

## DISCRETE SCHRÖDINGER OPERATORS ON A GRAPH

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In this paper, we study some spectral properties of the *discrete* Schrödinger operator  $-\Delta + q$  defined on a locally finite connected graph with an automorphism group whose orbit space is a finite graph.

The discrete Laplacian and its generalization have been explored from many different viewpoints (for instance, see [2] [4]). Our paper discusses the discrete analogue of the results on the bottom of the spectrum established by T. Kobayashi, K. Ono and T. Sunada [3] in the Riemannian-manifold-setting.

### § 1. Discrete Laplacians

Let  $X = (V, E)$  be a locally finite connected graph without loops and multiple edges. Here  $V$  and  $E$  are, respectively, the set of *vertices* and the set of *unoriented edges* of  $X$ . In a natural manner,  $X$  is regarded as a one-dimensional CW complex. We assign a positive *weight* to each vertex and also to each edge by giving mappings  $m : V \rightarrow \mathbb{R}_+$  and  $w : E \rightarrow \mathbb{R}_+$ . Let  $C_0(V)$  and  $C_0(E)$  be the space of all complex-valued functions on  $V$  and  $E$  with finite support, respectively. Define inner products on  $C_0(V)$  and  $C_0(E)$  by

$$(1.1) \quad \langle f, g \rangle = \sum_{x \in V} f(x) \overline{g(x)} m(x)$$

$$(1.2) \quad \langle \omega, \eta \rangle = \sum_{e \in E} \omega(e) \overline{\eta(e)} w(e).$$

The completions of  $C_0(V)$  and  $C_0(E)$  with respect to those inner products will be denoted by  $L^2(V)$  and  $L^2(E)$ , respectively.

Each edge has two orientations. We use the symbol  $E^{\text{or}}$  to represent the set of all *oriented* edges, so that forgetting orientation yields a two-to-one map  $p : E^{\text{or}} \rightarrow E$ . Reversing orientation gives rise to an involution on  $E^{\text{or}}$ , which we denote by  $e \mapsto \bar{e}$ . We shall use the same symbol  $w$  for

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the composition  $w \circ p$ , which is a function on  $E^{or}$ . For an oriented edge  $e$ ,  $o(e)$  and  $t(e)$  denote the origin and terminus point of  $e$ , respectively. Let  $\mathcal{O}_x = \{e \in E^{or}; o(e) = x\}$ .

We fix an orientation on each edge by giving a subset  $E_0$  of  $E^{or}$  such that  $E^{or} = E_0 \cup \bar{E}_0$  (disjoint) and we identify  $E_0$  with  $E$  by the map  $p$ . Define the operator  $d : C_0(V) \rightarrow C_0(E)$  by

$$(1.3) \quad df(e) = f(t(e)) - f(o(e)),$$

which is a natural analogue of the exterior derivation on a manifold.

A simple calculation gives the following formula for the formal adjoint  $d^*$  of  $d$ :

$$d^* \omega(x) = m(x)^{-1} \left\{ \sum_{\substack{e \in E_0 \\ t(e)=x}} \omega(e)w(e) - \sum_{\substack{e \in \bar{E}_0 \\ o(e)=x}} \omega(e)w(e) \right\}.$$

The discrete Laplacian  $\Delta = \Delta_x$  is now defined by

$$(1.4) \quad \Delta f(x) = -d^*df(x) = m(x)^{-1} \left\{ \sum_{e \in \mathcal{O}_x} f(t(e))w(e) - \left( \sum_{e \in \mathcal{O}_x} w(e) \right) f(x) \right\}.$$

Note that  $\Delta$  is independent of the choice of orientation on edges.

