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DISCRETE ω-SEQUENCES OF INDEX SETS[1]

BY

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ABSTRACT. We define a discrete ω-sequence of index sets to be a sequence $\{\theta A_n\}_{n\geq 0}$ of index sets of classes of recursively enumerable sets, such that for each n, θA_{n+1} is an immediate successor of θA_n in the partial order of degrees of index sets under one-one reducibility. The main result of this paper is that if S is any set to which the complete set K is not Turing-reducible, and A^S is the class of recursively enumerable subsets of S, then θA^S is at the bottom of c discrete ω-sequences. It follows that every complete Turing degree contains c discrete ω-sequences.

Introduction. Let $\{W_x\}_{x\geq 0}$ be a standard enumeration of all recursively enumerable (r.e.) sets. If A is any collection of r.e. sets, the *index set* of A is $\{x|\ W_x \in A\}$ and is denoted by θA. If $\{A_n\}_{n\geq 0}$ is a sequence of classes of r.e. sets, call the sequence $\{\theta A_n\}_{n\geq 0}$ a *discrete ω-sequence* of index sets if

(a) $\theta A_n <_1 \theta A_{n+1}$ for each n, and

(b) for every class B of r.e. sets, $\theta A_n \leq_1 \theta B \leq_1 \theta A_{n+1}$ implies $\theta B \cong \theta A_n$ or $\theta B \cong \theta A_{n+1}$.

That discrete ω-sequences exist was proved in [3]; it was shown there that if $\{Z_m\}_{m\geq 0}$ is the sequence of index sets of nonempty finite classes of finite sets (classified in [4] and, independently, in [2]), then $\{Z_m\}_{m\geq 0}$ is a discrete ω-sequence of index sets. Moreover, it easily follows from the results in [3] that the c nonisomorphic sequences $\{Y_m\}_{m\geq 0}$ satisfying $Y_m = Z_m$ or $\overline{Z}_m$ for each m are discrete ω-sequences of index sets. In this paper it is shown that discrete ω-sequences of index sets occur in great profusion. The fact that the sets Z_m are index sets of finite classes of finite sets appears not to be relevant; what generalizes is the fact that $Z_0 = \theta\{\emptyset\} \cong \{x|\ W_x \subseteq S\}$, where S is any co-r.e. set. The main results are as follows: (1) if $K \not\leq_T S$ (where K denotes Post's complete set) and $A^S = \{W_x|\ W_x \subseteq S\}$, then θA^S and $\overline{\theta A^S}$ are at the bottom of c discrete ω-se-

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quences of index sets; (2) every Turing degree $\mathbf{a} > \mathbf{0}'$ contains c discrete ω-sequences; (3) $\mathbf{0}'$ contains c discrete ω-sequences containing no sets recursively isomorphic to Z_m or $\overline{Z}_m$ for any m. We also prove a conjecture made in [3] that there exist sequences $\{\theta A_m\}_{m\geq 0}$ satisfying $Z_m <_1 \theta A_m$ and $\overline{Z}_m \not\leq_1 \theta A_m$, for each $m \geq 0$.

Notation. The terminology and notation is that of [6]. K denotes the complete set $= \{x \mid x \in W_x\}$. N denotes the set of natural numbers. For $X, Y \subseteq N$, $X \times Y$ denotes the recursive Cartesian product, via an effective pairing function $\langle x, y\rangle$ whose inverses are denoted by π_1, π_2; thus $z = \langle \pi_1(z), \pi_2(z)\rangle$. $\{D_n\}_{n\geq 0}$ is the canonical indexing of finite subsets of N, with $D_0 = \emptyset$. For $X, Y \subseteq N$, $X \leq Y$ means X is one-one reducible to Y. If $X \leq Y$ and $Y \leq X$, we invoke Myhill's isomorphism theorem [5] and write $X \cong Y$. $X \leq_T Y$ means X is Turing reducible to Y. $X \mid Y$ means X and Y are 1-1 incomparable.

0. **Required previous results.** We list here for more convenient reference some results of [3] which will be needed. The proofs can be found in [3]. In that paper, for each $m > 0$, $f_m: N^m \to N$ denotes a recursive one-one onto map with recursive inverses denoted by x_i^m, $0 \leq i < m$; i.e., $x = f_m(x_0^m, \cdots, x_{m-1}^m)$. For $m = 1$, f_1 is the identity and $x_0^1 = x$.

Lemma 0.1 (Lemma 10 of [3]). *If* $\overline{A}$ *is nonempty, then*
(a) $N \in A \to K \leq \theta A$,
(b) $\emptyset \in A \to \overline{K} \leq \theta A$.

Definition 0.2 (Definitions 1, 2 of [3]). For each x, let

$$k_m(x) = \text{cardinality}\{i \mid x_i^{m+1} \in K\}.$$

For each $n \geq 0$, let

$$Z_{2n} = \{x \mid k_{2n}(x) \text{ is even}\}, \qquad Z_{2n+1} = \{x \mid k_{2n+1}(x) \text{ is odd}\}.$$

Note that since $x = f_1(x)$, $x \in Z_0 \leftrightarrow x \notin K$, so that $Z_0 = \overline{K}$.

Lemma 0.3 (Theorem 2 of [3]). *For all* $n \geq 0$,
(a) $Z_{n+1} \cong \overline{K} \times \overline{Z}_n$,
(b) $Z_{2n+1} \cong K \times Z_{2n}$,
(c) $\overline{Z}_{2n+2} \cong K \times \overline{Z}_{2n+1}$.

Lemma 0.4 (Theorem 3(a), (b), (c) of [3]). *For all* $m \geq 0$,
(a) $Z_m < Z_{m+1}$, $\overline{Z}_m < \overline{Z}_{m+1}$,
(b) $Z_m < \overline{Z}_{m+1}$, $\overline{Z}_m < Z_{m+1}$,
(c) $Z_m \mid \overline{Z}_m$.

Lemma 0.5 (From Theorem 5 of [3]). *For all* $n \geq 0$,

(a) *if* $\theta A \cong Z_n$ *then* $N \notin A$,

(b) *if* $\theta A \cong \overline{Z}_{2n}$ *then* $\emptyset \notin A$,

(c) *if* $\theta A \cong \overline{Z}_{2n+1}$ *then* $\emptyset \in A$.

Lemma 0.6 (Theorem 3(d), (e) of [3]). *For all* $m \geq 0$,

(a) *there is no A satisfying* $\underline{Z}_m < \theta A < Z_{m+1}$ *or* $\overline{Z}_m < \theta A < \underline{\overline{Z}}_{m+1}$,

(b) *there is no A satisfying* $\overline{Z}_m < \theta A < Z_{m+1}$ *or* $Z_m < \theta A < \overline{Z}_{m+1}$.

Lemma 0.7 (Lemma 13 of [3]). *If* $\theta A \leq K \times \theta B$ *and* $\emptyset \in A$, *then* $\theta A \leq \theta B$.

Lemma 0.8 (Lemma 14 of [3]). *If* $\theta A \leq \overline{K} \times \theta B$ *and* $N \in A$, *then* $\theta A \leq \theta B$.

Lemma 0.9 (Lemma 15 of [3]). *If* $\theta A \leq \theta B \leq K \times \theta A$, *then* $\theta B \cong \theta A$ *or* $\theta B \cong K \times \theta A$.

Lemma 0.10 (Lemma 16 of [3]). *If* $\theta A \leq \theta B \leq \overline{K} \times \theta A$, *then* $\theta B \cong \theta A$ *or* $\theta B \cong \overline{K} \times \theta A$.

Lemma 0.11 (Lemma 9 of [3]). *For all* A, $\theta A \not\cong \theta \overline{A}$.

1. Index sequences.

Definition 1.1. Let $I_n = \{0, 1, \cdots, n\}$, $n \geq 0$, $J = \{0, \overline{0}, 1, \overline{1}, 2, \overline{2}\}$ where $\overline{0}$, $\overline{1}$, $\overline{2}$ are formal symbols introduced for notational purposes. An *index sequence* σ is any function $\sigma: I_n \rightarrow J$ such that

(a) $\sigma(0) \in \{0, \overline{0}\}$,

(b) $\sigma(i) \in \{1, \overline{1}, 2, \overline{2}\}$ for $0 < i \leq n$.

If σ is an index sequence and domain $\sigma = I_n$, σ has length $n + 1$, denoted by l_σ. In the following, σ will be freely identified with the concatenation $\sigma(0) * \sigma(1) * \cdots * \sigma(l_\sigma - 1)$ and $\sigma * i$ will be abbreviated to σi, $i = 1, \overline{1}, 2, \overline{2}$. In this notation, it is clear that $0, \overline{0}$ are index sequences, and that σi is an index sequence $\leftrightarrow$ σ is an index sequence and $i = 1, \overline{1}, 2, \overline{2}$.

Definition 1.2. If σ is an index sequence, its complementary sequence $\overline{\sigma}$ is defined inductively as follows:

(a) $0, \overline{0}$ are complementary,

(b) $\sigma 1$ and $\overline{\sigma}\overline{1}$ are complementary,

(c) $\sigma 2$ and $\overline{\sigma}\overline{2}$ are complementary.

It is easily seen by induction on l_σ that $\overline{\overline{\sigma}} = \sigma$ for all index sequences σ.

Definition 1.3. Suppose S is an infinite subset of N, $S = \{s_0, s_1, \cdots\}$ in any order, $s_i \neq s_j$ for $i \neq j$. For each index sequence σ, define a corresponding class A_σ^S of r.e. sets inductively on length σ, as follows:

(a) $A_0^S = \{W_x \mid W_x \subseteq S\}$,

(b) $A_{\bar{\sigma}}^S = \overline{A_\sigma^S}$,

(c) if σ has length $i+1$, $i \geq 0$,

$$A_{\sigma 1}^S = \{W_x \mid s_i \in W_x \text{ and } W_x \in A_\sigma^S\},$$

$$A_{\sigma 2}^S = \{W_x \mid s_i \notin W_x \text{ and } W_x \in A_\sigma^S\}.$$

Note that $A_{\sigma 1}^S, A_{\sigma 2}^S \subseteq A_\sigma^S$ for all σ, S.

