Ann. Inst. Statist. Math.
Vol. 56, No. 3, 497–510 (2004)

DISCRETE SEMI-STABLE DISTRIBUTIONS

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(Received December 16, 2002; revised August 19, 2003)

Abstract. The purpose of this paper is to introduce and study the concepts of discrete semi-stability and geometric semi-stability for distributions with support in $\boldsymbol{Z}_+$. We offer several properties, including characterizations, of discrete semi-stable distributions. We establish that these distributions possess the property of infinite divisibility and that their probability generating functions admit canonical representations that are analogous to those of their continuous counterparts. Properties of discrete geometric semi-stable distributions are deduced from the results obtained for discrete semi-stability. Several limit theorems are established and some examples are constructed.

Key words and phrases: Stability, geometric stability, infinite divisibility, discrete distributions, weak convergence.

1. Introduction

Lévy (1937) called a distribution on the real line semi-stable, with exponent $a \in \boldsymbol{R}$ and order $q \in \boldsymbol{R}$, if its characteristic function $f(t)$ satisfies for all $t \in \boldsymbol{R}$, $f(t) \neq 0$ and

$$\ln f(qt) = q^a \ln f(t). \tag{1.1}$$

It can be assumed that $q > 1$. Semi-stable distributions are infinitely divisible (or i.d.) and exist only for $a \in (0, 2)$. Their characteristic functions admit the canonical representation (see Lévy (1937)):

$$\ln f(t) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} (e^{itu} - 1) dN(u) & \text{if} \quad a \in (0,1), \\ \displaystyle\int_{-\infty}^{\infty} (e^{itu} - 1 - itu) dN(u) & \text{if} \quad a \in (1,2), \end{cases} \tag{1.2}$$

where

$$N(u) = \begin{cases} u^{-a} Q_1(\ln u) & \text{if} \quad u > 0, \\ |u|^{-a} Q_2(\ln |u|) & \text{if} \quad u < 0, \end{cases} \tag{1.3}$$

and Q_1 and Q_2 are periodic functions with period $\ln q$ and such that $N(u)$ is non-decreasing from $-\infty$ to 0 and from 0 to $+\infty$. In the case $a = 1$, the canonical representation becomes

$$\ln f(t) = mit + \int_0^{\infty} [(\cos tu - 1) dN_1(u) + i \sin tu dN_2(u)], \tag{1.4}$$

where $uN_1(u)$ and $uN_2(u)$ are periodic functions in $\ln u$ and m is a constant.

Limit theorems and other properties of semi-stable distributions were obtained by several authors (see for example Pillai (1971, 1985) and references therein).

Steutel and van Harn (1979) introduced the binomial thinning operator $\odot$ which they defined as follows:

$$\alpha \odot X = \sum_{i=1}^{X} X_i, \tag{1.5}$$

where $\alpha \in (0,1)$, X is a $\boldsymbol{Z}_+$-valued random variable (rv), here $\boldsymbol{Z}_+ := \{0,1,2,\dots\}$, and $\{X_i\}$ is a sequence of iid Bernoulli(α) rv's independent of X. The authors used the operator $\odot$ to introduce the concepts of discrete stability and discrete self-decomposability. Discrete stable distributions (with exponent $\gamma \in (0,1]$) are characterized by the following canonical representation of their probability generating functions (or pgf's):

$$P(z) = \exp\{-c(1-z)^\gamma\} \qquad z \in [0,1], \tag{1.6}$$

for some $c > 0$. Aly and Bouzar (2000) used the operator $\odot$ to offer the discrete analogue of the concept of geometric stability of Klebanov *et al.* (1984). Such distributions are characterized by pgf's of the form

$$P(z) = (1 + c(1-z)^\gamma)^{-1} \quad \text{for some} \quad c > 0. \tag{1.7}$$

The purpose of this paper is to introduce and study the concepts of discrete semi-stability and geometric semi-stability for distributions with support in $\boldsymbol{Z}_+$. In Section 2 we give a definition of discrete semi-stability that is analogous to (1.1). We offer several properties, including characterizations, of discrete semi-stable distributions. Notably, we establish that these distributions possess the property of infinite divisibility and that their pgf's admit canonical representations that are similar to those of their continuous counterparts (described in (1.2)–(1.4)). In Section 3 we study the concept of discrete geometric semi-stability. We deduce properties of discrete geometric semi-stable distributions from the results obtained for discrete semi-stability. We also establish that discrete geometric semi-stable distributions coincide with Jayakumar's (1995) semi-γ-geometric distributions. Several limit theorems are given in Section 4 and examples are developed in Section 5.

2. Discrete semi-stability

DEFINITION 2.1. A nondegenerate distribution on $\boldsymbol{Z}_+$ is said to be discrete semi-stable with exponent $\gamma > 0$ and order $\alpha \in (0,1)$ if its pgf $P(z)$ satisfies for all $|z| \le 1$, $P(z) \neq 0$ and

$$\ln P(1 - \alpha + \alpha z) = \alpha^\gamma \ln P(z). \tag{2.1}$$

We remark from the definition that a distribution on $\boldsymbol{Z}_+$ is discrete stable with exponent $\gamma > 0$ if and only if it is discrete semi-stable with exponent γ and of all orders $\alpha \in (0,1)$.

The next lemma gathers some basic properties of discrete semi-stability.

