: if these values are real and Γ and imaginary on its dual, the solution is real on Γ and pure imaginary on the dual so the contributions $f(x)$ and $f(y)$ are simply the real and imaginary parts of the contour summation respectively. Unfortunately, this pair of boundary values are not independant but related by a Dirichlet to Neumann problem [CdV96].

3. Criticality

The term criticality, as well as our motivation to investigate discrete holomorphic functions, comes from statistical mechanics, namely the Ising model. A critical temperature is defined that restrains the interaction constants, interpreted here as lengths. We will see these geometrical constraints in Sect. 3.3.

Technically, as far as the continuous limit theorem is concerned, a weaker property, called *semi-criticality* is sufficient, it gives us a product between functions and forms. Moreover, at criticality, this product will be compatible with holomorphy.

3.1. Semi-criticality. Define $C_\theta := \{(r,t) : r \geq 0, t \in \mathbb{R}/\theta\mathbb{Z}\}/(0,t) \sim (0,t')$ with the metric $ds^2 := dr^2 + r^2 dt^2$ as the **standard cone of angle** $\theta > 0$ [Tro].

The cones can be realized by cutting and pasting paper, demonstrating their local isometry with the euclidean complex plane.

Let Σ be a compact Riemann surface and $P \subset \Sigma$ a discrete set of points. A **flat Riemannian metric with P as conic singularities** is an atlas $\{Z_{U_x} : U_x \to U' \subset C_{\theta_x} >\}_{x \in P}$ of open sets $U_x \subset \Sigma$, a neighbourhood of a singularity $x \in P$, into open sets of a standard cone, such that the singularity is mapped to the vertex of the cone and the changes of coordinates $C_{U,V} : U \cap V \to \mathbb{C}$ are euclidean isometries.

There is a lot of freedom allowed in the choice of a flat metric for a given closed Riemann surface Σ: Any finite set P of points on Σ with a set of angles $\theta_x > 0$ for every $x \in P$ such that $2\pi\chi(\Sigma) = \sum_{x \in P}(2\pi - \theta_x)$, defines uniquely a Riemannian flat metric on Σ with these conic singularities and angles [Tro].

Consider such a flat riemannian metric on a compact Riemann surface Σ and (Λ, ℓ) a double cellular decomposition of Σ as before.

Definition 3. *(Λ, ℓ) is a **semi-critical map** for this flat metric if the conic singularities are among the vertices of Λ and $\diamond$ can be realized such that each face $(x, y, x', y') \in \diamond_2$ is mapped, by a local isometry Z preserving the orientation, to a four-sided polygon $(Z(x), Z(y), Z(x'), Z(y'))$ of the euclidean plane, the segments $[Z(x), Z(x')]$ and $[Z(y), Z(y')]$ being of lengths $\ell(x, x')$, $\ell(y, y')$ respectively and forming a direct orthogonal basis. We name $\delta(\Lambda, \ell)$ the supremum of the lengths of the edges of $\diamond$.*

The local isometric maps Z are discrete holomorphic.

Voronoï and Delaunay complexes [PS85] are interesting examples of semi-critical dual complexes. Any discrete set of points Q on a flat Riemannian surface, containing the conic singularities, defines such a pair:

We first define two partitions V and D of Σ into sets of three types: 2-sets, 1-sets and 0-sets, and then show that they are in fact dual cellular complexes. They are defined by a real positive function m_Q on the surface, the *multiplicity*.

Consider a point $x \in \Sigma$; as the set Q is discrete, the distance $d(x, Q)$ is realized by geodesics of minimal length, generically a single one. Let $m_Q(x) \in [1, \infty)$ be the number of such geodesics. If $m_Q(x) = 1$, there exists a vertex $\pi(x) \in Q$ such that the shortest geodesic from x to $\pi(x)$ is the only geodesic from x to Q with such a small length.

The Voronoï 2-set associated to a vertex v in Q, is $\pi^{-1}(v)$, that is to say the set of points of Σ closer to this vertex than to any other vertex in Q. Each 2-set of V is a connected component of $m_Q^{-1}(1)$.

Likewise, the 1-sets are the connected components of $m_Q^{-1}(2)$. They are associated to pairs of points in Q.

The 0-sets are the connected components of $m_Q^{-1}([3, +\infty))$. Generically, they are associated to three points in Q.

V is a cellular complex (see below) and the complex D is its dual (generically a triangulation), its vertices are the points in Q, its edges are segments (x, x') for $x, x' \in Q$ such that there exist points equidistant and closer to them.

Proposition 10. *The Voronoï partition, on a closed Riemann surface with a flat metric with conic singularities, of a given discrete set of points Q containing the conic singularities, is a cellular complex.*

We have to prove that 2-sets are homeomorphic to discs, 1-sets are segments and 0-sets are points.

First, 2-sets are star-shaped, for every point x closer to $v \in Q$ than to any other point in Q, along a unique portion of a geodesic, the whole segment $[x, v]$ has the same property.

2-sets are open, if x is closer to $v \in Q$ than to any other point in Q, as it is discrete, $d(x, Q \setminus v) - d(x, v) > 0$. By triangular inequality, every point in the open ball of this radius centred at x is closer to v than to any other point in Q.

So 2-sets are homeomorphic to discs.

Let x be a point in a 1-set. It is defined by exactly two portions of geodesics D, D' from x to $y, y' \in Q$ (they may coincide). By definition, the open sphere centred at x containing $D \cup D'$ doesn't contain any point of Q so it can be lifted to the universal covering, where the usual rules of euclidean geometry tell us that the set of points equidistant to y and y' around x is a submanifold of dimension 1. As the surface is compact, if it is not a segment, it can only be a circle. Then, it's easy to see that the surface is homeomorphic to a 2-sphere and that y and y' are the only points in Q. But this is impossible because an euclidean metric on a 2-sphere has at least three conic singularities [Tro].

The same type of arguments shows that 0-sets are isolated points.

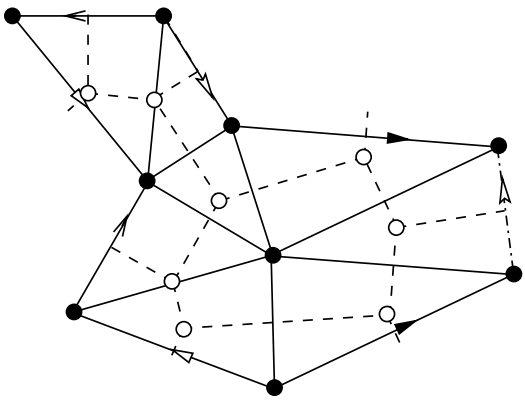


Fig. 6. The Voronoï/Delaunay decompositions associated to two points on a genus two surface

Proposition 11. *Such Delaunay/Voronoï dual complexes are semi-critical maps of the surface. Hence any Riemann surface admits semi-critical maps.*

The edge in V dual to $(x, x') \in D_1$ is a segment of their mediatrix so is orthogonal to (x, x'). Hence, equipped with the Euclidean length on the edges, (V, D) is a semi-critical map.

Remark 5. Apart from Voronoï/Delaunay maps, circle packings [CdV90] give another very large class of examples of interesting semi-critical decompositions (see Fig. 7).

The semi-criticality of a double map gives a coherent system of angles ϕ in $(0, \pi)$ on the oriented edges of Λ. An edge $(x, x') \in \Lambda_1$ is the diagonal of a certain diamond; $\phi(x, x')$ is the angle of that diamond at the vertex x. In particular, $\phi(x, x') \neq \phi(x', x)$ *a priori*. They verify that for every diamond, the sum of the angles on the four directions

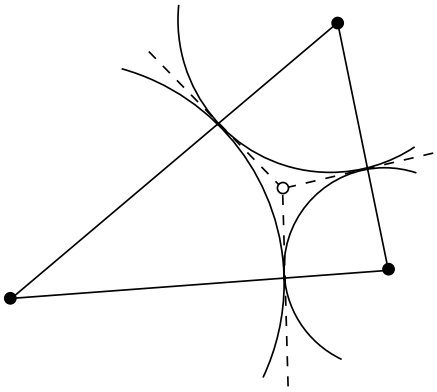


Fig. 7. Circle packing, the dual vertex to a face

of the two dual diagonals is 2π (see Fig. 8). Then the conic angle at a vertex is given by the sum of the angles over the incident edges.

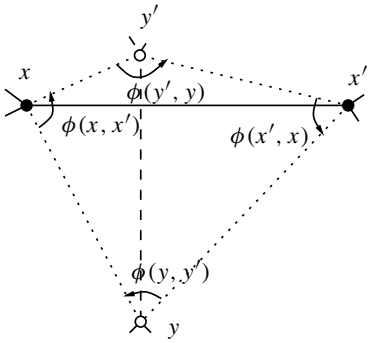


Fig. 8. A system of angles for a semi-critical map

3.2. Continuous limit. We state the main theorem, a converging sequence of discrete holomorphic functions on a refining sequence of semi-critical maps of the same Riemann surface, converges to a holomorphic function. Precisely:

Theorem 3. *Let Σ be a Riemann surface and $({}^k\Lambda, \ell_k)_{k\in\mathbb{N}}$ a sequence of semi-critical maps on it, with respect to the same flat metric with conic singularities. Assume that the lengths $\delta_k = \delta({}^k\Lambda)$ tend to zero and that the angles at the vertices of all the faces of the ${}^k\diamondsuit$ are in the interval $[\eta, 2\pi - \eta]$ with $\eta > 0$.*

Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of discrete holomorphic functions $f_k \in \Omega({}^k\Lambda)$, such that there exists a function f on Σ which verifies, for every converging sequence $(x_k)_{k\in\mathbb{N}}$ of points of Σ with each $x_k \in {}^k\Lambda_0$, $f\big(\lim_k(x_k)\big) = \lim_k\big(f_k(x_k)\big)$, then the function f is ***holomorphic*** *on Σ.*

Such a refining sequence is easy to produce (see Fig. 9) but the theorem takes into account more general sequences. A more natural refining sequence, which mixes the two

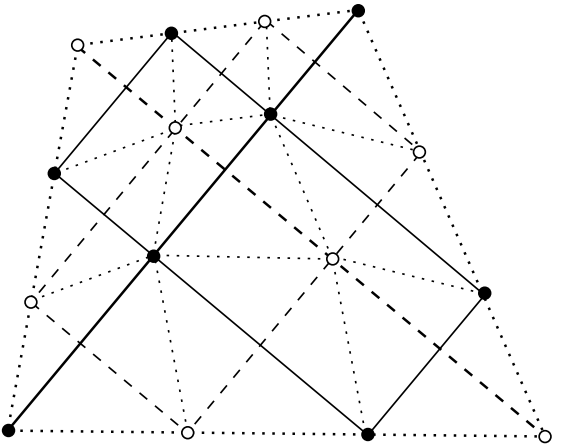


Fig. 9. Refining a semi-critical map

dual sequences is given by a series of tile centering procedures [GS87]: If one calls $\diamondsuit/2$ the cellular decomposition constructed from $\diamondsuit$ by replacing each tile by four smaller ones of half its size, and $\Gamma(\diamondsuit/2)$, $\Gamma^*(\diamondsuit/2)$ the double cellular decomposition it defines, one has $\Gamma(\diamondsuit/2) = \Gamma(\diamondsuit)\text{“}\cup\text{”}\Gamma^*(\diamondsuit)$ and the interesting following sequence:

$$\begin{array}{ccccccccccccc} \Gamma(\diamondsuit) & \to & \diamondsuit & \to & \Gamma(\diamondsuit/2) & \to & \diamondsuit/2 & \to \cdots \to & \diamondsuit/2^n & \to & \ldots \\ & \nearrow & & \searrow & & \nearrow & & \searrow \quad \nearrow & & \searrow & . \\ \Gamma^*(\diamondsuit) & & & & \Gamma^*(\diamondsuit/2) & & & \cdots & & & \cdots \end{array} \tag{3.1}$$

The horizontal arrows correspond to tile centering procedures, and the ascending, respectively descending arrows, to tile centering, resp. edge centering procedures. This sequence is not that exciting though since locally, the graph rapidly looks like a rectangular lattice. More interesting inflation rules staying at criticality can be considered too (see Fig. 21).