Remark. The classes A_σ^S are defined relative to a given enumeration of S. The notation makes no explicit reference to the enumeration, since it will shortly be shown that the index sets θA_σ^S corresponding to a given σ are unique up to recursive isomorphism.

Lemma 1.4. *Let A be any class of r.e. sets, and let $s \in N$. Then*
(a) *if $A_1 = \{x \mid s \in W_x$ and $W_x \in A\}$ then $\theta A_1 \leq K \times \theta A$,*
(b) *if $A_2 = \{x \mid s \notin W_x$ and $W_x \in A\}$ then $\theta A_2 \leq \overline{K} \times \theta A$.*

Proof. Let $g(x)$ be a recursive function which computes the index of an r.e. set generated according to the following instructions:

$$W_{g(x)} = \varnothing \quad \text{if } s \notin W_x,$$
$$= N \quad \text{if } s \in W_x.$$

Then $g(x) \in K \leftrightarrow s \in W_x$. Let $h(x) = \langle g(x), x \rangle$. Then

$$x \in \theta A_1 \leftrightarrow s \in W_x \text{ and } W_x \in A$$
$$\leftrightarrow g(x) \in K \text{ and } x \in \theta A$$
$$\leftrightarrow h(x) \in K \times \theta A,$$

and

$$x \in \theta A_2 \leftrightarrow s \notin W_x \text{ and } W_x \in A$$
$$\leftrightarrow g(x) \in \overline{K} \text{ and } x \in \theta A$$
$$\leftrightarrow h(x) \in \overline{K} \times \theta A.$$

So $\theta A_1 \leq K \times \theta A$ and $\theta A_2 \leq \overline{K} \times \theta A$, both via h. (As usual, we need not bother to make h one-one, since all sets in question are index sets and thus cylinders [6].)

Lemma 1.5. *Let S be any infinite subset of N, $S = \{s_0, s_1, \cdots\}$. Let $S_0 = \varnothing$, $S_i = \{s_0, s_1, \cdots s_{i-1}\}$ for $i \geq 1$. If σ is an index sequence, $l_\sigma = i+1$, $i \geq 0$ and T is any finite subset of $S - S_i$, then*

$$W_x \in A_\sigma^S \leftrightarrow W_x \cup T \in A_\sigma^S \leftrightarrow W_x - T \in A_\sigma^S.$$

Proof. By induction on i. It suffices to prove the result for the cases when $\sigma = 0$, $\tau 1$ or $\tau 2$. The complementary cases follow by symmetry since, e.g., $W_x \in A^S_{\overline{\tau 1}} \leftrightarrow W_x \notin \overline{A^S_{\overline{\tau 1}}} = A^S_{\overline{\tau 1}}$. If $i = 0$ then $l_\sigma = 1$, so $\sigma = 0$ and T is any finite subset of $S - S_0 = S$. Since $A_0^S = \{W_x \mid W_x \subseteq S\}$, it is clear that

$$W_x \in A_0^S \leftrightarrow W_x \subseteq S \leftrightarrow W_x \cup T \subseteq S \leftrightarrow W_x - T \subseteq S.$$

Now assume the lemma holds for all τ of length $i + 1$ and let $l_\sigma = i + 2$, $T \subset S - S_{i+1}$; then $\sigma = \tau 1$ or $\tau 2$ where τ has length $i + 1$. But $S_i \subset S_{i+1}$ implies $T \subset S - S_{i+1} \subset S - S_i$ so, by the induction hypothesis,

$$W_x \in A_\tau^S \leftrightarrow W_x \cup T \in A_\tau^S \leftrightarrow W_x - T \in A_\sigma^S.$$

Also, $s_i \in S_{i+1}$ implies $s_i \notin T$, so

$$s_i \in W_x \leftrightarrow s_i \in W_x \cup T \leftrightarrow s_i \in W_x - T.$$

These two sets of equivalences imply

$$\begin{aligned} s_i \in W_x \text{ and } W_x \in A_\tau^S &\leftrightarrow s_i \in W_x \cup T \text{ and } W_x - T \in A_\tau^S \\ &\leftrightarrow s_i \in W_x - T \text{ and } W_x - T \in A_\tau^S \end{aligned}$$

and

$$\begin{aligned} s_i \notin W_x \text{ and } W_x \in A_\tau^S &\leftrightarrow s_i \notin W_x \cup T \text{ and } W_x \cup T \in A_\tau^S \\ &\leftrightarrow s_i \notin W_x - T \text{ and } W_x - T \in A_\tau^S. \end{aligned}$$

Now if $\sigma = \tau 1$, $A_\sigma^S = \{x \mid s_i \in W_x$ and $W_x \in A_\tau^S\}$ while if $\sigma = \tau 2$, $A_\sigma^S = \{x \mid s_i \notin W_x$ and $W_x \in A_\tau^S\}$. In either case, it follows that

$$W_x \in A_\sigma^S \leftrightarrow W_x \cup T \in A_\sigma^S \leftrightarrow W_x - T \in A_\sigma^S.$$

Lemma 1.6. *If S is any infinite set and σ any index sequence of length $i > 0$, then*

$$K \times \theta A_\sigma^S \leq \theta A_{\sigma 1}^S.$$

Proof. $A_{\sigma 1}^S = \{W_x \mid s_i \in W_x$ and $W_x \in A_\sigma^S\}$. Let h be a recursive function which computes the index of an r.e. set generated according to the following instructions:

Let

$$W_{h(x)} = \emptyset \qquad \text{if } \pi_1(x) \notin K,$$
$$= W_{\pi_2(x)} \cup \{s_i\} \quad \text{if } \pi_1(x) \in K.$$

Then

$$h(x) \in \theta A^S_{\sigma 1} \leftrightarrow s_i \in W_{h(x)} \text{ and } W_{h(x)} \in A^S_\sigma$$
$$\leftrightarrow \pi_1(x) \in K \text{ and } W_{h(x)} = W_{\pi_2(x)} \cup \{s_i\} \in A^S_\sigma.$$

Since $s_i \in S - S_i$, Lemma 1.5 implies that

$$W_{\pi_2(x)} \cup \{s_i\} \in A^S_\sigma \leftrightarrow W_{\pi_2(x)} \in A^S_\sigma;$$

so $h(x) \in \theta A^S_{\sigma 1} \leftrightarrow \pi_1(x) \in K$ and $W_{\pi_2(x)} \in A^S_\sigma$, and $K \times \theta A^S_\sigma \leq \theta A^S_{\sigma 1}$ via h.

Lemma 1.7. *If S is any infinite set and σ any index sequence of length $i > 0$, then $\overline{K} \times \theta A^S_\sigma \leq \theta A^S_{\sigma 2}$.*

Proof. $A^S_{\sigma 2} = \{W_x \mid s_i \notin W_x \text{ and } W_x \in A^S_\sigma\}$. Let h be a recursive function which computes the index of an r.e. set generated according to the following instructions:

$$W_{h(x)} = W_{\pi_2(x)} - \{s_i\} \quad \text{if } \pi_1(x) \notin K,$$
$$= N \qquad \text{if } \pi_2(x) \in K.$$

Then

$$h(x) \in \theta A^S_{\sigma 2} \leftrightarrow s_i \notin W_{h(x)} \text{ and } W_{h(x)} \in A^S_\sigma$$
$$\leftrightarrow \pi_1(x) \notin K \text{ and } W_{h(x)} = W_{\pi_2(x)} - \{s_i\} \in A^S_\sigma$$
$$\leftrightarrow \pi_1(x) \notin K \text{ and } W_{\pi_2(x)} \in A^S_\sigma,$$

using Lemma 1.5 as in the previous lemma. So $\overline{K} \times \theta A^S_\sigma \leq \theta A^S_{\sigma 2}$ via h.

Definition 1.8. If S is any infinite set and σ any index sequence, let

$$X^S_\sigma = \theta A^S_\sigma, \qquad X^S_{\overline{\sigma}} = \theta A^S_{\overline{\sigma}} = \overline{\theta A^S_\sigma} = \overline{X^S_\sigma}.$$

Lemma 1.9. *For all infinite sets S and all index sequences σ,*

(a) $X^S_{\sigma 1} \cong K \times X^S_\sigma$,

(b) $X^S_{\sigma 2} \cong \overline{K} \times X^S_\sigma$,

(c) $X^S_\sigma \leq X^S_{\sigma i}$, $i = 1, \overline{1}, 2, \overline{2}$.

Proof. By the definitions of $A^S_{\sigma 1}$ and $A^S_{\sigma 2}$, Lemma 1.4 implies $X^S_{\sigma 1} \leq K \times X^S_\sigma$ and $X^S_{\sigma 2} \leq \overline{K} \times X^S_\sigma$. That $K \times X^S_\sigma \leq X^S_{\sigma 1}$ and $\overline{K} \times X^S_\sigma \leq X^S_{\sigma 2}$ is given by Lemmas 1.6 and 1.7. It follows immediately that $X^S_\sigma \leq X^S_{\sigma i}$ if $i = 1, 2$. For $i = \overline{1}, \overline{2}$, $X_{\sigma i} = \overline{X^S_{\sigma \overline{i}}}$, where $\overline{i} = 1, 2$ so $X^S_{\overline{\sigma}} \leq X^S_{\overline{\sigma i}}$ which implies $X^S_\sigma = \overline{X^S_{\overline{\sigma}}} \leq X^S_{\sigma i}$.

Remark. Lemma 1.9 justifies the claim made after Definition 1.3 that the sets θA^S_σ obtained from different enumerations of the set S are recursively isomorphic. For $l_\sigma = 1$ the sets θA^S_σ depend only on S, and for $l_\sigma > 1$, the isomorphism is easily obtained by induction, using Lemma 1.9 (a) and (b).