*Remark 1.* Let  $h : V \rightarrow \mathbb{R}$  be a function defined by

$$h(x) = (1/m(x)) \sum_{e \in \mathcal{O}_x} w(e).$$

Then the operator  $\Delta$  is bounded as an operator acting in  $L^2(V)$  if and only if  $h$  is bounded. For the sake of completeness, we shall give a proof. Suppose that  $h$  is bounded. Then for any  $f \in C_0(V)$ ,

$$\begin{aligned} \|df\|^2 &\leq 2 \sum_{e \in E_0} (|f(t(e))|^2 + |f(o(e))|^2)w(e) \\ &= 2 \left\{ \sum_{x \in V} \sum_{\substack{e \in E_0 \\ t(e)=x}} |f(t(e))|^2 w(e) + \sum_{x \in V} \sum_{\substack{e \in E_0 \\ o(e)=x}} |f(o(e))|^2 w(e) \right\} \\ &= 2 \left\{ \sum_{x \in V} |f(x)|^2 \left( \sum_{e \in \mathcal{O}_x} w(e) \right) \right\} \\ &\leq c \|f\|^2, \end{aligned}$$

where  $c = 2 \sup_{x \in V} \{ (1/m(x)) \sum_{e \in \mathcal{O}_x} w(e) \}$ . Thus  $\Delta$  is bounded. Conversely, assume that  $\Delta$  is bounded. If  $h$  is unbounded, then for every positive real number  $K$ , there is an  $x \in V$  such that  $(1/m(x)) \sum_{e \in \mathcal{O}_x} w(e) \geq K$ . We see that  $\|d\delta_x\|^2 = \sum_{e \in \mathcal{O}_x} w(e) \geq Km(x) = K\|\delta_x\|^2$ , where  $\delta_x(y)$  equals 1 when  $y = x$  and zero elsewhere. It follows that  $\|\Delta\delta_x\| \|\delta_x\| \geq |(d^*d\delta_x, \delta_x)| = \|d\delta_x\|^2 \geq K\|\delta_x\|^2$ . Thus  $\Delta$  is unbounded. This contradicts our hypothesis that  $\Delta$  is bounded.

*Remark 2.* The discrete Laplacian defined above is a bit generalized

one of [2].

**§ 2. Bottom of the spectrum**

Let  $M = (V, E)$  be a *finite* connected graph, and let  $\pi : X \rightarrow M$  be a normal covering map as CW complexes with the covering transformation group  $\Gamma$ . The covering space  $X$  has a graph structure  $(\tilde{V}, \tilde{E})$  such that  $\pi$  is a morphism of graphs. Then  $\Gamma$  acts freely on  $\tilde{E}$  and  $\tilde{V}$  and  $\Gamma \backslash \tilde{E} \simeq E$ ,  $\Gamma \backslash \tilde{V} \simeq V$ . We assume that  $M$  has weights on vertices and edges. The weights on vertices and edges of  $X$  are naturally assigned by using the map  $\pi$  so that they are left invariant under the  $\Gamma$ -action. If we fix orientation on edges of  $M$ , then the induced one on  $\tilde{E}$  is preserved by the  $\Gamma$ -action. Take any real-valued function  $q_M \in C(V)$ . We see that  $q = q_M \circ \pi$  is invariant under the  $\Gamma$ -action. Since  $M$  is finite,  $H_M = -\Delta_M + q_M$  is identified with a hermitian matrix of finite size and its spectrum consists of real eigenvalues.

The operator  $H_X = -\Delta_X + q$  is just the lift of the operator  $H_M$  on  $M$  by the map  $\pi$  and is therefore bounded (see Remark 1) and self-adjoint. We denote by  $\lambda_0(H)$  the greatest lower bound of the spectrum of a self-adjoint operator  $H$ . Note that  $\lambda_0(H_M)$  is just the minimal eigenvalue of  $H_M$ .

LEMMA 1.  $\lambda_0(H_M)$  is simple and has a positive eigenfunction.