LEMMA 2.1. (i) *If $P(z)$ is the pgf of a discrete semi-stable distribution with exponent $\gamma > 0$ and order $\alpha \in (0,1)$, then for any $n \geq 0$ and $|z| \leq 1$,*

$$\ln P(1 - \alpha^n + \alpha^n z) = \alpha^{n\gamma} \ln P(z). \tag{2.2}$$

(ii) *If there exists a discrete semi-stable distribution with exponent $\gamma > 0$ and order $\alpha \in (0,1)$, then necessarily $0 < \gamma \leq 1$. In addition, if this distribution has finite mean, then $\gamma = 1$.*

PROOF. (i) follows directly from (2.1). To prove (ii), assume that $P(z)$ satisfies (2.1) for $\gamma > 0$ and $\alpha \in (0,1)$. Then by differentiation,

$$\frac{P'(z)}{P'(1-\alpha+\alpha z)} = \alpha^{1-\gamma}[P(z)]^{1-\alpha^\gamma}.$$

Since $P'(z)$ is increasing over the interval $[0,1)$ and $1-\alpha+\alpha z > z$ for $z \in [0,1)$, we have

$$\lim_{z \uparrow 1} \frac{P'(z)}{P'(1-\alpha+\alpha z)} = \alpha^{1-\gamma} \leq 1,$$

which implies that $\gamma \leq 1$. The additional assumption of finite mean is equivalent to $P'(1) < \infty$ (see for example Feller (1968)). Using a first-order Taylor series expansion of $\ln P(x)$ around $x = 1$, we obtain for $z \in [0,1]$,

$$\ln P(1 - \alpha^n(1-z)) = -\alpha^n(1-z)P'(1) + o(\alpha^n) \qquad (\text{as} \quad n \to \infty),$$

which combined with (2.2) yields

$$\ln P(z) = -\alpha^{n(1-\gamma)} \left[(1-z)P'(1) + \frac{o(\alpha^n)}{\alpha^n}\right]. \tag{2.3}$$

If $\gamma < 1$, then letting $n \to \infty$ in (2.3) leads to $\ln P(z) = 0$ for all $z \in [0,1]$ and hence, by analytic continuation, for all $|z| \leq 1$. This implies that the distribution is degenerate (total mass at 0) which is a contradiction. □

The important property of infinite divisibility is established next. We recall a lemma first (see Feller (1968)).

LEMMA 2.2. *A distribution on $\boldsymbol{Z}_+$ is i.d. if and only if its pgf $P(z)$ has the form*

$$P(z) = \exp\{-\lambda(1 - Q(z))\} \qquad (|z| \leq 1),$$

where $\lambda > 0$ and $Q(z)$ is a pgf.

PROPOSITION 2.1. *A discrete semi-stable distribution is i.d.*

PROOF. Let $P(z)$ be the pgf of a discrete semi-stable distribution with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$. For any $n \geq 0$ and $|z| \leq 1$, let $P_n(z) = \exp\{-\alpha^{-n\gamma}(1 - [P(z)]^{\alpha^{n\gamma}})\}$. Since by Lemma 2.1 (i) $[P(z)]^{\alpha^{n\gamma}} = P(1 - \alpha^n + \alpha^n z)$ is a pgf, it follows by Lemma 2.2 that for any $n \geq 0$, $P_n(z)$ is the pgf of an i.d. distribution on $\boldsymbol{Z}_+$.

Moreover, we have $\lim_{n\to\infty} P_n(z) = P(z)$ for any $|z| \leq 1$. Hence, any discrete semi-stable distribution is the weak limit of a sequence of i.d. distributions and is therefore i.d. (see Feller (1971)). □

COROLLARY 2.1. *A distribution on $\mathbf{Z}_+$ is discrete semi-stable with exponent $\gamma = 1$ if and only if it is Poisson.*

PROOF. The 'if' part is trivial. To prove the 'only if' part, let $P(z)$ be the pgf of a semi-stable distribution with exponent $\gamma = 1$ and order $\alpha \in (0,1)$. By Proposition 2.1 and Lemma 2.2, $P(z)$ admits the representation $\ln P(z) = -\lambda(1 - Q(z))$ for some $\lambda > 0$ and some pgf $Q(z)$. It follows by (2.1) that $Q(1 - \alpha + \alpha z) = 1 - \alpha + \alpha Q(z)$, and therefore, by differentiation, $Q'(1 - \alpha + \alpha z) = Q'(z)$ for all $|z| < 1$. Using the power series expansion $Q(z) = \sum_{n=0}^{\infty} q_n z^n$ for some probability mass function $(q_n, n \geq 0)$ and by letting $z = 0$, we arrive at $\sum_{n=1}^{\infty}(n+1)q_{n+1}(1-\alpha)^n = 0$, which implies that $q_{n+1} = 0$ for all $n \geq 1$. Therefore, $P(z) = \exp\{-\lambda_1(1 - z)\}$ for some $\lambda_1 > 0$. □

Using a special combination of γ and α, one can obtain a representation result for $\mathbf{Z}_+$-valued semi-stable rv's.

PROPOSITION 2.2. *Let X be a $\mathbf{Z}_+$-valued rv with a semi-stable distribution with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$. Assume there exist two integers $n \geq 1$ and $N \geq 1$ such that $\alpha^{n\gamma} = \frac{1}{N}$. Then X satisfies the equation*

$$X \stackrel{d}{=} \frac{1}{N^{1/\gamma}} \odot (X_1 + \cdots + X_N), \tag{2.4}$$

where $X_1, \ldots, X_N$ are iid with $X_1 \stackrel{d}{=} X$. Conversely, if a $\mathbf{Z}_+$-valued rv X admits the representation (2.4) for some $\gamma \in (0,1]$, $N \geq 1$, and $X_1, \ldots, X_N$ iid with $X_1 \stackrel{d}{=} X$, then the distribution of X is discrete semi-stable with exponent γ and order $\alpha = N^{-1/\gamma}$.