Lemma 1.10. *Let S be any infinite set $\subsetneq N$. If σ is any index sequence, then X^S_σ is in the bounded truth-table degree of $X^S_0 = \theta A^S_0$.*

Proof. By induction on l_σ. If $\sigma = 0$ or $\overline{0}$, $X^S_\sigma = X^S_0$ or $X^S_{\overline{0}}$, so $X^S_\sigma \equiv_{\text{btt}} X^S_0$. Assume $l_\sigma = n + 1$ and that the result holds for all τ such that $l_\tau \leq n$. Then by Definition 1.1, $\sigma = \tau i$ for some $i = 1, \overline{1}, 2$ or $\overline{2}$ and τ such that $X^S_\tau \equiv_{\text{btt}} X^S_0$. So it suffices to show that $X^S_\tau \equiv_{\text{btt}} X^S_{\tau i}$.

Case 1. $i = 1$ *or* 2. By Lemma 1.9, $X^S_{\tau i} \cong K \times X^S_\tau$ or $\overline{K} \times X^S_\tau$. In either case, $X^S_\tau \leq X^S_{\tau i}$ so $X^S_\tau \leq_{\text{btt}} X^S_{\tau i}$. To show $X^S_{\tau i} \leq_{\text{btt}} X^S_\tau$ it suffices to have $K, \overline{K} \leq_{\text{btt}} X^S_\tau$. But by Lemma 0.1, since S and thus each A^S_τ is nontrivial, $K \leq X^S_\tau$ or $\overline{K} \leq X^S_\tau$. In either case, $K, \overline{K} \leq_{\text{btt}} X^S_\tau$ and $X^S_{\tau i} \leq_{\text{btt}} X^S_\tau$.

Case 2. $i = \overline{1}$ *or* $\overline{2}$. Then $X^S_{\tau i} = \overline{X^S_{\tau \overline{i}}}$ where $\overline{i} = 1$ or 2, so by Case 1, $X^S_{\tau \overline{i}} \equiv_{\text{btt}} X^S_{\overline{\tau}} = \overline{X^S_\tau}$. So by complementation, $X^S_{\tau i} \equiv_{\text{btt}} X^S_\tau$.

Definition 1.11. Let R be any (fixed) nonempty r.e. set such that $\overline{R}$ is infinite. The sets $X^{\overline{R}}_\sigma$ will be denoted by Y_σ.

Lemma 1.12. $Y_0 \leq \overline{K}$.

Proof. $Y_0 = \{x \mid W_x \subseteq \overline{R}\}$, so $\overline{Y}_0 = \{x \mid W_x \cap R \neq \emptyset\}$ which is r.e., since R is assumed to be r.e. So $\overline{Y}_0 \leq K$ and $Y_0 \leq \overline{K}$.

Lemma 1.13. *Let S be any infinite set $\subsetneq N$. Then*

(a) $\overline{K} \leq X^S_0$,

(b) *for all index sequences* σ, $Y_\sigma \leq X^S_\sigma$.

Proof. (a) $A^S_0 = \{W_x \mid W_x \subseteq S\}$ so $\emptyset \in A^S_0$ and $N \in \overline{A^S_0}$, so by Lemma 0.1, $\overline{K} \leq \theta A^S_0 = X^S_0$.

(b) By induction on l_σ. By Lemma 1.12 and part (a), $Y_0 \leq X^S_0$ and, complementing, $Y_{\overline{0}} = \overline{Y_0} \leq X^S_{\overline{0}}$. Now assume the lemma holds for all τ of length $k > 0$ and let $l_\tau = k + 1$. Then $\sigma = \tau 1, \tau 2, \tau\overline{1}$ or $\tau\overline{2}$ for some τ with $l_\tau = k$. By the induction hypothesis, $Y_\tau \le X^S_\tau$ which implies $K \times Y_\tau \le K \times X^S_\tau$ and $\overline{K} \times Y_\tau \le \overline{K} \times X^S_\tau$. If $\sigma = \tau 1$, then by Lemma 1.9(a), $Y_\sigma \cong K \times Y_\tau \le K \times X^S_\tau \cong X^S_\sigma$; if $\sigma = \tau 2$, then by Lemma 1.9(b), $Y_\sigma \cong \overline{K} \times Y_\tau \le \overline{K} \times X^S_\tau = X^S_\sigma$. So if $\sigma = \tau 1$ or $\tau 2$, $Y_\sigma \le X^S_\sigma$. If $\overline{\sigma} = \tau\overline{1}$ or $\tau\overline{2}$, the result follows by complementation, since $\overline{\sigma} = \overline{\tau}1$ or $\overline{\tau}2$ where $l_{\overline{\tau}} = k$, so that $Y_{\overline{\sigma}} \le X^S_{\overline{\sigma}}$ which implies $Y_\sigma \le X^S_\sigma$.

Remark. Lemma 1.13 justifies the lack of reference to R in the notation Y_σ, since if R' is any other nonempty r.e. set with $\overline{R}'$ infinite, it follows that $Y^R_\sigma \le X^{\overline{R}'}_\sigma = Y^{R'}_\sigma$ and $Y^{R'}_\sigma \le X^{\overline{R}}_\sigma = Y^R_\sigma$. Thus for every index sequence σ, $Y^R_\sigma \cong Y^{R'}_\sigma$, so Y_σ is independent of the choice of R.

2. Acceptable index sequences.

Definition 2.1. The subset $\mathfrak{A}$ of *acceptable* index sequences is defined inductively as follows.

(a) $0, \overline{0} \in \mathfrak{A}$.

(b) if l_σ is odd,

$$\sigma 1 \in \mathfrak{A} \leftrightarrow \sigma = 0 \text{ or } \sigma = \tau\overline{1} \text{ or } \tau 2 \quad \text{for some } \tau \in \mathfrak{A},$$

$$\sigma 2 \in \mathfrak{A} \leftrightarrow \sigma = \overline{0} \text{ or } \sigma = \tau 1 \text{ or } \tau\overline{2} \text{ for some } \tau \in \mathfrak{A},$$

$$\sigma\overline{1} \in \mathfrak{A} \leftrightarrow \sigma = \overline{0} \text{ or } \sigma = \tau 1 \text{ or } \tau\overline{2} \text{ for some } \tau \in \mathfrak{A},$$

$$\sigma\overline{2} \in \mathfrak{A} \leftrightarrow \sigma = 0 \text{ or } \sigma = \tau\overline{1} \text{ or } \tau 2 \text{ for some } \tau \in \mathfrak{A},$$

(c) if l_σ is even,

$$\sigma 1 \in \mathfrak{A} \leftrightarrow \sigma = \tau\overline{1} \text{ or } \tau\overline{2} \quad \text{for some } \tau \in \mathfrak{A},$$

$$\sigma 2 \in \mathfrak{A} \leftrightarrow \sigma = \tau\overline{1} \text{ or } \tau\overline{2} \quad \text{for some } \tau \in \mathfrak{A},$$

$$\sigma\overline{1} \in \mathfrak{A} \leftrightarrow \sigma = \tau 1 \text{ or } \tau 2 \quad \text{for some } \tau \in \mathfrak{A},$$

$$\sigma\overline{2} \in \mathfrak{A} \leftrightarrow \sigma = \tau 1 \text{ or } \tau 2 \quad \text{for some } \tau \in \mathfrak{A}.$$

It is clear that if $\sigma \in \mathfrak{A}$,

l_σ odd $\to$ one of $\sigma, \overline{\sigma}$ must have form

$$0,\ \tau\overline{1}1,\ \tau\overline{2}1,\ \tau\overline{1}2 \text{ or } \tau\overline{2}2 \quad \text{for some } \tau \in \mathfrak{A},$$

l_σ even $\to$ one of $\sigma, \overline{\sigma}$ must have form

$$01,\ \overline{0}2,\ \tau\overline{1}1,\ \tau 21,\ \tau 12 \text{ or } \tau\overline{2}2 \quad \text{for some } \tau \in \mathfrak{A}.$$

We note for later use that for each $\sigma \in \mathfrak{A}$, there are exactly two ways to extend σ to a sequence $\sigma i \in \mathfrak{A}$.

Lemma 2.2. *Let S be any infinite set $\subsetneqq N$ and let $\sigma \in \mathfrak{A}$. Then*

(a) *if l_σ is odd*,

$$\sigma = 0 \ \textit{or} \ \tau\overline{1} \ \textit{or} \ \tau 2 \longrightarrow \emptyset \in A^S_\sigma \ \textit{and} \ N \notin A^S_\sigma,$$

$$\sigma = \overline{0} \ \textit{or} \ \tau 1 \ \textit{or} \ \tau\overline{2} \longrightarrow \emptyset \notin A^S_\sigma \ \textit{and} \ N \in A^S_\sigma.$$

(b) *if* l_σ *is even,*

$$\sigma = \tau 1 \ \textit{or} \ \tau 2 \longrightarrow \emptyset \notin A^S_\sigma \ \textit{and} \ N \notin A^S_\sigma,$$

$$\sigma = \tau\overline{1} \ \textit{or} \ \tau\overline{2} \longrightarrow \emptyset \in A^S_\sigma \ \textit{and} \ N \in A^S_\sigma.$$

Proof. If $\sigma = 0$, $A^S_\sigma = \{W_x \mid W_x \subseteq S\}$, so clearly $\emptyset \in A^S_\sigma$ and $N \notin A^S_\sigma$. If $\sigma = \overline{0}$, $A^S_\sigma = \overline{A^S_0}$, so $\emptyset \notin A^S_\sigma$ and $N \in A^S_\sigma$. Now assume the lemma holds for all τ such that $1 \leq l_\tau < l_\sigma$.

Case 1. $l_\sigma = 2i + 2$.

Subcase 1.1. $\sigma = \tau 1$ for some $\tau \in \mathfrak{A}$. Then by Definition 2.1, $\tau = 0$ or $\lambda\overline{1}$ or $\lambda 2$ for some $\lambda \in \mathfrak{A}$. By the induction hypothesis, since $l_\tau = 2i + 1$, $N \notin A^S_\tau$. By Definition 1.3, $A^S_\sigma = \{W_x \mid s_{2i} \in W_x$ and $W_x \in A^S_\tau\}$. Clearly $\emptyset \notin A^S_\sigma$, and $N \notin A^S_\tau$ implies $N \notin A^S_\sigma$, since $A^S_\sigma = A_{\tau 1} \subseteq A^S_\tau$.