*Proof.* Let  $V = \{1, \dots, n\}$ . For  $1 \leq i \leq n$ , set

$$\begin{aligned} \varphi_i(x) &= \frac{1}{\sqrt{m(i)}} && \text{if } x = i \\ &= 0 && \text{otherwise.} \end{aligned}$$

Then  $\{\varphi_i\}$  is an orthonormal basis of  $L^2(V)$ . Let  $A = (a_{ij})$  be the matrix of  $\Delta_M$  with respect to this basis. If  $(i, j)$  is an edge of  $M$  with  $i \neq j$ , then  $a_{ij} = (\Delta_M \varphi_j, \varphi_i) = (1/\sqrt{m(i)m(j)})w(i, j)$ . Hence the off-diagonal entries of the matrix  $A$  are nonnegative real numbers. Let  $A' = (a'_{ij})$  be the matrix with  $a'_{ij} = a_{ij}$  for  $i \neq j$  and  $a'_{ii} = 0$ . Since  $M$  is connected, the matrix  $A'$  is irreducible. Thus the operator  $\Delta_M - q_M$  has the form  $A' + D$ , where  $D$  is a diagonal matrix with entries  $d_{ii} \in \mathbb{R}$ . The facts that the maximal eigenvalue  $-\lambda_0(A' + D)(= -\lambda_0(H_M))$  is simple and there exists a positive eigenfunction associated with it, follow readily by applying the Perron-Frobenius Theorem [5] to the matrix  $A' + D + xI$  for large enough  $x \in \mathbb{R}$ .

THEOREM 1.  $\lambda_0(H_M) \leq \lambda_0(H_X)$ . The equality holds if and only if the covering transformation group  $\Gamma$  is amenable.

To prove this, we will employ a representation-theoretic technique. We fix orientation  $\tilde{E}_0$  on  $\tilde{E}$  induced from an orientation of edges of  $M$ . We also identify  $\tilde{E}_0$  with  $\tilde{E}$ .

Let  $\rho$  be a unitary representation of  $\Gamma$  on a Hilbert space  $W$  and  $L^2_\rho(V) = \{s : \tilde{V} \mapsto W; s(\sigma x) = \rho(\sigma)s(x) \text{ for all } x \in \tilde{V} \text{ and } \sigma \in \Gamma\}$  with the natural inner product

$$\langle s_1, s_2 \rangle = \sum_{x \in \mathcal{D}_V} \langle s_1(x), s_2(x) \rangle_W m(x),$$

where  $\mathcal{D}_V$  is a finite fundamental subset in  $\tilde{V}$  for the  $\Gamma$ -action; i.e.,  $\mathcal{D}_V$  is a subset of  $\tilde{V}$  such that for every  $x \in \tilde{V}$ , there exists a unique pair  $(\sigma, x') \in \Gamma \times \mathcal{D}_V$  satisfying  $\sigma x = x'$ . Note that  $\tilde{V} = \bigcap_{\gamma \in \Gamma} \gamma \mathcal{D}_V$  and  $\gamma \mathcal{D}_V \cap \mathcal{D}_V = \emptyset$  for  $\gamma \neq \text{id}$ . One can easily check that the inner product is independent of the choice of  $\mathcal{D}_V$ . Let  $L^2_\rho(E) = \{\varphi : \tilde{E} \mapsto W; \varphi(\sigma e) = \rho(\sigma)\varphi(e) \text{ for all } e \in \tilde{E} \text{ and } \sigma \in \Gamma\}$  with the following inner product

$$\langle \varphi_1, \varphi_2 \rangle = \sum_{e \in \mathcal{D}_E} \langle \varphi_1(e), \varphi_2(e) \rangle_W w(e),$$

where  $\mathcal{D}_E$  is a finite fundamental subset in  $\tilde{E}$  for the  $\Gamma$ -action. This definition also does not depend on the choice of  $\mathcal{D}_E$ .