PROOF. Denote by $P(z)$ the pgf of X and by $P_1(z)$ that of $\frac{1}{N^{1/\gamma}} \odot (X_1 + \cdots + X_N)$. Then, by definition of the $\odot$ operation (see (1.5)), we have

$$P_1(z) = [P(1 - N^{-1/\gamma} + N^{-1/\gamma} z)]^N \qquad (|z| \leq 1). \tag{2.5}$$

Since $\alpha^{n\gamma} = \frac{1}{N}$, it follows that $P_1(z) = [P(1 - \alpha^n + \alpha^n z)]^{\alpha^{-n\gamma}}$. Therefore by (2.2), $P_1(z) = P(z)$. To prove the converse, we use (2.5) with $P_1(z) = P(z)$. This implies that $P(1 - N^{-1/\gamma} + N^{-1/\gamma} z) = [P(z)]^{1/N}$, or $P(1 - \alpha + \alpha z) = [P(z)]^{\alpha^\gamma}$, where $\alpha = N^{-1/\gamma}$. □

Just like their continuous counterparts (see (1.2)–(1.4)), discrete semi-stable distributions can be characterized by canonical representations of their pgf's.

PROPOSITION 2.3. *A distribution on $\mathbf{Z}_+$ is discrete semi-stable with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$ if and only if its pgf $P(z)$ admits the form*

$$P(z) = \exp\{-(1 - z)^\gamma h(z)\} \qquad (0 \leq z < 1), \tag{2.6}$$

where $h(\cdot)$, defined over $[0,1)$, satisfies $h(1 - \alpha + \alpha z) = h(z)$ for any $z \in [0,1)$, or, equivalently,

$$P(z) = \exp\{-(1 - z)^\gamma g(|\ln(1 - z)|)\} \qquad (0 \leq z < 1), \tag{2.7}$$

where $g(\cdot)$, defined over $[0, \infty)$, is a periodic function with period $-\ln\alpha$.

PROOF. It is easy to verify that if (2.6) holds, then $P(z)$ satisfies (2.1) for all $z \in [0,1)$, and hence, by analytic continuation, for all $|z| \le 1$. Conversely, assume $P(z)$ satisfies (2.1). Letting $h(z) = -(1-z)^{-\gamma} \ln P(z)$ for $z \in [0,1)$, we have by (2.1),

$$h(1-\alpha+\alpha z) = -(\alpha(1-z))^{-\gamma} \ln P(1-\alpha+\alpha z) = -\alpha^{-\gamma}(1-z)^{-\gamma}\alpha^{\gamma} \ln P(z) = h(z),$$

which implies (2.6). We conclude by showing that (2.6) and (2.7) are equivalent. If (2.6) holds, define $g(\tau) = h(1-e^{-\tau})$ for $\tau \ge 0$. Then $g(|\ln(1-z)|) = h(z)$ for any $z \in [0,1)$. Moreover, $g(\tau - \ln\alpha) = h(1-\alpha e^{-\tau}) = h(1-\alpha+\alpha(1-e^{-\tau})) = h(1-e^{-\tau}) = g(\tau)$, which implies that $g(\cdot)$ is periodic with period $-\ln\alpha$ and thus (2.7) is proven. If (2.7) holds, define $h(z) = g(|\ln(1-z)|)$ for $z \in [0,1)$. Then $h(1-\alpha+\alpha z) = g(-\ln\alpha + |\ln(1-z)|) = g(|\ln(1-z)|) = h(z)$, implying (2.6). □

Using the fact that discrete semi-stable distributions are i.d., one can obtain a modified canonical representation of their pgf's. We recall that a function $P(z)$ on $[0,1]$ is the pgf of an i.d. discrete distribution if and only if it admits the representation (see Steutel (1970)),

$$\ln P(z) = -\int_z^1 R(x)dx, \tag{2.8}$$

where $R(x) = \sum_{n=0}^{\infty} r_n x^n$, with $r_n \ge 0$ and, necessarily, $\sum_{n=0}^{\infty} r_n (n+1)^{-1} < \infty$.

PROPOSITION 2.4. *An i.d. distribution on $\boldsymbol{Z}_+$ with pgf $P(z)$ described by (2.8) is discrete semi-stable with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$ if and only if*

$$R(z) = (1-z)^{\gamma-1} r(z) \qquad (0 \le z < 1), \tag{2.9}$$

where $r(\cdot)$, defined over $[0,1)$, satisfies $r(1-\alpha+\alpha z) = r(z)$ for any $z \in [0,1)$, or, equivalently,

$$R(z) = (1-z)^{\gamma-1} r_1(|\ln(1-z)|) \qquad (0 \le z < 1), \tag{2.10}$$

where $r_1(\cdot)$, defined over $[0,\infty)$, is periodic with period $-\ln\alpha$.