The demonstration of the continuous limit theorem needs three lemmas:

Lemma 2. *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of functions on an open set $\Omega \subset \mathbb{C}$ such that there exists a function f on Ω verifying, for every converging sequence $(x_k)_{k\in\mathbb{N}}$ of points of Ω, $f\big(\lim_k(x_k)\big) = \lim_k\big(f_k(x_k)\big)$. Then the function f is continuous and uniform limit of (f_k) on any compact.*

Taking a constant sequence of points, we see that (f_k) converges to f pointwise. So with the notations of the theorem, $(f_k(x_k))$ converges to $f(x)$ and $(f_l(x_k))_{l\in\mathbb{N}}$ to $f(x_k)$. Combining the two, $(f(x_k))$ converges to $f(x)$ so f is continuous. If the convergence was not uniform on a compac sett, then there would exist a converging sequence (x_k) with $(f_k(x_k) - f(x_k))$ not converging to zero. But f is continuous in $x = \lim(x_k)$ and $(f_k(x_k))$ converges to $f(x)$, which, combined, contradicts the hypotheses.

Lemma 3. *Let $(ABCD)$ be a four sided polygon of the Euclidean plane such that its diagonals are orthogonal and the vertices angles are in $[\eta, 2\pi - \eta]$ with $\eta > 0$. Let (M, M') be a pair of points on the polygon. There exists a path on $(ABCD)$ from M to M' of minimal length ℓ. Then*

$$\frac{MM'}{\ell} \geq \frac{\sin\eta}{4}.$$

It is a straightforward study of a several variables function. If the two points are on the same side, $MM' = \ell$ and $\sin\eta \leq 1$. If they are on adjacent sides, the extremal position with MM' fixed is when the triangle $MM'P$, with P the vertex of $(ABCD)$ between them, is isocel. The angle in P being less than η, $\frac{MM'}{\ell} \geq \sin\frac{\eta}{2} > \frac{\sin\eta}{2}$. If the points are on opposite sides, the extremal configuration is given by Fig. 10.2., where $\frac{MM'}{\ell} = \frac{\sin\eta}{4}$.

1. M, M' on adjacent sides

2. M, M' on opposite sides

Fig. 10. The two extremal positions

Lemma 4. *Let (Λ, ℓ) be any double cellular decomposition and $\alpha \in C^1(\diamondsuit)$ a closed 1-form. The 1-form $f \cdot \alpha$ is closed for any holomorphic function $f \in \Omega(\Lambda)$ if and only if α is holomorphic.*

Just check.

Proof of Theorem 3. We interpolate each function f_k from the discrete set of points ${}^k\Lambda_0$ to a function $\bar{f}_k$ of the whole surface, linearly on the edges of ${}^k\diamondsuit$ and harmonicly in its faces.

Let (ζ_k) be a converging sequence of points in Σ. Each ζ_k is in the adherence of a face of ${}^k\diamondsuit$. Let x_k, y_k be the minimum and maximum of Re f_k around the face. By the maximum principle for the harmonic function Re $\bar{f}_k$,

$$\text{Re } f_k(x_k) \leq \text{Re } \bar{f}_k(\zeta_k) \leq \text{Re } f_k(y_k).$$

Moreover, the distance between x_k and ζ_k is at most $2\delta_k$, as well as for y_k. It implies that (x_k) and (y_k) converge to $x = \lim(\zeta_k)$, $(f_k(x_k))$ and $(f_k(y_k))$ to $f(x)$, and (Re $\bar{f}_k(\zeta_k)$) to Re $f(x)$; and similarly for its imaginary part. So, by Lemma 2, the function f is continuous, and is the uniform limit of $(\bar{f}_k)$ on every compact set. In particular, it is bounded on any compact.

By the theorem of inessential singularities, since f is continuous hence bounded on any compact set, and that conic singularities form a discrete set in Σ, to show that f is holomorphic, we can restrict ourselves to each element $U \subset \Sigma$ of a euclidean atlas of the *punctured* surface (without conic singularities). We have an explicit coordinate z on U.

Let γ be a homotopically trivial loop in U of finite length ℓ. We are going to prove that $\oint_\gamma f dz = 0$. The theorem of Morera then states that f is holomorphic.

Let us fix the integer k. By application of Lemma 3 on every face of ${}^k\diamondsuit$ crossed by γ, we construct a loop $\gamma_k \in C_1({}^k\diamondsuit)$, homotopic to γ, of length $\ell(\gamma_k) \leq \frac{4\ell}{\sin\eta}$ (see Fig. 11).

As the diameter of a face of ${}^k\diamondsuit$ is at most $2\delta_k$, all these faces are contained in the tubular neighbourhood of γ of diameter $4\delta_k$. Its area is $4\delta_k\ell$ and it contains the set C of Σ between γ and γ_k.

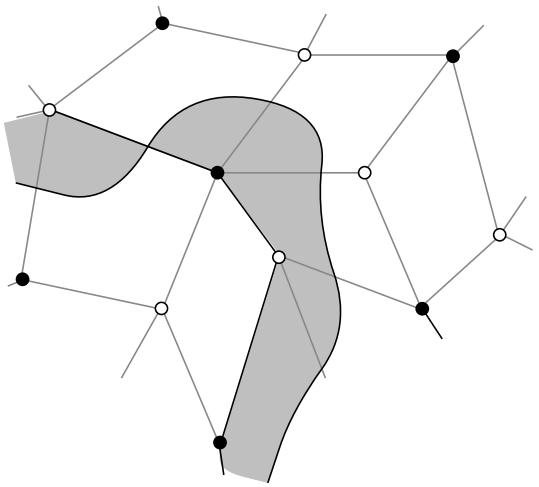


Fig. 11. The discretised path

Assume f is of class C^1, on the compact C, $|\bar\partial f|$ is bounded by a number M. Applying Stockes formula to $f\,dz$,

$$\Big|\oint_\gamma f(z)dz - \oint_{\gamma_k} f(z)dz\Big| \le \iint_C |\bar\partial f(z)|dz\wedge d\bar z \le M\times 4\delta_k\ell.$$

So $\oint_\gamma f(z)dz = \lim \oint_{\gamma_k} f(z)dz$. Taking a sequence of class C^1 functions converging uniformly to f on C, we prove the same result for f simply continuous because all the paths into account are of bounded lengths.

As $(\bar f_k)$ converges uniformly to f on C and the paths are of bounded lengths, we also have that $(|\oint_{\gamma_k}\big(\bar f_k(z)-f(z)\big)dz|)_{k\in\mathbb{N}}$ tends to zero. But because the interpolation is linear on edges of ${}^k\diamondsuit$, $\oint_{\gamma_k}\bar f_k(z)dz = \oint_{\gamma_k} f_k dZ$, the second integral being the coupling between a 1-chain and a 1-cochain of ${}^k\diamondsuit$. But since f_k and dZ are discrete holomorphic, $f_k dZ$ is a closed 1-form, and $\oint_{\gamma_k} f_k dZ = 0$. So $\oint_{\gamma_k} f_k(z)dz$ tends to zero and

$$\oint_\gamma f(z)dz = 0. \qquad \square$$

3.3. Criticality.

Proposition 12. *Let α be a holomorphic 1-form, $f\cdot\alpha$ is holomorphic for any holomorphic function if and only if $\int_{(y,x)}\alpha = \int_{(x',y')}\alpha$ for each pair of dual edges $(x,x'),(y,y')$.*

Let $(x, y, x', y') \in \diamondsuit_2$ be a face of $\diamondsuit$, the Cauchy–Riemann equation for $f \cdot \alpha$, on the couple (x, x') and (y, y') is the nullity of:

$$\begin{aligned}
\frac{\int_{(y,y')} f\cdot\alpha}{\ell(y,y')} - i\frac{\int_{(x,x')} f\cdot\alpha}{\ell(x,x')} &= \frac{1}{\ell(y,y')}\Bigg(\frac{f(x)+f(y)}{2}\int_{(y,x)}\alpha + \frac{f(x)+f(y')}{2}\int_{(x,y')}\alpha \\
&\quad + \frac{f(x')+f(y)}{2}\int_{(y,x')}\alpha + \frac{f(x')+f(y')}{2}\int_{(x',y')}\alpha\Bigg) \\
&\quad - i\frac{1}{\ell(x,x')}\Bigg(\frac{f(x)+f(y)}{2}\int_{(x,y)}\alpha + \frac{f(x')+f(y)}{2}\int_{(y,x')}\alpha \\
&\quad + \frac{f(x)+f(y')}{2}\int_{(x,y')}\alpha + \frac{f(x')+f(y')}{2}\int_{(y',x')}\alpha\Bigg) \\
&= \left(\int_{(y,x)}\alpha + \int_{(y',x')}\alpha\right)\frac{f(y')-f(y)}{\ell(y,y')},
\end{aligned}$$

after having developed, used the holomorphy of α, then the holomorphy of f.

So to be able to construct out of the holomorphic 1-forms dZ given by local flat isometries, and a holomorphic function a holomorphic 1-form $f\,dZ$, we have to impose that for each face $(x, y, x', y') \in \diamondsuit_2$, $Z(x) - Z(y) = Z(y') - Z(x')$. Geometrically, it means that each face of the graph $\diamondsuit$ is mapped by Z to a parallelogram in $\mathbb{C}$. But as the diagonals of this parallelogram are orthogonal, it is a lozenge (or rhombus, or diamond).

Definition 4. *A double (Λ, ℓ) of a Riemann surface Σ is* **critical** *if it is semi-critical and each face of $\diamondsuit_2$ are lozenges. Let $\delta(\Lambda)$ be the common length of their sides.*

Remark 6. This has an intrinsic meaning on Σ, the faces of $\diamondsuit$ are genuine lozenges on the surface and every edge of Λ can be realized by segments of length given by ℓ, two dual edges being orthogonal segments.

Another equivalent way to look at criticality can be useful: a double (Λ, ℓ) is critical if there exists an application $Z : \widetilde{\Sigma \setminus P} \to \mathbb{C}$ from the universal covering of the punctured surface $\Sigma \setminus P$ for a finite set $P \subset \Lambda_0$ into $\mathbb{C}$ identified to the oriented Euclidean plane $\mathbb{R}^2$ such that

– the image of an edge $a \in \tilde{\Lambda}_1$ is a linear segment of length $\ell(a)$,
– two dual edges are mapped to a direct orthogonal basis,
– Z is an embedding out of the vertices,
– there exists a representation ρ of the fundamental group $\pi_1(\Sigma \setminus P)$ into the group of isometries of the plane respecting orientation such that,

$$\forall \gamma \in \pi_1(\Sigma \setminus P),\ Z \circ \gamma = \rho(\gamma) \circ Z,$$

– and the lengths of all the segments corresponding to the edges of $\diamondsuit$ are all equal to the same $\delta > 0$.

