DISCRETE SERIES FOR SEMISIMPLE LIE GROUPS. II

EXPLICIT DETERMINATION OF THE CHARACTERS

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§ 1. Introduction

Let G be a connected semisimple Lie group and K a maximal compact subgroup of G. We shall show in this paper that G has a discrete series (see [4 (d), § 5]) if and only if it has a compact Cartan subgroup B. Let $\mathcal{E}_d$ denote the set of all equivalence classes of irreducible unitary representations of G, which are square-integrable. For any $\omega \in \mathcal{E}_d$, let Θ_ω denote the character, χ_ω the infinitesimal character and $d(\omega)$ the formal degree (see [4 (d), § 3]) of ω. Then it is known [4 (d), § 5] that the distribution

$$T = \sum_{\omega \in \mathcal{E}_d} d(\omega) \, \Theta_\omega$$

represents the contribution of the discrete series to the Plancherel formula of G. We intend to obtain explicit formulas for $d(\omega)$, Θ_ω and T.

Let $\mathfrak{g}$ and $\mathfrak{b}$ be the Lie algebras of G and B respectively. Then the character group of B may be identified with a lattice L in the space of all real-valued linear functions $\mathfrak{F}$ on $(-1)^{\frac{1}{2}} \mathfrak{b}$. Put

$$\varpi(\lambda) = \prod_{\alpha > 0} \langle \lambda, \alpha \rangle \quad (\lambda \in \mathfrak{F}),$$

where α runs over all positive roots of $(\mathfrak{g}, \mathfrak{b})$ and the scalar product is defined, as usual, by means of the Killing form. Let L' denote the set of all $\lambda \in L$ where $\varpi(\lambda) \neq 0$. Then, for every $\lambda \in L'$, we have constructed in [4, (p)], an invariant eigendistribution Θ_λ of $\mathfrak{Z}$ on G. Put $q = \frac{1}{2} \dim G/K$ and $\varepsilon(\lambda) = \operatorname{sign} \varpi(\lambda)$ $(\lambda \in L')$. Our main result (Theorem 16) states that the distributions $(-1)^q \varepsilon(\lambda) \Theta_\lambda$ $(\lambda \in L')$ are exactly the characters of the discrete series and

$$T = c^{-1}(-1)^q \sum_{\lambda \in L'} \varpi(\lambda) \, \Theta_\lambda$$

where c is a positive constant.

Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{b})$ and $\tilde{B}$ the normalizer of B in G. Then $W_G = \tilde{B}/B$ may be regarded as a subgroup of W. Define χ_λ $(\lambda \in L')$ as usual (see [4 (p), § 29]) and let $\mathcal{E}_d(\lambda)$ denote the set of all $\omega \in \mathcal{E}_d$ such that $\chi_\omega = \chi_\lambda$. Then[1]

$$[\mathcal{E}_d(\lambda)] = [W(\lambda)][W_G]^{-1},$$

where $W(\lambda)$ is the set of all $s \in W$ such that $s\lambda \in L$.

Fix $\lambda \in L'$ and let $\Theta_{\lambda, \mathfrak{b}}$ $(\mathfrak{b} \in \mathcal{E}_K)$ denote the Fourier components of Θ_λ (see [4 (q), § 17]). Then $\Theta_{\lambda, \mathfrak{b}}$ are analytic functions and it is one of the principal steps of the proof to show that they lie in $L_2(G)$ (Theorem 12). This is done by means of Lemma 67, whose proof is based on two key results (Lemmas 42 and 43), which are derived from a study of certain differential equations.

This paper is divided into four parts. After recalling some known facts about representations on a locally convex space, we prove Theorem 1, which seems to play an important role in harmonic analysis. Then we introduce the space $\mathcal{C}(G)$, which is the analogue of the Schwartz space over $\mathbf{R}^n$. Theorem 2 asserts that $C_c^\infty(G)$ is dense in $\mathcal{C}(G)$ and Theorem 3 allows us to reduce certain problems from G to a proper subgroup. Theorem 4 contains a general result which implies the convergence of certain integrals (Theorems 5 and 6) and thus enables us to define the mapping $f \to F_f (f \in \mathcal{C}(G))$ in § 18. A distribution on G is called tempered if it extends to a continuous linear function on $\mathcal{C}(G)$. Theorem 7 gives a

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[1] As usual, $[F]$ denotes the number of elements in a set F.

simple necessary and sufficient condition for an invariant eigendistribution to be tempered. This shows, in particular, that Θ_λ $(\lambda \in L')$ is tempered and Theorem 3 of [4 (q)] remains valid for $f \in C(G)$ (Theorem 8). This permits us to prove the second conjecture of [4 (k), § 16] and thus complete the proof of the Plancherel formula for G/K. Theorem 9 established a weak estimate for a Fourier component Θ_b of a tempered and invariant eigendistribution Θ. This will be required in Parts II and III.

The main problem in Part II is to determine the behaviour, at infinity, of a K-finite eigenfunction f of $\mathfrak{Z}$, which satisfies a weak inequality of the type mentioned above. The principal result (Lemma 43) is that f lies in $L_2(G)$ if and only if it lies in $C(G)$. This is proved by induction on dim G. By making use of the differential equations, one reduces the problem from G to a proper subgroup M (cf. [4 (k)]).

In Part III, we apply the above results to the task of determining the eigenfunctions of $\mathfrak{Z}$ in $C(G)$. Here Lemma 64 plays an important role. It enables us to show that such eigenfunctions do not exist unless rank $G =$ rank K. As an application we obtain in § 33 a proof of a conjecture of Selberg.

These results are then utilized to determine all the characters of the discrete series. Here the fact that we work in $C(G)$, rather than $C_c^\infty(G)$, is decisive. First we show that Θ_ω $(\omega \in \mathcal{E}_d)$ is tempered and there exists a $\lambda \in L'$ such that $\chi_\omega = \chi_\lambda$. Let $s_1, s_2, ..., s_r$ be a complete set of representatives for $W_G \backslash W(\lambda)$. Then we prove that

$$\Theta_\omega = \sum_{1 \leqslant i \leqslant r} c_i \Theta_{s_i \lambda},$$

where c_i are complex numbers. Moreover, by making use of the Schur orthogonality relations, it is possible to establish that

$$\sum_{1 \leqslant i \leqslant r} |c_i|^2 = 1.$$

On the other hand, one verifies that c_i are integers. This proves that

$$\Theta_\omega = (-1)^q \varepsilon(s_i, \lambda) \Theta_{s_i \lambda}$$

for some i and $d(\omega) = c^{-1}[W_G] |\varpi(\lambda)|$. It should be noticed that the entire discussion of §§ 40, 41 is quite similar to Weyl's original treatment of the same problem in the compact case (see [10, § 3]). The main task here is to relate the Fourier analysis on G, so far as the discrete series is concerned, to that on B (see Theorem 14 and Lemma 81, together with its corollaries). This can be done only by operating in $C(G)$.

Part IV deals with certain inequalities which are needed for the proof of Lemma 21. They will also be useful later, when we come to the continuous series for G in another paper.

Some of the results of this paper have been announced in [4 (l)].

Part I. Analysis in the space $C(G)$

§ 2. Representations on a locally convex space

In this section we recall some elementary and well-known facts about representations on locally convex spaces (see [2, p. 109]).

Let V be a Hausdorff, locally convex (real or complex) vector space and G a locally compact topological group. A representation π of G on V is a mapping, which assigns, to every $x \in G$, a continuous endomorphism $\pi(x)$ of V such that the following two conditions hold:

1) $\pi(xy) = \pi(x)\pi(y)$ $(x, y \in G)$ and $\pi(1) = I$.

2) $(x, v) \to \pi(x)v$ is a continuous mapping of $G \times V$ into V.

It is easy to verify that the above two conditions imply the following [2, p. 110].

3) Let C be a compact set in G. Then for any neighborhood U_0 of zero in V, we can choose another neighborhood U of zero such that $\pi(C)U \subset U_0$.

Let S be the set of all continuous seminorms on V. Then the following immediate consequence of 3) will be frequently useful.

4) Given a compact set C in G and an element $\nu_0 \in S$, we can choose $\nu \in S$ such that

$$\nu_0(\pi(x)\,v) \leqslant \nu(v)$$

for all $x \in C$ and $v \in V$.

Conversely we have the following result.

LEMMA 1. *Suppose π satisfies, in addition to* 1), *the following two conditions.*

2′) *The mapping $x \to \pi(x)v$ of G into V is continuous for every $v \in V$.*

3′) *There exists a neighborhood U of* 1 *in G with the following property. Given $\nu_0 \in S$, we can choose $\nu \in S$ such that*

$$\nu_0(\pi(u)v) \leqslant \nu(v)$$

for all $u \in U$ and $v \in V$.

Then π is a representation of G on V.

Fix $x_0 \in G$ and $v_0 \in V$. Then

$$\pi(x_0 u)v - \pi(x_0)v_0 = \pi(x_0)\pi(u)(v - v_0) + \pi(x_0)(\pi(u)v_0 - v_0)$$

for $u \in U$ and $v \in V$. Hence it is clear that $\pi(x_0 u)v \to \pi(x_0)v_0$ as $u \to 1$ and $v \to v_0$. This proves our assertion.

Now assume that V is complex and complete, G is unimodular and dx is a Haar measure on G. π being a representation of G on V, we define, for any $f \in C_c(G)$, a linear transformation $\pi(f)$ in V by

$$\pi(f)\,v = \int f(x)\,\pi(x)\,v\,dx \quad (v \in V).$$

It follows from 3) that $\pi(f)$ is continuous. Define the convolution $f \ast g$ $(f,\,g \in C_c(G))$ as usual, so that

$$(f \ast g)\,(x) = \int f(y)\,g(y^{-1}x)\,dy \quad (x \in G),$$

Then $\pi(f \ast g) = \pi(f)\pi(g)$.

Let us now suppose that G is a Lie group. A vector $v \in V$ is said to be differentiable (under π) if the mapping $x \to \pi(x)v$ of G into V is of class C^∞. Let V^∞ denote the space of all differentiable vectors in V. $\mathfrak{g}$ being the Lie algebra of G, define

$$\pi(X)\,v = \lim_{t \to 0} \; t^{-1}(\pi(\exp tX)\,v - v) \quad (t \in \mathbf{R},\, t \neq 0)$$

for $X \in \mathfrak{g}$ and $v \in V^\infty$. Then $\pi(X)$ is a linear transformation in V^∞ and $X \to \pi(X)$ is a representation of $\mathfrak{g}$ on V^∞. Let $\mathfrak{G}$ be the universal enveloping algebra of $\mathfrak{g}_c$. Then this extends uniquely to a representation of $\mathfrak{G}$ which we denote again by π.

As usual we regard elements of $\mathfrak{G}$ as left-invariant differential operators on G. Define the anti-isomorphism ϱ of $\mathfrak{G}$ onto the algebra of right-invariant differential operators on G as in [4 (o), § 2]. U being any open set in G, we write

$$f(g;\, x) = f(x;\, \varrho(g)), \quad f(g;\, x;\, g') = f(x;\, \varrho(g) \circ g') \quad (x \in U;\, g,\, g' \in \mathfrak{G})$$

for $f \in C^\infty(U)$.

LEMMA 2. *Let* $f \in C_c^\infty(G)$. *Then* $\pi(f)\,V \subset V^\infty$ *and*

$$\pi(X)\pi(f)\,v = -\pi(\varrho(X)f)\,v$$

for $X \in \mathfrak{g}$ *and* $v \in V$.

It is clear that

$$\pi(y)\pi(f)\,v = \pi(_yf)\,v,$$

where $_yf(x) = f(y^{-1}x)$. Fix $X \in \mathfrak{g}$ and put $y_t = \exp tX$ $(t \in \mathbf{R})$. Then

$$f(y_t^{-1}x) - f(x) = -t \int_0^1 f'(y_{st}^{-1}x)\,ds,$$

where $f' = \varrho(X)f$. Hence

$$t^{-1}(\pi(y_t) - 1)\,\pi(f)\,w = -\int_0^1 \pi(y_{st})\,v'\,ds,$$

where $v' = \pi(f')\,v$. The statements of the lemma are now obvious.

Let $g \to g^*$ be the anti-automorphism of $\mathfrak{G}$ such that $X^* = -X$ $(X \in \mathfrak{g})$.

COROLLARY 1. $\pi(g)\pi(f)v = \pi(\varrho(g^*)f)v$ for $g \in \mathfrak{G}$.

This is obvious from the lemma.

COROLLARY 2. *Let* $v \in V^\infty$ *and* $f \in C_c^\infty(G)$. *Then*

$$\pi(f)\pi(g)v = \pi(g^*f)v \quad (g \in \mathfrak{G}).$$

Define $f_y(x) = f(xy^{-1})$ $(x \in G)$ for a fixed $y \in G$. Then $\pi(f)\pi(y)v = \pi(f_y)v$ and from this we conclude, as in the proof of Lemma 2 that

$$\pi(f)\pi(X)v = -\pi(Xf)v \quad (X \in \mathfrak{g}).$$

Clearly this implies the required result.

Let f_j $(j \geqslant 1)$ be a sequence in $C_c^\infty(G)$. We say that it is a Dirac sequence if the following conditions hold. $f_j \geqslant 0$ and

$$\int f_j dx = 1$$

and, for any neighborhood U of 1, Supp $f_j \subset U$ for all j except a finite number. Let K be a compact subgroup of G. It is obvious that there always exists a Dirac sequence f_j $(j \geqslant 1)$ such that $f_j(kxk^{-1}) = f_j(x)$ $(k \in K, x \in G)$.

LEMMA 3. *Let* f_j $(j \geqslant 1)$ *be a Dirac sequence in* $C_c^\infty(G)$. *Then*

$$\lim_{j \to \infty} \pi(f_j)v = v$$

for every $v \in V$.

Fix $v \in V$, $\nu \in S$ and $\varepsilon > 0$. Then we can choose a neighborhood U of 1 in G such that $\nu(\pi(x)v - v) \leqslant \varepsilon$ for $x \in U$. Now

$$\pi(f_j)v - v = \int f_j(x)(\pi(x)v - v)\,dx,$$

and therefore $\nu(\pi(f_j)v - v) \leqslant \varepsilon$ if Supp $f_j \subset U$. This proves the lemma.

COROLLARY 1. V^∞ *is dense in* V.

This is obvious from Lemmas 2 and 3.

COROLLARY 2. *Fix* $v_0 \in V$ *and let* V_0 *be the smallest closed subspace of* V *containing* v_0 *which is stable under* $\pi(G)$. *Then elements of the form* $\pi(f)v_0$ $(f \in C_c^\infty(G))$ *are dense in* V_0.

Let W be the space consisting of all elements of the form $\pi(f)v_0$ $(f \in C_c^\infty(G))$. It is obvious that W is stable under $\pi(G)$. Therefore the same holds for $\mathrm{Cl}(W)$. But then since $v_0 \in \mathrm{Cl}(W)$ from Lemma 3, our assertion follows.

Let K be a compact subgroup of G and $\mathcal{E}_K$ the set of all equivalence classes of finite-dimensional irreducible representations of K. For any $\mathfrak{d} \in \mathcal{E}_K$, define a linear transformation $E_\mathfrak{d}$ in V by

$$E_\mathfrak{d} v = d(\mathfrak{d}) \int_K \mathrm{conj}\ \xi_\mathfrak{d}(k) \cdot \pi(k)\, v\, dk \quad (v \in V).$$

Here $d(\mathfrak{d})$ is the degree and $\xi_\mathfrak{d}$ the character of $\mathfrak{d}$ and dk is the normalized Haar measure of K. Then $E_\mathfrak{d}$ is a continuous projection. Put $V_\mathfrak{d} = E_\mathfrak{d} V$.

LEMMA 4. $\sum_{\mathfrak{d} \in \mathcal{E}_K} V_\mathfrak{d} \cap V^\infty$ *is dense in* V.

We shall give a proof of this lemma in § 6.

§ 3. Absolute convergence of the Fourier series

As before let S be the set of all continuous seminorms on a complex locally convex space V, which we assume to be complete. Let $\{v_j\}_{j \in J}$ be an indexed family of elements of V. We say that the series

$$\sum_{j \in J} v_j$$

converges, if the following condition holds. Define

$$s_F = \sum_{j \in F} v_j$$

for any finite subset F of J. Then for any neighborhood V_0 of zero in V, there should exist a finite subset F_0 of J such that $s_{F_1} - s_{F_2} \in V_0$ for any two finite subsets F_1, F_2 of J containing F_0. Since V is complete, the partial sums s_F then have a limit s in V. s is called the sum of the series and we write

$$s = \sum_{j \in J} v_j.$$

Moreover, the series is said to converge absolutely if

$$\sum_{j \in J} \nu(v_j) < \infty$$

for every $\nu \in S$. It is obvious that absolute convergence implies convergence.

Let π be a representation of a compact Lie group K on V and define $E_\mathfrak{d}\,(\mathfrak{d} \in \mathcal{E}_K)$ as in § 2. Then for any vector $v \in V$, we call $E_\mathfrak{d} v$ the $\mathfrak{d}$th Fourier component of v.

LEMMA 5. *Let v be a differentiable vector in V. Then the Fourier series*

$$\sum_{\mathfrak{b} \in \mathcal{E}_K} E_{\mathfrak{b}} v$$

converges absolutely to v.

Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{K}$ the universal enveloping algebra of $\mathfrak{k}_c$. Since K is compact, we can choose a positive-definite quadratic form Q on $\mathfrak{k}$ which is invariant under the adjoint representation of K. Let $X_1, ..., X_r$ be a base for $\mathfrak{k}$ over $\mathbf{R}$ orthonormal with respect to Q and put

$$\Omega = 1 - (X_1{}^2 + ... + X_r{}^2) \in \mathfrak{K}.$$

Also put $\alpha_{\mathfrak{b}} = d(\mathfrak{b}) \text{ conj } \xi_{\mathfrak{b}}\ (\mathfrak{b} \in \mathcal{E}_K)$ so that $E_{\mathfrak{b}} = \pi(\alpha_{\mathfrak{b}})$. It is obvious that Ω, regarded as a differential operator, commutes with both left and right translations of K. Fix a unitary representation σ in the class $\mathfrak{b}$. Then $\sigma(\Omega)$ commutes with $\sigma(k)\ (k \in K)$ and therefore, by Schur's lemma, $\sigma(\Omega) = c(\mathfrak{b}) \sigma(1)$ where $c(\mathfrak{b}) \in \mathbf{C}$. However $\sigma(X_i)\ (1 \leqslant i \leqslant r)$ are obviously skew-adjoint operators. Hence $c(\mathfrak{b})$ is real and $\geqslant 1$. Therefore

$$\xi_{\mathfrak{b}}(k; \Omega) = \text{tr } (\sigma(k)\sigma(\Omega)) = c(\mathfrak{b})\, \xi_{\mathfrak{b}}(k) \quad (k \in K)$$

and this shows that $\Omega\alpha_{\mathfrak{b}} = c(\mathfrak{b})\alpha_{\mathfrak{b}}$. Hence we conclude from Corollary 2 of Lemma 2 that

$$E_{\mathfrak{b}}\pi(\Omega)v = c(\mathfrak{b}) E_{\mathfrak{b}} v \quad (\mathfrak{b} \in \mathcal{E}_K).$$

LEMMA 6. *Fix $v \in S$. Then we can select $v_0 \in S$ such that*

$$\nu(E_{\mathfrak{b}}v) \leqslant c(\mathfrak{b})^{-m} d(\mathfrak{b})^2 \nu_0(\pi(\Omega^m)v)$$

for $\mathfrak{b} \in \mathcal{E}_K$, any integer $m \geqslant 0$ and any differentiable vector $v \in V$.

Since K is compact, we can choose $\nu_0 \in S$ such that

$$\nu(\pi(k)u) \leqslant \nu_0(u)$$

for $k \in K$ and $u \in V$. Therefore

$$\nu(E_{\mathfrak{b}}u) = \nu(\pi(\alpha_{\mathfrak{b}})u) \leqslant d(\mathfrak{b})^2 \nu_0(u)$$

since $\sup |\alpha_{\mathfrak{b}}| \leqslant d(\mathfrak{b})^2$. Now we have seen above that

$$E_{\mathfrak{b}}v = c(\mathfrak{b})^{-m} E_{\mathfrak{b}}\pi(\Omega^m)v \quad (m \geqslant 0),$$

if v is differentiable. Hence our assertion follows immediately.

LEMMA 7.
$$\sum_{\mathfrak{b} \in \mathcal{E}_K} d(\mathfrak{b})^2 c(\mathfrak{b})^{-m} < \infty$$

if m is sufficiently large.

Assuming this for a moment, we shall first finish the proof of Lemma 5. It is obvious from Lemmas 6 and 7 that the series

$$\sum_{\mathfrak{d}} E_{\mathfrak{d}} v$$

converges absolutely. Let v_0 denote its sum and put $u = v - v_0$. We have to show that $u = 0$. Fix $\mathfrak{d}_0 \in \mathcal{E}_K$. Since $E_{\mathfrak{d}_0}$ is continuous,

$$E_{\mathfrak{d}_0} v_0 = \sum_{\mathfrak{d}} E_{\mathfrak{d}_0} E_{\mathfrak{d}} v = E_{\mathfrak{d}_0} v$$

from the Schur orthogonality relations. This shows that $E_{\mathfrak{d}} u = 0$ for all $\mathfrak{d} \in \mathcal{E}_K$.