Subcase 1.2. $\sigma = \tau 2$ for some $\tau \in \mathfrak{A}$. Then by Definition 2.1, $\tau = \overline{0}$ or $\lambda 1$ or $\lambda\overline{2}$ for some $\lambda \in \mathfrak{A}$. By the induction hypothesis, since $l_\tau = 2i + 1$, $\emptyset \notin A^S_\tau$. By Definition 1.3, $A^S_\sigma = \{W_x \mid s_{2i} \notin W_x$ and $W_x \in A^S_\tau\}$. Clearly $N \notin A^S_\sigma$, and $\emptyset \notin A^S_\tau \longrightarrow \emptyset \notin A^S_\sigma$, since $A^S_\sigma = A^S_{\tau 2} \subseteq A^S_\tau$.

Subcase 1.3. $\sigma = \tau\overline{1}$ or $\tau\overline{2}$ for some $\tau \in \mathfrak{A}$. The result follows by complementation from the other subcases since $\overline{\sigma} = \overline{\tau}1$ or $\overline{\tau}2$ and $\emptyset, N \in A^S_\sigma \leftrightarrow \emptyset, N \notin A^S_{\overline{\sigma}}$.

Case 2. $l_\sigma = 2i + 3$.

Subcase 2.1. $\sigma = \tau 1$ for some $\lambda \in \mathfrak{A}$. Then by Definition 2.1, $\tau = \lambda\overline{1}$ or $\lambda\overline{2}$ for for some $\lambda \in \mathfrak{A}$, and by the induction hypothesis, since $l_\tau = 2i + 2$, $N \in A^S_\tau$. By Definition 1.3, $A^S_\sigma = \{W_x \mid s_{2i+1} \in W_x$ and $W_x \in A^S_\tau\}$. Clearly $\emptyset \notin A^S_\sigma$ and, since $N \in A^S_\tau$ and $s_{2i+1} \in N$, $N \in A^S_\sigma$.

Subcase 2.2. $\sigma = \tau 2$ for some $\tau \in \mathfrak{A}$. Then by Definition 2.1, $\tau = \lambda 2$ for some $\lambda \in \mathfrak{A}$, and by the induction hypothesis, since $l_\tau = 2i + 2$, $\emptyset \in A^S_\tau$. By Definition 1.3, $A^S_\sigma = \{W_x \mid s_{2i+1} \notin W_x$ and $W_x \in A^S_\tau\}$. Clearly $N \notin A^S_\tau$ and, since $\emptyset \in A^S_\tau$ and $s_{2i+1} \notin \emptyset, \emptyset \in A^S_\sigma$.

Subcase 2.3. $\sigma = \tau\overline{1}$ or $\tau\overline{2}$ for some $\tau \in \mathfrak{A}$. By complementation from Subcases 2.2 and 2.3.

Lemma 2.3. *Let* $\sigma \in \mathfrak{A}$. *Then*

(a) $\sigma = 0 \longrightarrow Y_\sigma \cong Z_0$,

$\sigma = \overline{0} \longrightarrow Y_\sigma \cong \overline{Z}_0$.

(b) *If* $l_\sigma = 2n + 2$, *then*

$\sigma = \tau 1$ *or* $\tau 2 \rightarrow Y_\sigma \cong Z_{2n+1}$,

$\sigma = \tau \overline{1}$ *or* $\tau \overline{2} \rightarrow Y_\sigma \cong Z_{2n+1}$.

(c) *If* $l_\sigma = 2n + 3$, *then*

$\sigma = \tau \overline{1}$ *or* $\tau 2 \rightarrow Y_\sigma \cong Z_{2n+2}$,

$\sigma = \tau 1$ *or* $\tau \overline{2} \rightarrow Y_\sigma \cong \overline{Z}_{2n+2}$.

Proof. By induction on l_σ. If $l_\sigma = 1$ then $\sigma = 0$ or $\overline{0}$. By Lemma 1.12, $Y_0 \leq \overline{K}$ and by Lemma 1.13 (a), $\overline{K} \leq X_0^{\overline{R}} = Y_0$. So $Y_0 \cong \overline{K} = Z_0$, by Definition 0.2, and $Y_{\overline{0}} = \overline{Y_0} \cong \overline{Z_0}$. Now assume the results hold for all $\tau \in \mathfrak{A}$ such that $1 \leq l_\gamma < l_\sigma$.

Case 1. $l_\sigma = 2n + 2$.

Subcase 1.1. $\sigma = \tau 1$ or $\tau \overline{1}$ for some $\tau \in \mathfrak{A}$. By Definition 2.1, $\tau 1 \in \mathfrak{A} \leftrightarrow \tau = 0$ or $\lambda \overline{1}$ or $\lambda 2$ for some $\lambda \in \mathfrak{A}$. By the induction hypothesis, since $l_\tau = 2n + 1$, $Y_\tau \cong Z_{2n}$. Then by Lemmas 1.9 and 0.3 $Y_{\tau 1} \cong K \times Y_\tau \cong K \times Z_{2n} \cong Z_{2n+1}$. Replacing τ by $\overline{\tau}$ in this argument gives $Y_{\overline{\tau} 1} \cong Z_{2n+1}$, so $Y_{\tau \overline{1}} = \overline{Y}_{\overline{\tau} 1} \cong \overline{Z}_{2n+1}$.

Subcase 1.2. $\sigma = \tau 2$ or $\tau \overline{2}$ for some $\tau \in \mathfrak{A}$. By Definition 2.1, $\tau 2 \in \mathfrak{A} \leftrightarrow \tau = \overline{0}$ or $\lambda 1$ or $\lambda \overline{2}$ for some $\lambda \in \mathfrak{A}$. By the induction hypothesis, $Y_\tau \cong \overline{Z}_{2n}$, so by Lemmas 1.9 and 0.3, $Y_{\tau 2} \cong \overline{K} \times Y_\tau \cong \overline{K} \times \overline{Z}_{2n} \cong Z_{2n+1}$. Similarly, $Y_{\overline{\tau} 2} \cong Z_{2n+1}$, so $Y_{\tau \overline{2}} = \overline{Y}_{\overline{\tau} 2} \cong \overline{Z}_{2n+1}$.

Case 2. $l_\sigma = 2n + 3$.

Subcase 2.1. $\sigma = \tau 1$ or $\tau \overline{1}$ for some $\tau \in \mathfrak{A}$. By Definition 2.1, $\tau 1 \in \mathfrak{A} \leftrightarrow \tau = \lambda \overline{1}$ or $\lambda \overline{2}$ for some $\lambda \in \mathfrak{A}$. By the induction hypothesis, $Y_\tau \cong \overline{Z}_{2n+1}$. Then by Lemmas 1.9 and 0.3, $Y_{\tau 1} \cong K \times Y_\tau \cong K \times \overline{Z}_{2n+1} \cong \overline{Z}_{2n+2}$. Similarly, $Y_{\overline{\tau} 1} \cong \overline{Z}_{2n+2}$, so $Y_{\tau \overline{1}} = \overline{Y}_{\overline{\tau} 1} \cong Z_{2n+2}$.

Subcase 2.2. $\sigma = \tau 2$ or $\tau \overline{2}$ for some $\tau \in \mathfrak{A}$. By Definition 2.1, $\tau 2 \in \mathfrak{A} \leftrightarrow \tau = \lambda \overline{1}$ or $\lambda \overline{2}$ for some $\lambda \in \mathfrak{A}$. By the induction hypothesis, $Y_\tau \cong \overline{Z}_{2n+1}$, so by Lemmas 1.9 and 0.3, $Y_{\tau 2} \cong \overline{K} \times Y_\tau \cong \overline{K} \times \overline{Z}_{2n+1} \cong Z_{2n+2}$. Similarly, $Y_{\overline{\tau} 2} \cong Z_{2n+2}$, so $Y_{\tau \overline{2}} = \overline{Y}_{\overline{\tau} 2} \cong \overline{Z}_{2n+2}$.

Lemma 2.4. (a) *If* $\sigma \in \mathfrak{A}$ *then* $Y_\sigma \cong Z_{l_\sigma - 1}$ *or* $\overline{Z_{l_\sigma - 1}}$.
(b) *If* $\sigma, \tau \in \mathfrak{A}$ *and* $l_\tau < l_\sigma$, *then* $Y_\tau < Y_\sigma$.

Proof. (a) follows from Lemma 2.3, since the various cases exhaust $\mathfrak{A}$. For (b), assume $l_\tau = m + 1$ and $l_\sigma = n + 1$ for $m < n$. Then by (a), $Y_\tau \cong Z_m$ or $\overline{Z}_m$ and $Y_\sigma \cong Z_n$ or $\overline{Z}_n$. Then by Lemma 0.4, $Y_\tau < Z_{m+1} \leq Z_n$ and $Y_\tau < \overline{Z}_{m+1} < \overline{Z}_n$. Thus in any case $Y_\tau < Y_\sigma$.

Lemma 2.5. *For all* m, n,
(a) $Z_m \ncong \overline{Z}_n$,
(b) $m \neq n \rightarrow Z_m \ncong Z_n$.

Proof. By Lemma 0.4, $m < n \to Z_m < \overline{Z}_{m+1} < \overline{Z}_n$, and $m = n \to Z_m \mid \overline{Z}_n$. Thus in either case (a) holds. Lemma 0.4 also implies (b), since, e.g., $m < n \to Z_m < Z_{m+1} \leq Z_n$.

Lemma 2.6. Let $\sigma \in \mathfrak{A}$. *Then*

(a) $Y_\sigma \cong Z_0 \to \sigma = 0$,
(b) $Y_\sigma \cong Z_{2n+1} \to l_\sigma = 2n + 2$ *and* $\sigma = \tau 1$ *or* $\tau 2$ *for some* $\tau \in \mathfrak{A}$,
(c) $Y_\sigma \cong Z_{2n+2} \to l_\sigma = 2n + 3$ *and* $\sigma = \tau\overline{1}$ *or* $\tau 2$ *for some* $\tau \in \mathfrak{A}$.