The criticality of a double map gives a coherent system of angles ϕ in $(0, \pi)$ on the unoriented edges of Λ, $\phi(x, x')$ is the angle in the lozenge for which (x, x') is a diagonal, at the vertex x (or x'). They verify that for every lozenge, the sum of the angles on the dual diagonals is π. Then the conic angle at a vertex is given by the sum of the angles over the incident edges.

Every discrete conformal structure (Λ, ℓ) defines a conformal structure on the associated topological surface by pasting lozenges together according to the combinatorial data (though most of the vertices will be conic singularities). Conversely,

Theorem 4. *Every closed Riemann surface accepts a critical map.*

Proof. We first produce critical maps for cylinders of any modulus: Consider a row of n squares and glue back its ends to obtain a cylinder, its modulus, the ratio of the square of the distance from top to bottom by its area is $\frac{1}{n}$.

Stacking m such rows upon each other, one gets a cylinder of modulus $\frac{m}{n}$.

Squares can be bent into lozenges yielding a continuous family of cylinders of moduli ranging from zero to $\frac{2}{n}$ (see Fig. 12). Hence we can get cylinders of any modulus.

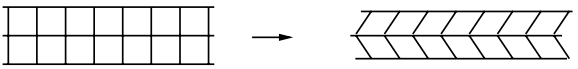


Fig. 12. Two bent rows

Dehn twists can be performed on these critical cylinders, see Fig. 13.

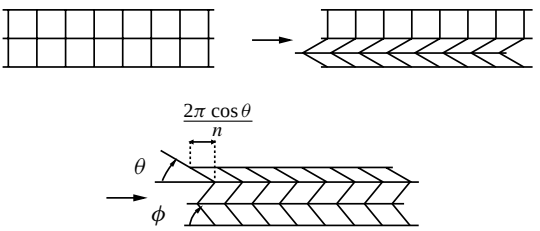


Fig. 13. Performing a Dehn twist

Gluing three cylinders together along their bottom (n has to be even), one can produce trinions of any modulus (see Fig. 14) and these trinions can be glued together according to any angle. Hence, every Riemann surface can be so produced [Bus]. □

Remark 7. An equilateral surface is a Riemann surface which can be triangulated by equilateral triangles with respect to a flat metric with conic singularities. Equilateral surfaces are the algebraic curves over $\bar{\mathbb{Q}}$ [VoSh] so are dense among the Riemann surfaces. Cutting every equilateral triangle into nine, three times smaller, triangles (see Fig. 15), one can couple these triangles by pairs so that they form lozenges, hence a critical map.

In Figs. 16–19 are some examples of critical decompositions of the plane. In Fig. 20, a higher genus example, found in Coxeter [Cox1], of the cellular decomposition of a collection of handlebodies (the genus depends on how the sides are glued pairwise) by

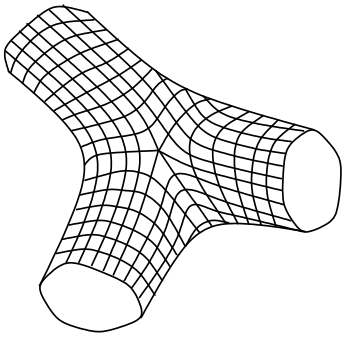


Fig. 14. Gluing three cylinders into a trinion

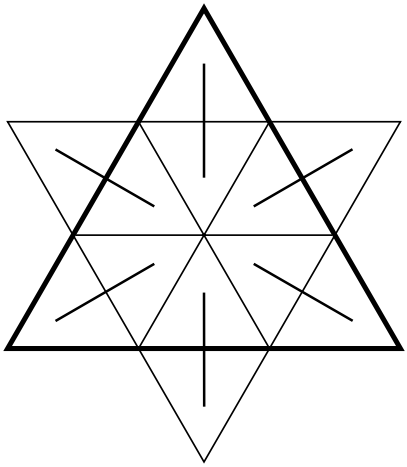


Fig. 15. An equilateral triangle cut in nine yielding lozenges

ten regular pentagons, the centre is a branched point of order three; together with its dual, they form a critical map. It is the case for any cellular decomposition by just one regular tile when its vertices are co-cyclic. This decomposition gives rise to a critical sequence using the Penrose inflation rule [GS87]. Figure 21 illustrates this inflation rule sequence on a simpler genus two example where each outer side has to be glued with the other parallel side.

3.4. Physical interpretation.

Theorem 5. *A translationally invariant discrete conformal structure* (Λ, ρ) *on* Λ *the double square or triangular/hexagonal lattices decomposition of the plane or the genus one torus, is critical and flat if and only if the Ising model defined by the interaction constants* $K_e := \frac{1}{2}\mathrm{Arcsinh}\rho_e$ *on each edge* $e \in \Lambda_1$ *is critical as usually defined in statistical mechanics [McCW].*

Proof. We prove it by solving another problem which contains these two particular cases, namely the translationally invariant square lattice with period two [Yam]. At a

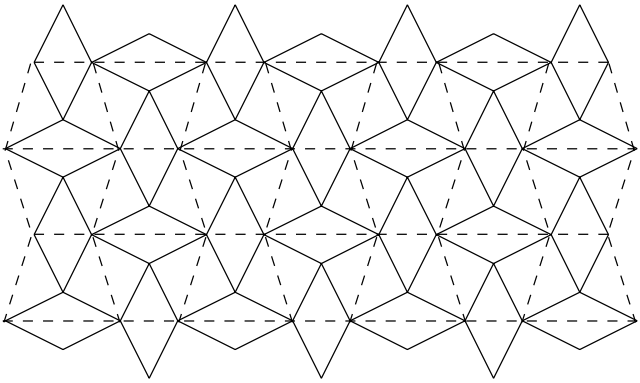


Fig. 16. A 1-parameter family of critical deformations of the square lattice

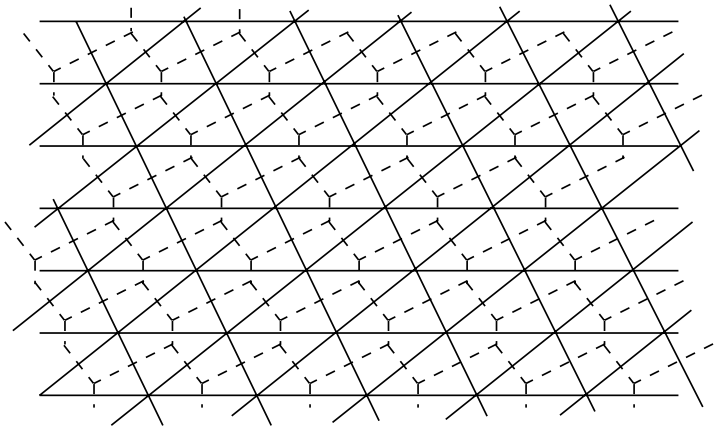


Fig. 17. A 2-parameters family of critical deformations of the triangular/hexagonal lattices. This family, key to the solution of the triangular Ising model, induced Baxter to set up the Yang–Baxter equation [Bax]. Our notion of criticality fits beautifully into this framework

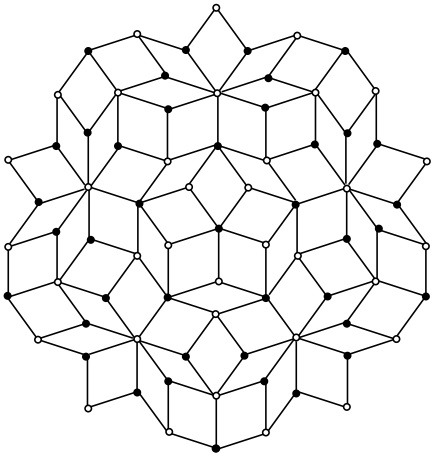


Fig. 18. The order 5 Penrose quasi crystal

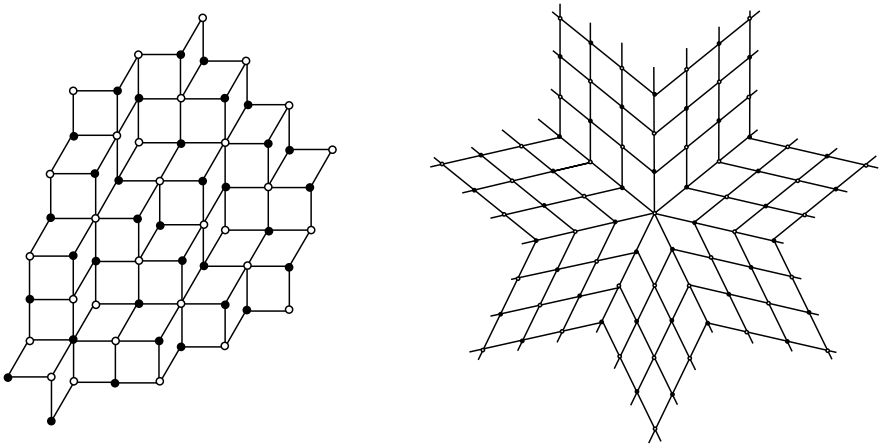


Fig. 19. Lozenge patchworks

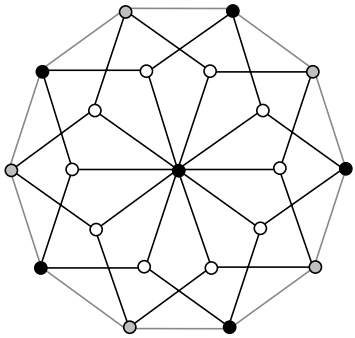
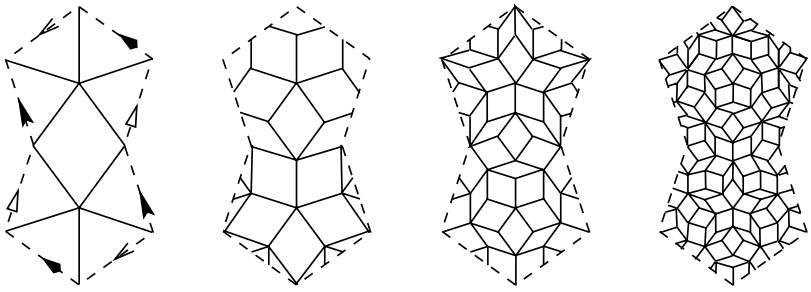


Fig. 20. Higher genus critical handlebody



a) **b)** **c)** **d)**

Fig. 21. Sequence of critical maps of a genus two handlebody using Penrose inflation rule

particular vertex, the flat critical condition on the four conformal parameters is:

$$\sum_{i=1}^{4} \arctan \rho_i = \pi,$$

which is obviously invariant by all the symmetries of the problem, including duality. When $\rho_i = \rho_{i+2}$, we get the usual period one Ising model criticality on the square lattice

$$\sinh 2K_h \sinh 2K_v = 1,$$

and likewise when one of the four parameters degenerates to zero or infinity, the three remaining coefficients fulfill

$$\sinh 2K_{\mathrm{I}} \sinh 2K_{\mathrm{II}} \sinh 2K_{\mathrm{III}} = \sinh 2K_I + \sinh 2K_{\mathrm{II}} + \sinh 2K_{\mathrm{III}}$$

which is (a form of) the criticality condition for the triangular/hexagonal Ising model. The case shown in Fig. 16 occurs when $\rho_1 = \rho_3 = 1$, implying $\rho_2 \rho_4 = 1$. □

We see here that flat criticality, when the angles at conic singularitites are multiples of 2π, is more meaningful than criticality in general. This theorem is important because it shows that statistical criticality is meaningful even at the finite size level. It is well known [KW] that for lattices, it corresponds to self-duality, which has a meaning for finite systems; here we see that self-duality corresponds to a compatibility with holomorphy. In a sense, our notion of criticality defines self-duality for more complex graphs than lattices. Furthermore, we will see in Sect. 4 that criticality implies the existence of a discrete massless Dirac spinor, which is the core of the Ising model. Although we saw that criticality implies a continuous limit theorem, the thermodynamic limit is not necessary for criticality to be detected, and to have an interesting meaning.