Now fix $\nu \in S$ and select $\nu_0 \in S$ as in the proof of Lemma 6. For a given $\varepsilon > 0$, choose a neighborhood K_0 of 1 in K such that $\nu(\pi(k) u - u) \leqslant \varepsilon$ for $k \in K_0$. Fix a function $f \geqslant 0$ in $C(K)$ such that $f = 0$ outiside K_0 and

$$\int f(k) \, dk = 1.$$

Then

$$\nu(\pi(f) u - u) \leqslant \int f(k) \, \nu(\pi(k) u - u) \, dk \leqslant \varepsilon.$$

Call a function $\beta \in C(K)$ K-finite, if the space spanned by the left and right translates of β under K has finite dimension. Then by the Peter–Weyl theorem, we can choose a K-finite function $\beta \in C(K)$ such that $\sup |\beta - f| \leqslant \varepsilon$. Then

$$\nu(\pi(f) u - \pi(\beta) u) \leqslant \varepsilon \sup_{k \in K} \nu(\pi(k) u) \leqslant \varepsilon \nu_0(u)$$

and therefore

$$\nu(\pi(\beta) u - u) \leqslant \varepsilon(\nu_0(u) + 1).$$

On the other hand, $\alpha_{\mathfrak{d}} * \beta = \beta * \alpha_{\mathfrak{d}}$ since $\alpha_{\mathfrak{d}}$ is a class function. Therefore

$$E_{\mathfrak{d}} \pi(\beta) u = \pi(\beta) E_{\mathfrak{d}} u = 0.$$

Moreover, since β is K-finite, we can choose a finite subset F of $\mathcal{E}_K$ such that

$$\beta = \sum_{\mathfrak{d} \in F} \alpha_{\mathfrak{d}} * \beta.$$

Therefore

$$\pi(\beta) u = \sum_{\mathfrak{d} \in F} E_{\mathfrak{d}} \pi(\beta) u = 0$$

and this shows that

$$\nu(u) \leqslant \varepsilon(\nu_0(u) + 1).$$

Making ε tend to zero, we get $\nu(u) = 0$. Since ν was an arbitrary element in S, this implies that $u = 0$.

$$\sum_{\mathfrak{b}\in {}^{c}F} c(\mathfrak{b})^{-m} d(\mathfrak{b})^2 \leqslant 2^m \sum_{\lambda\in L^*} (1+q(\lambda))^{-m} |p(\lambda)|^2 < \infty,$$

provided m is sufficiently large.

§ 5. Differentiable vectors and Fourier series in function spaces

If M is a differentiable manifold, the spaces $C_c^\infty(M)$ and $C^\infty(M)$, taken with their usual topologies, are locally convex and complete. Let us now return to the notation of § 2 and for any $f \in C^\infty(G)$ and $y \in G$, define $l(y)f$ to be the function $x \to f(y^{-1}x)$ $(x \in G)$. We claim l is a representation of G on $C^\infty(G)$. For any compact subset Ω of G and $g \in \mathfrak{G}$, put

$$\nu_{\Omega,g}(f) = \sup_{\Omega} |gf| \quad (f \in C^\infty(G)).$$

Then the seminorms $\nu_{\Omega,g}$, taken toghether for all Ω and g, define the topology of $C^\infty(G)$ and it is clear that

$$\nu_{\Omega,g}(l(y)f) = \nu_{y^{-1}\Omega,g}(f).$$

Moreover, g and $l(y)$ commute as linear transformations in $C^\infty(G)$. Therefore if $f_1 = gf$,

$$\nu_{\Omega,g}(l(y)f - f) = \sup_{\Omega} |l(y)f_1 - f_1| \to 0$$

as $y \to 1$. Hence we conclude from Lemma 1 that l is a representation.

LEMMA 8. *Fix $f \in C^\infty(G)$. Then f is a differentiable vector under l and $l(X)f = -\varrho(X)f$ for $X \in \mathfrak{g}$.*

Fix $X \in \mathfrak{g}$ and put $y_t = \exp tX$ $(t \in \mathbf{R})$. Then it would be enough to verify that

$$\lim_{t\to 0} t^{-1}(l(y_t)f - f) = -\varrho(X)f$$

in $C^\infty(G)$. Fix Ω and g as above and put $f_1 = gf$. Then since g and $\varrho(X)$ commute, we have

$$\nu_{\Omega,g}(t^{-1}\{l(y_t)f - f\} + \varrho(X)f) = \sup_{\Omega} |t^{-1}\{l(y_t)f_1 - f_1\} + f_2|,$$

where $f_2 = \varrho(X)f_1$. But we have seen during the proof of Lemma 2 that

$$f_1(y_t^{-1}x) - f_1(x) = -t \int_0^1 f_2(y_{st}^{-1}x)\,ds \quad (x \in G).$$

Therefore
$$\sup_{\Omega} |t^{-1}\{l(y_t)f_1 - f_1\} + f_2| \leqslant \int_0^1 \sup_{\Omega} |l(y_{st})f_2 - f_2|\,ds \to 0$$

as $t \to 0$.

§ 4. Proof of Lemma 7

It remains to verify Lemma 7. Let K_0 be the connected component of 1 in K. Since K is compact, the index $N = [K:K_0]$ is finite. Put $\mathcal{E} = \mathcal{E}_K$ and let $\mathcal{E}_0$ be the set of all equivalence classes of irreducible finite-dimensional representations of K_0. If $\mathfrak{b} \in \mathcal{E}$ and $\mathfrak{b}_0 \in \mathcal{E}_0$, we denote by $[\mathfrak{b}:\mathfrak{b}_0]$ the number of times $\mathfrak{b}_0$ occurs in the reduction of $\mathfrak{b}$ with respect to K_0. For a given $\mathfrak{b}_0 \in \mathcal{E}_0$, let $\mathcal{E}(\mathfrak{b}_0)$ denote the set of all $\mathfrak{b} \in \mathcal{E}$ such that $[\mathfrak{b}:\mathfrak{b}_0] \geqslant 1$. Then it is a simple consequence of the Frobenius reciprocity theorem that

$$\sum_{\mathfrak{b} \in \mathcal{E}(\mathfrak{b}_0)} [\mathfrak{b}:\mathfrak{b}_0] d(\mathfrak{b}) = N d(\mathfrak{b}_0),$$

where $d(\mathfrak{b}_0)$ is the degree of $\mathfrak{b}_0$. Let $\xi_{\mathfrak{b}_0}$ denote the character (on K_0) of a class $\mathfrak{b}_0 \in \mathcal{E}_0$. Then it is easy to see that

$$\Omega \xi_{\mathfrak{b}_0} = c(\mathfrak{b}_0) \xi_{\mathfrak{b}_0},$$

where $c(\mathfrak{b}_0) = c(\mathfrak{b})$ for any $\mathfrak{b} \in \mathcal{E}(\mathfrak{b}_0)$. Therefore

$$\sum_{\mathfrak{b} \in \mathcal{E}} c(\mathfrak{b})^{-m} d(\mathfrak{b})^2 \leqslant \sum_{\mathfrak{b}_0 \in \mathcal{E}_0} c(\mathfrak{b}_0)^{-m} \sum_{\mathfrak{b} \in \mathcal{E}(\mathfrak{b}_0)} d(\mathfrak{b}_0)^2 \leqslant N^2 \sum_{\mathfrak{b}_0 \in \mathcal{E}_0} c(\mathfrak{b}_0)^{-m} d(\mathfrak{b}_0)^2.$$

Hence it would be enough to consider the case when K is connected.

Fix a Cartan subgroup A of K with Lie algebra $\mathfrak{a}$. Then A is connected. Let L be the kernel of the exponential mapping of $\mathfrak{a}$ into A. Then L is a lattice in $\mathfrak{a}$. Consider the space $\mathfrak{F}$ of all real-valued linear functions on $(-1)^{\frac{1}{2}}\mathfrak{a}$ and the lattice L^* of all $\lambda \in \mathfrak{F}$ such that $e^{\lambda(H)} = 1$ for $H \in L$. Introduce an order in $\mathfrak{F}$ and put

$$\varrho = \tfrac{1}{2} \sum_{\alpha > 0} \alpha,$$

where α runs over all positive roots of $(\mathfrak{k}, \mathfrak{a})$. For any $\mathfrak{b} \in \mathcal{E}$, let $\lambda(\mathfrak{b})$ denote the highest weight of $\mathfrak{b}$ with respect to $\mathfrak{a}$. Then the following facts are well known. $\lambda(\mathfrak{b}) \in L^*$ and $\mathfrak{b}$ is completely determined by $\lambda(\mathfrak{b})$. Moreover there exists a polynomial function p on $\mathfrak{F}$ such that $d(\mathfrak{b}) = p(\lambda(\mathfrak{b}))$ for all $\mathfrak{b} \in \mathcal{E}$. Finally, there exists a positive-definite quadratic form q on $\mathfrak{F}$ such that

$$c(\mathfrak{b}) = 1 + q(\lambda(\mathfrak{b}) + \varrho) - q(\varrho) \quad (\mathfrak{b} \in \mathcal{E}).$$

We can obviously choose a compact set C in $\mathfrak{F}$ such that

$$q(\lambda + \varrho) - q(\varrho) \geqslant q(\lambda)/2$$

for $\lambda \in \mathfrak{F}$ outside C. Let F denote the set of all $\mathfrak{b} \in \mathcal{E}$ such that $\lambda(\mathfrak{b}) \in C$. Then F is finite since $L^* \cap C$ is finite. Let ^cF denote the complement of F in $\mathcal{E}$. Then

For any $\alpha \in C(K)$ and $f \in C^\infty(G)$, define

$$\alpha * f = \int_K \alpha(k)\, l(k)\, f \, dk$$

and let $\alpha_\mathfrak{b}(\mathfrak{b} \in \mathcal{E}_K)$ have the same meaning as in § 3.

COROLLARY 1. *The series*

$$\sum_{\mathfrak{b} \in \mathcal{E}_K} \alpha_\mathfrak{b} * f \quad (f \in C^\infty(G))$$

converges absolutely to f in $C^\infty(G)$.

This is an immediate consequence of Lemmas 5 and 8.

COROLLARY 2. *Fix $f \in C_c^\infty(G)$. Then the series*

$$\sum_{\mathfrak{b} \in \mathcal{E}_K} \alpha_\mathfrak{b} * f$$

converges absolutely to f in $C_c^\infty(G)$.

Fix a compact set Ω in G such that $K\Omega = \Omega$ and $\mathrm{Supp}\, f \subset \Omega$. As usual, let $C_\Omega^\infty(G)$ denote the space of all functions $\phi \in C_c^\infty(G)$ whose support lies in Ω. Then $C_\Omega^\infty(G)$ is a closed subspace of $C_c^\infty(G)$ and the two topologies induced on it from $C^\infty(G)$ and $C_c^\infty(G)$ coincide. Therefore our assertion follows from Corollary 1.

Let $r(y)f$ $(y \in G, f \in C^\infty(G))$ denote the function $x \to f(xy)$ $(x \in G)$. Then one proves in the same way that r is a representation of G on $C^\infty(G)$ and every $f \in C^\infty(G)$ is differentiable under r. Moreover $l(g)f = \varrho(g^*)f$ and $r(g)f = gf$ $(g \in \mathfrak{G})$ in the notation of the corollaries of Lemma 2. Define

$$f * \alpha = \int_K \alpha(k^{-1})\, r(k)\, f \, dk \quad (\alpha \in C(K)).$$

Then the analogues of the two corollaries of Lemma 8 hold also for r.

Note that $l(x)$ and $r(y)$ $(x, y \in G)$ commute and hence

$$\alpha * (f * \beta) = (\alpha * f) * \beta \quad (\alpha, \beta \in C(K)).$$

We may therefore simply write $\alpha * f * \beta$.

Define a representation π of $G \times G$ on $C^\infty(G)$ as follows.

$$\pi((x, y))f = l(x)\, r(y)\, f \quad (x, y \in G, f \in C^\infty(G)).$$

Since $\quad \pi((x, y))f - f = l(x)\,(r(y)f - f) + (l(x)f - f),$

it follows from Lemma 1 that π is indeed a representation. It is obvious from Lemma 8 and its analogue for r, that every $f \in C^\infty(G)$ is differentiable under π.

LEMMA 9. *Let V be either one of the two spaces $C^\infty(G)$ or $C_c^\infty(G)$, taken with its usual topology. Then for any $f \in V$, the series*

$$\sum_{b_1, b_2 \in \mathcal{E}_K} \alpha_{b_1} \ast f \ast \alpha_{b_2}$$

converges absolutely to f in V.

If $V = C^\infty(G)$, this follows immediately from Lemma 5. The rest is proved in the same way as Corollary 2 of Lemma 8.

§ 6. Proof of Lemma 4

We now come to the proof of Lemma 4. Fix $v \in V$, $\nu \in S$ and $\varepsilon > 0$. Then, by Lemma 3, we can select $f \in C_c^\infty(G)$ such that

$$\nu(\pi(f)v - v) \leqslant \varepsilon.$$

Choose a compact set Ω in G such that $K\Omega = \Omega$ and Supp $f \subset \Omega$. Put

$$\mu(g) = \int |g| \, dx \quad (g \in C_c^\infty(G))$$

and

$$\alpha_F = \sum_{b \in F} \alpha_b$$

for any finite subset F of $\mathcal{E}_K$. Then μ is a continuous seminorm on $C_c^\infty(G)$ and Supp $(f - \alpha_F \ast f) \subset \Omega$. Therefore

$$\nu(\pi(f - \alpha_F \ast f)v) \leqslant c\mu(f - \alpha_F \ast f),$$

where

$$c = \sup_{x \in \Omega} \nu(\pi(x)v) < \infty.$$

Hence we conclude from Corollary 2 of Lemma 8 that

$$\nu(\pi(f - \alpha_F \ast f)v) \leqslant \varepsilon$$

if F is sufficiently large. Therefore

$$\nu(\pi(\alpha_F \ast f)v - v) \leqslant \nu(\pi(f - \alpha_F \ast f)v) + \nu(\pi(f)v - v) \leqslant 2\varepsilon.$$

Since

$$\pi(\alpha_F \ast f)v \in \sum_{b \in \mathcal{E}_K} V_b \cap V^\infty$$

from Lemma 2, the assertion of Lemma 4 is now obvious.

§ 7. Some elementary facts about σ and Ξ

Let $\mathfrak{g}$ be a reductive Lie algebra over **R**. Then $\mathfrak{g} = \mathfrak{c} + \mathfrak{g}_1$ where $\mathfrak{c}$ is the center and $\mathfrak{g}_1$ the derived algebra of $\mathfrak{g}$. Let θ be an automorphism of $\mathfrak{g}$ such that $\theta^2 = 1$ and let $\mathfrak{k}$ and $\mathfrak{p}$ be the subspaces of $\mathfrak{g}$ corresponding to the eigenvalues 1 and -1 respectively of θ. We assume that the quadratic form

$$Q(X) = -\operatorname{tr}(\operatorname{ad} X \, \operatorname{ad} \theta(X)) \quad (X \in \mathfrak{g})$$

is positive-definite on $\mathfrak{g}_1$.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and K the analytic subgroup of G corresponding to $\mathfrak{k}$. We assume that:

1) K is compact,

2) The mapping $(k, X) \to k \exp X$ $(k \in K, X \in \mathfrak{p})$ defines an analytic diffeomorphism of $K \times \mathfrak{p}$ onto G.

Let $\log$ denote the inverse of the exponential mapping from $\mathfrak{p}$ to $\exp \mathfrak{p}$. Suppose we have a Euclidean norm on $\mathfrak{p}$ such that $\|X^k\| = \|X\|$ $(k \in K, X \in \mathfrak{p})$ and

$$\langle X, (\operatorname{ad} Z)^2 Y \rangle = \langle (\operatorname{ad} Z)^2 X, Y \rangle \quad (X, Y, Z \in \mathfrak{p})$$

for the corresponding scalar product. Then it is easy to see that $\mathfrak{c}_\mathfrak{p} = \mathfrak{c} \cap \mathfrak{p}$ and $\mathfrak{p}_1 = [\mathfrak{k}, \mathfrak{p}] = \mathfrak{p} \cap \mathfrak{g}_1$ are mutually orthogonal (under this norm). Put

$$\sigma(x) = \|X\|$$

for $x = k \exp X$ $(k \in K, X \in \mathfrak{p})$ and extend θ to an automorphism of G (see [4(o), § 16]). Then it is obvious that

$$\sigma(x) = \sigma(\theta(x)) = \sigma(x^{-1}) \quad (x \in G).$$

If $|C|$ $(C \in \mathfrak{c}_\mathfrak{p})$ is an arbitrary Euclidean norm on $\mathfrak{c}_\mathfrak{p}$ and we define

$$\|C + X\|^2 = |C|^2 - \operatorname{tr}(\operatorname{ad} X \, \operatorname{ad} \theta(X)) \quad (C \in \mathfrak{c}_\mathfrak{p}, X \in \mathfrak{p}_1)$$

then all the above conditions are fulfilled.

LEMMA 10. $\sigma(xy) \leqslant \sigma(x) + \sigma(y)$ for $x, y \in G$.

We may obviously assume that $x = \exp X$, $y = \exp Y$ $(X, Y \in \mathfrak{p})$. Then if $xy = k \exp Z$ $(k \in K, Z \in \mathfrak{p})$, it is clear that

$$\exp 2Z = \theta(xy)^{-1} xy = \exp Y \cdot \exp 2X \cdot \exp Y.$$

Now define $Z(t) \in \mathfrak{p}$ by

$$\exp 2Z(t) = \exp tY \cdot \exp 2X \cdot \exp tY \quad (t \in \mathbf{R}).$$

Then $Z(t)$ is an analytic function of t and it follows by differentiating with respect to t that (see [5(a), p. 95])

$$\{(1 - \exp(-2 \operatorname{ad} Z(t)))/2 \operatorname{ad} Z(t)\} Z'(t) = \tfrac{1}{2}(1 + \exp(-2 \operatorname{ad} Z(t))) Y,$$

where $Z'(t) = dZ(t)/dt$. Therefore

$$\langle Z(t), Z'(t) \rangle = \langle Z(t), Y \rangle.$$

Now suppose $Z(t) = 0$ for some $t \in \mathbf{R}$. Then

$$1 = \exp tY \cdot \exp 2X \cdot \exp tY$$

and therefore $X = -tY$. But then $Z = (1-t) Y$ and so

$$\|Z\| \leqslant \|Y\| + \|tY\| = \|Y\| + \|X\|$$

which is equivalent to the assertion of the lemma. Hence we may assume that $Z(t) \neq 0$ for every $t \in \mathbf{R}$. Then $\|Z(t)\|$ is analytic in t and

$$\langle Z(t), Z'(t) \rangle = \|Z(t)\| d\|Z(t)\|/dt.$$

This shows that
$$\|Z(t)\| d\|Z(t)\| / dt = \langle Z(t), Y \rangle$$

and hence
$$|d\|Z(t)\| / dt| \leqslant \|Y\|$$

But then by integrating we get

$$\|Z(1)\| - \|Z(0)\| \leqslant \|Y\|.$$

However $Z(0) = X$ and $Z(1) = Z$. Therefore

$$\|Z\| \leqslant \|X\| + \|Y\|$$

and this proves the lemma.

Put $\mathfrak{k}_1 = \mathfrak{k} + \mathfrak{c} = \mathfrak{k} + \mathfrak{c}_\mathfrak{p}$ and let K_1 be the analytic subgroup of G corresponding to $\mathfrak{k}_1$. Define the function Ξ on G corresponding to K_1 as in [4(q), § 16]. Then by [4 (q), Lemma 31] Ξ is everywhere positive.

LEMMA 11. *There exists a number* $r \geqslant 0$ *such that*

$$\int_G \Xi(x)^2 (1 + \sigma(x))^{-r} \, dx < \infty.$$

Let G_0 and $C_{\mathfrak{p}}$ be the analytic subgoups of G corresponding to $\dot{\mathfrak{g}}_0 = \mathfrak{k} + \mathfrak{p}_1$ and $\mathfrak{c}_{\mathfrak{p}}$ respectively. Then since $\Xi(yc) = \Xi(y)$ and

$$\sigma(yc) \geqslant \max\left(\sigma(y), \sigma(c)\right) \quad (y \in G_0, \, c \in C_{\mathfrak{p}}),$$

it is clear that the above integral is majorized by

$$\int_{C_{\mathfrak{p}}} (1 + \sigma(c))^{-r/2} \, dc \int_{G_0} \Xi(y)^2 (1 + \sigma(y))^{-r/2} \, dy,$$

where dc and dy are the Haar measures on $C_{\mathfrak{p}}$ and G_0 respectively. Now it is clear that

$$\int_{C_{\mathfrak{p}}} (1 + \sigma(c))^{-r/2} \, dc < \infty$$

if r is sufficiently large. Hence it would be enough to consider the case when $\mathfrak{c}_{\mathfrak{p}} = \{0\}$. Then we can obviously replace G by G/Z_G where Z_G is the center of G. So we may assume that G is semisimple. Define $\mathfrak{a}$, Σ, $\mathfrak{a}^+$, A and A^+ as in [4 (q), § 21] and let dh denote the Haar measure of A. Then it follows from [4 (d), Lemma 38] that

$$\int_G \Xi(x)^2 (1 + \sigma(x))^{-r} \, dx = c_1 \int_{A^+} D(h) \Xi(h)^2 (1 + \sigma(h))^{-r} \, dh,$$

where c_1 is a positive number and

$$D(\exp H) = \prod_{\alpha \in \Sigma} (e^{\alpha(H)} - e^{-\alpha(H)})^{m_\alpha} \quad (H \in \mathfrak{a}^+),$$

m_α being the multiplicity of α. Put $\varrho = \frac{1}{2} \sum_{\alpha \in \Sigma} m_\alpha$. Then we know from [4 (j), Theorem 3] that we can choose positive numbers c_2 and d such that

$$\Xi(h) \leqslant c_2 \, e^{-\varrho(\log h)} (1 + \sigma(h))^d \quad (h \in A^+).$$

Therefore, since it is clear that

$$D(h) \leqslant e^{2\varrho(\log h)} \quad (h \in A^+),$$

we conclude that

$$\int_G \Xi^2 (1 + \sigma)^{-r} \, dx \leqslant c_1 \, c_2^{\,2} \int_{A^+} (1 + \sigma(h))^{2d-r} \, dh < \infty,$$

if r is sufficiently large. This proves Lemma 11.