The bounded operator  $d_\rho : L^2_\rho(V) \rightarrow L^2_\rho(E)$  is defined by

$$d_\rho s(e) = s(\text{t}(e)) - s(\text{o}(e)).$$

LEMMA 2. *The adjoint operator of  $d_\rho$  is given by*

$$(d_\rho^* \varphi)(x) = m(x)^{-1} \left( \sum_{\substack{e \in \tilde{E}_0 \\ \text{t}(e) = x}} \varphi(e)w(e) - \sum_{\substack{e \in \tilde{E}_0 \\ \text{o}(e) = 0}} \varphi(e)w(e) \right).$$

*Proof.* First note that the correspondences

$$\begin{aligned} d_1 : s &\longmapsto \varphi_1 & \varphi_1(e) &= s(\text{t}(e)) \\ d_2 : s &\longmapsto \varphi_2 & \varphi_2(e) &= s(\text{o}(e)) \end{aligned}$$

give rise to operators of  $L^2_\rho(V)$  into  $L^2_\rho(E)$ , and  $d_\rho = d_1 - d_2$ . Let  $\mathcal{D}_V$  be a fundamental set in  $\tilde{V}$ , and put

$$\mathcal{D}_E = \{e \in \tilde{E}_0; \text{t}(e) \in \mathcal{D}_V\}.$$

Then  $\mathcal{D}_E$  is a fundamental set in  $\tilde{E} = \tilde{E}_0$ , and

$$\begin{aligned} \langle d_1 s, \varphi \rangle &= \sum_{e \in \mathcal{D}_E} \langle s(\text{t}(e)), \varphi(e) \rangle_W w(e) \\ &= \sum_{x \in \mathcal{D}_V} \sum_{\substack{e \in \tilde{E}_0 \\ \text{t}(e) = x}} \langle s(x), \varphi(e) \rangle_W w(e). \end{aligned}$$

Thus we have

$$d_1^* \varphi(x) = m(x)^{-1} \sum_{\substack{e \in E_0 \\ t(e) = x}} \varphi(e) w(e).$$

Similarly, we obtain

$$d_2^* \varphi(x) = m(x)^{-1} \sum_{\substack{e \in E_0 \\ s(e) = x}} \varphi(e) w(e).$$

This completes the proof.

The Laplacian  $\Delta_\rho$  acting on  $L_\rho^2(V)$  is now defined by  $-\Delta_\rho = d_\rho^* d_\rho$  which is equal to

$$\Delta_\rho s(x) = m(x)^{-1} \left\{ \sum_{e \in \theta_x} s(t(e)) w(e) - \left( \sum_{e \in \theta_x} w(e) \right) s(x) \right\}.$$

The twisted discrete Schrödinger operator is then defined as the self-adjoint operator  $H_\rho = -\Delta_\rho + q$ .

LEMMA 3. *If  $\rho$  is the right regular representation of  $\Gamma$ , then  $(H_\rho, L_\rho^2(V))$  is unitarily equivalent to  $(H_X, L^2(\tilde{V}))$ ; and if  $\rho$  is the trivial representation 1, then  $(H_\rho, L_\rho^2(V))$  is unitarily equivalent to  $(H_M, L^2(V))$ .*

*Proof.* Let  $W = L^2(\Gamma) = \{\varphi : \Gamma \rightarrow \mathbb{C} \mid \sum_{\sigma \in \Gamma} |\varphi(\sigma)|^2 < \infty\}$  and  $\rho$  be the right regular representation  $\rho_r$  of  $\Gamma$  on  $W$ . From now on, we simply write  $\rho$  for  $\rho_r$ . To prove that  $H_\rho$  and  $H_X$  are unitarily equivalent to each other, we have to show that there exists a unitary map  $\Phi : L^2(\tilde{V}) \mapsto L_\rho^2(V)$  such that  $H_\rho \circ \Phi = \Phi \circ H_X$ .