It is easy to produce higher genus flat critical maps and compute their critical temperature, the examples in Figs. 20–21 have four kinds of interactions corresponding to the diagonals of the two kinds of quadrilateral tiles. They are critical when the angles of the quadrilaterals are $\frac{\pi}{5}$, $\frac{2\pi}{5}$, $\frac{3\pi}{5}$, and $\frac{4\pi}{5}$, corresponding to Ising interactions

$$\sinh 2K_n = \tan \frac{n\pi}{10}. \tag{3.2}$$

The author had made no attempt to verify these values numerically.

A general way is, considering a critical genus one torus made up of a translationally invariant lattice, to cut two parallel segments of equal length and seam them back, interchanging their sides. This creates two conic singularities where an extra curvature of -2π is concentrated at each point, yielding a genus two handlebody. Repeating the process, we may produce critical handlebodies of arbitrarily large genus if we start with a very fine mesh. One has to beware that our continuous limit theorem applies only to fixed genus, it cannot grow with the refinement of the mesh. This explains why the union-jack lattice (the square lattice and its diagonals) or the three dimensional Ising model, which can be modelled as a genus mnp surface for a $2m \times 2n \times 2p$ cubic network, are beyond the scope of our technique as far as a continuous limit theorem is concerned. With this restriction in mind, we see that both the existence and the value of a critical temperature is essentially a local property and neither depends on the genus nor on the modulus of the handlebody. It is not the case for more interesting quantities such as the partition function, which can be obtained in principle from the discrete Dirac spinor that

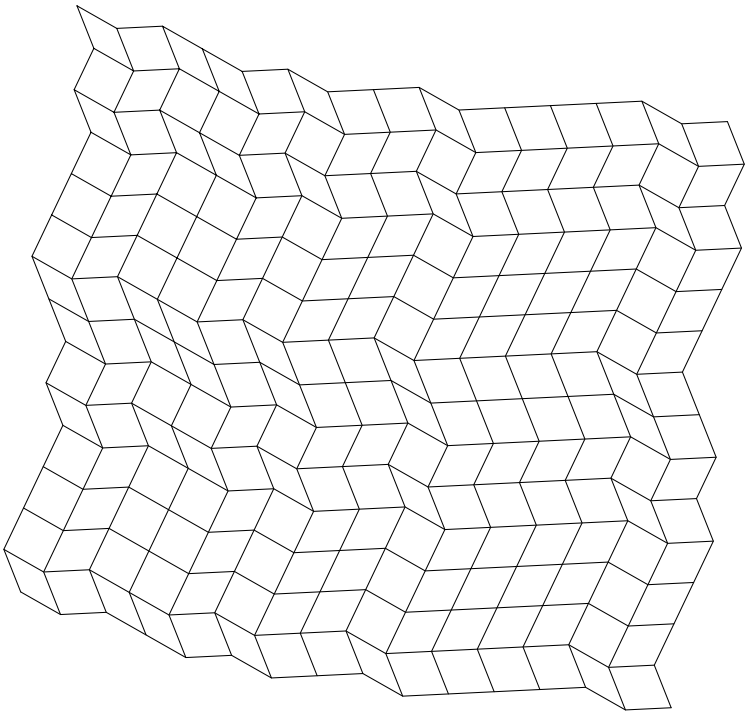


Fig. 22. The diamond graph of a critical labyrinth lattice

criticality provides, defined in Sect. 4. But such a calculus is beyond the scope of this article.

Apart from the standard lattices, the critical temperature of other well known graphs can be computed using our method, for example the labyrinth [BGB], whose diamond is pictured in Fig. 22, has the topology of the square lattice but has five different interactions strengths controlled by two binary words, labelling the columns and rows by 0's and 1's. And also new ones such as the "street graph" depicted in Fig. 23. Its double row transfer matrix appears to be the product of three commuting transfer matrices, two triangular and a square one.

Other cases such as the Kagomé [Syo] or more generally lattices of chequered type [Uti] can be handled using a technique called electrical moves [CdV96] which enables us to move around, and causes appearing or disappearing conic singularities of a flat metric. This will be the subject of a subsequent article, explaining the relationship between discrete holomorphy, electrical moves and knots and links. These electrical moves act in the space of all the graphs with discrete conformal structures in a similar way to that of the Baxterisation processes in the spectral parameter space of an integrable model (see [AdABM]). We are going to see that the link with statistical mechanics is even deeper than simply pointing out a submanifold of critical systems inside the huge space of all Ising models, as the similarity with the continuous case extends to the existence of a discrete Dirac spinor near criticality.

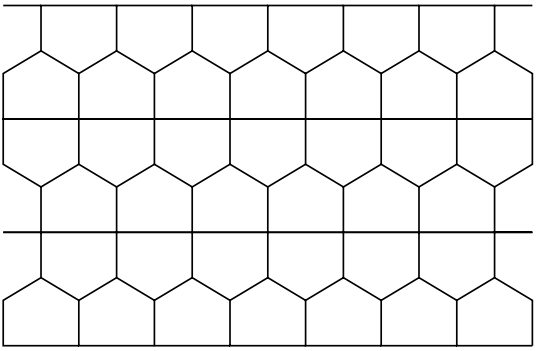


Fig. 23. The "street" lattice

3.5. Polynomial ring.

Definition 5. *Let (Λ, ℓ) be a critical map. In a given flat map $Z : U \to \mathbb{C}$ on the simply connected U, choose a vertex $z_0 \in \Lambda_0$, and for a holomorphic function f, define the holomorphic functions $f^\dagger$ and f' by the following formulae:*

$$f^\dagger(z) := \varepsilon(z)\bar{f}(z),$$

where $\bar{f}$ denotes the complex conjugate and $\varepsilon(\Gamma) = +1, \varepsilon(\Gamma^*) = -1$,

$$f'(z) := \frac{4}{\delta^2}\left(\int_{z_0}^{z} f^\dagger dZ\right)^\dagger.$$

See [Duf] for similar definitions. Notice that f' is defined up to ε if one changes the base point.

Proposition 13. *Let (Λ, ℓ)be a critical map. In a given flat map $Z : U \to \mathbb{C}$ on the simply connected U, for every holomorphic function $f \in \Omega(\Lambda)$, $df = f'dZ$. We hence call f' the derivative of f.*

Consider an edge $(x, y) \in \diamond_1$, $x \in \Gamma_0$, $y \in \Gamma_0^*$,

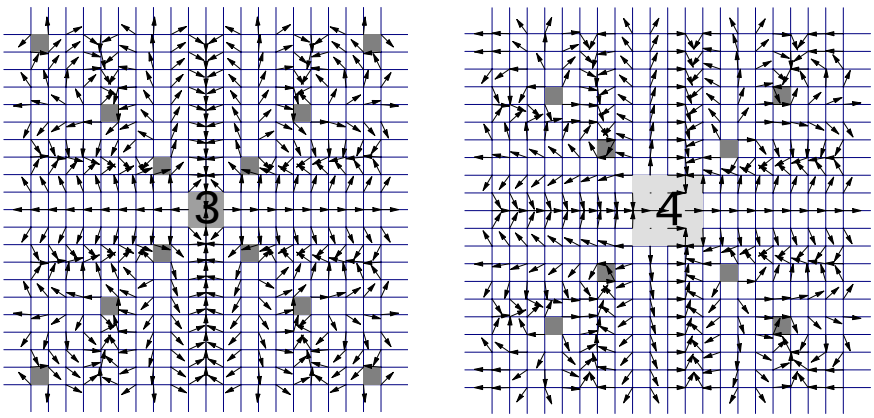
$$\begin{aligned} f'(y) &= \frac{4}{\delta^2}\left(\int_{z_0}^{x} f^\dagger dZ + \int_{x}^{y} f^\dagger dZ\right)^{\dagger_y} \\ &= -f'(x) + \frac{4}{\delta^2}\left(\frac{\bar{f}(x) - \bar{f}(y)}{2}(Z(y) - Z(x))\right)^{\dagger_y} \\ &= -f'(x) - \frac{2}{\delta^2}(f(x) - f(y))(\bar{Z}(y) - \bar{Z}(x)). \end{aligned}$$

So $\int_{(x,y)} f'dZ = -\frac{f(x)-f(y)}{\delta^2}(\bar{Z}(y) - \bar{Z}(x))(Z(y) - Z(x)) = f(y) - f(x)$.

Definition 6. *Let U be a simply connected flat region and $z_0 \in U$. Define inductively the holomorphic functions $Z^k(z) := \int_{z_0}^{z} \frac{1}{k} Z^{k-1} dZ$ given $Z^0 := 1$. As the space of holomorphic functions on U is finite dimensional, these functions are not free; let P_U be the minimal polynomial such that $P_U(Z) = Z^n + \ldots = 0$.*

Conjecture 6. The space of holomorphic functions on U, convex, is isomorphic to $\mathbb{C}[Z]/P_U$.

We won't define here the notion of convexity, see [CdV96]. The question is whether the set (Z^k) generates the space of holomorphic functions. The problem is that zeros are not localised, and as the power of Z^k increases, the set of its zeros spread on the plane and get out of U. Figure 24 is an example on the unit square lattice with U the square $[-10, 10] \oplus [-10, 10]i$, the degree increases with k until 16 where four zeros get out of the square. So a definition of the degree of a function by a Gauss formula is delicate.



Z^{15} and its zeros Z^{16} and its zeros

Fig. 24. The zeros of Z^{16} get out of the square $[-10, 10] \oplus [-10, 10]i$

4. Dirac Equation

Although we believe our theory can be applied to a lot of different problems, our motivation was to shed new light on statistical mechanics and the Ising model in particular. This statistical model has been linked with Dirac spinors since the work of Kaufman [K] and Onsager and Kaufman [KO]. We refer among others to [McCW81, SMJ, KC]. Hence we are interested in setting up a Dirac equation in the context of discrete holomorphy. To achieve this goal we first have to define the discrete analogue of the fibre bundle on which spinors live. We therefore have to define a discrete spin structure. Physics provides us with a geometric definition [KC] based on paths in a certain $\mathbb{Z}_2$-homology, that we generalise to our need (higher genus, boundary, arbitrary topology). We begin by showing that such an object in the continuum is indeed a spin structure, then define the discrete object. We then set up the Dirac equation for discrete spinors, show that

it implies holomorphy and that the existence of a solution is equivalent to criticality. The Ising model gives us an object which satisfies the discrete Dirac equation, namely the fermion, $\Psi = \sigma\mu$ as defined in [KC], corresponding to a similar object defined previously by Kaufman [K]. It fulfills the Dirac equation at criticality, but also off criticality, corresponding to a **massive** Dirac spinor. We will end this article by describing off-criticality, as defined by the author's Ph.D. advisor, Daniel Bennequin.