PROOF. By (2.1) and (2.8), $P(z)$ is the pgf of discrete semi-stable distribution with exponent γ and order $\alpha \in (0,1)$ if and only if

$$\int_{1-\alpha+\alpha z}^1 R(x)dx = \alpha^{\gamma} \int_z^1 R(x)dx \qquad (0 \le z \le 1)$$

which, by differentiation, is equivalent to

$$R(1-\alpha+\alpha z) = \alpha^{\gamma-1} R(z) \qquad (0 \le z \le 1). \tag{2.11}$$

It is easy to see that (2.9) implies (2.11). Conversely, assume that (2.11) holds. Letting $r(z) = (1-z)^{1-\gamma} R(z)$ for $z \in [0,1)$, we have

$$r(1-\alpha+\alpha z) = (\alpha(1-z))^{1-\gamma} R(1-\alpha+\alpha z) = \alpha^{1-\gamma}(1-z)^{1-\gamma}\alpha^{\gamma-1} R(z) = r(z),$$

which implies (2.9). The proof that (2.9) and (2.10) are equivalent is identical to the one used to establish that (2.6) and (2.7) are equivalent. The details are omitted. □

New characterizations of discrete stability are obtained as a corollary to Propositions 2.3 and 2.4.

COROLLARY 2.2. *Let $P(z)$ be the pgf of a discrete semi-stable distribution on $\boldsymbol{Z}_+$ with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$. The following assertions are equivalent.*

(i) *This distribution is discrete stable*;

(ii) $\lim_{z\uparrow 1} h(z) < \infty$, *where $h(z)$ is as in* (2.6);

(iii) $\lim_{z\uparrow 1} r(z) < \infty$, *where $r(z)$ is as in* (2.9).

PROOF. (i)⇔(ii): The 'only if' part is trivial since in this case (by (1.6)) $h(z)$ is constant. Conversely, we have by (2.6) that $h(1-\alpha^n+\alpha^n z) = h(z)$ for every $n \geq 1$ and $z \in [0,1)$. Letting $n \to \infty$, we conclude that $h(z) = \lim_{x\uparrow 1} h(x)$ for any $z \in [0,1)$. This implies that $h(z)$ is constant and hence $P(z)$ admits the representation (1.6). (i)⇔(iii) is proven along the same lines. □

We recall (see Steutel and van Harn (1979)) that a distribution on $\boldsymbol{Z}_+$ with pgf $P(z)$ is (discrete) self-decomposable if for any $\beta \in (0,1)$ there exists a pgf $P_\beta(z)$ such that for every $|z| \leq 1$,

$$P(z) = P(1-\beta+\beta z)P_\beta(z). \tag{2.12}$$

Unlike discrete stable distributions (see Steutel and van Harn (1979)), discrete semi-stable distributions are not necessarily self-decomposable (a counter-example will be provided in Section 5). The next result shows how one can construct a distribution on $\boldsymbol{Z}_+$ that is semi-stable and self-decomposable. We will need the following canonical representation of the pgf of a discrete self-decomposable distribution (see Steutel and van Harn (1979)):

$$\ln P(z) = \int_z^1 \frac{\ln Q(x)}{1-x}dx \tag{2.13}$$

where $Q(z)$ is the pgf of an i.d. distribution on $\boldsymbol{Z}_+$.

COROLLARY 2.3. *A discrete self-decomposable distribution with pgf $P(z)$ is discrete semi-stable with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$ if and only if $Q(z)$ of* (2.13) *is the pgf of a discrete semi-stable distribution with exponent γ and order α. In that case, for any $\beta \in (0,1)$, $P_\beta(z)$ of* (2.12) *is itself the pgf of a discrete semi-stable distribution* (*also with exponent γ and order α*).

PROOF. For the 'if' part, we note that by Proposition 2.3, $-\ln Q(z) = (1-z)^\gamma h(z)$ with $h(1-\alpha+\alpha z) = h(z)$. Since discrete self-decomposable distributions are i.d., this implies (see (2.8)) that $R(z) = \frac{-\ln Q(z)}{1-z} = (1-z)^{\gamma-1}h(z)$. The conclusion follows from Proposition 2.4. Conversely, we have by (2.8), (2.9), and (2.13), $\ln Q(z) = -(1-z)^\gamma r(z)$, with $r(1-\alpha+\alpha z) = r(z)$, which implies $\ln Q(1-\alpha+\alpha z) = \alpha^\gamma \ln Q(z)$. The last part of the corollary follows by noting that for any $\beta \in (0,1)$, $\ln P_\beta(1-\alpha+\alpha z) = \ln P(1-\alpha+\alpha z) - \ln P(1-\alpha+\alpha(1-\beta+\beta z)) = \alpha^\gamma \ln P(z) - \alpha^\gamma \ln P(1-\beta+\beta z) = \alpha^\gamma \ln P_\beta(z)$. □

Remark. van Harn *et al.* (1982) (see also van Harn and Steutel (1993)) used a continuous semi-group of pgf's $(F_t, t \geq 0)$ and a generalized multiplication they denoted by $\odot_F$ to introduce F-stability for discrete distributions. The semi-group $F_t(z) = 1 - e^{-t} + e^{-t}z$ leads to the discrete stability characterized by (1.7). F-semi-stability can be defined using the same approach. All the results in this section can be shown to have a version for F-semi-stable distribtutions. The details are omitted.