Define the map  $\Phi : C_0(\tilde{V}) \mapsto L_\rho^2(V)$  by

$$\Phi(f) = s,$$

where the function  $s$  is defined to be  $s(x)(\sigma) = f(\sigma x)$  for  $x \in \tilde{V}$ ,  $\sigma \in \Gamma$ . One can check that  $s(\mu x) = \rho(\mu)s(x)$  for any  $\mu \in \Gamma$ ,  $x \in \tilde{V}$ . By the definition of fundamental set, we have

$$\begin{aligned} \|s\|^2 &= \sum_{x \in \mathfrak{B}_V} \|s(x)\|_W^2 m(x) \\ &= \sum_{x \in \mathfrak{B}_V} \sum_{\sigma \in \Gamma} |f(\sigma x)|^2 m(x) \\ &= \|f\|^2 \end{aligned}$$

for any  $f \in C_0(\tilde{V})$ . Thus  $s \in L_\rho^2(V)$ . Hence the map  $\Phi$  is extended uniquely to an isometry of  $L^2(\tilde{V})$  into  $L_\rho^2(V)$ .

Next, we claim that  $\Phi$  is onto. Take any  $s \in L_\rho^2(V)$ , define  $f : \tilde{V} \rightarrow \mathbb{C}$

by  $f(x) = s(x)1$ , where  $1$  is the identity element of  $\Gamma$ . Since

$$\begin{aligned} \sum_{x \in \tilde{V}} |f(x)|^2 m(x) &= \sum_{x \in \tilde{V}} |s(x)1|^2 m(x) \\ &= \sum_{x \in \mathcal{D}_V} \sum_{\sigma \in \Gamma} |\rho(\sigma)s(x)1|^2 m(x) \\ &= \sum_{x \in \mathcal{D}_V} \sum_{\sigma \in \Gamma} |s(x)\sigma|^2 m(x) \\ &= \sum_{x \in \mathcal{D}_V} \|s(x)\|^2 m(x), \end{aligned}$$

therefore  $f \in L^2(\tilde{V})$ . Put  $s' = \Phi(f)$ . Then  $s'(x)(\sigma) = f(\sigma x) = s(\sigma x)1 = [\rho(\sigma)s(x)]1 = s(x)(\sigma)$  for every  $x \in \tilde{V}$  and  $\sigma \in \Gamma$ . Hence  $\Phi(f) = s' = s$ .

For any  $f \in L^2(\tilde{V})$ , we have

$$\begin{aligned} (\{H_\rho \circ \Phi(f)\}(x))(\sigma) &= (\{H_\rho \circ s(x)\}(\sigma)) \\ &= -\frac{1}{m(x)} \left\{ \sum_{e \in \mathcal{D}_x} s(t(e))\sigma w(e) - \left( \sum_{e \in \mathcal{D}_x} w(e) \right) s(x)\sigma \right\} + q(x)s(x)\sigma \\ &= -\frac{1}{m(x)} \left\{ \sum_{e \in \mathcal{D}_x} f(\sigma t(e))w(e) - \left( \sum_{e \in \mathcal{D}_x} w(e) \right) f(\sigma x) \right\} + q(x)f(\sigma x) \\ &= -\frac{1}{m(x)} \left\{ \sum_{e \in \mathcal{D}_{\sigma x}} f(t(e))w(e) - \left( \sum_{e \in \mathcal{D}_x} w(e) \right) f(\sigma x) \right\} + q(\sigma x)f(\sigma x) \\ &= H_x f(\sigma x) \\ &= (\{\Phi \circ H_x(f)\}(x))\sigma. \end{aligned}$$

This proves the first part of the theorem.

The second part of the theorem is easy to prove.

The Kazhdan distance  $\delta(\rho, \mathbf{1})$  (or  $\delta_A(\rho, \mathbf{1})$ ) between  $\rho$  and  $\mathbf{1}$  is defined by

$$\delta(\rho, \mathbf{1}) = \inf_{\substack{v \in W \\ \|v\|=1}} \sup_{\sigma \in A} \|\rho(\sigma)v - v\|,$$

where  $A$  is a fixed finite set of generators of  $\Gamma$ . The following lemma shows that the distance does not depend essentially on the choice of  $A$ .