4.1. Universal spin structure. A spin structure [Mil] on a principal fibre bundle (E, B) over a manifold B, with $\mathrm{SO}(n)$ as a structural group, is a principal fibre bundle (E', B), of structural group $\mathrm{Spin}(n)$, and a map $f : E' \to E$ such that the following diagram is commutative:

$$\begin{array}{ccc} E' \times \mathrm{Spin}(n) & \to & E' \\ \downarrow f \times \lambda & & \downarrow f \quad \searrow \\ & & B \\ E \times \mathrm{SO}(n) & \to & E \quad \nearrow \end{array}$$

where λ is the standard 2-fold covering homomorphism from $\mathrm{Spin}(n)$ to $\mathrm{SO}(n)$.

In this paper we consider only spin structures on the tangent bundle of a surface. On a generic Riemann surface Σ, there is not a canonical spin structure. We are going to describe a surface $\hat{\Sigma}$, $2^{2-\chi(\Sigma)}$-fold covering of Σ, on which there exists a preferred spin structure. It allows us to define every spin structure on Σ as a quotient of this universal spin structure. We will treat the continuous case and then the discrete case.

Definition 7. *Let Σ be a differentiable surface with a base-point y^0; $\hat{\Sigma}$ is the set of pairs $(z, [\lambda]_2)$, where $z \in \Sigma$ is a point and $[\lambda]_2$ the homology of a path λ from y^0 to z considered in the relative homology $H_1(\Sigma, \{y^0, z\}) \otimes \mathbb{Z}_2$.*

$\hat{\Sigma}$ is the $2^{2-\chi(\Sigma)}$ covering associated to the intersection H of the kernels of all the homomorphisms from $\pi_1(\Sigma)$ to $\mathbb{Z}_2$, that is to say the quotient of the universal covering by the subgroup $H \subset \pi_1(\Sigma)$ of loops whose homology is null modulo two.

Choose v_0 a tangent vector at y^0. For each point $z \in \Sigma$, define $\Sigma_z := \Sigma \setminus \{y^0, z\} \sqcup \mathbb{S}^1 \sqcup \mathbb{S}^1$, the blown up of Σ at y^0 and z (add only one circle in the case $y^0 = z$). Consider the set of oriented paths in Σ_z, from the point corresponding to the vector v_0 at y^0 to the directions at z (the vector v_0 is needed only when $z = y^0$). Define an equivalence relation $\sim_z$ (see Fig. 25) on this set by stating that two paths λ, λ' are equivalent if and only if $\lambda - \lambda'$ is a cycle and $[\lambda - \lambda']_2 = 0$ in the homology $H_1(\Sigma \setminus \{z\}, \mathbb{Z}_2)$.

1. $z \neq y^0$ 2. $z = y^0$

Fig. 25. Paths of different classes with respect to $\sim_z$ for $z \neq y^0$ and $z = y^0$

Definition 8. *The **universal spin structure** $\mathcal{S}$ of Σ is the set of pairs $(z, [\lambda]_{\sim_z})$, with $z \in \Sigma$ and $[\lambda]_{\sim_z}$ the $\sim_z$-equivalence class of the path λ from y^0 to z in Σ_z.*

Theorem 7. *$\mathcal{S}$ is a spin structure on $\hat{\Sigma}$ and is the only one such that the action of the fundamental group $\pi_1(\Sigma)$ on $\hat{\Sigma}$ can be lifted to. Moreover it is the pull-back of any spin structure on Σ.*

Proof. The proof is in three steps, we check that $\mathcal{S}$ is a spin structure, we define a spin structure $\mathcal{S}_0$ through group theory and we show that both are equal to a third spin structure $\mathcal{S}_1$.

There is an obvious projection from $\mathcal{S}$ to $\hat{\Sigma}$ defined by $(z, [\lambda]_{\sim_z}) \mapsto (z, [\lambda]_2)$. The fibre of this projection at $(z, [\lambda]_2)$ is the set of $\sim_z$-equivalence classes of paths from y^0 to the blown-up circle at z. To each class is associated the tangent direction at z so $\mathcal{S}_z$ is a covering of $ST_z(\hat{\Sigma})$. As $H_1(\Sigma \setminus \{z\}, \mathbb{Z}_2)$ is $2^{3-\chi(\Sigma)}$ dimensional (a loop around z is not homologically trivial), for each point in $ST(\hat{\Sigma})$, there are two different lifts. The path in $\mathcal{S}_z$ corresponding to turning around z once yields the $\mathbb{Z}_2$-deck transformation. Hence $\mathcal{S}$ is a spin structure on $\hat{\Sigma}$.

Let $G := \pi_1(\Sigma)$ and $G' := \pi_1(ST\Sigma)$; the $\mathbb{S}^1$-fibre bundle $ST(\Sigma) \to \Sigma$ induces a short exact sequence $\mathbb{Z} \hookrightarrow G' \twoheadrightarrow G$. Every double covering of $ST\Sigma$ is defined by the kernel S' of an homomorphism u from G' to $\mathbb{Z}/2$, moreover, for S' to be a spin structure, its intersection with the subgroup $\mathbb{Z}$ must be $2\mathbb{Z}$.

Likewise, the fibration $\hat{\Sigma} \to \Sigma$ implies that the fundamental group $H' := \pi_1(ST\hat{\Sigma})$ of the directions bundle of $\hat{\Sigma}$ is the subgroup of G' over $H := \pi_1(\hat{\Sigma})$,

$$\begin{array}{ccccc} \mathbb{Z} & \to & H' & \to & H \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \to & G' & \to & G \end{array}, \tag{4.1}$$

The intersection of the subgroups H' and S' is a well defined spin structure $\mathcal{S}_0$ on $\hat{\Sigma}$: Indeed, consider another spin structure $S'' = \mathrm{Ker}\ (v : G' \to \mathbb{Z}/2)$ on Σ, its intersection with $\mathbb{Z}$ is $2\mathbb{Z}$ hence the kernel of $u - v$ contains the whole subgroup $\mathbb{Z}$, that is to say $u - v$ comes from a homomorphism of G to $\mathbb{Z}/2$ and we have $S'' \cap H' = S' \cap H'$. In other words, $\mathcal{S}_0$ is the unique spin structure on $\hat{\Sigma}$ which is the pull-back of a spin structure on Σ and it is the pull-back of any spin structure.

Let $z \in \Sigma$ be a point, consider the set of paths in $ST\Sigma$ from the base point (y^0, v^0) to any direction at z. Consider on this set the equivalence relation $\sim'_z$ defined by fixed extremities $\mathbb{Z}/2$-homology. The class $[\lambda]_{\sim'_z}$ of a path λ from (y^0, v^0) to (z, v) is its homology class in $H_1(ST\Sigma, \{(y^0, v^0), (z, v)\}) \otimes \mathbb{Z}/2$. The projection $ST\Sigma \twoheadrightarrow \Sigma$ splits $H_1(ST\Sigma, \{(y^0, v^0), (z, v)\}) \otimes \mathbb{Z}/2$ into

$$\mathbb{Z}/2 \to H_1(ST\Sigma, \{(y^0, v^0), (z, v)\}) \otimes \mathbb{Z}/2 \to H_1(\Sigma, \{y^0, z\}) \otimes \mathbb{Z}/2, \tag{4.2}$$

hence the set $\mathcal{S}_1$ of pairs $(z, [\lambda]_{\sim'_z})$ for all points $z \in \Sigma$ and all paths λ, is a spin structure on $\hat{\Sigma}$.

Let S' be a spin structure on Σ, it defines an element in $\mathbb{Z}/2$ for each loop in $ST\Sigma$. So each path in $ST\Sigma$ beginning at (y^0, v^0) defines, through the splitting 4.2, an element in $\mathcal{S}_1$ which is then the pull-back of S' to $\hat{\Sigma}$, hence $\mathcal{S}_0 = \mathcal{S}_1$.

On the other hand $\mathcal{S} = \mathcal{S}_1$ because there is a continuous projection from $\mathcal{S}$ to $\mathcal{S}_1$: For an element $(z, [\lambda]_{\sim_z})$, consider a C^1-path $\lambda \in \Sigma$ representing the class. Lift it to a path

in $ST\Sigma$ by the tangent direction at each point, its class $[\lambda]_{\sim_z}$ only depends on $[\lambda]_{\sim_z}$ and gives us an element in $\mathcal{S}_1$. □

4.2. Discrete spin structure.

Definition 9. *Let Υ be a cellular complex of dimension two, a **spin structure** on Υ is a* graph Υ'*, double cover of the* 1*-skeleton of Υ such that the lift of the boundary of every face is a non-trivial double cover. They are considered up to isomorphisms. Let S_D be the set of such spin structures.*

*A **spinor** ψ on Υ' is an equivariant complex function on Υ' regarding the action of $\mathbb{Z}/2$, that is to say, for all $\xi \in \Upsilon'_0$, $\psi(\bar{\xi}) = -\psi(\xi)$ if $\bar{\xi}$ represents the other lift.*

Remark 8. Usually, a spinor field is a section of a spinor bundle, that is to say a square root of a tangent *vector* field. Here, we consider square roots of *covectors*; we should say *cospinors*.

A discrete spin structure is encoded by a representation of the cycles of Υ, $Z_1(\Upsilon) :=$ Ker $\partial \cap C_1(\Upsilon)$, into $\mathbb{Z}/2$ which associates to $\gamma \in Z_1(\Upsilon)$, the value $\mu(\gamma) = 0$ if it can be lifted in Υ' to a cycle and $\mu(\gamma) = 1$ if it can not. By construction, the value of the boundary of a face is 1 and the value of a cycle which is the boundary of a 2-chain of Υ is the number of faces enclosed, modulo two.

We are going to show that this structure is indeed a good notion of discrete spin structure. First, there are as many discrete spin structures on a surface as there are in the continuous case:

Proposition 14. *On a closed connected oriented genus g surface Σ, the set S_D of inequivalent discrete spin structures of a cellular decomposition Υ is of cardinal 2^{2g}. The space of representations of the fundamental group of the surface into $\mathbb{Z}/2$ acts freely and transitively on S_D.*

We explicitly build discrete spin structures and count them: Let T be a maximal tree of Υ, that is to say a sub-complex of dimension one containing all the vertices of Υ and a maximal subset of its edges such that there is no cycle in T. Choose $2g$ edges $(e_k)_{1\leq k\leq 2g}$ in $\Upsilon \setminus T$ such that the $2g$ cycles $(\gamma_k) \in Z_1(\Upsilon)^{2g}$ extracted from $(T \cup e_k)_{1\leq k\leq 2g}$ form a basis of the fundamental group of Σ (and Υ). Let $T_+ := T \cup_k e_k$ and consider T', the sub-complex of the dual Υ^* formed by all the edges in Υ^* not crossed by T_+. It is a maximal tree of Υ^*. Likewise we define $T'_+ := T' \cup_k e_k^*$.

We construct inductively a spin structure Υ': its first elements are a double copy of T and we add edges without any choice to make as we take leaves out of T'_+. When only cycles are left, a choice concerning an edge e_k has to be taken, opening a cycle in T'_+. The process goes on until T'_+ is empty.