3. Discrete geometric semi-stability

DEFINITION 3.1. A nondegenerate distribution on $\boldsymbol{Z}_+$ is said to be discrete geometric semi-stable with exponent $\gamma > 0$ and order $\alpha \in (0,1)$ if its pgf $P(z)$ satisfies

$$P(1 - \alpha + \alpha z) = \frac{P(z)}{\alpha^\gamma + (1 - \alpha^\gamma)P(z)} \qquad (|z| \leq 1). \tag{3.1}$$

As in the case of discrete semi-stability, one can deduce from Definition 3.1 that a distribution on $\boldsymbol{Z}_+$ is discrete geometric stable with exponent $\gamma > 0$ if and only if it is discrete geometric semi-stable with exponent γ and of all orders $\alpha \in (0,1)$.

Jayakumar (1995) called a distribution on $\boldsymbol{Z}_+$ a semi-γ-geometric distribution for some $0 < \gamma \leq 1$ if it has a pgf of the form

$$P(z) = (1 + \psi(1 - z))^{-1}, \tag{3.2}$$

where $\psi(z)$ is nondecreasing over $[0,1]$, $\psi(0) = 1$, and there exists $\alpha \in (0,1)$ such that $\psi(\alpha z) = \alpha^\gamma \psi(z)$ for any $|z| \leq 1$. The author showed that a semi-γ-geometric distribution arises as the unique marginal distribution of a $\boldsymbol{Z}_+$-valued first-order stationary autoregressive process with a specific innovation sequence.

It is easily seen that a semi-γ-geometric distribution is discrete geometric semi-stable with exponent γ (and some order $\alpha \in (0,1)$). We will show later in this section that the converse also holds.

The representation of a discrete geometric semi-stable distribution in terms of $\boldsymbol{Z}_+$-valued rv's follows immediately from the definition and is stated without proof.

PROPOSITION 3.1. *A $\boldsymbol{Z}_+$-valued rv X has a discrete geometric semi-stable distribution with exponent $\gamma > 0$ and order $\alpha \in (0,1)$ if and only if it admits the following representation:*

$$X \stackrel{d}{=} \alpha \odot \sum_{i=1}^{N} X_i, \tag{3.3}$$

where $\{X_i\}$ is a sequence of iid rv's, $X_i \stackrel{d}{=} X$, N has the geometric distribution with parameter α^γ, and $\{X_i\}$ and N are independent.

Next, we establish the property of geometric infinite divisiblity.

PROPOSITION 3.2. *Any discrete geometric semi-stable distribution is geometric i.d., and hence i.d.*

PROOF. Let $P(z)$ be the pgf of a discrete semi-stable distribution with exponent $\gamma > 0$ and order $\alpha \in (0,1)$. Using (3.1) and induction, we have for every $n \geq 1$

$$P(1-\alpha^n+\alpha^n z) = \frac{P(z)}{\alpha^{n\gamma}+(1-\alpha^{n\gamma})P(z)}, \tag{3.4}$$

from which it follows $P(z) = \lim_{n\to\infty} P_n(z)$, where $P_n(z) = (1+\alpha^{-n\gamma}(1-P(1-\alpha^n+\alpha^n z)))^{-1}$. By Aly and Bouzar (2000), $P_n(z)$ is the pgf of a geometric i.d. distribution for every $n \geq 1$. Therefore, any geometric semi-stable distribution is the weak limit of a sequence of geometric i.d. distributions and, hence, must itself be geometric i.d. (see Klebanov *et al.* (1984)). The second part follows by Aly and Bouzar (2000). □

The next result establishes the connection between discrete semi-stability and discrete geometric semi-stability.

PROPOSITION 3.3. *A distribution on Z_+ with pgf $P(z)$ is discrete geometric semi-stable with exponent $\gamma > 0$ and order $\alpha \in (0,1)$ if and only if*

$$H(z) = \exp\left\{1 - \frac{1}{P(z)}\right\} \tag{3.5}$$

is the pgf of a discrete semi-stable distribution with exponent $\gamma > 0$ and order $\alpha \in (0,1)$.

PROOF. If $P(z)$ satisfies (3.1), then we easily deduce that $\ln H(1-\alpha+\alpha z) = \alpha^\gamma \ln H(z)$. Since by Proposition 3.2 $P(z)$ is geometric i.d., it follows by Aly and Bouzar (2000) that $H(z)$ is a (i.d.) pgf which proves the 'only if' part. Conversely, if $H(z)$ is the pgf of a discrete semi-stable distribution with exponent $\gamma > 0$ and order $\alpha \in (0,1)$, then $P(z)$ can be easily shown to satisfy (3.1). □

The following three corollaries are a direct consequence of Proposition 3.3, Lemma 2.1, and Proposition 2.3.

COROLLARY 3.1. *If there exists a discrete geometric semi-stable distribution with exponent $\gamma > 0$ and order $\alpha \in (0,1)$, then necessarily $0 < \gamma \leq 1$. In addition, if this distribution has finite mean, then $\gamma = 1$.*

COROLLARY 3.2. *A distribution on Z_+ with pgf $P(z)$ is discrete geometric semi-stable with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$ if and only if*

$$P(z) = (1 - \ln H(z))^{-1} \tag{3.6}$$

where $H(z)$ is the pgf of a discrete semi-stable distribution with the same exponent and order.

COROLLARY 3.3. *A distribution on Z_+ is discrete geometric semi-stable with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$ if and only if its pgf $P(z)$ admits the form*

$$P(z) = (1+(1-z)^\gamma h(z))^{-1} \qquad (0 \leq z < 1), \tag{3.7}$$

where $h(\cdot)$, defined over $[0,1)$, satisfies $h(1-\alpha+\alpha z) = h(z)$ for any $z \in [0,1)$, or, equivalently,

$$P(z) = (1 + (1-z)^{\gamma} g(|\ln(1-z)|))^{-1} \qquad (0 \le z < 1), \tag{3.8}$$

where $g(\cdot)$, defined over $[0,\infty)$, is a periodic function with period $-\ln\alpha$.