Remark. Suppose $\mathfrak{c}_{\mathfrak{p}} = \{0\}$. Then one proves in the same way that $(1+\sigma)^r \Xi \in L_p(G)$ for $p > 2$ and $r \in \mathbf{R}$.

§ 8. Proof of Theorem 1

We keep to the notation of § 7 and define $\mathfrak{G}$ as in § 2. Let $\mathfrak{Z}$ be the center of $\mathfrak{G}$ and $\mathfrak{K}$ the subalgebra of $\mathfrak{G}$ generated by $(1, \mathfrak{k}_c)$. The following theorem will play an important role in the harmonic analysis on G.

THEOREM 1.[1] *Let V be a complex vector space of finite dimension and f a C^∞ function from G to V such that the functions zf $(z \in \mathfrak{Z}\mathfrak{K})$ span a finite-dimensional space. Fix a neighborhood U of 1 in G and let J be the space of all functions $\alpha \in C_c^\infty(G)$ such that $\mathrm{Supp}\ \alpha \subset U$ and $\alpha(kxk^{-1}) = \alpha(x)$ $(k \in K, x \in G)$. Then there exists an element $\alpha \in J$ such that $f * \alpha = f$.*

We regard f as an element of $C^\infty(G) \otimes V$ and extend the representation r of § 5 on this space by making G act trivially on V. Then, as we have seen in § 5, every element $\phi \in C^\infty(G) \otimes V$ is differentiable under r and $r(g)\phi = g\phi$ $(g \in \mathfrak{G})$. Let $\mathfrak{U}$ be the set of all $u \in \mathfrak{Z}\mathfrak{K}$ such that $uf = 0$. Then $\mathfrak{U}$ is a left ideal in $\mathfrak{Z}\mathfrak{K}$ of finite codimension. Let W be the smallest closed subspace of $C^\infty(G) \otimes V$ containing f, which is stable under $r(G)$. Then it is obvious that W contains $W_0 = r(\mathfrak{G})f$. We claim that $W = \mathrm{Cl}(W_0)$. For otherwise, by the Hahn-Banach theorem, we could choose a continuous linear function $\beta \neq 0$ on W such that $\beta = 0$ on W_0. Put

$$F(x) = \beta(r(x)f) \quad (x \in G).$$

Since f is differentiable under r, it is obious that $F \in C^\infty(G)$ and

$$F(x; g) = \beta(r(x)r(g)f) \quad (g \in \mathfrak{G}).$$

Therefore $uF = 0$ for $u \in \mathfrak{U}$. However $\mathfrak{U}$ contains elliptic differential operators (see the proof of Lemma 33 of [4 (q)]) and so we conclude that F is an analytic function. On the other hand,

$$F(1; g) = \beta(r(g)f) = 0 \quad (g \in \mathfrak{G})$$

since $\beta = 0$ on W_0. Hence $F = 0$ and this implies that $\beta = 0$ on W. This contradiction proves that $W = \mathrm{Cl}(W_0)$.

Put $W_1 = r(\mathfrak{K})f$. Then $\dim W_1 < \infty$ and therefore W_1 is closed in W. Moreover one proves in the same way as above that $r(K)f \subset W_1$ so that W_1 is stable under $r(K)$. Since $f \in W_1$, we can choose a finite subset F of $\mathcal{E}_K$ such that $f = f * \alpha_F$. (Here α_F has the same meaning as in § 6.) Put

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$$E_F = \sum_{\mathfrak{b} \in F} E_{\mathfrak{b}}$$

in the notation of § 2 (with $\pi = r$). We claim that $W_F = E_F W$ has finite dimension. Since W_1 is fully reducible under $r(K)$, it is obvious that the natural representation of $\mathfrak{K}$ on $\mathfrak{K}/\mathfrak{K} \cap \mathfrak{U}$ is semisimple. Moreover since dim $(\mathfrak{Z}\mathfrak{K}/\mathfrak{U}) < \infty$, it follows from [4 (a), Theorem 1, p. 195] that

$$W_0 = \sum_{\mathfrak{b} \in \mathcal{E}_K} E_{\mathfrak{b}} W_0$$

and dim $E_{\mathfrak{b}} W_0 < \infty$ for every $\mathfrak{b} \in \mathcal{E}_K$. On the other hand, W_0 is dense in W and therefore $E_{\mathfrak{b}} W_0$ is dense in $E_{\mathfrak{b}} W$. Hence $E_{\mathfrak{b}} W_0 = E_{\mathfrak{b}} W$ and this shows that dim $W_F < \infty$.

We have seen in § 2 that there exists a Dirac sequence α_j $(j \geqslant 1)$ with $\alpha_j \in J$. Then by Lemma 3, $f * \alpha_j \to f$ in W as $j \to \infty$. Let W_2 be the space of all elements in W of the form $f * \alpha$ $(\alpha \in J)$. Since $\alpha(kxk^{-1}) = \alpha(x)$ $(k \in K, x \in G)$ and $f \in W_F$, it is obvious that $W_2 \subset W_F$. Hence W_2 is a vector space of finite dimension and therefore it is closed in W. Therefore $f = \lim_{j \to \infty} f * \alpha_j \in W_2$ and this proves the theorem.

§ 9. The space $\mathcal{C}(G)$

Fix an open set U in G and let $\mathcal{C}^0(U)$ denote the space of all continuous functions f from U to $\mathbf{C}$ such that

$$\nu_r(f) = \sup_U (1 + \sigma)^r \Xi^{-1} |f| < \infty$$

for every $r \in \mathbf{R}$. Put

$$_{g_1}\nu_{r, g_2}(f) = \nu_r((\varrho(g_1) \circ g_2) f)$$

for $f \in C^\infty(U)$, g_1, $g_2 \in \mathfrak{G}$ and $r \in \mathbf{R}$. Let $\mathcal{C}(U)$ be the subspace of those $f \in C^\infty(U)$ for which $_{g_1}\nu_{r, g_2}(f) < \infty$ for all r and (g_1, g_2). We topologize $\mathcal{C}(U)$ by means of the seminorms $_{g_1}\nu_{r, g_2}$ $(g_1, g_2 \in \mathfrak{G}, r \in \mathbf{R})$. In this way $\mathcal{C}(U)$ becomes a locally convex Hausdorff space which is easily seen to be complete.[1]

LEMMA 12. *Fix a, $b \in G$ and, for any function f on U, let f' denote the function on aUb given by*

$$f'(x) = f(a^{-1}xb^{-1}) \quad (x \in aUb).$$

Then $f \to f'$ defines a topological mapping of $\mathcal{C}(U)$ onto $\mathcal{C}(aUb)$.

This is an easy consequence of Lemma 10 and [4(q), Lemma 32].

Now let G' be a Lie group such that G is the connected component of 1 in G'. Moreover let U be an open subset of G' which meets only a finite number of connected components of G'. Then we can choose $a_i \in G'$ and open sets U_i in G such that U is the disjoint union

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[1] $\mathcal{C}(U) = \{0\}$ by convention, if U is empty.

of $a_i U_i (1 \leqslant i \leqslant r)$. For any $f \in C^\infty(U)$, let f_i denote the function on U_i given by $f_i(x) = f(a_i x)$ $(x \in U_i)$. Consider the space $C(U)$ of all $f \in C^\infty(U)$ such that $f_i \in C(U_i)$ $(1 \leqslant i \leqslant r)$ and let V denote the Cartesian product of $C(U_i)$ $(1 \leqslant i \leqslant r)$ with the natural topology. We topologize $C(U)$ in such a way that the mapping $f \to (f_1, ..., f_r)$ of $C(U)$ onto V becomes an isomorphism. It follows from Lemma 12 that the structure of $C(U)$, as a locally convex space, is independent of the choice of a_i and U_i. Moreover it is obvious that the injection of $C_c^\infty(U)$ into $C(U)$ is continuous.

By a tempered distributon T on U, we mean a continuous linear mapping of $C(U)$ into $\mathfrak{C}$.

Now assume that G'/G is finite. Then U can be any open subset of G.

THEOREM 2. *Suppose G'/G is finite. Then $C_c^\infty(G')$ is dense in $C(G')$.*

In view of this theorem, we can identify tempered distributions on G' with those distributions which are continuous in the relative topology of $C_c^\infty(G')$ as a subspace of $C(G')$. Moreover, it is obviously enough to prove this theorem in case $G' = G$. This requires some preparation which will be undertaken in the next few sections.

§ 10. The left- and right-regular representations on $C(G)$

Let S denote the set of all continuous seminorms on $C(G)$. For any $f \in C(G)$ and $y \in G$, define $l(y)f$ and $r(y)f$ as in § 5.

LEMMA 13. *$l(y)f$ and $r(y)f$ are in $C(G)$. Moreover for a given compact set Ω in G and $\nu \in S$, we can choose $\nu' \in S$ such that*

$$\nu(l(y)f) + \nu(r(y)f) \leqslant \nu'(f)$$

for $y \in \Omega$ and $f \in C(G)$.

Put $r(y)f = f_y$ and fix $g, g' \in \mathfrak{G}$. Then

$$f_y(g'; x; g) = f(g'; xy; g^{y^{-1}})$$

for $x, y \in G$. We can choose linearly independent elements g_i $(1 \leqslant i \leqslant p)$ in $\mathfrak{G}$ and analytic functions a_i on G such that

$$g^{y^{-1}} = \sum_{1 \leqslant i \leqslant p} a_i(y) g_i \quad (y \in G).$$

Then $$f_y(g'; x; g) = \sum_i a_i(y) f(g'; xy; g_i).$$

If we apply a similar argument to $l(y)f$ and take into account Lemma 10 and [4 (q), Lemma 32], our assertions follow immediately.

LEMMA 14. *Fix $s \geqslant 0$ and put*

$$\nu_s(f) = \sup \, (1+\sigma)^s \, \Xi^{-1} |f| \quad (f \in C(G)).$$

Then for any $\varepsilon > 0$, we can choose a neighborhood U of 1 in G and an element $\nu' \in S$ such that

$$\nu_s(f_y - f) \leqslant \varepsilon \nu'(f)$$

for $y \in U$ and $f \in C(G)$. Moreover, ν' does not depend on ε.

Introduce a norm in $\mathfrak{g}$ and fix a base $X_1, \ldots, X_n$ for $\mathfrak{g}$ over $\mathbf{R}$. Then we can choose $c_0 > 0$ such that

$$\max_{1 \leqslant i \leqslant n} |c_i| \leqslant c_0 \Big| \sum_{1 \leqslant i \leqslant n} c_i X_i \Big|$$

for $c_i \in \mathbf{R}$ $(1 \leqslant i \leqslant n)$. Now let $f \in C(G)$ and $X \in \mathfrak{g}$. Then it is clear that

$$f(x \exp X) - f(x) = \int_0^1 f(x \exp tX; X) \, dt \quad (x \in G)$$

and therefore

$$|f(x \exp X) - f(x)| \leqslant c_0 |X| \sum_i \int_0^1 |f(x \exp tX; X_i)| \, dt.$$

But then it is obvious that we can choose $c > 0$ such that

$$\nu_s(f_{\exp X} - f) \leqslant c |X| \sum_{1 \leqslant i \leqslant n} \nu_s(X_i f)$$

for $|X| \leqslant 1$ $(X \in \mathfrak{g})$ and $f \in C(G)$. Clearly this implies the assertion of the lemma.

COROLLARY 1. *Fix $\nu \in S$ and $\varepsilon > 0$. Then we can choose a neighborhood U of 1 in G and $\nu' \in S$ such that*

$$\nu(l(y)f - f) + \nu(r(y)f - f) \leqslant \varepsilon \nu'(f)$$

for $y \in U$ and $f \in C(G)$. Moreover, ν' is independent of ε.

We use the notation of the proof of Lemma 13. Then

$$f_y(g'; x; g) - f(g'; x; g) = \sum_i (a_i(y) - 1) f(g'; xy; g_i) + \{f(g'; xy; g) - f(g'; x; g)\}.$$

Put $D = \varrho(g') \circ g$ and $D_i = \varrho(g') \circ g_i$. Then D and D_i define continuous endomorphisms of $C(G)$ and the above relation may be written as

$$D(r(y)f - f) = \sum_i (a_i(y) - 1) r(y) D_i f + (r(y) - 1) Df.$$

Therefore it is obvious from Lemmas 13 and 14 that, for a given $s \geqslant 0$, we can choose a neighborhood U of 1 in G and an element $\nu' \in S$ (independent of ε) such that

$$\nu_s(D(r(y)f - f)) \leqslant \varepsilon \nu'(f)$$

for $y \in U$ and $f \in (\mathcal{C}(G))$. Since a similar argument holds for $l(y)$, our assertion follows.

COROLLARY 2. *Both l and r define representations of G on $\mathcal{C}(G)$.*

This is obvious from Lemmas 1 and 13 and Corollary 1 above.

LEMMA 15. *Every element $f \in \mathcal{C}(G)$ is a differentiable vector under both l and r. Moreover, $r(X)f = Xf$ and $l(X)f = -\varrho(X)f$ for $X \in \mathfrak{g}$.*

Fix $f \in \mathcal{C}(G)$, $Y \in \mathfrak{g}$ and put

$$\phi_t = t^{-1}(r(y_t)f - f) - Yf \quad (t \in \mathbf{R}, \, t \neq 0),$$

where $y_t = \exp tY$. Then it follows from Lemma 8 that if D is any differential operator on G and Ω any compact subset of G, then

$$\sup_{\Omega} |D\phi_t| \to 0$$

as $t \to 0$. Fix $g, g' \in \mathfrak{G}$ and a number $m \geqslant 0$. Then it follows in particular that for any compact Ω,

$$\sup_{x \in \Omega} |\phi_t(g'; x; g)| \Xi(x)^{-1}(1 + \sigma(x))^m \to 0$$

as $t \to 0$.

On the other hand (see the proof of Lemma 2),

$$\phi_t(x) = \int_0^1 \{f(xy_{ts}; Y) - f(x; Y)\} \, ds.$$

Hence $\qquad \displaystyle \phi_t(g'; x; g) = \sum_i \int_0^1 a_i(y_{ts}) f'(xy_{ts}; g_i Y) \, ds - f'(x; gY),$

where $f' = \varrho(g')f$ and $a_i, g_i \ (1 \leqslant i \leqslant p)$ have the same meaning as in the proof of Lemma 13. Fix a compact neighborhood $U = U^{-1}$ of 1 in G. Then we can choose a number c such that $1 + \sigma(y) \leqslant c$, $|a_i(y)| \leqslant c \ (1 \leqslant i \leqslant p)$ and $\Xi(xy) \leqslant c\Xi(x)$ for $y \in U$ and $x \in G$. Fix $\delta > 0$ such that $y_t \in U$ for $|t| \leqslant \delta$. Then

$$|\phi_t(g'; x; g)| \Xi(x)^{-1}(1 + \sigma(x))^m$$

$$\leqslant c^{m+2} \sum_i \sup_{u \in U} |f'(xu; g_i Y)| \Xi(xu)^{-1}(1 + \sigma(xu))^m + |f'(x; gY)| \Xi(x)^{-1}(1 + \sigma(x))^m$$

for $|t| \leqslant \delta$. Now fix $\varepsilon > 0$. Since $f' \in \mathcal{C}(G)$, we can choose a compact set Ω_0 in G such that

$$|f'(x; g_i Y)| \Xi(x)^{-1}(1+\sigma(x))^m \leqslant \varepsilon$$

outside Ω_0 for $0 \leqslant i \leqslant p$. (Here $g_0 = g$.) Put $\Omega = \Omega_0 U$. Then Ω is compact and it is clear that

$$|\phi_t(g'; x; g)| \Xi(x)^{-1}(1+\sigma(x))^m \leqslant (p+1)c^{m+2}\varepsilon$$

if $x \notin \Omega$ and $|t| \leqslant \delta$. Therefore, in view of our earlier result, we can now conclude that $\phi_t \to 0$ in $C(G)$ as $t \to 0$. This shows that f is differentiable under r and $r(Y)f = Yf$. The proof for l is similar.

Define $\alpha_\mathfrak{b}$ $(\mathfrak{b} \in \mathcal{E}_K)$ as in § 5.

LEMMA 16. *For any $f \in C(G)$, the series*

$$\sum_{\mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{E}_K} \alpha_{\mathfrak{b}_1} * f * \alpha_{\mathfrak{b}_2}$$

converges absolutely to f in $C(G)$.

This is proved in the same way as Lemma 9.

§ 11. Spherical functions

Let $\mu = (\mu_1, \mu_2)$ be a (continuous) double representation [1] of K on a (complex) vector space V of finite dimension. Then by a μ-spherical function we mean a function ϕ from G to V such that

$$\phi(k_1 x k_2) = \mu_1(k_1)\phi(x)\mu_2(k_2) \quad (k_1, k_2 \in K; x \in G).$$

Fix a norm on V.

LEMMA 17. *For any two elements $g, g' \in \mathfrak{G}$, we can choose a finite number of $g_i \in \mathfrak{G}$ $(1 \leqslant i \leqslant p)$ with the following property. If ϕ is any C^∞ μ-spherical function, then*

$$|\phi(g; x; g')| \leqslant \sum_{1 \leqslant j \leqslant p} |\phi(x; g_j)|$$

and

$$|\phi(g; x; g')| \leqslant \sum_{1 \leqslant j \leqslant p} |\phi(g_j; x)|$$

for $x \in G$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Introduce an order in the space of real linear functions α on $\mathfrak{a}$ and, for any such α, let $\mathfrak{g}_\alpha$ denote the subspace of those $X \in \mathfrak{g}$ for which $[H, X] = \alpha(H)X$ for $H \in \mathfrak{a}$. Let Σ be the set of all positive roots of $(\mathfrak{g}, \mathfrak{a})$ and $(\alpha_1, ..., \alpha_l)$ the set of simple roots in Σ. Put

$$\mathfrak{n} = \sum_{\alpha > 0} \mathfrak{g}_\alpha.$$

Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ and $\mathfrak{G} = \mathfrak{K}\mathfrak{A}\mathfrak{N}$ where [2] $\mathfrak{K} = \mathfrak{S}(\mathfrak{k}_c)$, $\mathfrak{A} = \mathfrak{S}(\mathfrak{a}_c)$ and $\mathfrak{N} = \mathfrak{S}(\mathfrak{n}_c)$.

---

[1] This means that V is a left K-module under μ_1 and a right K-module under μ_2. Moreover, the operations of K on the left, commute with those on the right.

[2] We use here the notation of [4 (m), p. 280].

Fix an integer $d \geqslant 0$ such that $g, g' \in {}_d\mathfrak{G}$ (see [4 (o), § 2] for the notation). Then we can choose a base B for ${}_d\mathfrak{G}$ such that every element $b \in B$ has the form $b = \varkappa u \nu$ where $\varkappa \in \mathfrak{K}$, $u \in \mathfrak{A}$, $\nu \in \mathfrak{N}$ and

$$\nu^a = \exp \left(\sum_{1 \leqslant i \leqslant l} m_i \, \alpha_i \, (\log a) \right) \nu \quad (a \in A = \exp \mathfrak{a}),$$

m_i being nonnegative integers. Then

$$g^k = \sum_{b \in B} a_b(k) b, \, g'^k = \sum_{b \in B} a_b'(k) b \quad (k \in K),$$

where a_b and a_b' are continuous functions on K.

Now since any two norms on V are equivalent, we may assume that $\left| \mu_1(k_1) v \mu_2(k_2) \right| = \left| v \right|$ for $k_1, k_2 \in K$ and $v \in V$. Put $A^+ = \exp \mathfrak{a}^+$ where $\mathfrak{a}^+$ is the set of those points $H \in \mathfrak{a}$ where $\alpha(H) \geqslant 0$ for $\alpha \in \Sigma$. Then $G = KA^+K$. Put

$$c = \sup_{k \in K} \max_{b \in B} \left(\left| a_b(k) \right|, \left| a_b'(k) \right| \right).$$

Then if $x = k_1 h k_2$ $(k_1, k_2 \in K; h \in A^+)$, it is clear that

$$\left| \phi(g; x; g') \right| \leqslant \left| \phi(g^{k_1^{-1}}; h; g'^{k_2}) \right| \leqslant c^2 \sum_{b, b' \in B} \left| \phi(b; h; b') \right|.$$

Now $b = \varkappa_b u_b \nu_b$ $(b \in B)$ as above. Let us denote the representation of $\mathfrak{K}_c$ corresponding to μ_i again by μ_i $(i = 1, 2)$. For any endomorphism T of V, define

$$\left| T \right| = \sup_{\left| v \right| \leqslant 1} \left| Tv \right| \quad (v \in V)$$

as usual and put
$$c_1 = \sup_{b \in B} \left| \mu_1(\varkappa_b) \right|.$$

Then
$$\left| \phi(b; h; b') \right| = \left| \mu_1(\varkappa_b) \phi(h; u_b \nu_b^{h^{-1}} b') \right| \leqslant c_1 \left| \phi(h; u_b \nu_b b') \right|$$

since $\alpha_i(\log h) \geqslant 0$ $(1 \leqslant i \leqslant l)$. Hence

$$\left| \phi(g; x; g') \right| \leqslant c^2 c_1 \sum_{b, b' \in B} \left| \phi(h; u_b \nu_b b') \right|.$$

Now let g_j $(1 \leqslant j \leqslant p)$ be a base for the subspace of $\mathfrak{G}$ spanned by $(u_b \nu_b b')^k$ $(b, b' \in B, k \in K)$. Then it is clear that we can choose a number $c_2 \geqslant 0$ with the following property. If $b, b' \in B$, $k \in K$ and

$$(u_b \nu_b b')^k = \sum_{1 \leqslant j \leqslant p} \gamma_j g_j \quad (\gamma_j \in \mathbb{C}),$$

then $\left| \gamma_j \right| \leqslant c_2$. This shows that

$$|\phi(g; \, x; g')| \leqslant c_3 \sum_{1 \leqslant j \leqslant p} |\phi(x; g_j)|$$

for $x \in G$ where $c_3 = c^2 c_1 c_2$. Since our hypotheses are symmetrical with respect to left and right, the assertion of the lemma is now obvious.