Proof. (a) Assume $\sigma \neq 0$. Then $\sigma = \overline{0}$ or τi for some $\tau \in \mathfrak{A}$, $i = 1, 2, \overline{1}$ or $\overline{2}$. By Lemma 2.3, this implies $Y_\sigma = Z_{\overline{0}}$ or $Y_\sigma = Z_m$ or $\overline{Z}_m$ for some $m > 0$. In any case, by Lemma 2.5, $Y_\sigma \not\cong Z_0$.

(b) Let $m = l_\sigma - 1$. If $l_\sigma \neq 2n + 2$, then $m \neq 2n + 1$ and, by Lemma 2.4(a), $Y_\sigma \cong Z_m$ or $\overline{Z}_m$. By Lemma 2.5, this implies $Y_\sigma \not\cong Z_{2n+1}$. If $l_\sigma = 2n + 2$ but $\sigma \neq \tau 1$ or $\tau 2$ for some $\tau \in \mathfrak{A}$, then $\sigma = \tau\overline{1}$ or $\tau\overline{2}$. Then by Lemma 2.3(b), $Y_\sigma \cong \overline{Z}_{2n+1} \not\cong Z_{2n+1}$.

(c) Let $m = l_\sigma - 1$. If $l_\sigma \neq 2n + 3$, then $m \neq 2n + 2$ and, by Lemma 2.4(a), $Y_\sigma \cong Z_m$ or $\overline{Z}_m$ So by Lemma 2.5, $Y_\sigma \not\cong Z_{2n+2}$. If $l_\sigma = 2n + 3$ but $\sigma \neq \tau\overline{1}$ or $\tau 2$ for some $\tau \in \mathfrak{A}$ then $\sigma = \tau 1$ or $\tau\overline{2}$, so by Lemma 2.3(c), $Y_\sigma \cong \overline{Z}_{2n+2} \not\cong Z_{2n+2}$.

Theorem 2.7. *Let* $\sigma \in \mathfrak{A}$. *Then*

(a) $Y_\sigma \cong Z_0 \leftrightarrow \sigma = 0$,
(b) $Y_\sigma \cong \overline{Z}_0 \leftrightarrow \sigma = \overline{0}$,
(c) $Y_\sigma \cong Z_{2n+1} \leftrightarrow l_\sigma = 2n + 2$ *and* $\sigma = \tau 1$ *or* $\tau 2$ *for some* $\tau \in \mathfrak{A}$,
(d) $Y_\sigma \cong \overline{Z}_{2n+1} \leftrightarrow l_\sigma = 2n + 2$ *and* $\sigma = \tau\overline{1}$ *or* $\tau\overline{2}$ *for some* $\tau \in \mathfrak{A}$,
(e) $Y_\sigma \cong Z_{2n+2} \leftrightarrow l_\sigma = 2n + 3$ *and* $\sigma = \tau\overline{1}$ *or* $\tau 2$ *for some* $\tau \in \mathfrak{A}$,
(f) $Y_\sigma \cong \overline{Z}_{2n+2} \leftrightarrow l_\sigma = 2n + 3$ *and* $\sigma = \tau 1$ *or* $\tau\overline{2}$ *for some* $\tau \in \mathfrak{A}$.

Proof. (a), (c) and (e) follow from Lemmas 2.3 and 2.6. The other parts are obtained by complementation, since $l_\sigma = l_{\overline{\sigma}}$, $\overline{\tau i} = \tau\overline{i}$ and $Y_\sigma \cong Z_m \leftrightarrow Y_{\overline{\sigma}} \cong \overline{Z}_m$.

Lemma 2.8. *If* $\sigma i, \sigma j \in \mathfrak{A}$ $(i, j = 1, 2, \overline{1}$ or $\overline{2})$ *then* $i \neq j \to Y_{\sigma i} \cong \overline{Y}_{\sigma j}$.

Proof. Assume $i \neq j$ and $\sigma i, \sigma j \in \mathfrak{A}$.

Case 1. $l_\sigma = 2n + 1$. If $\sigma = 0$ or $\tau\overline{1}$ or $\tau 2$ for some $\tau \in \mathfrak{A}$, then, by Definition 2.1, $\sigma i, \sigma j \in \mathfrak{A} \leftrightarrow i, j = 1$ or $\overline{2}$, say $i = 1$ and $j = \overline{2}$. Since $l_{\sigma i} = l_{\sigma j} = 2n + 2$, it follows by Theorem 2.7 that $Y_{\sigma 1} \cong Z_{2n+1}$ and $Y_{\sigma\overline{2}} \cong \overline{Z}_{2n+1}$, so $Y_{\sigma i} \cong \overline{Y}_{\sigma j}$. If $\sigma = \overline{0}$ or $\tau 1$ or $\tau\overline{2}$, the result follows by consideration of complements.

Case 2. $l_\sigma = 2n + 2$. If $\sigma = \tau 1$ or $\tau 2$ for some $\tau \in \mathfrak{A}$ then, by Definition 2.1, $\sigma i, \sigma j \in \mathfrak{A} \leftrightarrow i, j = \overline{1}$ or $\overline{2}$, say $i = \overline{1}$ and $j = \overline{2}$. Since $l_{\sigma i} = l_{\sigma j} = 2n + 3$, it follows by Theorem 2.7 that $Y_{\sigma\overline{1}} \cong Z_{2n+2}$ and $Y_{\sigma\overline{2}} \cong \overline{Z}_{2n+2}$, so $Y_{\sigma i} = \overline{Y_{\sigma j}}$. If $\sigma = \tau\overline{1}$ or $\tau\overline{2}$, the result again follows by considering complements.

Lemma 2.9. *Let S be any infinite set $\subsetneq N$. Then for all $\sigma \in \mathfrak{A}$, if $i = 1, \bar{1}, 2, \bar{2}$ and $\sigma i \in \mathfrak{A}$, $X_\sigma \not\cong Y_{\sigma i}$.*

Proof. *Case* 1. $l_\sigma = 2n+1$, $\sigma = 0$ or $\tau\bar{1}$ or $\tau 2$ for some $\tau \in \mathfrak{A}$. Then by Lemma 2.2, $\varnothing \in A_\sigma^S$ and $N \notin A_\sigma^S$. By Definition 2.1, $\sigma i \in \mathfrak{A} \rightarrow i = 1$ or $\bar{2}$.

Subcase 1.1. $i = 1$. Then by Theorem 2.7, since $l_{\sigma i} = 2n+2$, $Y_{\sigma i} \cong Z_{2n+1}$. It follows by Lemma 0.5 that $\theta A \cong Y_{\sigma i} \rightarrow \overline{\theta A} \cong \overline{Z}_{2n+1} \rightarrow \varnothing \in \bar{A}$. But this implies $X_\sigma^S = \theta A_\sigma^S \not\cong Y_{\sigma i}$ since $\varnothing \in A_\sigma^S$.

Subcase 1.2. $i = \bar{2}$. Then by Theorem 2.7, $Y_{\sigma i} = \overline{Z}_{2n+1}$, so by Lemma 0.5, $\theta A = Y_{\sigma i} \rightarrow \theta\bar{A} = Z_{2n+1} \rightarrow N \in A$. But this implies $X_\sigma^S = \theta A_\sigma^S \not\cong Y_{\sigma i}$, since $N \notin A_\sigma^S$.

Case 2. $l_\sigma = 2n+1$, $\sigma = \bar{0}$ or $\tau 1$ or $\tau\bar{2}$ for some $\tau \in \mathfrak{A}$. Then $\bar{\sigma} = 0$ or $\tau\bar{1}$ or $\tau\bar{2}$ so, by Case 1, $X_{\bar{\sigma}}^S \not\cong Y_{\bar{\sigma}\bar{i}} = Y_{\overline{\sigma i}}$. But this implies $X_\sigma^S = \overline{X}_{\bar{\sigma}}^S \not\cong \overline{Y}_{\overline{\sigma i}} = Y_{\sigma i}$.

Case 3. $l_\sigma = 2n+2$, $\sigma = \tau 1$ or $\tau 2$ for some $\tau \in \mathfrak{A}$. Then by Lemma 2.2, $\varnothing \notin A_\sigma^S$ and $N \notin A_\sigma^S$. By Definition 2.1, $\sigma i \in \mathfrak{A} \rightarrow i = \bar{1}$ or $\bar{2}$.

Subcase 3.1. $i = \bar{1}$. Then by Theorem 2.7, $Y_{\sigma i} \cong Z_{2n+2}$, so by Lemma 0.5, $\theta A = Y_{\sigma i} \rightarrow \theta\bar{A} = \overline{Z}_{2n+2} \rightarrow \varnothing \in A$. It follows that $X_\sigma^S = \theta A_\sigma^S \not\cong Y_{\sigma i}$, since $\varnothing \notin A_\sigma^S$.

Subcase 3.2. $i = \bar{2}$. Then by Theorem 2.7, $Y_{\sigma i} \cong \overline{Z}_{2n+2}$, so by Lemma 0.5, $\theta A \cong Y_{\sigma i} \rightarrow \theta\bar{A} = Z_{2n+2} \rightarrow N \in A$. If follows that $X_\sigma^S = \theta A_\sigma^S \not\cong Y_{\sigma i}$, since $N \notin A_\sigma^S$.

Case 4. $l_\sigma = 2n+2$, $\sigma = \tau\bar{1}$ or $\tau\bar{2}$ for some $\tau \in \mathfrak{A}$. Then $\bar{\sigma} = \tau\bar{1}$ or $\tau 2$, so by Case 3, $X_{\bar{\sigma}}^S \not\cong Y_{\bar{\sigma}\bar{i}} = Y_{\overline{\sigma i}}$. It follows that $X_\sigma^S = \overline{X_{\bar{\sigma}}^S} \not\cong \overline{Y}_{\overline{\sigma i}} = Y_{\sigma i}$.

Lemma 2.10. *For all S, $S \leq X_0^S$.*

Proof. Recall that $X_0^S = \{x \mid W_x \subseteq S\}$, and let g be a recursive function such that $\{n\} = W_{g(n)}$, for all n. Then $n \in S \leftrightarrow \{n\} \subseteq S \leftrightarrow g(n) \in X_0^S$.