These choices are completely encoded by a representation μ such as in the remark, and the $2g$ values $(\mu(\gamma_k))_{1\leq k\leq 2g}$ determine the spin structure. On the other hand, this representation defines the spin structure and there are 2^{2g} such different representations. Hence the choices of the maximal tree and the edges e_k are irrelevant.

Because a cycle in Υ belongs to a class in the fundamental group of the surface (up to a choice of a path to the base point, irrelevant for our matter), the representations of the fundamental group into $\mathbb{Z}/2$ obviously act on spin structures: A representation $\rho : \pi_1(\Sigma) \to \mathbb{Z}/2$ associates to a spin structure defined by a representation $\mu : Z_1(\Upsilon) \to \mathbb{Z}/2$, the spin structure defined by the representation $\rho(\mu)$ such that $\rho(\mu)(\gamma) := \mu(\gamma) +$

$\rho([\gamma])$, where $[\gamma] \in \pi_1(\Sigma)$ is the class of the cycle γ in the fundamental group. This action is clearly free, and transitive because the set of representations is of cardinal 2^{2g}.

Given $\Lambda = \Gamma \sqcup \Gamma^*$ a double cellular decomposition, we introduce a cellular decomposition which is the discretised version of the tangent directions bundle of both Γ and Γ^*:

Definition 10. *The* **triple graph** Υ *is a cellular complex whose vertices are unoriented edges of* $\diamond$, $\Upsilon_0 = \{\{x, y\}/(x, y) \in \diamond_1\}$. *Two vertices* $\{x, y\}, \{x', y'\} \in \Upsilon_0$ *are neighbours in* Υ *iff the edges* (x, y) *and* (x', y') *are incident (that is to say* $x = x'$ *or* $x = y'$ *or* $y = x'$ *or* $y = y'$*), and they bound a common face of* $\diamond$. *There are two edges in* Υ *for each edge in* Λ. *For this to be a cellular decomposition of the surface in the empty boundary case, one needs to add faces of three types, centred on vertices of* Γ, *of* Γ^* *and on faces of* $\diamond$ *(see Fig. 26).*

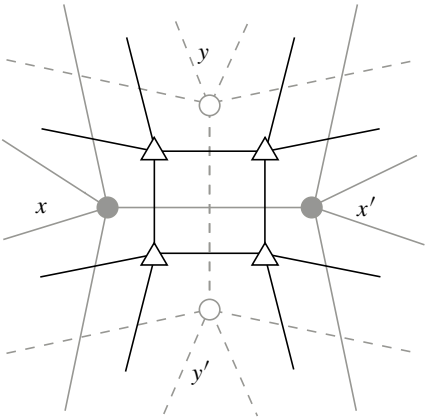


Fig. 26. The triple graph Υ

Remark 9. The topology of the usual tangent directions bundle is not at all mimicked by the incidence relations of Υ, the former is 3 dimensional and the latter is a 2-cellular complex.

Let $(x^0, y^0) \in \diamond_1$ be a given edge. All the complexes Γ, Γ^*, $\diamond$, Υ are lifted to $\hat{\Sigma}$.

Definition 11. *The* ***discrete universal spin structure*** $\hat{\Upsilon}'$ *is the following* 1*-complex: Its vertices are of the form* $((x, y), [\gamma^{y}_{y^0}])$, *where* $(x, y) \in \Upsilon_0$ *is a pair of neighbours in* $\diamond$ *and* $\gamma^{y}_{y^0}$ *is a path from* y^0 *to* y *on* Γ^*, *avoiding the faces* x^* *and* x^{0*}. *We are interested only in its relative homology class modulo two, that is to say* $[\gamma^{y}_{y_0}] \in H_1(\Gamma^* \setminus x^*, \{y^0, y\}) \otimes \mathbb{Z}_2$. *We will denote a point by* $((x, y), \gamma^{y}_{y^0})$ *and identify it with* $((x, y), \gamma'^{y}_{y^0})$ *whenever* $\gamma^{y}_{y^0}$ *and* $\gamma'^{y}_{y^0}$ *are homologous.*

Two points $((x, y), \gamma^{y}_{y^0})$ *and* $((x', y'), \gamma^{y^0}_{y'})$ *are neighbours in* $\hat{\Upsilon}'$ *if*

– $x = x'$, $(y, y') \in \Gamma_1^*$ *and* $\gamma^{y^0}_{y} - \gamma^{y^0}_{y'} + (y, y')$ *is homologous to zero in* $H_1(\Gamma^* \setminus x^*) \otimes \mathbb{Z}_2$,

– $y = y'$, $(x, x') \in \Gamma_1^*$ *and* $\gamma^{y^0}_{y} - \gamma^{y^0}_{y'}$ *is homologous to zero in* $H_1(\Gamma^* \setminus x^*) \otimes \mathbb{Z}_2$.

$\hat{\Upsilon}'$ is a double covering of $\hat{\Upsilon}$ and it is connected around each face (see Fig. 27). It is a discrete spin structure on $\hat{\Upsilon}$ in the sense defined above. Once a basis of the fundamental group $\pi_1(\Upsilon)$ is chosen, every representation of the homology group of Σ into $\mathbb{Z}_2$ allows us to quotient this universal spin structure into a double covering of Υ, yielding a usual spin structure Υ'.

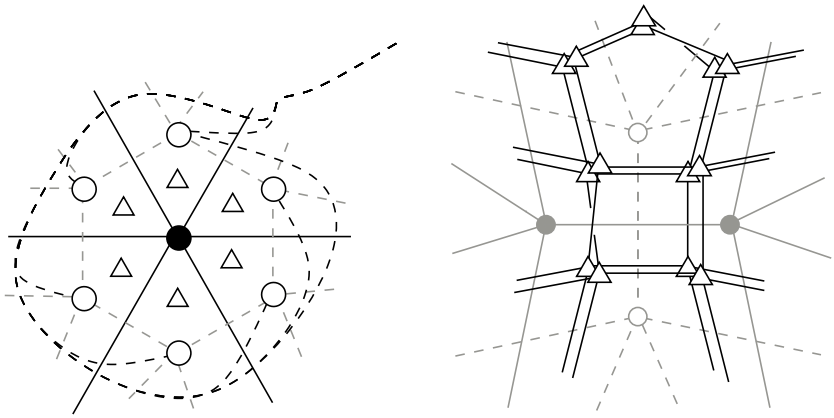


Fig. 27. Double covering around faces of Υ

4.3. Dirac equation. A spinor changes sign between the two lifts in Υ' of a vertex of Υ, in other words it is multiplied by -1 when it turns around a face. The faces of Υ which are centred on diamonds are four sided. We set up the **spin symmetry** equation for a function ζ on Υ'_0, on a positively oriented face $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Upsilon_2$ around a diamond, lifted to an 8-term cycle $(\xi_1^+, \xi_2^+, \xi_3^+, \xi_4^+, \xi_1^-, \xi_2^-, \xi_3^-, \xi_4^-) \in Z_1(\Upsilon')$:

$$\zeta(\xi_3^+) = i\zeta(\xi_1^+). \tag{4.3}$$

It implies obviously that ζ is a spinor, that is to say $\zeta(\xi_\bullet^-) = -\zeta(\xi_\bullet^+)$.

The coherent system of angles ϕ given by a semi-critical structure locally provides a spinor respecting the spin symmetry away from conic singularities: Define half angles θ on oriented edges of Υ in the following way: Each edge $(\xi, \xi') \in \Upsilon_1$ cuts an edge $a \in \Lambda_1$, set $\theta(\xi, \xi') := \pm\frac{\phi(a)}{2}$ whether (ξ, ξ') turns in the positive or negative direction around the diamond. Choose a base point $\xi_0 \in \Upsilon'_0$, define ζ by $\zeta(\xi_0) = 1$ and

$$\zeta(\xi) := \exp i \sum_{\lambda \in \gamma} \theta(\lambda) \tag{4.4}$$

for any path γ from ξ_0 to ξ. The sum of the half angles are equal to π around the faces of $\diamond$ and half the conic angle around a vertex, so if it is a regular flat point, we get $\frac{2\pi}{2} = \pi$ again, hence ζ is a well defined spinor. As diagonals of the faces of $\diamond$ are orthogonal, ζ fulfills the spin symmetry. Moreover, if the conic angles are congruous to 2π modulo 4π, ζ can be extended to any simply connected region; if the fundamental group acts by translations, ζ is defined on the whole Υ'.

We are going to define a propagation equation which comes from the Ising model. It is fulfilled by the fermion defined by Kaufman [K] which is known to converge to a Dirac spinor near criticality. We will use the definition $\psi = \sigma\mu$ given by Kadanoff and Ceva [KC]. The Dirac equation has a long history in the Ising model, beginning with the work of Kaufman [K] and Onsager and Kaufman [KO], we refer among others to [McCW81, SMJ, KC]. The equation that we need is defined explicitly in [DD], hence we will name it the **Dotsenko** equation, even though it might be found elsewhere in other forms. It is fulfilled by the fermion at criticality as well as off criticality. But this equation is only a part of the full Dirac equation. For a function ζ on Υ'_0, with the same notations as before, and if $a \in \Lambda_1$ is the diagonal of the diamond, between (ζ_2, ζ_3) and (ζ_4, ζ_1) (see Fig. 28):

$$\zeta(\xi_1^+) = \sqrt{1+\rho(a)^2}\,\zeta(\xi_2^+) - \rho(a)\zeta(\xi_3^+). \tag{4.5}$$

A check around the diamond shows that it also implies that ζ is a spinor: We write the Dotsenko equation in ξ_2^+ and ξ_3^+,

$$\begin{aligned}\zeta(\xi_2^+) =& \sqrt{1+\rho(a^*)^2}\,\zeta(\xi_3^+) - \rho(a^*)\zeta(\xi_4^+),\\ \zeta(\xi_3^+) =& \sqrt{1+\rho(a)^2}\,\zeta(\xi_4^+) - \rho(a)\zeta(\xi_1^-),\end{aligned}$$

hence, as $\sqrt{1+\rho(a)^2}\sqrt{1+\rho(a^*)^2} = \rho(a) + \rho(a^*)$,

$$\begin{aligned}\zeta(\xi_1^+) =& \rho(a^*)\zeta(\xi_3^+) - \sqrt{1+\rho(a^*)^2}\,\zeta(\xi_4^+)\\ =& \rho(a^*)(\sqrt{1+\rho(a)^2}\,\zeta(\xi_4^+) - \rho(a)\zeta(\xi_1^-)) - \sqrt{1+\rho(a^*)^2}\,\zeta(\xi_4^+)\\ =& -\zeta(\xi_1^-).\end{aligned}$$

The **Dirac equation** is the conjunction of the symmetry (4.3) and the Dotsenko (4.5) equations. We will see that this same equation describes the massive and massless Dirac equation, the mass measuring the distance from criticality.

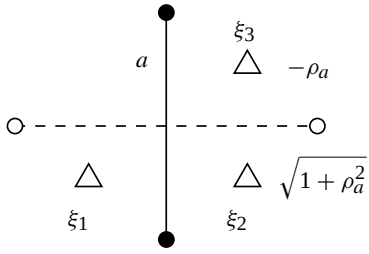


Fig. 28. The Dotsenko equation

Given two spinors ζ, ζ', their pointwise product is no longer a spinor but a regular function on Υ. As there are two edges in Υ for each edge in Λ, there is an obvious averaging map from 1-forms on Υ to 1-forms on Λ: We define $d_\Upsilon\zeta\zeta' \in C^1(\Lambda)$ by the following formula, with the same notation as before,

$$2\int_a d_\Upsilon\zeta\zeta' := \zeta(\xi_3)\zeta'(\xi_3) - \zeta(\xi_2)\zeta'(\xi_2) + \zeta(\xi_4)\zeta'(\xi_4) - \zeta(\xi_1)\zeta'(\xi_1).$$

$d_\Upsilon\zeta\zeta'$ is by definition an exact 1-form on Υ but its average *is not a priori* exact on Λ.