§ 12. Application to $C_F(G)$

For any finite subset F of $\mathcal{E}_K$, define α_F as in § 6 and let $C_F(G)$ denote the subspace of all elements in $C(G)$ of the form $\alpha_F * f * \alpha_F$ ($f \in C(G)$). It is clear that an element $f \in C(G)$ lies in $C_F(G)$ if and only if $\alpha_F * f * \alpha_F = f$. Hence $C_F(G)$ is closed in $C(G)$.

Put
$$\nu_{m, g}(f) = \sup \, (1 + \sigma)^m \Xi^{-1} |gf|,$$
$$_g\nu_m(f) = \sup \, (1 + \sigma)^m \Xi^{-1} |\varrho(g)f|$$

for $m \geqslant 0$, $g \in \mathfrak{G}$ and $f \in C(G)$ and let $S_1 = \{\nu_{m, g}; \, m \geqslant 0, \, g \in \mathfrak{G}\}$ and $S_2 = \{_g\nu_m; \, m \geqslant 0, \, g \in \mathfrak{G}\}$.

LEMMA 18. *Let F be a finite subset of $\mathcal{E}_K$. Then each of the three sets of seminorms S_1, S_2 and S define the same topology on $C_F(G)$.*

Consider $C(K \times K)$ as a Banach space with the norm

$$|f| = \sup_{k_1, k_2 \in K} |f(k_1, k_2)| \quad (f \in C(K \times K))$$

and let $\mu_1(k)f$ and $f\mu_2(k)$ ($k \in K$) respectively denote the functions

$$(k_1, k_2) \to f(k^{-1}k_1, k_2) \quad \text{and} \quad (k_1, k_2) \to f(k_1, k_2 k^{-1}) \quad (k_1, k_2 \in K).$$

Then $\mu = (\mu_1, \mu_2)$ is a double representation of K on $C \, (K \times K)$. Let C_F be the subspace of all $f \in C(K \times K)$ such that

$$f = \int_K \alpha_F(k) \mu_1(k) f \, dk = \int_K \alpha_F(k) f \mu_2(k) \, dk.$$

Then C_F is a finite-dimensional space invariant under μ. We denote the restriction of μ on C_F by μ_F.

For any $f \in C_F(G)$, define the μ_F-spherical function f^* from G to C_F as follows. If $x \in G$, $f^*(x)$ is the function
$$(k_1, k_2) \to f(k_1^{-1} x k_2^{-1}) \quad (k_1, k_2 \in K)$$
in C_F. It is clear that

$$|f^*(g_1; \, x; g_2)| = \sup_{k_1, k_2 \in K} |f(g_1^{k_1}; \, k_1 \, x k_2^{-1}; \, g_2^{k_2})|$$

for $g_1, g_2 \in \mathfrak{G}$ and $x \in G$. Therefore if we apply Lemma 17 with $V = C_F$, Lemma 18 follows immediately.

§ 13. Density of $C_c^\infty(G)$ in $C(G)$

Now we come to the proof of Theorem 2. We have to show that $C_c^\infty(G)$ is dense in $C(G)$. Fix a finite subset F of $\mathcal{E}_K$. In view of Lemma 16, it would be enough to verify the following result.

LEMMA 19. $C_c^\infty(G) \cap C_F(G)$ *is dense in* $C_F(G)$.

For any $t > 0$, let G_t denote the open set consisting of all $x \in G$ with $\sigma(x) < t$. Also let ξ_t denote the characteristic function of G_t. Fix $a > 0$ and an element $\alpha \in C_c^\infty(G_a)$ such that $\alpha(k_1 x k_2) = \alpha(x)$ $(k_1, k_2 \in K; x \in G)$ and

$$\int_G \alpha(x)\,dx = 1.$$

Put
$$u_t = (1 - \xi_t) * \alpha = 1 - \xi_t * \alpha,$$

where the star denotes convolution on G as usual. It is clear thas $u_t \in C_c^\infty(G)$.

LEMMA 20. *We have*
$$u_t(x) = \begin{cases} 0 & \text{if } \sigma(x) \leqslant t - a, \\ 1 & \text{if } \sigma(x) \geqslant t + a, \end{cases}$$

and
$$|u_t(x; g)| \leqslant \int_G |\alpha(y; g)|\,dy \quad (x \in G)$$

for $g \in \mathfrak{G}$.

It is clear that
$$u_t(x) = \int_{G_a} (1 - \xi_t(xy^{-1}))\,\alpha(y)\,dy$$

and if we fix $y \in G_a$, it follows from Lemma 10 that

$$\xi_t(xy^{-1}) = \begin{cases} 0 & \text{if } \sigma(x) \geqslant t + a, \\ 1 & \text{if } \sigma(x) \leqslant t - a. \end{cases}$$

This gives the first statement of the lemma. Now fix $g \in \mathfrak{G}$. Since g is left-invariant, we have

$$gu_t = (1 - \xi_t) * g\alpha.$$

Therefore
$$u_t(x; g) = \int (1 - \xi_t(xy^{-1}))\,\alpha(y; g)\,dy$$

and this implies the desired inequality.

Now fix $f \in C_F(G)$ and put

$$f_t = (1 - u_t)f = (\xi_t \ast \alpha)f.$$

Since α and ξ_t are both invariant under left and right translations of K, it is obvious that $f_t \in C_c^\infty(G) \cap C_F(G)$. We now claim that $f_t \to f$ in $C_F(G)$ as $t \to +\infty$. Fix $g \in \mathfrak{G}$. Then

$$g(f - f_t) = g(u_t f) = \sum_{1 \leqslant i \leqslant p} g_i' u_t \cdot g_i f,$$

where g_i, g_i' are suitable elements in $\mathfrak{G}$. Moreover,

$$f(x) - f_t(x) = u_t(x)f(x) = f(x)$$

if $\sigma(x) \geqslant t + a$ and

$$(1 + \sigma(x))^m \Xi(x)^{-1}|f(x; g)| \leqslant (1 + t)^{-1}\nu_{m+1, g}(f)$$

for $\sigma(x) \geqslant t$ and $m \geqslant 0$ in the notation of § 12. Hence

$$(1 + \sigma(x))^m \Xi(x)^{-1}|f(x; g) - f_t(x; g)| \leqslant (1 + t)^{-1}\nu_{m+1, g}(f)$$

for $\sigma(x) \geqslant t + a$. Now suppose $\sigma(x) < t + a$. By Lemma 20, $f(x) - f_t(x) = 0$ if $\sigma(x) \leqslant t - a$. So let us assume that $t - a < \sigma(x) < t + a$. Then if $t > a$, we conclude from Lemma 20 that

$$(1 + \sigma(x))^m \Xi(x)^{-1}|f(x; g) - f_t(x; g)| \leqslant \sum_{1 \leqslant i \leqslant p} c_i(1 + \sigma(x))^m \Xi(x)^{-1}|f(x; g_i)|$$
$$\leqslant \sum_i c_i(1 + t - a)^{-1}\nu_{m+1, g_i}(f),$$

where

$$c_i = \int_G |\alpha(y; g_i')|\, dy \quad (1 \leqslant i \leqslant p).$$

This shows that $\nu_{m, g}(f - f_t) \to 0$ as $t \to +\infty$ and therefore by Lemma 18, f_t converges to f in $C_F(G)$. This proves Lemma 19 and therefore also Theorem 2.

§ 14. An inequality

Let $\mathfrak{h} = \theta(\mathfrak{h})$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{m}$ and $\tilde{M}$ the centralizers of $\mathfrak{h} \cap \mathfrak{p}$ in $\mathfrak{g}$ and G respectively. Let M denote the connected component of 1 in $\tilde{M}$.

Fix compatible orders on the spaces of real-valued linear functions on $\mathfrak{h} \cap \mathfrak{p}$ and $\mathfrak{h} \cap \mathfrak{p} + (-1)^{\frac{1}{2}}\mathfrak{h} \cap \mathfrak{k}$ and let P be the set of all positive roots of $(\mathfrak{g}, \mathfrak{h})$. Let P_M be the subset of those roots in P which vanish identically on $\mathfrak{h} \cap \mathfrak{p}$ and P_+ the complement of P_M in P. Put

$$\mathfrak{n}_{\mathfrak{h}} = \mathfrak{g} \cap (\sum_{\alpha \in P_+} \mathbf{C}X_\alpha)$$

in the notation of (4 (n), § 4]. Then $\mathfrak{n}_{\mathfrak{h}}$ is a nilpotent subalgebra of $\mathfrak{g}$ and $\tilde{M}$ normalizes $\mathfrak{n}_{\mathfrak{h}}$. Put

$$d(m) = |\det (\mathrm{Ad}\,(m))_{\mathfrak{n}_{\mathfrak{h}}}|^{\frac{1}{2}} \quad (m \in \tilde{M})$$

and let $N_{\mathfrak{h}}$ denote the analytic subgroup of G corresponding to $\mathfrak{n}_{\mathfrak{h}}$.

Extend $\mathfrak{h} \cap \mathfrak{p}$ to a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and introduce an order on the dual of $\mathfrak{a}$ which is compatible with that chosen above on the dual of $\mathfrak{h} \cap \mathfrak{p}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be the corresponding Iwasawa decomposition. Then $\mathfrak{n}_{\mathfrak{h}} \subset \mathfrak{n}$. Put

$$\varrho(H) = \tfrac{1}{2}\,\mathrm{tr}\,(\mathrm{ad}\,H)_{\mathfrak{n}} \quad (H \in \mathfrak{a})$$

and define $\mathfrak{a}^+$ and $A^+ = \exp \mathfrak{a}^+$ as usual (see [4 (q), § 21]). Then by [4 (j), Theorem 3], we can choose a number $d \geqslant 0$ such that

$$\sup_{h \in A^+} (1 + \sigma(h))^{-d}\, e^{\varrho(\log h)}\, \Xi(h) < \infty.$$

Let Ξ_M denote the function on M corresponding to Ξ if we replace (G, K) by $(M, K \cap M)$.

LEMMA 21. *Let dn denote the Haar measure of $N_{\mathfrak{h}}$ and fix $r > r' \geqslant 0$. Then we can choose a number $c > 0$ such that*

$$d(m) \int_{N_{\mathfrak{h}}} (1 + \sigma(mn))^{-(r + 2\,d)}\, \Xi(mn)\, dn \leqslant c(1 + \sigma(m))^{-r'}\, \Xi_M(m)$$

for $m \in M$. Moreover, there exists a number $c_0 \geqslant 1$ such that

$$1 + \sigma(m) \leqslant c_0(1 + \sigma(mn)) \quad (m \in M, n \in N_{\mathfrak{h}}).$$

It is clear that for a fixed [1] $m \in M$, $dn^m = d(m)^2 dn$. Therefore since $mn = n^m \cdot m$, we get the following result immediately.

COROLLARY. *In the above notation, we also have*

$$d(m)^{-1} \int_{N_{\mathfrak{h}}} (1 + \sigma(nm))^{-(r + 2\,d)}\, \Xi(nm)\, dn \leqslant c(1 + \sigma(m))^{-r'}\, \Xi_M(m)$$

for $m \in M$.

We shall give a proof of Lemma 21 in § 44.

---

[1] $y^x = xyx^{-1}\ (x, y \in G)$ as usual.

§ 15. The mapping of $C(G)$ into $C(\tilde{M})$

Let us keep to the above notation. Then $M \cap K$ has finite index in $\tilde{M} \cap K$ and therefore $\tilde{M}/M$ is finite. Hence the space $C(\tilde{M})$ is well defined (see § 9). Let Ω be a compact subset of G. Then it follows from Lemma 10, [4 (q), Lemma 32] and § 14 that, for any $f \in C(G)$, the integrals

$$\int_{N_{\mathfrak{h}}} |f(xn)| \, dn, \quad \int_{N_{\mathfrak{h}}} |f(nx)| \, dn$$

converge uniformly for $x \in \Omega$ (see also Corollary 2 of Lemma 90). Put

$$u_f(m) = d(m) \int_{N_{\mathfrak{h}}} f(mn) \, dn = d(m^{-1}) \int_{N_{\mathfrak{h}}} f(nm) \, dn \quad (m \in M),$$

and [1] $\mathfrak{M} = \mathfrak{S}(\mathfrak{m}_c)$. It is easy to verify (see [4 (q), § 10]) that there exist two automorphisms $\mu \to \mu'$ and $\mu \to {}^{\backprime}\mu$ of $\mathfrak{M}$ such that

$$\mu' = d^{-1}\mu \circ d, \quad {}^{\backprime}\mu = d\mu \circ d^{-1} \quad (\mu \in \mathfrak{M}).$$

Then it follows from what we have said above that $u_f \in C^\infty(M)$ and

$$u_f(\mu_1; m) = d(m) \int_{N_{\mathfrak{h}}} f(\mu_1'; mn) \, dn,$$

$$u_f(m; \mu_2) = d(m^{-1}) \int f(nm; \mu_2) \, dn = d(m) \int f(mn; {}^{\backprime}\mu_2) \, dn$$

for $\mu_1, \mu_2 \in \mathfrak{M}$. Since ${}^{\backprime}\mu_2 f \in C(G)$, we conclude that

$$u_f(\mu_1; m; \mu_2) = d(m) \int f(\mu_1'; mn; {}^{\backprime}\mu_2) \, dn.$$

In view of Lemma 21, the following result is now obvious.

LEMMA 22. $f \to u_f$ is a continuous linear mapping of $C(G)$ into $C(M)$.

Put
$$\tilde{f}(x) = \int_K f(x^k) \, dk \quad (x \in G)$$

for $f \in C(G)$. Then it follows from Lemma 13 that $f \to \tilde{f}$ is a continuous endomorphism of $C(G)$. Now define

$$g_f(m) = d(m) \int_{N_{\mathfrak{h}}} \tilde{f}(mn) \, dn \quad (m \in \tilde{M}).$$

---

[1] We use here the notation of [4 (m), p. 280].

THEOREM 3. $f \to g_f$ *is a continuous linear mapping of* $C(G)$ *into* $C(\tilde{M})$.

This is obvious from Lemma 22.

As usual let $\mathfrak{Z}$ denote the center of $\mathfrak{G}$ and $\mu = \mu_{\mathfrak{g}/\mathfrak{m}}$ the homomorphism of [4 (o), § 12]. Since $C_c^\infty(G)$ is dense in $C(G)$, we get the following result from the corollary of [4 (q), Lemma 14].

COROLLARY. *Let* $z \in \mathfrak{Z}$ *and* $f \in C(G)$. *Then*

$$g_{zf} = \mu(z) g_f.$$

§ 16. Proof of Theorem 4

Let $I(G)$ be the space of all continuous functions f on G which are *bi*-invariant under K, and $I^+(G)$ the subset of all real $f \geqslant 0$. Put $I_c^\infty(G) = I(G) \cap C_c^\infty(G)$ and

$$I_c^+(G) = C_c(G) \cap I^+(G).$$

Let $\mathfrak{Q}$ be the centralizer of $\mathfrak{k}$ in $\mathfrak{G}$ and μ a seminorm[1] on $I_c^\infty(G)$ satisfying the following two conditions.

1) There exist elements $q_1, \ldots, q_p \in \mathfrak{Q}$ such that

$$\mu(f) \leqslant \sum_{1 \leqslant i \leqslant p} \int_G |q_i f| \, \Xi \, dx \quad (f \in I_c^\infty(G)).$$

2) If f_1, f_2 are two elements in $I_c^\infty(G)$ such that $f_1 \geqslant f_2 \geqslant 0$, then $\mu(f_1) \geqslant \mu(f_2)$.

For any $\phi \in I^+(G)$, put

$$\mu(\phi) = \sup \mu(f),$$

where f runs over all functions in $I^+(G) \cap I_c^\infty(G)$ such that $\phi \geqslant f$. Fix a number $r \geqslant 0$ as in Lemma 11.

THEOREM 4. *Under the above conditions*

$$\mu(\Xi(1+\sigma)^{-r}) < \infty.$$

Fix $\alpha \in I^+(G) \cap I_c^\infty(G)$. Then $\alpha * f \in I_c^\infty(G)$ for $f \in I_c^+(G)$ and

$$\mu(\alpha * f) \leqslant \sum_{1 \leqslant i \leqslant p} \int |q_i \alpha * f| \, \Xi \, dx$$

---

[1] Here we take the space $I_c^\infty(G)$ without topology.

since $q_i(\alpha \divideontimes f) = q_i(f \divideontimes \alpha) = f \divideontimes q_i \alpha = q_i \alpha \divideontimes f$, the convolution being abelian in the present case (see [5 (a), Theorem 4.1, p. 408]). Now

$$\int |q_i \alpha \divideontimes f| \, \Xi \, dx \leqslant \int |\alpha(y; q_i)| \, f(y^{-1} x) \, \Xi(x) \, dy \, dx.$$

Therefore if $V = \text{Supp } \alpha$ and $c_1 = \sum_i \sup |q_i \alpha|$.

we get $$\mu(\alpha \divideontimes f) \leqslant c_1 \int_{V \times G} f(x) \, \Xi(yx) \, dy \, dx.$$

Since V is compact, we conclude from [4 (q), Lemma 32] that there exists a number $c_2 \geqslant 0$ such that

$$\mu(\alpha \divideontimes f) \leqslant c_2 \int_G f \, \Xi \, dx$$

for all $f \in I_c^+(G)$. It follows without difficulty from condition 2) on μ that the same inequality continues to hold for $f \in I^+(G)$.

Now take $f = \Xi(1 + \sigma)^{-r}$. Then $f \in I^+(G)$. Moreover we know that $\Xi > 0$ everywhere and (see[5 (a), p. 399])

$$\int_K \Xi(xky) \, dk = \Xi(x) \, \Xi(y)   (x, y \in G).$$

Therefore if we choose $\alpha \in I^+(G) \cap I_c^\infty(G)$ such that

$$\int \alpha \, \Xi \, dx = 1,$$

it is obvious that $\alpha \divideontimes \Xi = \Xi$. Then

$$(\alpha \divideontimes f)(x) = \int \alpha(y) \, \Xi(y^{-1} x) \, (1 + \sigma(y^{-1} x))^{-r} \, dy.$$

Now $$1 + \sigma(y^{-1} x) \leqslant (1 + \sigma(y)) \, (1 + \sigma(x))$$

from Lemma 10. Hence $$(\alpha \divideontimes f)(x) \geqslant c_0^{-1} f(x)$$

where $$c_0 = \sup_{y \in V} (1 + \sigma(y))^r$$

and $V = \text{Supp } \alpha$ as before. Therefore

$$\mu(f) \leqslant c_0 \mu(\alpha \divideontimes f) \leqslant c_0 \, c_2 \int f \, \Xi \, dx.$$

Since $$\int f \, \Xi \, dx = \int \Xi^2 (1 + \sigma)^{-r} \, dx < \infty,$$

we get the assertion of the theorem.

§ 17. Convergence of certain integrals

We shall now derive some consequences of Theorem 4. Assume that G is acceptable (see [4 (o), § 18]) and let A be a Cartan subgroup of G. We use the notation of [4 (o), § 23].

THEOREM 5. *Fix r as in § 16. Then*

$$\sup_{a \in A'} |\Delta(a)| \int_{G^*} \Xi(a^{x^*})(1 + \sigma(a^{x^*}))^{-r} dx^* < \infty.$$

Let $\mathfrak{h}$ be the Lie algebra of A. Put

$$\mu(f) = \sup_{a \in A'} |F_f(a)|, \, \nu(f) = \int_A |\Delta_M(a) F_f(a)| \, da \quad (f \in I_c^\infty(G))$$

in the notation of [4 (o), Theorem 3]. Then it follows from [4 (q), Theorem 4] that there exists a number $c \geqslant 0$ such that

$$\nu(f) \leqslant c \int_G |f| \, \Xi \, dx \quad (f \in I_c^\infty(G)).$$

Moreover, by [4 (o), Theorem 3], we can select $z_1, \ldots, z_p \in \mathfrak{Z}$ such that

$$\mu(f) \leqslant \sum_{1 \leqslant i \leqslant p} \nu(z_i f) \quad (f \in I_c^\infty(G)).$$

Hence it is obvious that μ satisfies the two conditions of § 16. Moreover, it follows from the elementary properties of an integral that if $\phi \in I^+(G)$ and $a \in A'$, then

$$\int_{G^*} \phi(a^{x^*}) \, dx^* = \sup_f \int_{G^*} f(a^{x^*}) \, dx^*,$$

where f runs over all elements in $I^+(G) \cap I_c^\infty(G)$ such that $f \leqslant \phi$. Therefore the assertion of Theorem 5 is now an immediate consequence of Theorem 4.