LEMMA 4. *Suppose that  $A$  and  $B$  are any finite sets of generators of  $\Gamma$ . Then there exist positive constants  $k_1$  and  $k_2$  such that*

$$k_1 \delta_B(\rho, \mathbf{1}) \leq \delta_A(\rho, \mathbf{1}) \leq k_2 \delta_B(\rho, \mathbf{1}).$$

*Proof.* Let  $C = A \cup B$ . Choose an integer  $N$  large enough such that every  $\sigma \in C$  can be expressed as

$$\sigma = \mu_1 \mu_2 \cdots \mu_n,$$

where  $\mu_i \in A$  and  $n \leq N$ . Then

$$\begin{aligned} \|\rho(\sigma)v - v\| &\leq \|\rho(\mu_1)\cdots\rho(\mu_n)v - \rho(\mu_1)\cdots\rho(\mu_{n-1})v\| \\ &\quad + \|\rho(\mu_1)\cdots\rho(\mu_{n-1})v - v\| \\ &\leq \|\rho(\mu_n)v - v\| + \|\rho(\mu_1)\cdots\rho(\mu_{n-1})v - v\| \\ &\leq \sum_{i=1}^n \|\rho(\mu_i)v - v\| \\ &\leq N \sup_{\mu \in A} \|\rho(\mu)v - v\|. \end{aligned}$$

It follows that  $\delta_A \geq c_1 \delta_C$  for some constant  $c_1$ . Similarly, one can also show that  $\delta_B \geq c_2 \delta_C$  for some constant  $c_2$ . On the other hand, since  $A, B \subset C$ , we have  $\delta_C \geq \delta_A, \delta_B$ . These inequalities together prove the result.

To prove Theorem 1, it suffices to establish the following Theorem (cf. [4] [7]). For, in the next theorem, when  $\rho$  is the right regular representation  $\rho_r$ , Theorem 1 follows from the fact that  $\delta(\rho_r, \mathbf{1}) = 0$  if and only if  $\Gamma$  is amenable.

**THEOREM 2.** *There exist positive constants  $c_1$  and  $c_2$  such that*

$$c_1 \delta(\rho, \mathbf{1})^2 \leq \lambda_0(H_\rho) - \lambda_0(H_{\mathbf{1}}) \leq c_2 \delta(\rho, \mathbf{1})^2$$

for all  $\rho$ . In particular,  $\lambda_0(H_\rho) = \lambda_0(H_{\mathbf{1}})$  if and only if  $\delta(\rho, \mathbf{1}) = 0$ .

*Proof.* Note that

$$\lambda_0(H_\rho) = \inf_{s \in L^2_\rho(V)} \frac{\langle H_\rho s, s \rangle}{\|s\|^2}.$$

By Lemma 1, we may take a positive solution  $f \in L^2(V)$  to the equation  $H_M f = \lambda_0(H_{\mathbf{1}})f$ . We have

$$(2.1) \quad \begin{aligned} \langle \Delta_\rho(fs), fs \rangle &= \sum_{x \in \mathcal{D}_V} \langle \sum_{e \in \sigma_x} f(t(e))s(t(e))w(e) \\ &\quad - (\sum_{e \in \sigma_x} w(e))f(x)s(x), f(x)s(x) \rangle. \end{aligned}$$

Substituting the following equality

$$\sum_{e \in \sigma_x} w(e)f(x) = \lambda_0(H_{\mathbf{1}})f(x)m(x) - q(x)f(x)m(x) + \sum_{e \in \sigma_x} f(t(e))w(e)$$

into (2.1), we obtain

$$(2.2) \quad \begin{aligned} \langle \Delta_\rho(fs), fs \rangle &= \sum_{x \in \mathcal{D}_V} \langle \sum_{e \in \sigma_x} f(t(e))(s(t(e)) - s(x))w(e), f(x)s(x) \rangle \\ &\quad - \lambda_0(H_{\mathbf{1}}) \langle fs, fs \rangle + \langle qfs, fs \rangle. \end{aligned}$$

We now set  $\mathcal{D} = \{e \in \tilde{E}^{or}; e \in \mathcal{O}_x \text{ for some } x \in \mathcal{D}_V\}$ . It is easy to check that  $\mathcal{D}$  and  $\bar{\mathcal{D}} = \{\bar{e}; e \in \mathcal{D}\}$  are fundamental sets in  $\tilde{E}^{or}$  for the natural  $\Gamma$ -action.