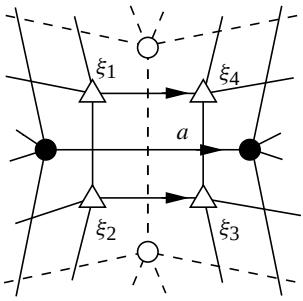


Fig. 29. The 1-form on Λ associated to two spinors.

Proposition 15. *If ζ and ζ' respect whether the spin symmetry or the Dotsenko equation, then $d_\Upsilon \zeta\zeta'$ is a closed 1-form. If ζ is a Dirac spinor and ζ' fulfills the Dotsenko equation, then $d_\Upsilon \zeta\zeta'$ is holomorphic, $d_\Upsilon \bar{\zeta}\zeta'$ anti-holomorphic and every holomorphic 1-form on Λ can be written this way on a simply connected domain, uniquely up to a constant.*

A sufficient condition for $d_\Upsilon \zeta\zeta'$ to be closed on Λ is that, with the same notations as above, $\zeta(\xi_3)\zeta'(\xi_3) - \zeta(\xi_2)\zeta'(\xi_2) = \zeta(\xi_4)\zeta'(\xi_4) - \zeta(\xi_1)\zeta'(\xi_1)$ because $\oint_{\partial y^*} d_\Upsilon \zeta\zeta'$ for a vertex $y \in \Lambda_0$ is a sum of such differences on the edges of Υ around y. This is so if there exists a 2×2-matrix A such that

$$\begin{pmatrix} \zeta(\xi_3^+) \\ \zeta(\xi_2^+) \end{pmatrix} = A \begin{pmatrix} \zeta(\xi_4^+) \\ \zeta(\xi_1^+) \end{pmatrix},$$

a similar formula for ζ', and ${}^t A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The solutions are of the form $A = \begin{pmatrix} \epsilon\sqrt{1+\lambda^2} & \lambda \\ \epsilon\lambda & \sqrt{1+\lambda^2} \end{pmatrix}$ for a complex number $\lambda \in \mathbb{C}$, $\epsilon = \pm 1$ and a determination of $\sqrt{1+\lambda^2}$. This is the case for the spin symmetry, $\lambda = -i$, $\epsilon = +1$ and for the Dotsenko equation, $\lambda = \rho(a)$, $\epsilon = -1$, $\sqrt{1+\lambda^2} > 0$.

If ζ is a Dirac spinor and ζ' fulfills the Dotsenko equation, then

$$\begin{aligned} \int_{a^*} d_\Upsilon \zeta\zeta' &= \zeta(\xi_4^+)\zeta'(\xi_4^+) - \zeta(\xi_3^+)\zeta'(\xi_3^+) \\ &= i\zeta(\xi_2^+)(\sqrt{1+\rho(a)^2}\zeta'(\xi_3^+) - \rho(a)\zeta'(\xi_2^+)) - i\zeta(\xi_1^+)\zeta'(\xi_3^+) \\ &= i\zeta(\xi_2^+)(\sqrt{1+\rho(a)^2}\zeta'(\xi_3^+) - \rho(a)\zeta'(\xi_2^+)) \\ &\quad - i(\sqrt{1+\rho(a)^2}\zeta(\xi_2^+) - \rho(a)\zeta(\xi_3^+))\zeta'(\xi_3^+) \\ &= i\rho(a)\left(\zeta(\xi_3^+)\zeta'(\xi_3^+) - \zeta(\xi_2^+)\zeta'(\xi_2^+)\right) = i\rho(a)\int_a d_\Upsilon \zeta\zeta'. \end{aligned}$$

So $d_\Upsilon \zeta\zeta'$ is holomorphic. Of course, $d\bar{\zeta}\zeta'$ is anti-holomorphic. Conversely, if $d_\Upsilon \zeta\zeta'$ is holomorphic with ζ a Dirac spinor, then ζ' fulfills the Dotsenko equation.

Given a holomorphic 1-form $\alpha \in \Omega^{(1,0)}(\Lambda)$, define α_Υ on Υ_1 by the obvious map $\int_{(\{x,y\},\{y,x'\})} \alpha_\Upsilon := \int_{(x,x')} \alpha$. It is a closed 1-form on Υ because α is closed on Λ, so there exists a function a on any simply connected domain of Υ_0, unique up to an additive constant, such that $d_\Upsilon a = \alpha_\Upsilon$. A check shows that the only spinors ζ'' such

that $d_\Upsilon \zeta \zeta'' = 0$ on Λ are the one proportional to $\bar{\zeta}$. It is consistent with the fact that the Dirac spinor is of constant modulus (see Eq. (4.6)). Hence the function $\zeta' := a/\zeta$ on Υ' is the unique spinor (up to a constant times $1/\zeta \sim \bar{\zeta}$) such that $d_\Upsilon \zeta \zeta' = \alpha$.

Notice that for ζ a Dirac spinor, the holomorphic 1-form associated to it on Λ is locally, for a given flat coordinate Z, $d_\Upsilon \zeta \zeta = \lambda dZ$, with $\lambda \in \mathbb{C}$ a certain constant.

4.4. Existence of a Dirac spinor.

Theorem 8. *There exists a Dirac spinor on a double map iff it is critical for a given flat metric with conic angles congruous to 2π modulo 4π and such that the fundamental group acts by translations. The Dirac spinor is unique up to a multiplicative constant.*

Proof. Let ζ be a non-zero Dirac spinor. Consider a positively oriented face $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Upsilon_2$ around a diamond with diagonals a, a^* as in Fig. 28, lifted to an 8-term cycle $(\xi_1^+, \xi_2^+, \xi_3^+, \xi_4^+, \xi_1^-, \xi_2^-, \xi_3^-, \xi_4^-) \in C_1(\Upsilon')$. The equation

$$e^{i\frac{\phi(a)}{2}} = \frac{\rho(a^*) + i}{\sqrt{1 + \rho(a^*)^2}}$$

defines an angle $\phi(a) \in (0, \pi)$ for every edge $a \in \Lambda_1$.

The Dotsenko and symmetry equations combine into

$$\zeta(\xi_2^+) = \frac{\rho(a) + i}{\sqrt{1 + \rho(a)^2}} \zeta(\xi_1^+). \tag{4.6}$$

The fact that ζ is a spinor implies that, summing the four angles around the diamond, we get $e^{i(\phi(a)+\phi(a^*))} = -1$. As each angle is less than π, their sum is equal to π. The same consideration around a vertex $x \in \Lambda_0$, yields $\exp i \sum_{(x,x')\in\Lambda_1} \frac{\phi(x,x')}{2} = -1$. So ϕ is a coherent system of angles and the map is critical with conic angles congruous to 2π modulo 4π.

Conversely, given ϕ a coherent system of angles with conic angles congruous to 2π modulo 4π, the preceding construction described by Eq. (4.4) gives the only Dirac spinor. □

In this case, dZ is a well defined holomorphic 1-form on the whole surface.

Corollary 9. *Let (Λ, ρ) be a discrete conformal structure and P a set of vertices, containing among others the vertices v such that the sum $\sum_{e\sim v} \mathrm{Arctan}\rho(e)$, summed over all edges e incident to v, is greater than 2π. The discrete conformal structure is critical with P as conic singularities if and only if there exist Dirac spinors on every simply connected domain containing no point of P.*

We define in which sense a discrete spinor converges to a continuous spinor. We don't define these spinors on specific spin structures but rather on the universal spin structure $\mathcal{S}$.

Consider a sequence of finer and finer critical maps such as in Theorem 3. Choose a converging sequence of base points $(x_k^0, y_k^0) \in {}^k\Upsilon_0$ on each critical map such that the direction sequence $(\frac{\overrightarrow{x_k^0 y_k^0}}{d(x_k^0, y_k^0)})$ converges to a tangent vector (x^0, v^0).

Consider a sequence of points $(x_k, y_k) \in {}^k\Upsilon_0$, defining a sequence of points (x_k) converging to x in Σ and a converging sequence of directions $v = \lim \frac{\overrightarrow{x_k y_k}}{d(x_k, y_k)}$. By compacity of the circle, there exist such sequences for every point $x \in \Sigma$ and the criticality implies that it is in at least three directions for flat points, separated by angles less than π.

The different limits allow us to identify, after a certain rank, the relative homology groups $H_1({}^k\Gamma^* \setminus x_k^*, \{y_k^0, y_k\}) \otimes \mathbb{Z}_2$ with $H_1(\Sigma_x, \{(x^0, v^0), (x, v)\}) \otimes \mathbb{Z}_2$, the classes of paths in the blown-up of Σ at x^0 and x.

Definition 12. *We will say that a sequence $(\zeta_k)_{k\in\mathbb{N}}$ of spinors converges if and only if, for any converging sequences, $\left((x_k, y_k) \in {}^k\Upsilon_0\right)_{k\in\mathbb{N}}$ defining a limit tangent vector, and $([\lambda_k])_{k\in\mathbb{N}}$ of classes of paths in ${}^k\Gamma^*$ from y_k^0 to y, avoiding the face x_k^*, the sequence of values $(\zeta_k(x_k, [\lambda_k]))$ converges.*

Remark 10. It defines a continuous limit spinor ζ by equivariance: Let $x \in \hat{\Sigma}$, the set D_x of directions in which there exist converging sequences of discrete directions is by definition a closed set. Let u, v two boundary directions of D_x such that the entire arc A of directions between them is not in D_x. Consider $[(x, [\lambda]_x), (x, [\lambda']_x)] \subset \mathcal{S}$ a lift of A. The circle $\mathbb{S}^1$ acts on the directions, hence on the $\sim_x$-classes, let $\psi \in (0, \pi)$ the angle such that $(x, e^{i\psi}[\lambda]_x) = (x, [\lambda']_x)$. Define

$$\zeta(x, e^{i\phi}[\lambda]_x) := e^{iv(\phi)}\zeta(x, [\lambda]_x),$$

where $v(\phi) = \frac{\phi}{\psi}\frac{\zeta(x,[\lambda']_x)}{\zeta(x,[\lambda]_x)}$ ($v(\phi) = \frac{\phi}{2}$ for Dirac spinors).

Theorem 10. *Given a sequence of critical maps such as in Theorem 3 with Dirac spinors on all of them, they can be normed so that they converge to the usual Dirac spinor on the Riemann surface.*

In a local flat map Z, the square of the discrete Dirac spinor on ${}^k\hat{\Upsilon}'$ is (up to a multiplicative constant) the 1-form dZ evaluated on the edges. Hence their sequence converges.

4.5. Massive Dirac equation, discrete fusion algebra and conclusions. For completeness and motivation, we describe below the situation off-criticality where elliptic integrals come into play, and investigate a form of the discrete fusion algebra in the Ising model. This work was done by Daniel Bennequin and will be the subject of a subsequent article.