Let γ be a semisimple element in G and G_γ the centralizer of γ in G. Then G_γ is unimodular and therefore the factor space $\bar{G} = G/G_\gamma$ has an invariant measure $d\bar{x}$. Let $x \to \bar{x}$ denote the projection of G on $\bar{G}$ and put

$$\gamma^{\bar{x}} = \gamma^x = x\gamma x^{-1} \quad (x \in G).$$

THEOREM 6. $$\int_{G/G_\gamma} \Xi(\gamma^{\bar{x}})(1 + \sigma(\gamma^{\bar{x}}))^{-r} \, d\bar{x} < \infty.$$

Let $\mathfrak{z}$ be the centralizer of γ in $\mathfrak{g}$. Since γ is semisimple, $\mathfrak{z}$ is reductive in $\mathfrak{g}$ and rank $\mathfrak{z}$ = rank $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{z}$ which is fundamental in $\mathfrak{z}$ (see [4 (n),

§ 11]) and A the Cartan subgroup of G corresponding to $\mathfrak{h}$. Then $\gamma \in A$. As usual let P and $P_{\mathfrak{z}}$ denote the sets of positive roots of $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{z}, \mathfrak{h})$ respectively and $P_{\mathfrak{g}/\mathfrak{z}}$ the complement of $P_{\mathfrak{z}}$ in P. Put

$$\varpi_{\mathfrak{z}} = \prod_{\alpha \in P_{\mathfrak{z}}} H_\alpha$$

in the notation of [4 (n), § 4].

LEMMA 23.([1]) *There exists a number $c \neq 0$ such that*

$$F_f(\gamma; \varpi_{\mathfrak{z}}) = c \int_{G/G_\gamma} f(\gamma^{\bar{x}}) \, d\bar{x}$$

for all $f \in C_c^\infty(G)$.

We observe that in view of [4 (o), Lemma 40], the left side has a well-defined meaning. Moreover, since γ is semisimple, the orbit γ^G is closed (see [1, § 10.1]) and therefore [1, § 5.1] the integral on the right is also well defined.

Normalize the invariant measure dy^* on G_γ/A_0 in such a way that $dx^* = d\bar{x} \, dy^*$. Let U be an open, connected neighborhood of 1 in A such that $\det (\mathrm{Ad}\,(a) - 1)_{\mathfrak{g}/\mathfrak{z}} \neq 0$ for $a \in U$. Put $U' = U \cap (\gamma^{-1} A')$. Then an element $a \in U$ lies in U' if and only if $\det (\mathrm{Ad}\,(\gamma a) - 1)_{\mathfrak{z}/\mathfrak{h}} \neq 0$. Moreover, we may assume (see [4 (i), Theorem 1]) that U has the following property. For any compact set Ω in G, there exists a compact subset C of $\bar{G}$ such that $xUx^{-1} \cap \Omega = \emptyset$ ($x \in G$) unless $\bar{x} \in C$.

Fix $f \in C_c^\infty(G)$ and select C as above corresponding to $\Omega = \mathrm{Supp}\, f$. Then if $a \in U'$,

$$\int_{G^*} f((\gamma a)^{x^*}) \, dx^* = \int_C d\bar{x} \int_{G_\gamma^*} f(x(\gamma a)^{y^*} x^{-1}) \, dy^*.$$

Let G_γ^0 denote the connected component of 1 in G_γ and Z the center of G. Then ZG_γ^0 has finite index in G_γ (see [4 (h), Lemma 15]). Let N denote this index and choose y_i ($1 \leq i \leq N$) in G_γ such that

$$G_\gamma = \bigcup_{1 \leq i \leq N} y_i Z G_\gamma^0.$$

Define $\qquad\qquad g_x(y) = \sum_{1 \leq i \leq N} f(x\gamma y_i y y_i^{-1} x^{-1}) \quad (y \in G_\gamma^0)$

for $x \in G$. Then it is clear that

$$\int_{G^*} f((\gamma a)^{x^*}) \, dx^* = \int_C d\bar{x} \int_{(G_\gamma^0)^*} g_x(a^{y^*}) \, dy^*$$

for $a \in U'$.

Choose an open and convex neighborhood V of zero in $\mathfrak{h}$ such that $\exp V \subset U$ and $|\alpha(H)| < 1$ for $\alpha \in P$ and $H \in V$. Let V' denote the set of all points $H \in V$ where

([1]) Cf. Langlands [6, p. 114].

$$\pi_{\mathfrak{z}}(H) = \prod_{\alpha \in P_{\mathfrak{z}}} \alpha(H) \neq 0.$$

Then $\exp V' \subset U'$ and

$$\Delta(\gamma \exp H) = \xi_{\varrho}(\gamma) \prod_{\alpha \in P_{\mathfrak{g}/\mathfrak{z}}} (e^{\alpha(H)/2} - \xi_{\alpha}(\gamma)^{-1} e^{-\alpha(H)/2}) \Delta_{\mathfrak{z}}(H)$$

for $H \in V$. Here

$$\Delta_{\mathfrak{z}}(H) = \prod_{\alpha \in P_{\mathfrak{z}}} (e^{\alpha(H)/2} - e^{-\alpha(H)/2}).$$

Let D denote the differential operator on $\mathfrak{h}$ given by

$$D = \xi_{\varrho}(\gamma) \, \partial(\varpi_{\mathfrak{z}}) \circ \prod_{\alpha \in P_{\mathfrak{g}/\mathfrak{z}}} (e^{\alpha/2} - \xi_{\alpha}(\gamma)^{-1} e^{-\alpha/2}).$$

As usual D_0 denotes the local expression of D at the origin (see [4 (f), p. 90]).

LEMMA 24. $D_0 = \xi_{\varrho}(\gamma) \prod_{\alpha \in P_{\mathfrak{g}/\mathfrak{z}}} (1 - \xi_{\alpha}(\gamma)^{-1}) \partial(\varpi_{\mathfrak{z}}).$

Fix $q \in S(\mathfrak{h}_c)$ such that

$$\partial(q) = D_0 - \xi_{\varrho}(\gamma) \prod_{\alpha \in P_{\mathfrak{g}/\mathfrak{z}}} (1 - \xi_{\alpha}(\gamma)^{-1}) \partial(\varpi_{\mathfrak{z}}).$$

If p is the number of roots in $P_{\mathfrak{z}}$, it is clear that $d^0 \varpi_{\mathfrak{z}} = p$ and $d^0 q < p$. On the other hand, it is easy to see that $D_0^{s\alpha} = -D_0$ for any $\alpha \in P_{\mathfrak{z}}$. Therefore $\varpi_{\mathfrak{z}}$ divides q in $S(\mathfrak{h}_c)$ (see [4 (f), Lemma 10]) and this shows that $q = 0$.

For any function $g \in C_c^\infty(G_\gamma^0)$, define

$$\phi_g(H) = \Delta_{\mathfrak{z}}(H) \int_{(G_\gamma^0)^*} f((\exp H)^{y^*}) dy^* \quad (H \in V').$$

Then by [4 (n) Theorem 3], and [4 (i), Lemma 19], there exists a number $c_0 \neq 0$ such that

$$\lim_{H \to 0} \phi_g(H; \partial(\varpi_{\mathfrak{z}})) = c_0 g(1) \quad (H \in V')$$

for every $g \in C_c^\infty(G_\gamma^0)$. Hence it follows from Lemma 24 that

$$F_f(\gamma; \varpi_{\mathfrak{z}}) = c_1 \int_C g_x(1) d\bar{x} = c_1 N \int_{G/G_\gamma} f(\gamma^{\bar{x}}) d\bar{x},$$

where $c_1 = c_0 \varepsilon_R(\gamma) \xi_{\varrho}(\gamma) \prod_{\alpha \in P_{\mathfrak{g}/\mathfrak{z}}} (1 - \xi_{\alpha}(\gamma)^{-1}).$

This proves Lemma 23.

Now we come to the proof of Theorem 6. Put

$$\mu(f) = \left| \int_{G/G_\gamma} f(\gamma^{\bar{x}}) \, d\bar{x} \right| = |c^{-1} F_f(\gamma; \varpi_{\mathfrak{z}})| \quad (f \in I_c^\infty(G))$$

and define $\nu(f)$ as in the proof of Theorem 5. By [4 (o), Theorem 3], we can choose $z_1, \ldots, z_p \in \mathfrak{Z}$ such that

$$\mu(f) \leqslant \sum_{1 \leqslant i \leqslant p} \nu(z_i f) \quad (f \in I_c^\infty(G)).$$

This shows (see the proof of Theorem 5) that μ fulfills the two conditions of § 16. Moreover, it is clear that

$$\int_{G/G_\gamma} \phi(\gamma^{\bar{x}}) d\bar{x} = \sup_f \int_{G/G_\gamma} f(\gamma^{\bar{x}}) d\bar{x} \quad (\phi \in I^+(G)),$$

where f runs over all functions in $I^+(G) \cap I_c^\infty(G)$ such that $f \leqslant \phi$. Therefore Theorem 6 follows from Theorem 4.

§ 18. The mapping $f \to F_f$

We return to the notation of § 15 and define the function $D_{\mathfrak{h}}$ on G as in [4 (q), Lemma 35]. Also we recall that S is the set of all continuous seminorms on $C(G)$.

LEMMA 25. *Put*

$$\nu_1(f) = \int_G |f| \, |D_{\mathfrak{h}}|^{-\frac{1}{2}} dx \quad (f \in C(G)).$$

Then $\nu_1 \in S$.

Fix r as in Lemma 11 and put

$$\nu(f) = \sup |f| \, \Xi^{-1} (1 + \sigma)^r \quad (f \in C(G)).$$

Then it is clear that

$$\nu_1(f) \leqslant \nu(f) \int_G \Xi(1 + \sigma)^{-r} |D_{\mathfrak{h}}|^{-\frac{1}{2}} dx$$

and therefore our assertion follows from Lemma 11 and [4 (q), Lemma 35].

Now assume that G is acceptable and let $A = A_{\mathfrak{h}}$ be the Cartan subgroup of G corresponding to $\mathfrak{h}$. Since K is compact, A has only a finite number of connected components. Define $A'(I)$ as in [4 (o), § 22]. Then the space $C(A'(I))$ is well defined (see § 9). For any $f \in C_c^\infty(G)$, define the function $F_f \in C^\infty(A'(I))$ as in [4 (o), § 22].

LEMMA 26. *Let* $S(A'(I))$ *be the set of all continuous seminorms on* $C(A'(I))$. *Then* $F_f \in C(A'(I))$ *and for a given* $\nu_0 \in S(A'(I))$, *we can choose* $\nu \in S$ *such that*

$$\nu_0(F_f) \leqslant \nu(f) \quad (f \in C_c^\infty(G)).$$

We use induction on $\dim \mathfrak{g}$. Let $\mathfrak{c}$ be the center of $\mathfrak{g}$ and first assume that $\mathfrak{h} \cap \mathfrak{p} \not\subset \mathfrak{c}$. Then $\dim \mathfrak{m} < \dim \mathfrak{g}$ and the induction hypothesis is applicable to M and therefore our assertion follows immediately from Theorem 3 and [4 (o), Lemma 52] (see also [4 (q), § 10]). Hence we may suppose that $\mathfrak{h} \cap \mathfrak{p} = \mathfrak{c}_\mathfrak{p}$ where $\mathfrak{c}_\mathfrak{p} = \mathfrak{c} \cap \mathfrak{p}$ as before. Let us assume further that $\mathfrak{c}_\mathfrak{p} \neq \{0\}$ and put $\mathfrak{g}_1 = \mathfrak{k} + [\mathfrak{k}, \mathfrak{p}]$. Then $\mathfrak{g}$ is the direct sum of $\mathfrak{c}_\mathfrak{p}$ and $\mathfrak{g}_1$ and G is the direct product of the corresponding subgroups $C_\mathfrak{p}$ and G_1. Put([1]) $\mathfrak{C}_\mathfrak{p} = \mathfrak{S}(\mathfrak{c}_{\mathfrak{p}c})$ and $\mathfrak{G}_1 = \mathfrak{S}(\mathfrak{g}_{1c})$. Then we may assume, without loss of generality, that

$$\nu_0(g) = \sup_{h \in A'(I)} (1 + \sigma(h))^r |g(h; \gamma u)| \quad (g \in \mathcal{C}(A'(I)))$$

for some $r \geqslant 0$, $\gamma \in \mathfrak{C}_\mathfrak{p}$ and $u \in \mathfrak{S}(\mathfrak{h}_{1c})$ where $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{g}_1$. For any $f \in \mathcal{C}(G)$ and $c \in C_\mathfrak{p}$, let f_c denote the function $x \to f(cx; \gamma)$ $(x \in G_1)$ on G_1. Put $A_1'(I) = G_1 \cap A'(I)$ and let F_g $(g \in C_c^\infty(G_1))$ denote the function on $A_1'(I)$ corresponding to [4 (o), § 22]. Then it is obvious that

$$(1 + \sigma(ch))^r |F_f(ch; \gamma u)| \leqslant (1 + \sigma(c))^r (1 + \sigma(h))^r |F_{f_c}(h; u)|$$

for $c \in C_\mathfrak{p}$ and $h \in A_1'(I)$. Since $\dim \mathfrak{g}_1 < \dim \mathfrak{g}$, we can, by induction hypothesis, choose a continuous seminorm ν_1 on $\mathcal{C}(G_1)$ such that

$$(1 + \sigma(h))^r |F_g(h; u)| \leqslant \nu_1(g)$$

for $g \in C_c^\infty(G_1)$ and $h \in A_1'(I)$. Then it follows that

$$\nu_0(F_f) \leqslant \sup_{c \in C_\mathfrak{p}} (1 + \sigma(c))^r \nu_1(f_c) \quad (f \in C_c^\infty(G)).$$

Now put
$$\nu(f) = \sup_{c \in C_\mathfrak{p}} (1 + \sigma(c))^r \nu_1(f_c) \quad (f \in \mathcal{C}(G)).$$

Since $\Xi(cx) = \Xi(x)$ and $\sigma(cx) \geqslant \max(\sigma(c), \sigma(x))$ (see § 7) for $c \in C_\mathfrak{p}$ and $x \in G_1$, it is easy to verify that $\nu \in S$.

So now we may suppose that $\mathfrak{c}_\mathfrak{p} = \mathfrak{h} \cap \mathfrak{p} = \{0\}$ and therefore $\mathfrak{h} \subset \mathfrak{k}$. Put

$$\nu_1(f) = \int_G |f| \, |D_\mathfrak{h}|^{-\frac{1}{2}} \, dx \quad (f \in \mathcal{C}(G)).$$

Then $\nu_1 \in S$ by Lemma 25. Therefore since $A \subset K$ in the present case, our assertion follows from [4 (o), Theorem 3]. This completes the proof of Lemma 26.

Since $C_c^\infty(G)$ is dense in $\mathcal{C}(G)$ (Theorem 2) and $\mathcal{C}(A'(I))$ is complete, it is clear that that $f \to F_f$ can be extended uniquely to a continuous mapping of $\mathcal{C}(G)$ into $\mathcal{C}(A'(I))$. Thus, for every $f \in \mathcal{C}(G)$, we get a function $F_f \in \mathcal{C}(A'(I))$.

([1]) We use here the notation of [4 (m), p. 280].

LEMMA 27. *Let* $f \in C(G)$. *Then*

$$F_f(a) = \varepsilon_R(a)\,\Delta(a) \int_{G^*} f(a^{x^*})\,dx^* \quad (a \in A')$$

in the notation of [4 (o), § 23].

It is obvious from Theorem 5 that the integral on the right is well defined. Now choose a sequence $f_j \in C_c^\infty(G)$ ($j \geqslant 1$) such that $f_j \to f$ in $C(G)$ and put $\phi_j = f - f_j$. Then, in view of the definition of F_f, it would be enough to verify that

$$\sup_{A'} |\Delta(a)| \int_{G^*} |\phi_j(a^{x^*})|\,dx^* \to 0.$$

But since $\phi_j \to 0$ in $C(G)$, this is obvious from Theorem 5.

Let B be another Cartan subgroup of G conjugate to A. Fix $x \in G$ such that $B = A^x$. Then the isomorphism $a \to a^x$ defines a linear bijection of $C(A'(I))$ on a subspace $C(B'(I))$ of $C^\infty(B'(I))$. We topologize $C(B'(I))$ so as to make this bijection a homeomorphism. It is easy to verify that this topology is independent of the choice of x.

Now let us drop the condition that $\mathfrak{h} = \theta(\mathfrak{h})$ and define

$$F_f(a) = \varepsilon_R(a)\,\Delta(a) \int_{G^*} f(a^{x^*})\,dx^* \quad (f \in C(G),\ a \in A'(I)).$$

It follows from Theorem 5 that this integral exists. Since $\mathfrak{h}$ is conjugate to some Cartan subalgebra which is stable under θ, it is obvious from Lemmas 26 and 27 that $f \to F_f$ is a continuous mapping of $C(G)$ into $C(A'(I))$.

LEMMA 28.
$$F_f(\gamma; \varpi_\delta) = c \int_{G/G_\gamma} f(\gamma^{\bar{x}})\,d\bar{x}$$

for $f \in C(G)$ *in the notation of Lemma* 23.

It follows from Theorem 6 that the integral on the right is well defined. The rest of the argument is similar to that given above for Lemma 27.

Let us now return to the notation of § 15. If we replace G by M, we get the corresponding mapping $g \to F_g^M$ of $C(M)$ into $C(A_0'(I))$ where $A_0 = A \cap M$ and $A_0'(I) = A_0 \cap A'(I)$. Define $Z_A = Z(A)$ as in [4 (q), § 12]. Then the following result is obvious from [4 (o), Lemma 52] (see also [4 (q), § 10]), Theorem 3 and Lemma 26.

LEMMA 29. *For any* $a \in Z_A$, *put*

$$g_{f,a}(m) = g_f(am) \quad (m \in M,\ f \in C(G))$$

in the notation of Theorem 3. Then there exists a number $c > 0$ such that

$$F_f(ah) = c\xi_\varrho(a) F_{g_{f,a}}{}^M(h) \quad (a \in Z_A, f \in \mathcal{C}(G))$$

for $h \in A_0'(I)$.

§ 19. A criterion for an invariant eigendistribution to be tempered

Let Θ be a distribution on G. Then Θ is said to be $\mathfrak{Z}$-finite, if the space of all distributions of the form $z\Theta$ $(z \in \mathfrak{Z})$ has finite dimension. We recall (see [4 (o), Theorem 2]) that an invariant and $\mathfrak{Z}$-finite distribution is actually a locally summable function which is analytic on the regular set G'.

Define D as in [4 (o), § 28] and let $A_i \, (1 \leqslant i \leqslant r)$ be a complete set of Cartan subgroups of G, no two of which are conjugate in G. As usual put $A_i' = A_i \cap G'$.

THEOREM 7. *Let Θ be an invariant and $\mathfrak{Z}$-finite distribution on G. Then Θ is tempered if and only if there exists a number $s \geqslant 0$ such that*

$$\sup_{a \in A_i'} (1 + \sigma(a))^{-s} |D(a)|^{\frac{1}{2}} |\Theta(a)| < \infty \quad (1 \leqslant i \leqslant r).$$

Let $\mathfrak{a}_i$ be the Lie algebra of A_i. In view of Lemma 10, we can obviously assume that $\theta(\mathfrak{a}_i) = \mathfrak{a}_i$. Let us now use the notation of [4 (p), Lemma 63]. Then

$$\int_G f \, dx = \sum_{1 \leqslant i \leqslant r} c_i \int_{A_i} \varepsilon_{i,R} \operatorname{conj} \Delta_i \cdot F_{f,i} \, d_i a \quad (f \in C_c^\infty(G)),$$

where $F_{f,i} = F_f$ and $\varepsilon_{i,R} = \varepsilon_R$ for $A = A_i$ (see [4 (o), § 22]). Now choose $c \geqslant 0$ such that

$$|D(a)|^{\frac{1}{2}} |\Theta(a)| \leqslant c(1 + \sigma(a))^s \quad (a \in A_i', \ 1 \leqslant i \leqslant r).$$

Since $|D(a)| = |\Delta_i(a)|^2 \ (a \in A_i)$, it is clear that

$$|\Theta(f)| = \left| \int_G \Theta f \, dx \right| \leqslant c \sum_{1 \leqslant i \leqslant r} c_i \int_{A_i} (1 + \sigma(a))^s |F_{f,i}(a)| \, d_i a$$

for $f \in C_c^\infty(G)$. Hence it follows immediately from Lemma 26, that Θ is tempered.

Before proving the converse, we shall derive the following consequence of the theorem.

COROLLARY. *Suppose Θ is tempered. Then, in the above notation,*

$$\Theta(f) = \sum_{1 \leqslant i \leqslant r} c_i \int_{A_i} \varepsilon_{i,R} \Phi_i F_{f,i} \, d_i a \quad (f \in \mathcal{C}(G)),$$

where
$$\Phi_i(a) = \Theta(a) \cdot \operatorname{conj} \Delta_i(a) \quad (a \in A_i').$$

Fix $f \in \mathcal{C}(G)$ and choose a variable element $\alpha \in C_c^\infty(G)$, which converges to f in $\mathcal{C}(G)$. Then

$$\Theta(f) = \lim_{\alpha \to f} \Theta(\alpha).$$

But
$$\Theta(\alpha) = \int \Theta \alpha \, dx = \sum_{1 \leqslant i \leqslant r} c_i \int_{A_i} \varepsilon_{i,R} \Phi_i F_{\alpha,i} \, d_i a$$

and so our assertion is an immediate consequence of Lemma 26 and the above theorem.