Note that, if  $g_i(\sigma e) = \rho(\sigma)g_i(e)$ ,  $i = 1, 2$ , for every  $\sigma \in \Gamma$  and  $e \in \tilde{E}^{or}$ , then the summation

$$\sum_{e \in \mathcal{D}} \langle g_1(e), g_2(e) \rangle$$

does not depend on the choice of a fundamental set  $\mathcal{D}$ . Therefore we find

$$\begin{aligned} & \sum_{x \in \mathcal{D}_Y} \langle \sum_{e \in \mathcal{D}_x} f(t(e))(s(t(e)) - s(x))w(e), f(x)s(x) \rangle \\ &= \sum_{e \in \mathcal{D}} \langle f(t(e))(s(t(e)) - s(o(e)))w(e), f(o(e))s(o(e)) \rangle \\ &= \sum_{\bar{e} \in \mathcal{D}} \langle f(t(\bar{e}))(s(t(\bar{e})) - s(o(\bar{e})))w(\bar{e}), f(o(\bar{e}))s(o(\bar{e})) \rangle \\ &= \sum_{e \in \mathcal{D}} \langle f(o(e))(s(o(e)) - s(t(e)))w(e), f(t(e))s(t(e)) \rangle \\ &= \sum_{x \in \mathcal{D}_Y} \langle \sum_{e \in \mathcal{D}_x} f(x)(s(x) - s(t(e)))w(e), f(t(e))s(t(e)) \rangle, \end{aligned}$$

so that

$$\begin{aligned} & \sum_{x \in \mathcal{D}_Y} \sum_{e \in \mathcal{D}_x} f(t(e))f(x) \|s(t(e)) - s(x)\|_W^2 w(e) \\ &= \sum_{x \in \mathcal{D}_Y} \sum_{e \in \mathcal{D}_x} \{ \langle f(x)(s(t(e)) - s(x))w(e), f(t(e))s(t(e)) \rangle \\ & \quad - \langle f(x)(s(t(e)) - s(x))w(e), f(t(e))s(x) \rangle \} \\ &= -2 \sum_{x \in \mathcal{D}_Y} \langle \sum_{e \in \mathcal{D}_x} f(t(e))(s(t(e)) - s(x))w(e), f(x)s(x) \rangle. \end{aligned}$$

Combining this with (2.2), we deduce

$$\frac{\langle -\Delta_\rho fs, fs \rangle + \langle q(fs), fs \rangle}{\|fs\|^2} = \lambda_0(H_1) + \frac{1}{2}P,$$

where

$$P = \frac{\sum_{x \in \mathcal{D}_Y} \sum_{e \in \mathcal{D}_x} f(t(e))f(x) \|d_\rho s(e)\|_W^2 w(e)}{\sum_{x \in \mathcal{D}_Y} f(x)^2 \|s(x)\|_W^2 m(x)}.$$

There are positive constants  $k_1, k_2$  such that

$$k_1 P' \leq \inf_{fs \in L^2_\rho(V)} P \leq k_2 P',$$

where

$$P' = \frac{\sum_{x \in \mathcal{D}_Y} \sum_{e \in \mathcal{D}_x} \|d_\rho s(e)\|_W^2 w(e)}{\sum_{x \in \mathcal{D}_Y} \|s(x)\|_W^2 m(x)}.$$

Thus, it is enough to show that

$$c_1 \delta(\rho, \mathbf{1})^2 \leq \inf P' \leq c_2 \delta(\rho, \mathbf{1})^2.$$