Lemma 2.11. *Let S be any set such that $\bar{S}$ is not r.e. Then for all $\sigma \in \mathfrak{A}$, $Y_\sigma < X_\sigma^S$.*

Proof. By Lemma 1.13, $Y_\sigma \leq X_\sigma^S$, so it suffices to prove $X_\sigma^S \not\leq Y_\sigma$, by induction on l_σ.

Case 1. $l_\sigma = 1$. Then $\sigma = 0$ or $\bar{0}$. If $\sigma = \bar{0}$, $Y_\sigma \cong Z_{\bar{0}} = K$, by Lemma 2.3; also $\bar{S} \leq X_0^{\bar{S}} = X_{\bar{0}}^S$, by Lemma 2.10. Then $X_{\bar{0}}^S \leq Y_{\bar{0}} \rightarrow \bar{S} \leq X_{\bar{0}}^S \leq K$ which implies $\bar{S}$ is r.e., contrary to hypothesis. The result for $\sigma = 0$ follows by symmetry.

Case 2. $l_\sigma = k+2$, $k \geq 0$. Assume the result holds for all $\tau \in \mathfrak{A}$ such that $l_\tau < l_\sigma$, but that $X_\sigma^S \leq Y_\sigma$.

Since $l_\sigma > 1$, $\sigma = \tau i$ for some $\tau \in \mathfrak{A}$. By Lemmas 1.13 and 1.9(c), $Y_\tau \leq X_\tau^S \leq X_{\tau i}^S \leq Y_{\tau i}$. Since $l_\tau = k+1$, it follows by Lemma 2.4(a) that $Y_\tau \cong Z_k$ or $\overline{Z}_k$ and $Y_{\tau i} \cong Z_{k+1}$ or $\overline{Z}_{k+1}$. Then by Lemma 0.6, $Y_\tau \leq X_\tau^S = \theta A_\tau^S \leq Y_{\tau i}$ implies $Y_\tau \cong X_\tau^S$

or $Y_{\tau i} \cong X^S_\tau$. But the first of these contradicts the induction hypothesis and the latter contradicts Lemma 2.9.

Theorem 2.12. *Let S be any infinite set $\subsetneqq N$ and let $\sigma \in \mathfrak{A}$. Then*

(a) $Z_0 \leq X^S_0$;

(b) $\overline{Z}_0 \leq X^S_{\overline{0}}$;

(c) *if* $l_\sigma = 2n + 2$ *and* $\sigma = \tau 1$ *or* $\tau 2$ *for some* $\tau \in \mathfrak{A}$, *then* $Z_{2n+1} \leq X^S_\sigma$;

(d) *if* $l_\sigma = 2n + 2$ *and* $\sigma = \tau\overline{1}$ *or* $\tau\overline{1}$ *for some* $\tau \in \mathfrak{A}$, *then* $\overline{Z}_{2n+1} \leq X^S_\sigma$;

(e) *if* $l_\sigma = 2n + 3$ *and* $\sigma = \tau\overline{1}$ *or* $\tau 2$ *for some* $\tau \in \mathfrak{A}$, *then* $Z_{2n+2} \leq X^S_\sigma$;

(f) *if* $l_\sigma = 2n + 3$ *and* $\sigma = \tau 1$ *or* $\tau\overline{2}$ *for some* $\tau \in \mathfrak{A}$, *then* $\overline{Z}_{2n+2} \leq X^S_\sigma$.

If, in addition, $\overline{S}$ is not r.e., all the inequalities are strict.

Proof. By Lemma 1.13, Theorem 2.7 and Lemma 2.11.

Lemma 2.13. *Let S be any infinite set such that $K \not\leq_T S$. Then for all $\sigma \in \mathfrak{A}$, $Y_{\overline{\sigma}} \not\leq X^S_\sigma$.*

Proof. By induction on l_σ. For $\sigma = 0$, $Y_0 \cong \overline{K}$ by Lemma 2.3. That $\overline{K} \not\leq X^S_0 = \{x \mid W_x \cap \overline{S} \neq \emptyset\}$ if $K \not\leq_T S$ was proved in [1, Theorem 3.5], by observing that $X^S_{\overline{0}}$ is r.e. in S, so that $\overline{K} \leq X^S_0 \rightarrow \overline{K}$ r.e. in $S \rightarrow K \leq_T S$, contrary to hypothesis. By symmetry, $Y_{\overline{0}} \not\leq X^S_0$. Now assume the result for all $\tau \in \mathfrak{A}$ such that $l_\tau < l_\sigma$.

Case 1. $\sigma = \tau 1$ or $\tau\overline{1}$ for some $\tau \in \mathfrak{A}$. If $\sigma = \tau 1$ then, by Lemma 1.9, $X^S_\sigma \cong K \times X^S_\tau$ and, by Lemma 2.2, $\emptyset \in A^{\overline{R}}_{\overline{\sigma}} = \overline{A^R_\sigma}$. So $Y_{\overline{\sigma}} \leq X^S_\sigma \rightarrow Y_{\overline{\sigma}} = \theta A^{\overline{R}}_{\overline{\sigma}} \leq K \times X^S_\tau$ which by Lemma 0.7 implies $Y_{\overline{\sigma}} \leq X^S_\tau$. It follows by Lemma 2.4(b), since $l_\tau < l_\sigma$, that $Y_{\overline{\tau}} < Y_{\overline{\sigma}} \leq X^S_\tau$, which contradicts the induction hypothesis. If $\sigma = \tau\overline{1}$ the result follows by complementation.

Case 2. $\sigma = \tau 2$ or $\tau\overline{2}$ for some $\tau \in \mathfrak{A}$. If $\sigma = \tau 2$, then by Lemma 1.9, $X^S_\sigma \cong \overline{K} \times X^S_\tau$ and, by Lemma 2.2, $N \in A^{\overline{R}}_{\overline{\sigma}} = \overline{A^R_\sigma}$. So $Y_{\overline{\sigma}} \leq X^S_\sigma \rightarrow Y_{\overline{\sigma}} = \theta A^{\overline{R}}_{\overline{\sigma}} \leq \overline{K} \times X^S_\tau$ which by Lemma 0.8 implies $Y_{\overline{\sigma}} \leq X^S_\tau$. It follows by Lemma 2.4(b) that $Y_{\overline{\tau}} < Y_{\overline{\sigma}} \leq X^S_\tau$, which contradicts the induction hypothesis. The result for $\sigma = \tau\overline{2}$ follows by complementation.

Lemma 2.14. *Let S be any infinite set $\subsetneqq N$ such that $K \not\leq_T S$. Then for any index sequence σ and $i = 1, \overline{1}, 2$ or $\overline{2}$, $\sigma \in \mathfrak{A}$ and $\sigma i \in \mathfrak{A} \rightarrow X^S_\sigma < X^S_{\sigma i}$.*

Proof. By Lemma 1.9, $X^S_\sigma \leq X^S_{\sigma i}$ so it suffices to prove $X^S_{\sigma i} \not\leq X^S_\sigma$. Now by Lemma 2.4(b), $Y_{\overline{\sigma}} \leq Y_{\sigma i}$ and, by Lemma 1.13, $Y_{\sigma i} \leq X^S_{\sigma i}$. Then $X^S_{\sigma i} \leq X^S_\sigma$ implies $Y_{\overline{\sigma}} \leq Y_{\sigma i} \leq X^S_{\sigma i} \leq X^S_\sigma$, which contradicts Lemma 2.13.

Lemma 2.15. *Let S be any infinite set $\subsetneqq N$ such that $K \not\leq_T S$, and let $\sigma \in \mathfrak{A}$. If for $i, j = 1, \overline{1}, 2$ or $\overline{2}$, $\sigma i \in \mathfrak{A}$ and $\sigma j \in \mathfrak{A}$, then $i \neq j \rightarrow X^S_{\sigma i} \mid X^S_{\sigma j}$.*

Proof. By Lemma 1.13, $Y_{\sigma i} \leq X^S_{\sigma i}$, so to show $X^S_{\sigma i} \not\leq X^S_{\sigma j}$ it suffices to prove $Y_{\sigma i} \not\leq X^S_{\sigma j}$ for $j \neq i$. But, by Lemma 2.8, $Y_{\sigma i} = \overline{Y}_{\sigma j} = Y_{\overline{\sigma j}}$, and by Lemma 2.13, $Y_{\overline{\sigma j}} \not\leq X^S_{\sigma j}$, which implies $X^S_{\sigma i} \not\leq X^S_{\sigma j}$. The other half follows by symmetry.

3. Acceptable index functions.

Definition 3.1. Let f be a function, $f\colon N \to \{0, \overline{0}, 1, \overline{1}, 2, \overline{2}\}$. For each $i \in N$, let $\sigma(i, f)$ be defined inductively as follows:

(a) $\sigma(0, f) = f(0)$,

(b) $\sigma(i + 1, f) = \sigma(i, f) * f(i + 1)$.

If $\sigma(i, f) = \sigma$ then $\overline{\sigma}(i, f)$ denotes $\overline{\sigma}$.

Definition 3.2. Let f be a function, $f\colon N \to \{0, \overline{0}, 1, \overline{1}, 2, \overline{2}\}$. f is an *acceptable index function* (a.i.f.) if, for every $i \in N$, $\sigma(i, f) \in \mathfrak{A}$.

Note that by this definition f is an a.i.f. only if $f(0) \in \{0, \overline{0}\}$ and $f(i) \in \{1, \overline{1}, 2, \overline{2}\}$ for all $i > 0$.

Remark 1. There exist continuum–many acceptable index functions such that $f(0) = 0$ and continuum–many such that $f(0) = \overline{0}$. This is easily seen as follows: By Definition 2.1, 0 and $\overline{0}$ are both in $\mathfrak{A}$, and as noted after Definition 2.1, for each $\sigma \in \mathfrak{A}$ there are exactly two ways to extend σ to a sequence $\sigma i \in \mathfrak{A}$; and there are c paths through an infinite tree which branches twice at each node.