Finally, if a pgf $P(z)$ admits the representation (3.6), then it can be rewritten in the form (3.2) where $\psi(z) = -\ln H(1-z)$ is nondecreasing on $[0,1]$, $\psi(0) = 1$, and $\psi(\alpha z) = \alpha^{\gamma}\psi(z)$. We have thus established the following result.

COROLLARY 3.4. *Let $0 < \gamma \le 1$. A distribution on $\boldsymbol{Z}_+$ is semi-γ-geometric if and only if it is discrete geometric semi-stable (with exponent γ).*

Remarks. 1) The characterization ((i)$\Leftrightarrow$(ii)) of discrete stability in Corollary 2.2 extends almost verbatim to discrete geometic stability. The statements and details are omitted.

2) Proposition 3.1 is equivalent to Theorem 2.2 in Jayakumar (1995). Incidentally, Definition 2.1 in Jayakumar (1995) should read as follows: two distributions on $\boldsymbol{Z}_+$ with respective pgf's $P_1(z)$ and $P_2(z)$ are said to be of the same type if for some $\alpha \in (0,1)$, $P_1(z) = P_2(1-\alpha+\alpha z)$ for any $|z| \le 1$.

3) One can use the semi-group approach of van Harn *et al.* (1982) (see the remark at the end of Section 2) to define F-geometric-semi-stable distributions (see also Bouzar (1999)).

4) A more general notion of stability, based on random summations and called $\mathcal{N}$-stability, was studied by several authors (see Gnedenko and Korolev (1996), Subsection 4.6, for details). Following Aly and Bouzar (2000), one can define discrete $\mathcal{N}$-semi-stability that will contain discrete semi-stability and geometric semi-stability as special cases.

4. Limit theorems

Pillai (1971) showed that semi-stable distributions can arise as limiting distributions. We proceed to establish the discrete versions of Pillai's results.

PROPOSITION 4.1. *Let $(X_n, n \ge 1)$ be a sequence of $\boldsymbol{Z}_+$-valued, iid rv's with a common discrete semi-stable distribution with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$. For $n \ge 1$, let $k_n = [\alpha^{-n\gamma}]$ (where $[x]$ denotes the largest integer smaller than or equal to x) and let*

$$\zeta_n = \alpha^n \odot \sum_{i=1}^{k_n} X_i. \tag{4.1}$$

Then ζ_n converges weakly to a discrete semi-stable distribution with exponent γ and order α.

PROOF. Denote by $P(z)$ the common pgf of the X_i's and by $P_n(z)$ the pgf of ζ_n $(n \ge 1)$. By (4.1)

$$P_n(z) = [P(1-\alpha^n+\alpha^n z)]^{k_n}. \tag{4.2}$$

By assumption and by (2.2), $P(z) = [P(1-\alpha^n+\alpha^n z)]^{\alpha^{-n\gamma}}$. For every $n \geq 1$, $\alpha^{-n\gamma} = k_n + \theta_n$ for some $0 \leq \theta_n < 1$. Therefore,

$$|P_n(z) - P(z)| = |P(1-\alpha^n+\alpha^n z)|^{k_n} |1 - P^{\theta_n}(1-\alpha^n+\alpha^n z)| \leq |1 - P^{\theta_n}(1-\alpha^n+\alpha^n z)|.$$

Since $0 < \alpha < 1$ and $0 \leq \theta_n < 1$, it follows that $\lim_{n\to\infty} |1 - P^{\theta_n}(1-\alpha^n+\alpha^n z)| = 0$, which in turn implies that $\lim_{n\to\infty} P_n(z) = P(z)$. □

The converse holds in a slightly more general setting.

PROPOSITION 4.2. *Let $(X_n, n \geq 1)$ be a sequence of $\boldsymbol{Z}_+$-valued, iid rv's. Assume that for some $\gamma \in (0,1]$ and $\alpha \in (0,1)$, the sequence $(\zeta_n, n \geq 1)$ defined in* (4.1) (*again with $k_n = [\alpha^{-n\gamma}]$*) *converges weakly to a $\boldsymbol{Z}_+$-valued rv, then the limiting distribution is necessarily discrete semi-stable with exponent γ and order α.*

PROOF. Let $P(z)$ be the common pgf of the X_i's and let $P_n(z)$ be the pgf of ζ_n for $n \geq 1$. By assumption and by (4.2), there exists a pgf $Q(z)$ such that

$$Q(z) = \lim_{n\to\infty} [P(1-\alpha^n+\alpha^n z)]^{k_n}. \tag{4.3}$$

For every $n \geq 1$, $\alpha^{-n\gamma} = k_n + \theta_n$ for some $0 \leq \theta_n < 1$. Since $0 < \alpha < 1$ and $|P(z)| \leq 1$ for any $|z| \leq 1$, we have

$$\lim_{n\to\infty} |[P(1-\alpha^n+\alpha^n z)]^{k_n} - [P(1-\alpha^n+\alpha^n z)]^{\alpha^{-n\gamma}}| \leq \lim_{n\to\infty} |1 - [P(1-\alpha^n+\alpha^n z)]^{\theta_n}| = 0,$$

which implies

$$Q(z) = \lim_{n\to\infty} [P(1-\alpha^n+\alpha^n z)]^{\alpha^{-n\gamma}}. \tag{4.4}$$