We now let  $\mathcal{U}(\mathcal{D})$  be the set of vertices  $x \in \tilde{V}$  such that there exists  $e \in \mathcal{D}$  with  $t(e) = x$ . It follows from the definition of fundamental set that for every  $y \in \mathcal{U}(\mathcal{D})$ , there is a unique  $\sigma_y \in \Gamma$  with  $y \in \sigma_y \mathcal{D}_V$ . Consider  $B = \{\sigma_y; y \in \mathcal{U}(\mathcal{D})\} \cup A$ , another finite set of generators of  $\Gamma$ . From the definition of  $\delta_B(\rho, \mathbf{1})$ , it follows that for every  $\varepsilon > 0$ , there exists a  $v \in W$  with  $\|v\| = 1$  such that  $\|\rho(\sigma)v - v\| \leq \delta_B(\rho, \mathbf{1}) + \varepsilon$  for all  $\sigma \in B$ . For this fixed  $v$ , we define a function  $s : V \rightarrow W$  by setting  $s(x) = v$  for all  $x \in \mathcal{D}_V$  and  $s(\sigma x) = \rho(\sigma)v$  for every  $\sigma x \in \sigma \mathcal{D}_V$ . It is clear that  $s \in L^2_\rho(V)$ . Thus

$$\sum_{x \in \mathcal{D}_V} \|s(x)\|^2 m(x) = \sum_{x \in \mathcal{D}_V} m(x)$$

and

$$\sum_{x \in \mathcal{D}_V} \sum_{e \in \mathcal{D}_x} \|d_\rho s(e)\|^2 w(e) \leq \{\max_{e \in E} w(e)\} \sum_{\sigma \in B} \|\rho(\sigma)v - v\|^2 \leq C\{\delta_B(\rho, \mathbf{1}) + \varepsilon\}^2.$$

Since  $\varepsilon$  is arbitrary, we obtain

$$\inf P' \leq c_2 \delta(\rho, \mathbf{1})^2$$

for some positive constant  $c_2$ .

We next show the inequality  $c_1 \delta(\rho, \mathbf{1})^2 \leq \inf P'$  for some positive constant  $c_1$ . Since for a unit vector  $v$ ,

$$\delta(\rho, \mathbf{1})^2 \leq \sum_{\sigma \in A} \|\rho(\sigma)v - v\|^2,$$

by substituting  $v = s(x)/\|s(x)\|$ , we have

$$(2.3) \quad \delta(\rho, \mathbf{1})^2 \sum_{x \in \mathcal{D}_V} \|s(x)\|^2 m(x) \leq \sum_{x \in \mathcal{D}_V} \sum_{\sigma \in A} \|s(\sigma x) - s(x)\|^2 m(x)$$

for every  $s \in L^2_\rho(V)$ . For each  $x \in \mathcal{D}_V$  and  $\sigma \in \Gamma$ , we choose a path  $C(x, \sigma x)$  in  $X$  joining  $x$  and  $\sigma x$ . Let  $|C(x, \sigma x)| = \# \{\text{edges in the path } C(x, \sigma x)\}$  and  $K = \max_{x \in \mathcal{D}_V} \max_{\sigma \in A} |C(x, \sigma x)|$ . The inequality (2.3) and

$$\|s(\sigma x) - s(x)\|^2 \leq K \sum_{e \in C(x, \sigma x)} \|s(t(e)) - s(o(e))\|^2$$

imply

$$\delta(\rho, \mathbf{1})^2 \sum_{x \in \mathcal{D}_V} \|s(x)\|^2 m(x) \leq c(\#\mathcal{D})K^2 \sum_{x \in \mathcal{D}_V} \sum_{e \in \mathcal{D}_x} \|d_\rho s(e)\|^2 w(e),$$

where  $c = \max_{x \in V} m(x) \times (\min_{e \in E} w(e))^{-1} \times (\#A)$ . Thus the proof of the theorem is complete.

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