Now in order to prove the second part of Theorem 7, we need some preparation. Define S and S_1 as in Lemma 18.

LEMMA 30. *Let Θ be a tempered and invariant distribution on G. Then Θ is continuous in the topology defined on $\mathcal{C}(G)$ by S_1.*

Fix a function $\alpha \in C_c^\infty(G)$, such that $\int \alpha \, dx = 1$ and, for any $f \in C_c^\infty(G)$, put

$$f_0(x) = \int_G \alpha(y) f(x^y) \, dy \quad (x \in G).$$

Then
$$f_0(z_1 x z_2) = \int_G \alpha(y z_1^{-1}) f(y x z_2 z_1 y^{-1}) \, dy \quad (z_1, z_2 \in G).$$

Fix $g', g'' \in \mathfrak{G}$. Then it is clear that we can select $g_1, \ldots, g_p \in \mathfrak{G}$ such that

$$\left| f_0(g'; x; g'') \right| \leqslant \sum_{1 \leqslant i \leqslant p} \sup_{y \in \Omega} \left| f(y x y^{-1}; g_i) \right| \quad (x \in G)$$

for all $f \in C_c^\infty(G)$. Here $\Omega = \operatorname{Supp} \alpha$. Now fix $m \geqslant 0$. Then by Lemma 10 and [4 (q), Lemma 32], we can choose $c \geqslant 0$ such that

$$\sup_G \left| f_0(g'; x; g'') \right| \Xi(x)^{-1}(1 + \sigma(x))^m \leqslant c \sum_{1 \leqslant i \leqslant p} \sup_G \left| f(x; g_i) \right| \Xi(x)^{-1}(1 + \sigma(x))^m.$$

This shows that, for a given $\nu \in S$, there exists a number $c \geqslant 0$ and a finite set $(\nu_1, \ldots, \nu_p)$ of elements in S_1, such that

$$\nu(f_0) \leqslant c \sum_{1 \leqslant i \leqslant p} \nu_i(f) \quad (f \in C_c^\infty(G)).$$

On the other hand, since Θ is tempered, we can choose $\nu \in S$ such that

$$\left| \Theta(f) \right| \leqslant \nu(f) \quad (f \in C_c^\infty(G)).$$

Moreover,
$$\Theta(f) = \Theta(f_0),$$
since Θ is invariant. Hence

$$|\Theta(f)| = |\Theta(f_0)| \leqslant \nu(f_0) \leqslant c \sum_{1 \leqslant i \leqslant p} \nu_i(f) \quad (f \in C_c^\infty(G))$$

and this proves our assertion.

Now fix a Cartan subalgebra $\mathfrak{h} = \theta(\mathfrak{h})$ of $\mathfrak{g}$ and let A be the corresponding Cartan subgroup of G. Let A_0 be the center of A and $\tilde{A}$ the normalizer of A in G. Put $W_A = \tilde{A}/A_0$ and let $x \to x^*$ denote the natural projection of G on $G^* = G/A_0$. Then W_A is a finite group and a^s and x^*s $(s \in W_A, a \in A, x^* \in G^*)$ are defined as usual (see [4 (o), §20]). If β is any function on A', we denote by β^s the function $a \to \beta(a^{s^{-1}})$ $(a \in A', s \in W_A)$.

Put $G_A = (A')^G$ as usual and normalize the measures dx, da and dx^* in accordance with Lemma 91. Fix a function $\alpha^* \in C_c^\infty(G^*)$ such that $\alpha^*(x^*s) = \alpha^*(x^*)$ $(x^* \in G^*, s \in W_A)$ and

$$\int_{G^*} \alpha^* \, dx^* = 1 .$$

Then, for any $\beta \in C_c^\infty(A')$, we define $f_\beta \in C_c^\infty(G_A)$ as follows.

$$f_\beta(a^x) = \alpha^*(x^*) \Delta(a)^{-1} \sum_{s \in W_A} \varepsilon(s) \beta(a^s) \quad (a \in A', x \in G).$$

Here $\varepsilon(s)$ has the usual meaning so that $\Delta^s = \varepsilon(s) \Delta$. Then it follows from Lemma 91 that

$$\Theta(f_\beta) = \int_A \Phi \beta \, da \quad (\beta \in C_c^\infty(A')),$$

where $\Phi(a) = \Theta(a) \cdot \mathrm{conj}\, \Delta(a)$ $(a \in A')$.

Let Q be the set of all roots of $(\mathfrak{g}, \mathfrak{h})$. Then for each $\alpha \in Q$, we have the character ξ_α of A (see [4 (o), §18]). Let η_α denote the function $(1 - \xi_\alpha^{-1})^{-1}$ on A' and let $\mathcal{R}$ be the ring of analytic functions on A' generated over $\mathbb{C}$ by 1 and η_α $(\alpha \in Q)$. It is obvious that $\mathcal{R}$ is stable under W_A. Moreover, one finds directly by differentiation that

$$H\eta_\alpha = \alpha(H) \eta_\alpha(1 - \eta_\alpha) \quad (\alpha \in Q, H \in \mathfrak{h}).$$

This shows that $\mathcal{R}$ is also stable under the differential operators in $\mathfrak{S}(\mathfrak{h}_c)$.

Fix a connected component A^+ of A'. Since K is compact, A' has only a finite number of connected components (see [4 (e), Lemma 9]) and so, in order to complete the proof of Theorem 7, it would be enough to show that there exist numbers c, $m \geqslant 0$ such that

$$|\Phi(a)| \leqslant c(1 + \sigma(a))^m \quad (a \in A^+).$$

Put $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{k}$, $\mathfrak{h}_2 = \mathfrak{h} \cap \mathfrak{p}$, $A_1 = A \cap K$, $A_1^0 = \exp \mathfrak{h}_1$ and $A_2 = \exp \mathfrak{h}_2$. Then A_1^0 is the connected component of 1 in A_1. For any $a \in A$, let a_i denote the component of a in A_i $(i = 1, 2)$ so that $a = a_1 a_2$. Let $\mathfrak{h}_2'$ be the set of all $H \in \mathfrak{h}_2$ such that $\alpha(H) \neq 0$ for any root α of $(\mathfrak{g}, \mathfrak{h})$ which

is not identically zero on $\mathfrak{h}_2$. Then $\mathfrak{h}_2'$ has only a finite number of connected components. Fix a connected component $\mathfrak{h}_2^+$ of $\mathfrak{h}_2'$ such that A^+ meets $A_1 A_2^+$. (Here $A_2^+ = \exp \mathfrak{h}_2^+$.) It would be sufficient to prove the required inequality for $a \in A^{++} = A^+ \cap (A_1 A_2^+)$.

Fix an element $b \exp H_0$ $(b \in A_1, H_0 \in \mathfrak{h}_2^+)$ in A^{++} and let $\mathfrak{z}$ denote the centralizer of $b A_1^0$ in $\mathfrak{g}$. Let $Q_{\mathfrak{z}}$ be the set of all roots of $(\mathfrak{z}, \mathfrak{h})$. Then every root in $Q_{\mathfrak{z}}$ is real (see [4 (n), § 4]) and $\alpha(H_0) \neq 0$ for $\alpha \in Q_{\mathfrak{z}}$.

Let $\mathfrak{F}$ and $\mathfrak{F}_2$ be the spaces of real linear functions on $(-1)^{\frac{1}{2}}\mathfrak{h}_1 + \mathfrak{h}_2$ and $\mathfrak{h}_2$ respectively. Introduce compatible orders in $\mathfrak{F}$ and $\mathfrak{F}_2$ such that an element $\lambda \in \mathfrak{F}_2$ is positive whenever $\lambda(H_0) > 0$. Let P be the set of all positive roots of $(\mathfrak{g}, \mathfrak{h})$ under this order and put $P_{\mathfrak{z}} = P \cap Q_{\mathfrak{z}}$. We may use this order for the definition of Δ and ξ_ϱ (see [4 (o), § 19]). Then

$$\Delta^{-1} = \xi_\varrho^{-1} \eta,$$

where $\eta = \prod_{\alpha \in P} \eta_\alpha \in \mathcal{R}$.

Fix a compact set $C = C^{-1}$ in G such that Supp $\alpha^* \subset C^*$.

LEMMA 31. *For any $g \in \mathfrak{G}$, we can select $u_i \in \mathfrak{S}(\mathfrak{h}_c)$ and $\eta_i \in \mathcal{R}$ $(1 \leqslant i \leqslant p)$ such that*

$$\left| f_\beta(a^x; g) \right| \leqslant \left| \xi_\varrho(a) \right|^{-1} \sum_{1 \leqslant i \leqslant p} \sum_{s \in W_\Delta} \left| \eta_i(a) \beta^s(a; u_i) \right|$$

for $\beta \in C_c^\infty(A')$, $a \in A'$ and $x \in C$.

Assuming this lemma, we shall first finish the proof of Theorem 7. By Lemma 10 and [4 (q), Lemma 32], we can choose $c \geqslant 1$ such that

$$\Xi(y^x) \leqslant c \Xi(y), \; 1 + \sigma(y^x) \leqslant c(1 + \sigma(y)) \quad (x \in C, \; y \in G).$$

Since $C = C^{-1}$, this implies that

$$\left| f_\beta(a^x; g) \right| \Xi(a^x)^{-1} (1 + \sigma(a^x))^m$$

$$\leqslant c^{m+1} \left| f_\beta(a^x; g) \right| \Xi(a)^{-1} (1 + \sigma(a))^m$$

$$\leqslant c^{m+1} \left| \xi_\varrho(a) \right|^{-1} \Xi(a)^{-1} (1 + \sigma(a))^m \sum_{1 \leqslant i \leqslant p} \sum_{s \in W_\Delta} \left| \eta_i(a) \beta^s(a; u_i) \right|$$

for $\beta \in C_c^\infty(A')$, $a \in A^{++}$, $x \in C$ and $m \geqslant 0$. Put $A^0 = (A^{++})^{W_\Delta}$. If $\beta \in C_c^\infty(A^0)$, it is clear that Supp $f_\beta \subset (A^{++})^C$. Hence

$$\sup_{x \in G} \left| f_\beta(x; g) \right| \Xi(x)^{-1} (1 + \sigma(x))^m$$

$$\leqslant c^{m+1} \sup_{a \in A^{++}} \left| \xi_\varrho(a) \Xi(a) \right|^{-1} (1 + \sigma(a))^m \sum_{1 \leqslant i \leqslant p} \sum_{s \in W_\Delta} \left| \eta_i(a) \beta^s(a; u_i) \right|,$$

for $\beta \in C_c^\infty(A^0)$ and $m \geqslant 0$.

It is clear that $A^{++} \subset bA_1{}^0A_2{}^+$. Extend $\mathfrak{h}_2$ to a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and define an order on the dual of $\mathfrak{a}$, which is compatible with the one chosen on $\mathfrak{F}_2$. It is clear that

$$\left| \xi_\varrho(a) \right| \Xi(a) = \Xi(a_2) e^{\varrho(\log a_2)} \quad (a \in A).$$

Since $a_2 \in A_2{}^+$ for $a \in A^{++}$, it follows easily from [4 (j), Lemma 36] that

$$\left| \xi_\varrho(a) \right| \Xi(a) \geqslant 1$$

for $a \in A^{++}$. Therefore since $\mathcal{R}$ is stable under W_A, we obtain the following result from Lemma 30.

LEMMA 32. *We can choose* $m \geqslant 0$, $\eta_i \in \mathcal{R}$ *and* $u_i \in \mathfrak{S}(\mathfrak{h}_c)$ $(1 \leqslant i \leqslant p)$ *such that*

$$\left| \int_A \beta \Phi \, da \right| \leqslant \sum_{1 \leqslant i \leqslant p} \sup_{a \in A'} (1 + \sigma(a))^m \left| \eta_i(a) \beta(a; u_i) \right|$$

for $\beta \in C_c^\infty(A^0)$.

Moreover, we have the following lemma.

LEMMA 33. *Let* $\alpha \in Q$. *Then*

$$\left| \eta_\alpha^{\cdot}(a) \alpha(\log a_2) \right| \leqslant 1 + \left| \alpha(\log a_2) \right| \quad (a \in A').$$

Since $\left| c_1 - c_2 \right| \geqslant \left| \left| c_1 \right| - \left| c_2 \right| \right|$, for two complex numbers c_1, c_2, it is clear that

$$\left| \eta_\alpha(a) \alpha(\log a_2) \right| \leqslant \left| \alpha(\log a_2) \right| \left| 1 - e^{-\alpha(\log a_2)} \right|^{-1}.$$

On the other hand, $t(1 - e^{-t})^{-1} = t + t(e^t - 1)^{-1}$

and $t(e^t - 1)^{-1} \leqslant 1 \quad (t \geqslant 0).$

Hence $\left| t \right| \left| 1 - e^{-t} \right|^{-1} \leqslant 1 + \left| t \right| \quad (t \in \mathbf{R})$

and this implies our assertion.

Fix a non-empty, open and connected subset U of $bA_1{}^0$ such that UA_2 meets A^{++} and ξ_α ($\alpha \in P$) never takes the value 1 on U unless $\alpha \in P_{\frac{1}{2}}$. Clearly this is possible. Then $UA_2{}^+ \subset A^{++} \subset bA_1{}^0A_2{}^+ \subset A'(R)$, where $A'(R)$ is defined as in [4 (o), § 19]. Also we know that Φ extends to an analytic function on $A'(R)$ (see [4 (o), Lemma 31] and [4 (p), Lemma 64]). For $\delta \in C_c^\infty(U)$ and $\gamma \in C_c^\infty(\mathfrak{h}_2{}^+)$, define the function $\delta \times \gamma$ in $C_c^\infty(A^{++})$ by

$$(\delta \times \gamma)(a) = \delta(a_1) \gamma(\log a_2) \quad (a \in A).$$

Let da_1 denote the Haar measure on A_1 and dH the Euclidean measure on $\mathfrak{h}_2$. We normalize them in such a way that

$$da = da_1 \, dH$$

for $a = a_1 \exp H \in A$. Put

$$\Phi_\delta(H) = \int_{A_1} \Phi(a_1 \exp H) \delta(a_1) \, da_1 \quad (H \in \mathfrak{h}_2{}^+).$$

Then it is clear that
$$\int_A (\delta \times \gamma) \cdot \Phi \, da = \int_{\mathfrak{h}_2{}^+} \Phi_\delta \, \gamma \, dH.$$

Now fix $\delta \in C_c^\infty(U)$ and put $V = \mathrm{Supp}\,\delta$. Then V is a compact subset of U. Let Q_I be the set of all imaginary roots (see [4 (n), §4]) of $(\mathfrak{g}, \mathfrak{h})$. Then if $\alpha \in Q_I$, it follows from the definition of U, that η_α remains bounded on V. Let P' denote the complement of $P_I = P \cap Q_I$ in P. Put

$$q(H) = \prod_{\alpha \in P'} \alpha(H) \quad (H \in \mathfrak{h}_2).$$

Then q is a polynomial function on $\mathfrak{h}_2$, which is not identically zero, and we conclude easily from Lemmas 32 and 33 that there exists an integer $m \geq 0$ such that the distribution

$$T_\delta : \gamma \to \int_{\mathfrak{h}_2{}^+} \Phi_\delta \, q^m \gamma \, dH \quad (\gamma \in C_c^\infty(\mathfrak{h}_2{}^+))$$

on $\mathfrak{h}_2{}^+$ is tempered.

On the other hand, since $bA_1{}^0 A_2{}^+ \subset A'(R)$ and Φ is analytic on $A'(R)$, we conclude from [4 (e), Theorem 2], [4 (p), Lemma 64], and the $\mathfrak{Z}$-finiteness of Θ, that

$$\Phi(ba_1 \exp H) = \sum_{1 \leq i \leq N} \xi_i(a_1) \sum_{1 \leq j \leq n} p_{ij}(H) e^{\lambda_j(H)} \quad (a_1 \in A_1{}^0, \ H \in \mathfrak{h}_2{}^+),$$

where $\xi_i \ (1 \leq i \leq N)$ are distinct characters of $A_1{}^0$, $\lambda_j \ (1 \leq j \leq n)$ distinct linear functions and p_{ij} polynomial functions on $\mathfrak{h}_{2c}$.

LEMMA 34. *Fix j $(1 \leq j \leq n)$. Then $p_{ij} = 0$ $(1 \leq i \leq N)$ unless[1] $\Re\lambda_j(H) \leq 0$ for all $H \in \mathfrak{h}_2{}^+$.*

For otherwise suppose that $\Re\lambda_j(H) > 0$ for some $H \in \mathfrak{h}_2{}^+$. Then it follows from [4 (p), Lemma 15] that

$$\sum_{1 \leq i \leq N} p_{ij} \int_{A_1} \xi_i(a_1) \delta(b^{-1} a_1) \, da_1 = 0.$$

Since this holds for every $\delta \in C_c^\infty(U)$, we conclude from [4 (h), Lemma 20] that $p_{ij} = 0 \ (1 \leq i \leq N)$.

It is now obvious from Lemma 34 that we can choose numbers c, $m \geq 0$ such that

$$|\Phi(a)| \leq c(1 + \sigma(a))^m \quad (a \in A^{++})$$

and this proves Theorem 7.

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[1] $\Re c$ denotes the real part of a complex number c.

We have still to prove Lemma 31. Put $\mathfrak{q} = [\mathfrak{g}, \mathfrak{h}]$ and for any $x \in G$, define the linear mapping Γ_x of $\mathfrak{G} \otimes \mathfrak{G}$ into $\mathfrak{G}$ as in [4 (o), § 2]. Then if $a \in A'$, Γ_a defines a bijection of $\mathfrak{S}(\mathfrak{q}_c) \otimes \mathfrak{S}(\mathfrak{h}_c)$ onto $\mathfrak{G}$ (see [4 (o), Lemma 10]). Let γ_a denote the inverse of this mapping.

LEMMA 35. *Fix* $g \in \mathfrak{G}$. *Then we can choose* $q_i \in \mathfrak{S}(\mathfrak{q}_c)$, $u_i \in \mathfrak{S}(\mathfrak{h}_c)$ *and* $\eta_i \in \mathcal{R}$ $(1 \leqslant i \leqslant p)$ *such that*

$$\gamma_a(g) = \sum_{1 \leqslant i \leqslant p} \eta_i(a) \, (q_i \otimes u_i) \quad (a \in A').$$

We use the notation of [4 (o), § 2] and put $d_X = L_X - R_X$ $(X \in \mathfrak{g}_c)$. Then d_X is a derivation of $\mathfrak{G}$. Define X_α $(\alpha \in Q)$ as in [4 (n), § 4]. Then

$$\sigma_a(X_\alpha) = \xi_\alpha(a)^{-1} L_{X_\alpha} - R_{X_\alpha} = -(1 - \xi_\alpha(a)^{-1}) L_{X_\alpha} + d_{X_\alpha} \quad (a \in A)$$

and therefore $\qquad \eta_\alpha(a) \, \sigma_a(X_a) = -(L_{X_\alpha} - \eta_\alpha(a) \, d_{X_\alpha}) \quad (a \in A').$

This shows that

$$(-1)^r \eta_{\alpha_1}(a) \ldots \eta_{\alpha_r}(a) \, \Gamma_a(X_{\alpha_1} X_{\alpha_2} \ldots X_{\alpha_r} \otimes u) = (L_{X_{\alpha_1}} - \eta_{\alpha_1}(a) \, d_{X_{\alpha_1}}) \ldots (L_{X_{\alpha_r}} - \eta_{\alpha_r}(a) \, d_{X_{\alpha_r}}) \, u$$

for $a \in A'$, $u \in \mathfrak{S}(\mathfrak{h}_c)$ and $\alpha_1, \ldots, \alpha_r \in Q$. The assertion of the lemma now follows by an easy induction on the degree of g.

For any $f \in C^\infty(G)$, put $f(x : a) = f(a^x) \, (x \in G, a \in A)$.

COROLLARY. *Let* $f \in C^\infty(G)$. *Then*

$$f(a^x; g^x) = \sum_{1 \leqslant i \leqslant p} \eta_i(a) \, f(x; q_i : a; u_i) \quad (x \in G, a \in A')$$

in the above notation.

This is obvious from [4 (o), Lemma 4].