A massive system in the continuous theory is no longer conformal. In the same fashion, Daniel Bennequin defined a massive discrete system of modulus k as a discrete double graph (Λ, ρ) such that, for each pair (a, a^*) of dual edges,

$$\rho(a)\rho(a^*) = \frac{1}{k}. \tag{4.7}$$

The massless case corresponds to $k = 1$. We showed that criticality was equivalent to a coherent system of angles $\phi(a)$ such as shown in Fig. 8, defined by $\tan\frac{\phi(a)}{2} = \rho(a)$, and adding up to 2π at each vertex of the double, except at conic singularities. The Dirac

spinor was constructed using the half angles $\frac{\phi(a)}{2}$. Similarly, for every edge, we define the massive "half angle" $u(a)$ as the elliptic integral

$$u(a) := \int_0^{\frac{\phi(a)}{2}} \frac{d\varphi}{\Delta'(\varphi)}, \tag{4.8}$$

where the measure is deformed by

$$k^2 + k'^2 = 1, \tag{4.9}$$

$$\Delta(\varphi) := \sqrt{1 - k^2 \sin^2 \varphi}, \tag{4.10}$$

$$\Delta'(\varphi) := \sqrt{1 - k'^2 \sin^2 \varphi}. \tag{4.11}$$

Using these non-circular half angles, and the corresponding "square angle" $I_{k'} := \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\Delta'(\varphi)}$, one can construct a massive Dirac spinor wherever the following "flatness" condition is fulfilled:

$$\sum_{a \in \partial F} (I_{k'} - u(a)) = I_{k'} \mod 4I_{k'} \text{ for each face } F \in \Lambda_2, \tag{4.12}$$

$$\sum_{a \ni v} (u(a)) = I_{k'} \mod 4I_{k'} \text{ for each vertex } v \in \Lambda_0. \tag{4.13}$$

Daniel Bennequin noticed that the fusion algebra of the Ising model could be understood at the finite level: Consider a trinion made of cylinders of a square lattice, of width m and n, glued into a cylinder of width $m+n$. It has been known since Kaufman [K] that, in the transfer matrix description of the Ising model, the configuration space of the Ising model on each of the three boundaries is a representation of spin groups $\mathrm{spin}(m)$, $\mathrm{spin}(n)$ and $\mathrm{spin}(m+n)$ respectively. If m is odd, there exists a unique irreducible representation Δ of $\mathrm{spin}(m)$ but when m is even, there are two irreducible representations, Δ^+ and Δ^-. A pair of pants gives us a map $\mathrm{spin}(m) \times \mathrm{spin}(n) \to \mathrm{spin}(m+n)$, in the case of a pair of pants of height zero, it's the inclusion given by the usual product. The representations of $\mathrm{spin}(m+n)$ induce representations of the product group that can be split into irreducible representations. If the three numbers are even,

$$\Delta^+ \to \Delta^+ \otimes \Delta^+ + \Delta^- \otimes \Delta^-, \tag{4.14}$$

$$\Delta^- \to \Delta^+ \otimes \Delta^- + \Delta^- \otimes \Delta^+, \tag{4.15}$$

while if only one of them is even,

$$\Delta \to \Delta^+ \otimes \Delta + \Delta^- \otimes \Delta, \tag{4.16}$$

and if m and n are both odd,

$$\Delta^+ \to \Delta \otimes \Delta, \tag{4.17}$$

$$\Delta^- \to \Delta \otimes \Delta. \tag{4.18}$$

Let us compile these data in an array and relabel Δ by σ, Δ^+ by 1 and Δ^- by ϵ:

$$\begin{array}{c|ccc} & 1 & \epsilon & \sigma \\ \hline 1 & 1 & \epsilon & \sigma \\ \epsilon & \epsilon & 1 & \sigma \\ \sigma & \sigma & \sigma & 1+\epsilon. \end{array} \tag{4.19}$$

This is read as follows, the $1+\epsilon$ in the slot $\sigma \otimes \sigma$ for example, means that the representation 1 and the representation ϵ of spin$(m+n)$ both induce a factor $\sigma \otimes \sigma$ in the representation in the product group spin$(m) \times$ spin(n).

We get exactly the fusion rules of the Ising model. The only difference compared with the continuous case is that the algebra is not closed at a finite level. The columns, rows and entries are not representations of the same group, rather we have a product of representations of spin(n) and spin(m) as a factor of a representation of spin$(n+m)$.

These results provide evidence that a discrete conformal field theory might be looked for: the discrete Dirac spinor at criticality is the discrete version of the conformal block associated with the field Ψ and some sort of fusion algebra can be identified at the finite level. The program we contemplate is, first to investigate other statistical models and see if there are such patterns. If that is the case, we must then mimic in the discrete setup the vertex operator algebra of the continuous conformal theory. This can be attempted by defining a discrete operator algebra, in a similar fashion to Kadanoff and Ceva [KC], and splitting this algebra according to its discrete holomorphic and anti-holomorphic parts. The hope is that some aspects of the powerful results and techniques defined by Belavin, Polyakov and Zamolodchikov [BPZ] will still hold. A very interesting issue would be, as we have done for the Ising model, to realize the fusion rules of a theory in the discrete setup, yielding its Verlinde algebra.

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References

[AdABM] Anglès d'Auriac, J.-Ch., Boukraa, S. and Maillard, J.M.: Let's baxterise. J. Stat. Phys. **102**, 641–700 (2001). hepth/0003212

[BGB] Baake, M, Grimm, U. and Baxter, R.J.: A critical Ising model on the labyrinth. Internat. J. Modern Phys. B **8**, 2526 (1994): Perspectives on solvable models. 3579–3600

[Bax] Baxter, R.J.: *Exactly solved models in statistical mechanics*. London: Academic Press Inc. [Harcourt Brace Jovanovich Publishers], 1989; Reprint of the 1982 original

[BPZ] Belavin, A.A., Polyakov, A.M. and Zamolodchikov, A.B.: Infinite conformal symmetry in two-dimensional quantum field theory. Nucl. Phys. B **241**, 2, 333–380 (1984)

[BS96] Benjamini, I. and Schramm, O.: Random walks and harmonic functions on infinite planar graphs using square tilings. Ann. Probab. **24**, 3, 1219–1238 (1996)

[Bus] Buser, P.: *Geometry and spectra of compact Riemann surfaces*. Boston, MA: Birkhäuser Boston Inc., 1992

[CdV90] Colin de Verdière, Y.: Un principe variationnel pour les empilements de cercles. Invent. Math. **104**, 655–669 (1991)

[CdV96] Colin de Verdière, Y. Gitler, I. and Vertigan, D.: Réseaux électriques planaires, I,I. Comment. Math. Helv. **71**, 1, 144–167 (1996)
[Cox1] Coxeter, H.S.M.: *Introduction to geometry*. Wiley Classics Library, New York: John Wiley & Sons Inc., 1989; Reprint of the 1969 edition
[DD] Dotsenko, V.S. and Dotsenko, V.S.: Critical behaviour of the phase transition in the 2d Ising model with impurities. Adv. in Phys.**32**, 2, 129–172 (1983)
[Duf] Duffin, R.J.: Basic properties of discrete analytic functions. Duke Math. J. **23**, 335–363 (1956)
[GS87] Grünbaum, B. and Shephard, G.C.: *Tilings and patterns*. New York: W. H. Freeman and Company, 1987
[Hug] Hughes, B.D.: *Random walks and random environments, Vol. 1*. New York: The Clarendon Press Oxford University Press, 1995
[ID] Itzykson, C. and Drouffe, J.-M.: *Statistical field theory, Vol. 2*. Cambridge Monographs on Mathematical Physics, Cambridge: Cambridge University Press, 1989
[KC] Kadanoff, L.P. and Ceva, H.: Determination of an operator algebra for the two-dimensional Ising model. Phys. Rev. B (3) **3**, 3918–3939 (1971)
[K] Kaufman, B: Crystal statistics ii. Partition function evaluated by spinor analysis. Phys. Rev. **76**, 1232 (1949)
[KO] Kaufman, B and Onsager, L: Crystal statistics iii. Short range order in a binary Ising lattice. Phys. Rev. **76**, 1244 (1949)
[Ken] Kenyon, R.: Tilings and discrete Dirichlet problems. Israel J. Math. **105**, 61–84 (1998)
[KW] Kramers, H.A. and Wannier, G.H.: Statistics of the two-dimensional ferromagnet, I. Phys. Rev. (2) **60**, 252–262 (1941)
[LF] Lelong-Ferrand, J.: *Représentation conforme et transformations à intégrale de Dirichlet bornée*. Paris: Gauthier-Villars, 1955
[McCW] McCoy, B.M. and Wu, T.T.: *The two-dimensional Ising model*. Cambridge, Massachusetts: Harvard University Press, 1973
[McCW81] McCoy, B.M. and Wu, T.T.: Non-linear partial difference equations for the two-spin correlation function of the two-dimensional Ising model. Nucl. Phys. B **180**, 89–115 (1981)
[M] Mercat, C.: *Holomorphie discrète et modèle d'Ising*, PhD thesis, Université Louis Pasteur, Strasbourg, France, 1998, under the direction of Daniel Bennequin, Prépublication de l'IRMA, available at http://www-irma.u-strasbg.fr/irma/publications/1998/98014.shtml
[Mil] Milnor, J: Spin structures on manifolds. Enseign. Math., II. Ser. **9**, 198–203 (1963)
[Ons] Onsager, L.: Crystal statistics, I. A two-dimensional model with an order-disorder transition. Phys. Rev. (2) **65**, 117–149 (1944)
[PS85] Preparata, F.P. and Shamos, M.I.: *Computational geometry*. Texts and Monographs in Computer Science, New York: Springer-Verlag, 1985
[SMJ] Sato, M, Miwa, T and Jimbo, M: Studies on holonomic quantum fields i–iv. Proc. Japan Acad. **53 (A)**, 1–6 (1977)
[Sie] Siegel, C.L.: *Topics in complex function theory. Vol. II*, Wiley Classics Library, New York: John Wiley & Sons Inc., 1988. Automorphic functions and abelian integrals, translated from the German by A. Shenitzer and M. Tretkoff, with a preface by Wilhelm Magnus, Reprint of the 1971 edition. A Wiley-Interscience Publication
[Syo] Syôzi, Itiro: Statistics of kagomé lattice. Progress Theoret. Phys. **6**, 306–308 (1951)
[Tro] Troyanov, M.: Les surfaces euclidiennes à singularités coniques. Enseign. Math. (2) **32** 1, 2, 79–94 (1986)
[Uti] Utiyama, T.: Statistics of two-dimensional Ising lattices of chequered types. Progress Theoret. Phys. **6**, 907–909 (1951); Letter to the Editor
[Veb] Veblen, O.: *Analysis situs. 2. edit.*, New York: American Mathematical Society X, 1931
[VoSh] Voevodskiĭ, V.A. and Shabat, G.B.: Equilateral triangulations of Riemann surfaces, and curves over algebraic number fields. Dokl. Akad. Nauk SSSR **304** 2, 265–268 (1989)
[Wan50] Wannier, G.H.: Antiferromagnetism. The triangular Ising net. Physical Rev. (2) **79**, 357–364 (1950)
[Whit] Whitney, H.: Product on complexes. Annals of Math. (2) **39**, 397–432 (1938)
[Yam] Yamamoto, T.: On the crystal statistics of two-dimensional Ising ferromagnets. Progress Theoret. Phys. **6**, 533–542 (1951)

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