Lemma 3.3. *Let f be defined by $f(0) = 0$, $f(2n + 1) = 1$, $f(2n + 2) = \overline{1}$. Then f is an acceptable index function and, for each m, $Z_m \cong Y_{\sigma(m,f)}$.*

Proof. We show by induction on m that $\sigma(m, f) \in \mathfrak{A}$ and $Z_m \cong Y_{\sigma(m,f)}$. For $m = 0$, the result holds since $\sigma(0, f) = 0 \in \mathfrak{A}$ and $Z_0 \cong Y_0$ by Theorem 2.7. Now assume the result holds for m.

Case 1. $m + 1$ is odd. Then $\sigma(m + 1, f) = \sigma(m, f) * f(m + 1) = \sigma(m, f) * 1$, and $Y_{\sigma(m,f)} \cong Z_m$ implies $\sigma(m, f) = 0$ or $\tau\overline{1}$ or $\tau 2$ for some $\tau \in \mathfrak{A}$, by Theorem 2.7(a) and (e). Since $l_{\sigma(m,f)} = m + 1$ is odd, it follows by Definition 2.1 that $\sigma(m + 1, f) = \sigma(m, f) * 1 \in \mathfrak{A}$, and by Theorem 2.7(c) that $Y_{\sigma(m+1,f)} \cong Z_{m+1}$.

Case 2. $m + 1$ is even. Then $\sigma(m + 1, f) = \sigma(m, f) * f(m + 1) = \sigma(m, f) * \overline{1}$, and $Y_{\sigma(m,f)} \cong Z_m$ implies $\sigma(m, f) = \tau 1$ or $\tau 2$ for some $\tau \in \mathfrak{A}$, by Theorem 2.7(c). Since $l_{\sigma(m,f)} = m + 1$ is even, it follows by Definition 2.1 that $\sigma(m + 1, f) = \sigma(m, f) * \overline{1} \in \mathfrak{A}$, and by Theorem 2.7(e) that $Y_{\sigma(m+1,f)} \cong Z_{m+1}$.

4. Discrete ω-sequences.

Definition 4.1. Let $\{A_n\}_{n \geq 0}$ be a sequence of classes of r.e. sets. The sequence $\{\theta A_n\}_{n \geq 0}$ is a *discrete ω-sequence of index sets* iff

(a) $\theta A_n < \theta A_{n+1}$ for each n;

(b) for any class B of r.e. sets and each n, $\theta A_n \leq \theta B \leq \theta A_{n+1}$ implies $\theta B \cong \theta A_n$ or $\theta B \cong \theta A_{n+1}$.

Definition 4.2. *A discrete ω-sequence of 1-degrees* denotes the sequence of 1-degrees of a discrete ω-sequence of index sets.

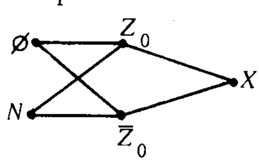
Evidently two different sequences of sets may determine the same sequence of 1-degrees.

Definition 4.3. If S is a set and f an acceptable index function, the sequence $\{\theta A_{\sigma(n,f)}\}_{n\geq 0}$ is the *S-sequence of index sets determined by f.* The corresponding sequence of 1-degrees is the *S-sequence of 1-degrees determined by f.*

Lemma 4.4. *The 1-degrees of Z_0 and $\overline{Z_0}$ are each at the bottom of c discrete ω-sequences of 1-degrees, each contained in the bounded truth-table degree of Z_0.*

Proof. Let $\{X_m\}_{m\geq 0}$ be any sequence such that $X_0 = Z_0$, $X_m = Z_m$ or $\overline{Z}_m$ for each $m > 0$. Then by Lemmas 0.4 and 0.6, each such sequence is a discrete ω-sequence. Since $Z_m \not\cong \overline{Z}_m$, it follows as in Remark 1 that there are c such sequences and that distinct sequences determine distinct sequences of degrees; similarly if $X_0 = \overline{Z}_0$. That the sequences are contained in the btt-degree of Z_0 follows from Lemma 0.3 and the fact that $Z_0 = \overline{K}$, as in the proof of Lemma 1.10.

Lemma 4.5. *Let $X = Z_1$ or $\overline{Z}_1$. Then*



is an initial segment of the partial ordering of index sets under one-one reducibility.

Proof. Let B be any class of r.e. sets. It is well known that $\emptyset$, $N < K$, $\overline{K}$, which together with Lemma 0.4, implies $\emptyset$, $N < Z_0$, $\overline{Z}_0 < X$. Assume B is a class of r.e. sets such that $\theta B \leq X$. We will show that $\theta B \cong \emptyset$, N, Z_0 or $\overline{Z_0}$.

Case 1. $B = \emptyset$ or $\overline{B} = \emptyset$. Then $\theta B = \emptyset$ or $\theta B = N$, respectively.

Case 2. $B \neq \emptyset$ and $\overline{B} \neq \emptyset$. If $\emptyset \in B$, then by Lemma 0.1, $\overline{K} = Z_0 \leq \theta B \leq X$. So by Lemma 0.6, $\theta B \cong Z_0$ or $\theta B \cong X$. If $\emptyset \notin B$, then by Lemma 0.1, $\overline{K} = Z_0 \leq \theta\overline{B} \leq \overline{X}$ so, by Lemma 0.6, $\theta\overline{B} \cong Z_0$ or $\theta\overline{B} \cong \overline{X}$. It follows that $\theta B \cong \overline{Z_0}$ or $\theta B \cong X$, which completes the proof.

Remark 2. Lemma 4.5 cannot be strengthened by replacing Z_1 by Z_m for $m > 1$; i.e., we can prove that, for all $m > 1$, Z_m has a predecessor θB such that $\theta B \not\cong Z_k$ or $\overline{Z}_k$ for any $k < m$. The proof will appear elsewhere [7].

Theorem 4.6. *If S is co-r.e., then the 1-degrees of θA_0^S and $\theta \overline{A}_0^S$ are at the bottom of c discrete ω-sequences of 1-degrees. If $S \neq N$, these sequences are all contained in the bounded truth-table degree of θA_0^S.*

Proof. *Case* 1. $S \neq N$. Then $\theta\overline{A}_0^S = \{x \mid W_x \cap \overline{S} \neq \emptyset\}$ is r.e. and $N \in \theta\overline{A}_0^S$, so $\theta\overline{A}_0^S \leq K$ and, by Lemma 0.1, $K \leq \theta\overline{A}_0^S$. So $\theta\overline{A}_0^S \cong K = \overline{Z}_0$ and $\theta A_0^S \cong Z_0$. The conclusion then follows from Lemma 4.4.

Case 2. $S = N$. Then $\theta A_0^S = N$ and $\theta\overline{A}_0^S = \emptyset$. Let $\{X_m\}_{m\geq 0}$ be any sequence such that $X_0 = N$, $X_{m+1} = Z_m$ or $\overline{Z}_m$ for all m. As in the proof of Lemma 4.4, there are c such sequences, and similarly if $X_0 = \emptyset$. These sequences are discrete, by Lemmas 4.4 and 4.5, and determine distinct sequences of degrees since $X_m \cong \overline{X}_m$, for all m.

Theorem 4.7. *Let S be any infinite set $\subsetneq N$ such that $K \not\leq_T S$. If f is any acceptable index function, the S-sequence of index sets determined by f is a discrete ω-sequence of index sets, all contained in the bounded truth-table degree of θA_0^S.*

Proof. Let θA_n denote $\theta A^S_{\sigma(n,f)}$. By Lemma 1.10, each θA_n is in the btt-degree of θA_0^S. By Lemma 2.14, $\theta A_n = X^S_{\sigma(n,f)} < X^S_{\sigma(n+1,f)} = \theta A_{n+1}$, since $\sigma(n+1, f) = \sigma(n, f) * i$ and $\sigma(n, f), \sigma(n+1, f) \in \mathfrak{A}$. It remains to show the sequence is discrete. Assume $\theta A_n \leq \theta B \leq \theta A_{n+1}$.

Case 1. $f(n+1) = 1$. Then $\sigma(n+1, f) = \sigma(n, f) * 1$, so by Lemma 1.9, $\theta A_{n+1} = X^S_{\sigma(n+1,f)} \cong K \times X^S_{\sigma(n,f)} = K \times \theta A_n$. But by Lemma 0.9, $\theta A_n \leq \theta B \leq \theta A_{n+1} \cong K \times \theta A_n$ implies $\theta B \cong \theta A_n$ or $\theta B \cong \theta A_{n+1}$.

Case 2. $f(n+1) = 2$. Then $\sigma(n+1, f) = \sigma(n, f) * 2$ so by Lemma 1.9, $\theta A_{n+1} = X^S_{\sigma(n+1,f)} \cong \overline{K} \times X^S_{\sigma(n,f)} = \overline{K} \times \theta A_n$. Then by Lemma 0.10, $\theta A_n \leq \theta B \leq \theta A_{n+1} \cong \overline{K} \times \theta A_n$ implies $\theta B \cong \theta A_n$ or $\theta B \cong \theta A_{n+1}$.

Case 3. $f(n+1) = \overline{1}$ or $\overline{2}$. Then $\sigma(n+1, f) = \sigma(n, f) * i$ where $i = 1$ or 2, and $\theta A_n \leq \theta B \leq \theta A_{n+1}$ implies $\theta\overline{A}_n \leq \theta\overline{B} \leq \theta\overline{A}_{n+1}$ where $\theta\overline{A}_n = X_{\overline{\sigma}(n,f)}$ and $\theta\overline{A}_{n+1} = X_{\overline{\sigma}(n+1,f)} = X_{\overline{\sigma}(n,f)*i}$, where $i = 1$ or 2. By Cases 1 and 2, replacing σ by $\overline{\sigma}$ (since $\sigma \in \mathfrak{A} \leftrightarrow \overline{\sigma} \in \mathfrak{A}$), $\theta\overline{A}_n \leq \theta\overline{B} \leq \theta\overline{A}_{n+1}$ implies $\theta\overline{B} \cong \theta\overline{A}_n$ or $\theta\overline{B} \cong \theta A_{n+1}$. The result follows by complementation.