We are now ready to prove Lemma 31. Let $g_1, \ldots, g_N$ be a base for the vector space over $\mathbf{C}$ spanned by g^x $(x \in G)$. Then

$$g^{x^{-1}} = \sum_{1 \leqslant i \leqslant N} c_i'(x) \, g_i \quad (x \in G),$$

where c_i' are analyti functions on G. Hence

$$f(a^x; g) = \sum_{1 \leqslant i \leqslant N} c_i'(x) \, f(a^x; g_i^x) \quad (x \in G, \ a \in A)$$

for $f \in C^\infty(G)$. Therefore it follows from the corollary of Lemma 35, that we can choose analytic functions c_i on G and $q_i \in \mathfrak{S}(\mathfrak{q}_c)$, $u_i \in \mathfrak{S}(\mathfrak{h}_c)$, $\eta_i \in \mathcal{R}$ $(1 \leqslant i \leqslant p)$ such that

$$f(a^x; g) = \sum_{1 \leqslant i \leqslant p} c_i(x) \eta_i(a) f(x; q_i : a; u_i) \quad (x \in G, \ a \in A')$$

for any $f \in C^\infty(G)$. Put $\alpha(x) = \alpha^*(x^*)$ and $\alpha_i(x) = c_i(x)\alpha(x; q_i)$ $(x \in G)$. Then it is clear that

$$f_\beta(a^x; g) = \sum_{1 \leqslant i \leqslant p} \alpha_i(x) \eta_i(a) \sum_{s \in W_A} \varepsilon(s) \beta^s(a; u_i \circ \Delta^{-1}) \quad (x \in C, \ a \in A')$$

for $\beta \in C_c^\infty(A')$. Let $u \to {}'u$ denote the automorphism of $\mathfrak{S}(\mathfrak{h}_c)$ such that ${}'H = H - \varrho(H)$ $(H \in \mathfrak{h}_c)$. Then since $\Delta^{-1} = \xi_\varrho^{-1}\eta$, it is clear that

$$u \circ \Delta^{-1} = \xi_\varrho^{-1} {}'u \circ \eta \quad (u \in \mathfrak{S}(\mathfrak{h}_c)),$$

as a differential operator on A'. Therefore since $\mathcal{R}$ is stable under both $\mathfrak{S}(\mathfrak{h}_c)$ and W_A, Lemma 31 follows immediately from the compactness of C.

§ 20. Proof of Theorem 8

Now suppose rank $G = $ rank K and fix a Cartan subgroup B of G such that $B \subset K$. Define L and Θ_λ $(\lambda \in L)$ as in [4 (q), § 8]. Then it follows from Theorem 7 and [4 (p), Lemma 52] that Θ_λ is tempered. Similarly we conclude from [4 (p), § 29] that

$$f \to \sum_{\lambda \in L} |\varpi(\lambda) \Theta_\lambda(f)| \quad (f \in C(G))$$

is a continuous mapping of $C(G)$ into $\mathbf{C}$. Therefore

$$f \to \sum_{\lambda \in L} \varpi(\lambda) \Theta_\lambda(f) \quad (f \in C(G))$$

is a tempered distribution.

Now we use the notation of [4 (q), § 14]. It follows from [4 (q), Lemma 27 and Cor. 1 of Lemma 57] that T_A is a tempered distribution on $C(\tilde{A}_R)$. Define

$$\phi_f(h) = F_f(h; \varpi_0) \quad (h \in \tilde{A}_R)$$

for $f \in C(G)$. Then it follows from § 18 that $f \to \phi_f$ is a continuous mapping of $C(G)$ into $C(\tilde{A}_R)$ and therefore

$$f \to T_A(\phi_f) \quad (f \in C(G))$$

is a tempered distribution on G. Hence we obtain the following result from [4 (q), Theorem 3].

THEOREM 8. *Let* $f \in C(G)$. *Then*

$$cf(1) = (-1)^q \sum_{\lambda \in L} \varpi(\lambda) \Theta_\lambda(f) + \sum_{2 \leqslant i \leqslant r} T_i(\phi_{f,i})$$

in the notation of [4 (q), *Theorem 3*].

Let $\mathcal{J}(G)$ denote the subspace of all $f \in \mathcal{C}(G)$ which are invariant under left and right translations of K. It follows from Lemma 19 that $I_c^\infty(G) = \mathcal{J}(G) \cap C_c^\infty(G)$ is dense in $\mathcal{J}(G)$. Moreover if we take into account Lemma 18 and [4 (j), Theorem 3 and Lemma 36], it follows without difficulty that the space $\mathcal{J}(G)$, including its topology, is the same as $I(G)$ of [4(k), § 12].

COROLLARY. *Let $f \in \mathcal{J}(G)$. Then, if G is not compact,*

$$cf(1) = \sum_{2 \leqslant i \leqslant r} T_i(\phi_{f,i}).$$

This is an immediate consequence of [4 (q), Theorem 6].

§ 21. Proof of an earlier conjecture

We now drop the assumption that rank G = rank K. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Define A, N, $\mathfrak{n}$ and σ as in [4 (q), § 16] and fix a Haar measure dn on N. Put

$$\Phi_f(a) = e^{\sigma(\log a)} \int_N f(an)\, dn \quad (a \in A)$$

for $f \in \mathcal{J}(G)$. Then we know from Theorem 3 that $f \to \Phi_f$ is a continuous mapping of $\mathcal{J}(G)$ into $\mathcal{C}(A)$.

LEMMA 36. *Let f be an element in $\mathcal{J}(G)$ such that $\Phi_f = 0$. Then $f = 0$.*

We shall prove this by induction on dim G. However we first verify the following result.

LEMMA 37. *It is sufficient to show that, under the conditions of Lemma 36, $f(1) = 0$.*

Fix an element $\alpha \in I_c^\infty(G)$ and consider $f_0 = \alpha * f$. Then

$$f_0 = \int_G \alpha(x)\, l(x) f\, dx \in \mathcal{J}(G)$$

from the results of § 10. Let da denote the Haar measure on A. Then we can assume that

$$dx = e^{2\,\sigma(\log a)}\, dk\, da\, dn \quad (x = kan,\ k \in K,\ a \in K,\ n \in N)$$

and therefore

$$f_0(y) = \int_G \alpha(x^{-1}) f(xy)\, dx = \int e^{2\sigma(\log a)} \alpha(n^{-1} a^{-1}) f(any)\, da\, dn \quad (y \in G).$$

Hence it follows without difficulty that

$$\Phi_{f_0}(a_0) = \int_A \Phi_\alpha(a)\,\Phi_f(a^{-1}a_0)\,da = 0 \quad (a_0 \in A)$$

and so we conclude that

$$\int \alpha(x^{-1})f(x)\,dx = f_0(1) = 0.$$

Since this holds for every $\alpha \in I_c^\infty(G)$, it is obvious that $f = 0$.

We shall now undertake some preparation for the proof of Lemma 36. Let $B = \theta(B)$ be a Cartan subgroup of G and $\mathfrak{b}$ its Lie algebra. For any $f \in C(G)$, define F_f corresponding to B as in § 18. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{b} \cap \mathfrak{p}$ in $\mathfrak{g}$ and M the analytic subgroup of G corresponding to $\mathfrak{m}$. Define the differential operator ϖ on B as in [4, (o), § 22].

LEMMA 38. *The statements of* [4 (o), Lemma 52] *remain true for* $f \in C(G)$. *Moreover, if* $\mathfrak{b}$ *is fundamental in* $\mathfrak{g}$, *there exists a positive number* c *such that*

$$cf(1) = (-1)^q F_f(1; \varpi)$$

for $f \in C(G)$. *Here* $q = \frac{1}{2}(\dim G/K - \operatorname{rank} G + \operatorname{rank} K)$.

The first part follows from Theorem 3, its corollary and Lemma 29. The second is a consequence of [4 (o), § 22], [4 (i), p. 759] and [4 (n), Lemmas 17 and 18].

Since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$, we can choose $k \in K$ such that $(\mathfrak{b} \cap \mathfrak{p})^k \subset \mathfrak{a}$. Hence we may assume that $\mathfrak{b} \cap \mathfrak{p} \subset \mathfrak{a}$. Put $\mathfrak{n}_1 = \mathfrak{m} \cap \mathfrak{n}$, $N_1 = M \cap N$, $K_1 = M \cap K$ and

$$\sigma_1(H) = \operatorname{tr}(\operatorname{ad} H)_{\mathfrak{n}_1} \quad (H \in \mathfrak{a}).$$

For any $f \in \mathcal{J}(G)$, define u_f as in Lemma 22 (for $\mathfrak{h} = \mathfrak{b}$). Then it is clear that u_f is bi-invariant under K_1 and therefore $u_f \in \mathcal{J}(M)$. The following lemma is a simple consequence of the definition of u_f.

LEMMA 39. *The Haar measure* dn_1 *on* N_1 *can be so normalized that*

$$\Phi_f(a) = e^{\sigma_1(\log a)} \int_{N_1} u_f(an_1)\,dn_1 \quad (a \in A)$$

for $f \in \mathcal{J}(G)$.

Now we come to the proof of Lemma 36. We may obviously assume that G is not compact. Fix $f \in \mathcal{J}(G)$ such that $\Phi_f = 0$ and first suppose that $\mathfrak{m} \neq \mathfrak{g}$. Then the induction hypothesis is applicable to M and therefore we conclude from Lemma 39 that $u_f = 0$. But then $F_f = 0$ from Lemma 29.

In view of Lemma 37, it would be sufficient to verify that $f(1) = 0$. We now use the notation of the proof of Lemma 26. Let f_1 be the restriction of f on G_1. If $\mathfrak{c}_\mathfrak{p} \neq \{0\}$, our induction hypothesis is applicable to G_1 and therefore $f(1) = f_1(1) = 0$. So we may assume

that $c_p = \{0\}$. Choose $\mathfrak{b}$ so that it is fundamental in $\mathfrak{g}$. If rank $\mathfrak{g} >$ rank $\mathfrak{k}$, it is clear that $\mathfrak{b} \cap \mathfrak{p} \neq \{0\}$. Therefore since $c_p = \{0\}$, it follows that $\mathfrak{m} \neq \mathfrak{g}$. But as we have seen above, this implies that $F_f = 0$ and therefore again $f(1) = 0$ from Lemma 38.

So we may now suppose that rank $\mathfrak{k} =$ rank $\mathfrak{g}$. Then $\mathfrak{b} \cap \mathfrak{p} \not\subset c$, if $\mathfrak{b}$ is not fundamental in $\mathfrak{g}$, and therefore $F_f = 0$. Hence it follows immediately from the corollary of Theorem 8 that $f(1) = 0$. This completes the proof of Lemma 36.

In [4 (k), § 16] the proof of the Plancherel formula for G/K was reduced to two conjectures. The first of these has been verified by Gindikin and Karpelevič [3] (see also [5 (b), § 3]) while Lemma 36 proves the second. Hence in particular, [4 (k), Corollary 2, p. 611] holds for all $f \in \mathcal{J}(G)$.

§ 22. Proof of Lemma 40 (first part)

We return to the notation of § 18 except that we write $B = A_{\mathfrak{h}}$. Let dh denote the Haar measure on B. Fix a finite subset F of $\mathcal{E}_K$ and define α_F as in § 12. Put $_F C(G) = \alpha_F \divideontimes C(G)$. It is obvious that $_F C(G)$ consists exactly of those $f \in C(G)$ for which $\alpha_F \divideontimes f = f$.

LEMMA 40. *Given $r_0 \geqslant 0$ and $u \in \mathfrak{S}(\mathfrak{h}_c)$, we can choose $r \geqslant 0$ and a finite set of elements $g_1, ..., g_p$ in $\mathfrak{G}$ such that*

$$\int_B (1 + \sigma(h))^{r_0} |F_f(h; u)| \, dh \leqslant \sum_{1 \leqslant i \leqslant p} \int_G |g_i f| \, \Xi (1 + \sigma)^r \, dx$$

for all $f \in {}_F C(G)$.

First suppose $\mathfrak{h} \cap \mathfrak{p} = \{0\}$. Then $B \subset K$ and

$$\int_B |F_f(h; u)| \, dh \leqslant \sup_{h \in B'} |F_f(h; u)|,$$

provided the total measure of B is 1. Let ϕ_B denote the characteristic function of $G_B = (B')^G$. Then it follows from [4 (o), Lemma 41 and Theorem 3] and Lemma 25 that there exist elements $z_1, ..., z_p \in \mathfrak{Z}$ such that

$$\sup_{h \in B'} |F_f(h; u)| \leqslant \sum_{1 \leqslant i \leqslant p} \int_G |z_i f| \, \phi_B \, dx \quad (f \in C(G)).$$

Now suppose $g \in {}_F C(G)$. Then $g = \alpha_F \divideontimes g$ and therefore

$$|g(x)| \leqslant |\alpha_F|_\infty g_1(x) \quad (x \in G),$$

where $|\alpha_F|_\infty = \sup |\alpha_F|$ and $\qquad g_1(x) = \int_K |g(kx)| \, dk.$

Therefore
$$\int |g| \phi_B \, dx \leqslant |\alpha_F|_\infty \int g_1 \phi_B \, dx \leqslant c \int |g| \, \Xi \, dx \quad (g \in {}_F C(G))$$

from [4 (q), Theorem 5], where c is a positive number independent of g. Hence

$$\sup_{h \in B'} |F_f(h; u)| \leqslant c \sum_{1 \leqslant i \leqslant p} \int |z_i f| \, \Xi \, dx \quad (f \in {}_F C(G))$$

and this implies our assertion in this case.

Now, in order to prove the lemma in general, we use induction on dim G. Let us keep to the notation of the proof of Lemma 26 and first assume that $\mathfrak{c}_\mathfrak{p} \neq \{0\}$. We can obviously suppose that $u = \gamma u_1$ where $\gamma \in \mathfrak{C}_\mathfrak{p}$ and $u_1 \in \mathfrak{S}(\mathfrak{h}_{1c})$. For $f \in C(G)$ and $c \in C_\mathfrak{p}$, let f_c denote the function on G_1 given by
$$f_c(x) = (1 + \sigma(c))^{r_0} f(cx; \gamma) \quad (x \in G_1).$$

Then $f_c \in C(G_1)$. Moreover, since $K \subset G_1$, it is obvious that $f_c \in {}_F C(G_1)$ if $f \in {}_F C(G)$. Finally
$$(1 + \sigma(ch))^{r_0} |F_f(ch; u)| \leqslant (1 + \sigma(h))^{r_0} |F_{f_c}(h; u_1)|$$

for $c \in C_\mathfrak{p}$ and $h \in B_1' = B' \cap G_1$. Therefore our assertion follows immediately by applying the induction hypothesis to G_1 and observing that

$$\max (\sigma(c), \sigma(x)) \leqslant \sigma(cx) \quad (c \in C_\mathfrak{p}, x \in G_1).$$

So we may now suppose that $\mathfrak{c}_\mathfrak{p} = \{0\}$ and $\mathfrak{h} \cap \mathfrak{p} \neq \{0\}$. Then the induction hypothesis is applicable to M.

§ 23. Proof of Lemma 40 (second part)

Let dm denote the Haar measure on M. Define Ξ_M, u_f ($f \in C(G)$) and Z_B as in §§ 14, 15 and Lemma 29 respectively. For $b \in Z_B$ and $f \in C(G)$, put

$$f_b(x) = \int_K f(kbxk^{-1}) \, dk \quad (x \in G).$$

Since $Z_B \subset K$, it is obvious that if $f \in {}_F C(G)$, the same holds for f_b.

LEMMA 41. *Fix $r \geqslant 0$ and $\zeta \in \mathfrak{S}(\mathfrak{m}_c)$. Then we can choose $v_1, \ldots, v_p$ in $\mathfrak{S}$ such that*

$$\int_M |\zeta u_{f_b}| \Xi_M (1 + \sigma)^r \, dm \leqslant \sum_{1 \leqslant i \leqslant p} \int_G |v_i f| \, \Xi (1 + \sigma)^r \, dx$$

for $f \in {}_F C(G)$ and $b \in Z_B$.

Put $N_2 = N_{\mathfrak{h}}$ in the notation of § 14. We have seen in § 15 that $\zeta u_g = u_{\,{}^{\backprime}\zeta g}$ $(g \in \mathcal{C}(G))$ where ${}^{\backprime}\zeta = d\zeta \circ d^{-1}$. Fix $b \in Z_B$. Then if $f \in {}_F\mathcal{C}(G)$, it is obvious that $f' = {}^{\backprime}\zeta f_b$ is also in ${}_F\mathcal{C}(G)$ and

$$\left| f'(x) \right| \leqslant |\alpha_F|_\infty \int_K \left| f'(kx) \right| dk = |\alpha_F|_\infty \int_{K \times K} \left| f(k_1 x k_2^{-1}; {}^{\backprime}\zeta^{k_2}) \right| dk_1 dk_2.$$

Let $\zeta_1, ..., \zeta_p$ be a base for the subspace of $\mathfrak{G}$ spanned by ${}^{\backprime}\zeta^k$ $(k \in K)$. Then

$${}^{\backprime}\zeta^k = \sum_{1 \leqslant i \leqslant p} a_i(k) \zeta_i \quad (k \in K),$$

where a_i are continuous functions on K. Put

$$c_1 = \max_i \ \sup \ |a_i|$$

and $c_2 = |\alpha_F|_\infty c_1$. Then it is clear that

$$\left| f'(x) \right| \leqslant c_2 \sum_i \int_{K \times K} \left| f_i(k_1 x k_2) \right| dk_1 dk_2,$$

where $f_i = \zeta_i f$.

On the other hand, it is clear that

$$\int \left| \zeta u_{f_b} \right| \Xi_M (1 + \sigma)^r \, dm \leqslant \int_{M \times N_2} \left| f'(mn) \right| d(m) \, \Xi_M(m) \, (1 + \sigma(m))^r \, dm \, dn.$$

By Lemma 21, there exists a number $c_0 \geqslant 1$ such that

$$1 + \sigma(m) \leqslant c_0 (1 + \sigma(mn)) \quad (m \in M, \ n \in N_2).$$

Put $c_3 = c_0^r c_2$. Then

$$\int \left| \zeta u_{f_b} \right| \Xi_M (1 + \sigma)^r \, dm \leqslant c_3 \int f_0(x) \, e^{-\varrho(H_2(x))} \Xi_M(\mu(x)) \, (1 + \sigma(x))^r \, dx$$

in the notation of § 42, where

$$f_0(x) = \sum_{1 \leqslant i \leqslant p} \int_{K \times K} \left| f_i(k_1 x k_2) \right| dk_1 dk_2.$$

But we know from the corollary of Lemma 84 that

$$\int_K e^{-\varrho(H_2(xk))} \Xi_M(\mu(xk)) \, dk = \Xi(x) \quad (x \in G).$$

Hence it is clear that

$$\int |\zeta u_{f_{\mathfrak{b}}}| \, \Xi_M (1+\sigma)^r \, dm \leqslant c_3 \sum_i \int |\zeta_i f| \, \Xi (1+\sigma)^r \, dx$$

and this implies the assertion of the lemma.

We can now finish the proof of Lemma 40. Let F_1 be the set of all irreducible classes of $K_1 = K \cap M$, which occur in the reduction, with respect to K_1, of some element of F. It is clear from Lemma 22 that $u_f \in_{F_1} C(M)$ if $f \in_F C(G)$. Since the induction hypothesis is applicable to M, we can choose $\zeta_1, \ldots, \zeta_q$ in $\mathfrak{S}(\mathfrak{m}_c)$ and $r \geqslant 0$ such that

$$\int_{B_0} |F_g^M(h; u)| \, (1+\sigma(h))^{r_0} dh \leqslant \sum_{1 \leqslant i \leqslant p} \int_M |\zeta_i g| \, \Xi_M (1+\sigma)^r \, dm$$

for $g \in_{F_1} C(M)$. (Here $B_0 = B \cap M$ and F_g^M is defined as in Lemma 29.) Therefore the required result follows immediately from Lemmas 29 and 41.

§ 24. Proof of Theorem 9

Let Θ be an invariant and $\mathfrak{Z}$-finite distribution on G. Fix $\mathfrak{b} \in \mathcal{E}_K$ and let $\Theta_{\mathfrak{b}}$ denote the corresponding Fourier component of Θ (see [4 (q), § 17]). Then we know from [4 (q), Lemma 33] that $\Theta_{\mathfrak{b}}$ is an analytic function on G.

THEOREM 9. *Suppose Θ is tempered. Then we can choose c, $m \geqslant 0$ such that*

$$|\Theta_{\mathfrak{b}}(x)| \leqslant c\Xi(x)(1+\sigma(x))^m \quad (x \in G).$$

It follows from Theorem 1 that we can choose $\beta' \in C_c^\infty(G)$ such that $\Theta_{\mathfrak{b}} = \Theta_{\mathfrak{b}} * \beta'$. Put $\beta(x) = \beta'(x^{-1})$. Then

$$\Theta_{\mathfrak{b}}(f) = \Theta_{\mathfrak{b}}(f * \beta) \quad (f \in C_c^\infty(G)).$$

On the other hand, from Theorem 7 and its corollary, we can choose numbers c_0, $s \geqslant 0$ such that

$$|\Theta(g)| \leqslant c_0 \sum_{1 \leqslant i \leqslant r} \int_{A_i} (1+\sigma(a))^s |F_{g,i}(a)| \, d_i a \quad (g \in C_c^\infty(G)).$$

Therefore from Lemma 40, there exist elements $v_1, \ldots, v_p \in \mathfrak{G}$ and $m \geqslant 0$ such that

$$|\Theta_{\mathfrak{b}}(g)| = |\Theta(g')| \leqslant \sum_{1 \leqslant i \leqslant p} \int |v_i g'| \, \Xi (1+\sigma)^m \, dx \leqslant d(\mathfrak{b})^2 \sum_i \int |v_i g| \, \Xi (1+\sigma)^m \, dx.$$

Here $g' = \alpha_{\mathfrak{b}} * g$ and we have made use of the fact that

$$\Theta_{\mathfrak{b}}(g) = \Theta(g * \alpha_{\mathfrak{b}}) = \Theta(\alpha_{\mathfrak{b}} * g)$$

which follows from the invariance of Θ. Therefore if we put $\beta_i = v_i\beta$, we get

$$|\Theta_b(f)| = |\Theta_b(f * \beta)| \leqslant \sum_i d(\mathfrak{b})^2 \int |f * \beta_i| \Xi(1+\sigma)^m dx \quad (f \in C_c^\infty(G)).$$

Now put $\Omega = \mathrm{Supp}\ \beta$ and $\qquad c_1 = d(\mathfrak{b})^2 \sum_i \sup |\beta_i|.$

Then $\qquad |\Theta_b(f)| \leqslant c_1 \int_\Omega (1+\sigma(y))^m dy \int_G |f(x)| \Xi(xy)(1+\sigma(x))^m dx.$

But we can choose (see [4 (q), Lemma 32]) $c_2 \geqslant 0$ such that

$$\Xi(xy) \leqslant c_2 \Xi(x) \quad (x \in G,\ y \in \Omega).$$

Then $\qquad c = c_1 c_2 \int_\Omega (1+\sigma(y))^m dy < \infty$

and $\qquad |\Theta_b(f)| \leqslant c \int_G |f| \Xi(1+\sigma)^m dx$

for $f \in C_c^\infty(G)$. The assertion of Theorem 9 is now obvious.