Moreover, for every $n \geq 1$,

$$\begin{aligned}[P(1-\alpha^{n+1}+\alpha^{n+1}z)]^{k_{n+1}-k_n} &= [P(1-\alpha^{n+1}+\alpha^{n+1}z)]^{\alpha^{-n\gamma}(\alpha^{-\gamma}-1)} \\ &\quad \times [P(1-\alpha^{n+1}+\alpha^{n+1}z)]^{\theta_{n+1}-\theta_n}.\end{aligned} \tag{4.5}$$

Now, by (4.4),

$$\lim_{n\to\infty} [P(1-\alpha^{n+1}+\alpha^{n+1}z)]^{\alpha^{-n\gamma}(\alpha^{-\gamma}-1)} = [Q(1-\alpha+\alpha z)]^{\alpha^{-\gamma}-1}, \tag{4.6}$$

and (again, since $0 < \alpha < 1$ and $0 \leq \theta_n < 1$ for every $n \geq 1$),

$$\lim_{n\to\infty} [P(1-\alpha^{n+1}+\alpha^{n+1}z)]^{\theta_{n+1}-\theta_n} = 1. \tag{4.7}$$

By (4.2), we have for every $n \geq 1$,

$$P_{n+1}(z) = [P_n(1-\alpha+\alpha z)]^{k_n} [P(1-\alpha^{n+1}+\alpha^{n+1}z)]^{k_{n+1}-k_n}. \tag{4.8}$$

Therefore, by combining (4.3), (4.5)–(4.8), we conclude $Q(z) = [Q(1-\alpha+\alpha z)]^{\alpha^{-\gamma}}$. □

The following result is the discrete analogue of Pillai's result (1985) obtained for semi-Laplace distributions.

PROPOSITION 4.3. *Let $(X_n, n \geq 1)$ be a sequence of $\boldsymbol{Z}_+$-valued, iid rv's with a common discrete geometric semi-stable distribution with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$ and let $(\zeta_n, n \geq 1)$ be the sequence defined in (4.1) (with $k_n = [\alpha^{-n\gamma}]$). Then ζ_n converges weakly to a discrete semi-stable distribution with exponent γ and order α.*

PROOF. By Corollary 3.2 and by (2.2), the pgf $P_n(z)$ of ζ_n $(n \geq 1)$ satisfies

$$\ln P_n(z) = -k_n \ln(1 - \ln H(1 - \alpha^n + \alpha^n z)) = -k_n \ln(1 - \alpha^{n\gamma} \ln H(z)), \tag{4.9}$$

where $H(z)$ is the pgf of some discrete semi-stable distribution. Since for every $n \geq 1$, $\alpha^{-n\gamma} = k_n + \theta_n$ for some $0 \leq \theta_n < 1$,

$$\lim_{n\to\infty} \ln P_n(z) = \ln H(z) \lim_{n\to\infty} \left[\frac{\ln(1 - \alpha^{n\gamma} \ln H(z))}{-\alpha^{n\gamma} \ln H(z)} \alpha^{n\gamma} k_n\right] = \ln H(z),$$

or, $\lim_{n\to\infty} P_n(z) = H(z)$. □

5. Examples

We start out by giving an example of a semi-stable distribution on $\boldsymbol{R}_+$. Let $\gamma, \alpha \in (0,1)$ and $A > 0$. Let $m = -\frac{2\pi}{\gamma \ln \alpha}$ and define

$$f(x) = x^{-1/\gamma}(1 - A\cos(m \ln x)) \qquad x > 0. \tag{5.1}$$

PROPOSITION 5.1. *If $0 < A \leq (1 + m\gamma)^{-1}$, then*

$$\phi(\tau) = \exp\left\{-\int_0^\infty (1 - e^{-\tau f(x)}) dx\right\} \qquad \tau \geq 0, \tag{5.2}$$

is the Laplace-Stieljes transform (LST) of a semi-stable distribution with exponent γ and order α.

PROOF. First, we note that f is continuous and nonnegative on $(0,\infty)$ with $\int_1^\infty f(x)dx < \infty$. Moreover, straightforward calculations and the assumption $0 < A \leq (1 + m\gamma)^{-1}$ imply $f'(x) < 0$ for every $x > 0$. Consequently, f is strictly decreasing and its inverse f^{-1} satisfies $\int_0^1 f^{-1}(x)dx < \infty$. Therefore, by Theorem 4.5 in Vervaat (1979), $\phi(\tau)$ of (5.2) is the LST of an i.d. distribution on $\boldsymbol{R}_+$ (in this case the sequence of rv's $(C_k, k \geq 1)$ in Vervaat's theorem is taken to be $C_k \equiv 1$ for every $k \geq 1$). Using a suitable change of variable, it can be shown that

$$\ln \phi(\alpha\tau) = \alpha^\gamma \ln \phi(\tau) \qquad (\tau \geq 0), \tag{5.3}$$

which completes the proof. □

We use Poisson mixtures and Proposition 5.1 to construct an example of a discrete semi-stable distribution. We recall that if $N_\lambda(\cdot)$ is a Poisson process with intensity $\lambda > 0$ and X is an $\boldsymbol{R}_+$-valued rv independent of $N_\lambda(\cdot)$, then the $\boldsymbol{Z}_+$-valued rv $N_\lambda(X)$ is called a Poisson mixture with mixing rv X. Its pgf $P_{\lambda,X}(z)$ is given by

$$P_{\lambda,X}(z) = \phi_X(\lambda(1-z)) \qquad (|z| \leq 1), \tag{5.4}$$

where ϕ_X is the LST of X.