Lemma 4.8. *Let S be any infinite set $\subsetneq N$ such that $K \leq_T S$. If f and g are acceptable index functions which determine the same S-sequence of 1-degrees, then $f = g$.*

Proof. Assume $f \neq g$. Then $f(k) \neq g(k)$ for some $k \in N$. Let n be the least such k. It will suffice to show that $\theta A^S_{\sigma(n,f)} \not\cong \theta A^S_{\sigma(n,g)}$.

Case 1. $n = 0$. Since f, g are a.i.f.'s, $f(0) \in \{0, \overline{0}\}$ and $g(0) \in \{0, \overline{0}\}$; since $f(0) \neq g(0)$, assume $f(0) = 0$ and $g(0) = \overline{0}$. Then, $\theta A^S_{\sigma(0,f)} = X_0^S$ and $\theta A^S_{\sigma(0,g)} = X^S_{\overline{0}} = \overline{X^S_0}$, and, by Lemma 1.11, $X_0^S \not\cong X^S_{\overline{0}}$.

Case 2. $n = m + 1$ for some $m \geq 0$. Then since n is the least k such that $f(k) \neq g(k)$, $\sigma(m, f) = \sigma(m, g)$; let τ denote this common index sequence. Then $\sigma(n, f) = \sigma(m, f) * f(m+1) = \tau i$ and $\sigma(n, g) = \sigma(m, g) * g(m+1) = \tau j$ where $i \neq j$ by hypothesis. Since f, g are a.i.f.'s, $\tau i \in \mathfrak{A}$ and $\tau j \in \mathfrak{A}$. Then by Lemma 2.15, $\theta A_{\sigma(n,f)} = X^S_{\tau i} \not\equiv X^S_{\tau j} = \theta A^S_{\sigma(n,g)}$.

Theorem 4.9. *Let S be any set such that $K \not\leq_T S$. Then the 1-degrees of θA^S_0 and $\overline{\theta A^S_0}$ are at the bottom of c discrete ω-sequences of 1-degrees. If $S \neq N$, these sequences are contained in the bounded truth-table degree of θA^S_0.*

Proof. *Case* 1. S is finite or $S = N$. Then S is co-r.e., so the result follows from Theorem 4.6.

Case 2. S is infinite, $S \subsetneq N$. By Remark 1, there are c acceptable index functions such that $f(0) = 0$, and c such that $f(0) = \bar{0}$. By Lemma 4.8, these functions determine different S-sequences of 1-degrees. By Theorem 4.7, these sequences are discrete are contained in the btt-degree of θA^S_0.

Definition 4.10. For any set P, let

(a) $P' = \{x \mid x \in W^P_x\}$,

(b) $P_0 = \{\langle u, v\rangle \mid D_u \subseteq P \text{ and } D_v \subseteq \bar{P}\}$,

(c) $P^* = X^{\overline{P0}}_{\bar{0}} = \{x \mid W_x \cap P_0 \neq \varnothing\}$.

Lemma 4.11. *For all sets P, $P \leq P_0$ and $P_0 \leq_{tt} P$.*

Proof. Let $D_0 = \varnothing$ and let $g(n)$ be a recursive function such that $D_{g(n)} = \{n\}$ for each n. Then $P \leq P_0$ via $h(n) = \langle g(n), 0\rangle$. It is easily seen that $P_0 \leq_{tt} P$ via (unbounded) truth-tables.

Lemma 4.12. *For all sets P, $P' \cong P^*$.*

Proof. P^* is r.e. in P, which implies $P^* \leq P'$. Let $g(x)$ be a recursive function defined by $W_{g(x)} = \{\langle u, v\rangle \mid (\exists y)\ (\langle x, y, u, v\rangle \in W_{\rho(x)})\}$ where $\rho(x)$ is as in [6, p. 132]. It is easily verified that $P' \leq P^*$ via g.

Lemma 4.13. *Let* $\mathbf{a}$ *be any Turing degree such that* $\mathbf{0}' \leq \mathbf{a}$. *Then there exists a set P such that*

(a) *P is not r.e,*

(b) *$K \not\leq_T P$,*

(c) *$P' \in \mathbf{a}$.*

Proof. Assume $\mathbf{0}' \leq \mathbf{a}$.

Case 1. $\mathbf{0}' < \mathbf{a}$. By Friedberg's Theorem [6, Corollary 13–IX(a)], there exists $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{b} \cup \mathbf{0}' = \mathbf{a}$. Clearly $\mathbf{0}' \not\leq \mathbf{b}$, while $\mathbf{b} \leq \mathbf{0}'$ implies $\mathbf{a} = \mathbf{0}'$. So $\mathbf{b} \mid \mathbf{0}'$, and any $P \in \mathbf{b}$ will satisfy the conditions of the lemma.

Case 2. $\mathbf{0}' = \mathbf{a}$. It is a well-known fact (proved by Friedberg) that there exists $\mathbf{d}$ such that $\mathbf{0} < \mathbf{d}$ and $\mathbf{d}' = \mathbf{0}'$; any such $\mathbf{d}$ contains a non-r.e. set P, and for such a P, $K \not\leq_T P$.

Lemma 4.14. *Let S be any infinite set such that $\overline{S}$ is not r.e. and $K \not\leq_T S$, and let f be any acceptable index function. Then the S-sequence of* 1*-degrees determined by f does not contain the* 1*-degree of Z_m or $\overline{Z}_m$ for any $m \geq 0$.*

Proof. It must be shown that for all m, n, $X^S_{\sigma(n,f)} \not\cong Z_m$ or $\overline{Z}_m$. Since by Theorem 2.7, each Z_m, $\overline{Z}_m \cong Y_\tau$ for some $\tau \in \mathfrak{A}$, it suffices to show that $X^S_\sigma \not\cong Y_\tau$ for any $\sigma, \tau \in \mathfrak{A}$.

Case 1. $l_\sigma < l_\tau$. Then by Lemma 2.4, $Y_{\overline{\sigma}} < Y_\tau$, so $X^S_\sigma \cong Y_\tau$ implies $Y_{\overline{\sigma}} < X^S_\sigma$, contradicting Lemma 2.13.

Case 2. $l_\sigma > l_\tau$. Then by Lemma 2.4, $Y_\tau < Y_\sigma$, and by Lemma 2.11, $Y_\sigma < X^S_\sigma$. So $Y_\tau < X^S_\sigma$ which implies $X^S_\sigma \not\cong Y_\tau$.

Case 3. $l_\sigma = l_\tau$. Assume $X^S_\sigma \cong Y_\tau$. By Lemma 2.4(a), $Y_\tau \cong Z_{l_\tau - 1}$ or $\overline{Z}_{l_\tau - 1}$, so $X^S_\sigma \cong Z_{l_\tau - 1}$ or $\overline{Z}_{l_\sigma - 1}$. Also by Lemma 2.4(a), $Y_\sigma \cong Z_{l_\sigma - 1}$ or $\overline{Z}_{l_\sigma - 1}$ and, since $l_\sigma = l_\tau$, $Z_{l_\sigma - 1} = Z_{l_\tau - 1}$. It follows that $X^S_\sigma \cong Y_\sigma$ or $\overline{Y}_\sigma$. But, by Lemma 2.11, $Y_\sigma < X^S_\sigma$ and, by Lemma 2.13, $\overline{Y}_\sigma = Y_{\overline{\sigma}} \not\leq X^S_\sigma$. Thus either way we get a contradiction.

Theorem 4.15. *Let* $\mathbf{a}$ *be any Turing degree such that* $\mathbf{0}' \leq \mathbf{a}$. *Then* $\mathbf{a}$ *contains c discrete ω-sequence of* 1*-degrees, none of whose elements are* 1*-degrees of Z_m or $\overline{Z}_m$ for any m.*

Proof. Assume $\mathbf{a} \geq \mathbf{0}'$. By Lemma 4.13, there is a set P such that P is not r.e., $K \leq_T P$ and $P' \in \mathbf{a}$. Now by Lemma 4.12, $P^* = X_{\overline{0}}^{\overline{P_0}} \cong P'$, so $X_{\overline{0}}^{\overline{P_0}} \in \mathbf{a}$. By Lemma 4.11, $P \leq P_0$ and $P_0 \leq_T P$. It follows that P not r.e. $\rightarrow P_0$ not r.e., and that $K \not\leq_T P \rightarrow K \leq_T P_0$. The bounded truth-table degree of $\theta A_0^{\overline{P_0}}$ is then contained in a, so that by Theorem 4.9, a contains c discrete ω-sequences of 1-degrees and by Lemma 4.14, these ω-sequences do not contain the 1-degree of Z_m or $\overline{Z}_m$ for any m.

In [3] it was conjectured that for each $m \geq 0$, there exists a class A with $Z_m < \theta A$ and $\overline{Z}_m \not\leq \theta A$. The present technique yields the following stronger result:

Theorem 4.16. *Every Turing degree* $\mathbf{a} \geq \mathbf{0}'$ *contains a discrete ω-sequence $\{\theta A_m\}_{m \geq 0}$ of index sets such that, for each m, $Z_m < \theta A_m$ and $\overline{Z}_m \not\leq \theta A_m$.*

Proof. Assume $\mathbf{a} \geq \mathbf{0}'$, and let P_0 be as in Theorem 4.15, i.e., P_0 is not r.e., $K \not\leq_T P_0$ and $X_{\overline{0}}^{\overline{P_0}} \in \mathbf{a}$. By Lemma 3.4, there exists an acceptable index function f such that, for all $m \geq 0$, $Z_m \cong Y_{\sigma(m,f)}$. Let $A_m = A^{P_0}_{\sigma(m,f)}$. Then by Theorem 4.7, $\{\theta A_m\}_{m \geq 0}$ is a discrete ω-sequence of index sets contained in a; for each m, $\underline{Z_m} \cong Y_{\sigma(m,f)} < \theta A^{\overline{P_0}}_{\sigma(m,f)} = \theta A_m$, by Lemma 2.11; and $\overline{Z}_m \cong Y_{\overline{\sigma}(m,f)} \not\leq \theta A^{\overline{P_0}}_{\sigma(m,f)} = \theta A_m$, by Lemma 2.13.

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