PROPOSITION 5.2. *If an $\boldsymbol{R}_+$-valued rv X has a semi-stable distribution, then for any $\lambda > 0$, the corresponding Poisson mixture $N_\lambda(X)$ has a discrete semi-stable distribution with the same exponent and the same order.*

PROOF. The LST ϕ_X of X satisfies (5.3) for some γ and α in $(0,1)$. Then by (5.3) and (5.4), the pgf of $N_\lambda(X)$ satisfies

$$\begin{aligned}\ln P_{\lambda,X}(1-\alpha+\alpha z) &= \ln \phi_X(\lambda\alpha(1-z))\\ &= \alpha^\gamma \ln \phi_X(\lambda(1-z)) = \alpha^\gamma \ln P_{\lambda,X}(z) \qquad (z \in (0,1)).\end{aligned}$$

Hence, $N_\lambda(X)$ has a discrete semi-stable distribution with exponent γ and order α. □

The next two results follow straightforwardly from Propositions 5.1 and 5.2.

COROLLARY 5.1. *Let γ and α be in $(0,1)$, $0 < A \le (1+m\gamma)^{-1}$, and f be as in* (5.1). *Then for any $\lambda > 0$,*

$$P_\lambda(z) = \exp\left\{-\int_0^\infty (1-e^{-\lambda(1-z)f(x)})dx\right\} \qquad (0 \le z \le 1) \tag{5.5}$$

is the pgf of a discrete semi-stable distribution with exponent γ and order α.

COROLLARY 5.2. *Let γ and α be in $(0,1)$, $0 < A \le (1+m\gamma)^{-1}$, and f be as in* (5.1). *Then for any $\lambda > 0$,*

$$G_\lambda(z) = \left(1+\int_0^\infty (1-e^{-\lambda(1-z)f(x)})dx\right)^{-1} \qquad (0 \le z \le 1) \tag{5.6}$$

is the pgf of a discrete geometric semi-stable distribution with exponent γ and order α.

We conclude with two counter-examples.

First we prove the existence of a discrete semi-stable distribution that is not discrete self-decomposable (see (2.12)). By Vervaat (1979), the semi-stable distribution on $\boldsymbol{R}$ with LST $\phi(\tau)$ given by (5.2) is self-decomposable if and only if $f(x)$ of (5.1) is log-convex. For $\gamma = 1/2$ and $\alpha = e^{-2\pi}$, $f(x) = x^{-2}(1-\frac{1}{2}\cos(2\ln x))$ is not log-convex, which implies that $\phi(\tau)$ is not self-decomposable. Therefore, by Theorem 5.2 in Steutel and van Harn (1993), there exists $\lambda > 0$ such that the discrete semi-stable distribution with pgf given by (5.5) ($\gamma = 1/2$ and $\alpha = e^{-2\pi}$) is not discrete self-decomposable.

The next example shows that discrete geometric semi-stable distributions are not necessarily discrete self-decomposable. This disproves Theorem 2.1 in Jayakumar (1995).

Consider the pgf $G(z)$ described by (5.6) with $\lambda = 10^4$, $\gamma = 1/2$, and $\alpha = e^{-2\pi}$ (and $f(x)$ as above). We show that for $\beta = 0.9$, $H(z) = G(z)/G(1-\beta+\beta z)$ is not a pgf. Straightforward calculations lead to

$$H'(0) = \lambda\frac{I_3(1+I_2) - I_4(1+I_1)}{(1+I_1)^2},$$

where $I_1 = \int_0^\infty (1 - e^{-\lambda f(x)})dx$, $I_2 = \int_0^\infty (1 - e^{-\beta\lambda f(x)})dx$, $I_3 = \int_0^\infty f(x)e^{-\lambda f(x)}dx$, and $I_4 = \int_0^\infty \beta f(x)e^{-\beta\lambda f(x)}dx$. Using the computer algebra system MATHEMATICA (command NIntegrate), we obtain the numerical approximations (with 14-digit accuracy)

$$I_1 \approx 184.156, \qquad I_2 \approx 175.603, \qquad I_3 \approx 0.00826099, \qquad I_4 \approx 0.00797862.$$

This implies that $I_3(1 + I_2) - I_4(1 + I_1) \approx -0.0183786$ (with a 10-digit accuracy), or $H'(0) < 0$. Therefore $H(z)$ is not a pgf.

Remark. Let $(T_k, k \geq 1)$ denote a sequence of successive arrival times of a Poisson process with intensity 1, i.e., $(T_{k+1} - T_k, k \geq 0)$ $(T_0 = 0)$ is a sequence of iid rv's with an exponential distribution with mean 1. By Theorem 4.5 in Vervaat (1979) and Proposition 5.1, the $\boldsymbol{R}_+$-valued rv $X = \sum_{k=1}^\infty f(T_k)$ (where f is given by (5.1), with $0 < A \leq (1 + m\gamma)^{-1}$) has a discrete semi-stable distribution with LST ϕ of (5.2). The corresponding Poisson mixture $N_\lambda(X)$ has a discrete semi-stable distribution with pgf P_λ of (5.5).

Acknowledgements

The author wishes to thank the referees for their comments which led to a significant improvement in the presentation of the paper.

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