COROLLARY. *Let $f \in C(G)$. Then*

$$\Theta_b(f) = \int_G \Theta_b f\, dx.$$

Let α be a variable element in $C_c^\infty(G)$ which converges to f in $C(G)$. Then

$$\Theta_b(f) = \lim \Theta_b(\alpha) = \lim \int \Theta_b \alpha\, dx = \int \Theta_b f\, dx$$

from Lemma 11.

§ 25. Application to tempered representations

Let π be a representation of G on a locally convex space $\mathfrak{H}$. We say that an element $\phi \in \mathfrak{H}$ is K-finite (under π), if the space spanned by the vectors $\pi(k)\phi$ $(k \in K)$ has finite dimension. Now suppose $\mathfrak{H}$ is a Hilbert space and π is unitary and irreducible. Let Θ_π denote the character of π (see [4 (b), § 5] and [8]). We say that π is tempered if Θ_π is tempered.

THEOREM 10. *Let π be an irreducible unitary representation of G on a Hilbert space $\mathfrak{H}$, which is tempered. Then there exists a number $m \geqslant 0$ with the following property. For any two K-finite vectors $\phi, \psi \in \mathfrak{H}$, we can choose a constant $c \geqslant 0$ such that*

$$\left|(\phi, \pi(x)\psi)\right| \leqslant c\Xi(x)(1+\sigma(x))^m \quad (x \in G).$$

Let Θ denote the character of π. Then Θ is an invariant eigendistribution of $\mathfrak{Z}$ on G. Define $E_{\mathfrak{b}}$ ($\mathfrak{b} \in \mathcal{E}_K$) as in § 2 and put $\mathfrak{H}_{\mathfrak{b}} = E_{\mathfrak{b}}\mathfrak{H}$. Choose an orthonormal base ψ_i ($i \in J$) for $\mathfrak{H}$ such that every ψ_i lies in $\mathfrak{H}_{\mathfrak{b}}$ for some $\mathfrak{b}$. Let $J_{\mathfrak{b}}$ be the set of all i such that $\psi_i \in \mathfrak{H}_{\mathfrak{b}}$. Then[1]

$$[J_{\mathfrak{b}}] = \dim \mathfrak{H}_{\mathfrak{b}} \leqslant Nd(\mathfrak{b})^2 \quad (d \in \mathcal{E}_K),$$

where N is a positive integer independent of $\mathfrak{b}$ (see [4 (b), Theorem 4]). It follows from the definition of Θ that

$$\Theta(f) = \sum_{i \in J} \int f(x)\,(\psi_i, \pi(x)\psi_i)\,dx \quad (f \in C_c^\infty(G)),$$

and therefore it is clear that

$$\Theta_{\mathfrak{b}}(x) = \operatorname{tr}\,(E_{\mathfrak{b}}\pi(x)E_{\mathfrak{b}}) \quad (x \in G)$$

for $\mathfrak{b} \in \mathcal{E}_K$. Now fix $\mathfrak{b}_0 \in \mathcal{E}_K$ such that $\mathfrak{H}_{\mathfrak{b}_0} \neq \{0\}$. Then, by Theorem 9, we can choose numbers c_0, $m \geqslant 0$ such that

$$\left|\Theta_{\mathfrak{b}_0}(x)\right| \leqslant c_0\Xi(x)(1+\sigma(x))^m \quad (x \in G).$$

Let ϕ and ψ be two K-finite elements in $\mathfrak{H}$. For any finite subset F of $\mathcal{E}_K$, put

$$E_F = \sum_{\mathfrak{b} \in F} E_{\mathfrak{b}}$$

and $\mathfrak{H}_F = E_F\mathfrak{H}$. Also define α_F as in § 6. We can obviously choose F so large that $\mathfrak{b}_0 \in F$ and ϕ, ψ lie in $\mathfrak{H}_F$. Let $\mathcal{L}_F(G)$ be the space of all functions $f \in C_c^\infty(G)$ such that $f = \alpha_F \ast f \ast \alpha_F$. Then $\mathcal{L}_F(G)$ is an algebra under convolution and it is clear that $\mathfrak{H}_F$ is stable under $\pi(f)$ for $f \in \mathcal{L}_F(G)$. Let $\pi_F(f)$ denote the restriction of $\pi(f)$ on $\mathfrak{H}_F$. Then π_F is a representation of $\mathcal{L}_F(G)$ on $\mathfrak{H}_F$. We claim that this representation is irreducible. Fix an element $\psi_0 \neq 0$ in $\mathfrak{H}_F$. Since $\mathfrak{H}$ is irreducible under π, elements of the form $\pi(f)\psi_0$ ($f \in C_c^\infty(G)$) are dense in $\mathfrak{H}$. Therefore, since

$$E_F\pi(f)\psi_0 = \pi(\alpha_F \ast f \ast \alpha_F)\psi_0$$

and $\dim \mathfrak{H}_F < \infty$, it is clear that $\mathfrak{H}_F = \pi(\mathcal{L}_F(G))\psi_0$ and this shows that π_F is irreducible. Hence, by the Burnside theorem, we can choose $\alpha, \beta \in \mathcal{L}_F(G)$ such that

$$\phi = \pi(\alpha)\psi_i,\ \psi = \pi(\beta)\psi_i \quad (i \in J_{\mathfrak{b}_0}).$$

Therefore $\quad (\phi, \pi(x)\psi) = (\psi_i, \pi(\check{\alpha})\pi(x)\pi(\beta)\psi_i) \quad (x \in G,\ i \in J_{\mathfrak{b}_0}),$

where $\check{\alpha}(y) = \operatorname{conj} \alpha(y^{-1})$ ($y \in G$). This shows that

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$$(\phi, \pi(x)\,\psi).\ \dim \mathfrak{H}_{\mathfrak{b}_\circ} = \operatorname{tr}\,(E_{\mathfrak{b}_\circ}\pi(\dot{\alpha})\,\pi(x)\,\pi(\beta)\,E_{\mathfrak{b}_\circ}) = \int_{G\times G} \tilde{\alpha}(y)\,\Theta_{\mathfrak{b}_\circ}\,(yxz)\,\beta(z)\,dy\,dz,$$

and the required result now follows immediately from Lemma 10 and [4 (q), Lemma 32], if we observe that $\mathfrak{H}_{\mathfrak{b}_\circ} \neq \{0\}$.

Part II. Spherical functions and differential equations

§ 26. Two key lemmas and their first reduction

Let V be a (complex) Hilbert space of finite dimension and $\mu = (\mu_1, \mu_2)$ a continuous and unitary double representation[1] of K on V. Define $\mathfrak{c}_\mathfrak{p}$ as in § 7.

LEMMA 42. *Let $\phi \neq 0$ be a C^∞ μ-spherical function (see § 11) from G to V such that:*

1) *The space of functions of the form $z\phi$ $(z \in \mathfrak{Z})$ has finite dimension.*

2) *There exist numbers c, $r \geq 0$ such that*

$$|\phi(x)| \leq c\,\Xi(x)\,(1+\sigma(x))^r \quad (x \in G).$$

Put
$$\|\phi\|_t = \left\{\int_{\sigma(x)\leq t} |\phi(x)|^2\,dx\right\}^{\frac{1}{2}} \quad (t \geq 0).$$

Then there exists a unique integer $\nu \geq 0$ such that

$$0 < \liminf_{t\to\infty} t^{-\nu/2}\,\|\phi\|_t \leq \limsup_{t\to\infty} t^{-\nu/2}\,\|\phi\|_t < \infty.$$

Moreover, $\nu \geq \dim \mathfrak{c}_\mathfrak{p}$ and, for any $\varepsilon > 0$, we can choose t_0, $\delta > 0$ such that

$$t_1^{-\nu}\{\|\phi\|_{t_2}^2 - \|\phi\|_{t_1}^2\} \leq \varepsilon$$

for $t_0 \leq t_1 \leq t_2 \leq (1+\delta)t_1$.

Fix a maximal abelian subspace $\mathfrak{a}_\mathfrak{p}$ of $\mathfrak{p}$ and let $A_\mathfrak{p} = \exp \mathfrak{a}_\mathfrak{p}$ be the corresponding subgroup of G. Introduce an order in the space of (real) linear functions on $\mathfrak{a}_\mathfrak{p}$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_\mathfrak{p} + \mathfrak{n}$ be the corresponding Iwasawa decomposition of $\mathfrak{g}$. As usual put

$$\varrho(H) = \tfrac{1}{2}\operatorname{tr}\,(\operatorname{ad} H)_\mathfrak{n} \quad (H \in \mathfrak{a}_\mathfrak{p})$$

and let $\mathfrak{a}_\mathfrak{p}^+$ be the set of all $H \in \mathfrak{a}_\mathfrak{p}$ where $\alpha(H) \geq 0$ for every positive root α of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$. We recall (see § 7) that $\mathfrak{p}$ is a Hilbert space with respect to the norm $\|X\|$ $(X \in \mathfrak{p})$. Consider the set S^+ of all points $H \in \mathfrak{a}_\mathfrak{p}^+$ with $\|H\| = 1$ and put $A_\mathfrak{p}^+ = \exp \mathfrak{a}_\mathfrak{p}^+$.

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[1] This means that V is a left K-module under μ_1 and a right K-module under μ_2. Moreover, the operations of K on the left, commute with those on the right.

LEMMA 43. *We keep to the notation of Lemma* 42. *Then the following three conditions are mutually equivalent.*

1) $\nu = 0$.

2) $\lim\limits_{t \to +\infty} e^{t\varrho(H)}\phi(h \exp tH) = 0$ *for* $H \in S^+$ *and* $h \in A_{\mathfrak{p}}{}^+$.

3) $\phi \in \mathsf{C}(G) \otimes V$.

It is convenient to prove the above two lemmas together. We shall call ν the index of ϕ. The uniqueness of ν is obvious from its definition. For the rest we use induction on dim $\mathfrak{g}$. If $\mathfrak{p} = \{0\}$, G is compact and all our assertions are true trivially with $\nu = 0$. So we may suppose that dim $\mathfrak{p} \geqslant 1$. First assume that $\mathfrak{c}_{\mathfrak{p}} \neq \{0\}$.

Fix an element $H_0 \in \mathfrak{c}_{\mathfrak{p}}$ with $\|H_0\| = 1$ and let $\mathfrak{g}_1 = \mathfrak{k} + \mathfrak{p}_1$ where $\mathfrak{p}_1$ is the orthogonal complement (see § 7) of $\mathbf{R}H_0$ in $\mathfrak{p}$. Then $\mathfrak{g}_1$ is an ideal in $\mathfrak{g}$. Let G_1 be the analytic subgroup of G corresponding to $\mathfrak{g}_1$. Then the mapping $(t, y) \to \exp tH_0 \cdot y$ $(t \in \mathbf{R}, y \in G_1)$ defines an analytic diffeomorphism of $\mathbf{R} \times G_1$ onto G. Put $\phi(t:y) = \phi(\exp tH_0 \cdot y)$. Since H_0 lies in the center of $\mathfrak{g}$, we can, in view of condition 1) of Lemma 42, choose complex numbers c_i $(0 \leqslant i \leqslant m)$ such that $c_0 = 1$ and

$$\sum_{0 \leqslant i \leqslant m} c_i H_0{}^{m-i}\phi = 0.$$

Therefore it is clear that

$$\phi(t:y) = \sum_{1 \leqslant i \leqslant p} e^{(-1)^{\frac{1}{2}}\lambda_i t}\phi_i(t:y) \quad (t \in \mathbf{R}, y \in G_1),$$

where $\lambda_1, \ldots, \lambda_p$ are distinct complex numbers,

$$\phi_i(t:y) = \sum_{0 \leqslant j \leqslant d} t^j \phi_{ij}(y) \quad (1 \leqslant i \leqslant p)$$

and ϕ_{ij} are([1]) C^∞ functions from G_1 to V. We may assume that $\phi_i \neq 0$. Then it follows (see [4 (j), § 15]) from condition 2) of Lemma 42 that $\lambda_1, \ldots, \lambda_p$ are all real. Now $K \subset G_1$ and dim $G_1 <$ dim G. Therefore it is easy to see that, if $\phi_{ij} \neq 0$, Lemmas 42 and 43 are applicable to ϕ_{ij} by induction hypothesis. Let dy denote the Haar measure on G_1 and ν_{ij} the index of the function ϕ_{ij} on G_1. Moreover we put $\nu_{ij} = -\infty$ if $\phi_{ij} = 0$. Now let

$$\nu = 1 + \max_{i, j} (2j + \nu_{ij}).$$

We shall prove that ν is the index of ϕ.

We may obviously assume that $dx = dt\,dy$. Then for $T \geqslant 0$,

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([1]) Here we make use of the fact that the functions $f_{ij}(t) = t^j e^{(-1)^{\frac{1}{2}}\lambda_i t}(1 \leqslant i \leqslant p, j \geqslant 0)$ are linearly independent over $\mathbf{C}$.

$$\|\phi\|_T \leqslant \left\{\int_0^T dt \int_{\sigma(y)\leqslant T} |\phi(t:y)|^2 dy\right\}^{\frac{1}{2}} \leqslant \sum_{i,j}\left\{\int_0^T t^{2j}dt \int_{\sigma(y)\leqslant T} |\phi_{ij}(y)|^2 dy\right\}^{\frac{1}{2}}$$

by the triangle inequality. Now fix i, j such that $\phi_{ij}\neq 0$. Then by the definition of ν_{ij}, we can choose a number $b_{ij}>0$ such that

$$\int_{\sigma(y)\leqslant T} |\phi_{ij}(y)|^2 dy \leqslant b_{ij}(1+T)^{\nu_{ij}}$$

for all $T\geqslant 0$. Therefore since $\nu\geqslant 1+2j+\nu_{ij}$, it is clear that

$$\limsup_{T\to\infty} T^{-\nu/2}\|\phi\|_T < \infty.$$

On the other hand, in order to show that

$$\liminf_{T\to\infty} T^{-\nu/2}\|\phi\|_T > 0,$$

it would be sufficient to obtain the following result.

LEMMA 44. *Fix $\delta>0$. Then*

$$\liminf_{T\to\infty} T^{-\nu} \int_{\substack{t^2+\sigma(y)^2\leqslant T^2\\ \sigma(y)\leqslant\delta t}} |\phi(t:y)|^2 dt\,dy > 0.$$

We may obviously assume that $\delta\leqslant 1$. For $T\geqslant 0$, let $J(T)$ denote the interval $T/2\leqslant t\leqslant T/\sqrt{2}$ and $G_1(T)$ the set of all points $y\in G_1$ with $\sigma(y)\leqslant T$. If $t\in J(T)$ and $y\in G_1(\delta T/2)$, it is obvious that $t^2+\sigma(y)^2\leqslant T^2$ and $\sigma(y)\leqslant\delta t$. Hence it would be enough to show that

$$\liminf_{T\to\infty} T^{-\nu} I(T) > 0,$$

where
$$I(T) = \int_{J(T)} dt \int_{G_1(\delta T/2)} |\phi(t:y)|^2 dy.$$

Let $\langle v_1, v_2\rangle$ $(v_1, v_2\in V)$ denote the scalar product in V and put

$$I_{ij}(T) = \int_{J(T)} dt \int_{G_1(\delta T/2)} e^{(-1)^{\frac{1}{2}}(\lambda_j-\lambda_i)t}\langle\phi_i(t:y),\ \phi_j(t:y)\rangle\,dy$$

and $I_i(T)=I_{ii}(T)$ $(1\leqslant i,j\leqslant p)$. Then it is clear that[1]

$$I(T) = \sum_{1\leqslant i\leqslant p} I_i(T) + 2\,\Re \sum_{1\leqslant i<j\leqslant p} I_{ij}(T).$$

Fix m, n $(0\leqslant m,n\leqslant d)$ and put

---

[1] $\Re c$ denotes the real part of a complex number c.

$$I_{im,jn}(T) = \int_{J(T)} dt \int_{G_1(\delta T/2)} e^{(-1)^{\frac{1}{2}}(\lambda_j - \lambda_i)t} t^{m+n} \langle \phi_{im}(y), \phi_{jn}(y) \rangle \, dy$$

for $1 \leqslant i, j \leqslant p$. Then if $i \neq j$, there exists a constant $a(im, jn) \geqslant 0$ such that

$$\left| \int_{J(T)} t^{m+n} e^{(-1)^{\frac{1}{2}}(\lambda_j - \lambda_i)t} \, dt \right| \leqslant a(im, jn)(1+T)^{m+n} \quad (T \geqslant 0).$$

(This follows by integrating by parts and using induction on $m + n$.) Hence it is clear that

$$\left| I_{im,jn}(T) \right| \leqslant a(im, jn)(1+T)^{m+n} \| \phi_{im} \|_{\delta T/2} \| \phi_{jn} \|_{\delta T/2},$$

where

$$\| \phi_{kl} \|_t^2 = \int_{G_1(t)} | \phi_{kl}(y) |^2 \, dy \quad (t \geqslant 0)$$

for $1 \leqslant k \leqslant p$, $1 \leqslant l \leqslant d$. Therefore if $\phi_{im} \neq 0$, $\phi_{jn} \neq 0$, we get

$$\left| I_{im,jn}(T) \right| \leqslant b(im, jn)(1+T)^{m+n+(\nu_{im}+\nu_{jn})/2} \leqslant b(im, jn)(1+T)^{\nu-1} \quad (T \geqslant 0),$$

where $b(im, jn)$ is a positive number independent of T. Therefore it would be enough to verify that

$$\liminf_{T \to \infty} T^{-\nu} \sum_{1 \leqslant i \leqslant p} I_i(T) > 0.$$

Fix i such that $\nu_{ij} + 2j + 1 = \nu$ for some j. Then it would suffice to show that

$$\liminf_{T \to \infty} T^{-\nu} I_i(T) > 0.$$

Let Q be the set of all j $(0 \leqslant j \leqslant d)$ such that $\nu_{ij} + 2j + 1 = \nu$. Then if $j \notin Q$, we have $\nu_{ij} + 2j + 1 < \nu$. Hence

$$T^{-\nu} \int_{J(T)} t^{2j} \, dt \int_{G_1(\delta T/2)} | \phi_{ij}(y) |^2 \, dy \leqslant T^{\nu_{ij}+2j+1-\nu} T^{-\nu_{ij}} \| \phi_{ij} \|_T^2 \to 0$$

as $T \to \infty$. Put

$$\phi_i^0(t:y) = \sum_{j \in Q} t^j \phi_{ij}(y) \quad \text{and} \quad I_i^0(T) = \int_{J(T)} dt \int_{G_1(\delta T/2)} | \phi_i^0(t:y) |^2 \, dy.$$

Then it would obviously be enough to prove that

$$\liminf_{T \to \infty} T^{-\nu} I_i^0(T) > 0.$$

Now fix $\varepsilon (0 < \varepsilon \leqslant \delta/2)$ and put

$$I(\varepsilon:T) = \int_{J(T)} dt \int_{G_1(\varepsilon T)} |\phi_i^0(t:y)|^2 \, dy.$$

Then $I(\varepsilon:T) \leqslant I_i^0(T)$ and so it is sufficient to verify that

$$\liminf_{T\to\infty} T^{-\nu} I(\varepsilon:T) > 0.$$

Put
$$I_{mn}(\varepsilon:T) = \int_{J(T)} dt \int_{G_1(\varepsilon T)} t^{m+n} \langle \phi_{im}(y), \phi_{in}(y) \rangle \, dy$$

and
$$I_m(\varepsilon:T) = I_{mm}(\varepsilon:T) \quad (m,n \in Q).$$

Then
$$I(\varepsilon:T) = \sum_{j \in Q} I_j(\varepsilon:T) + 2\Re \sum_{\substack{m,n \in Q \\ m < n}} I_{mn}(\varepsilon:T).$$

Moreover, since $\phi_{ij} \neq 0$ for $j \in Q$, we can, by induction hypothesis, choose positive numbers a, b such that

$$\|\phi_{ij}\|_T \leqslant b(1+T)^{\nu_{ij}/2} \quad (T \geqslant 0, \, j \in Q)$$

and
$$\|\phi_{ij}\|_T \geqslant aT^{\nu_{ij}/2} \quad (j \in Q)$$

for T sufficiently large. Then

$$|I_{mn}(\varepsilon:T)| \leqslant b^2 T^{m+n+1}(1+\varepsilon T)^{(\nu_{im}+\nu_{in})/2} \quad (m,n \in Q).$$

Hence
$$I(\varepsilon:T) \geqslant \sum_{j \in Q} c_j T^{2j+1}(\varepsilon T)^{\nu_{ij}} - 2\sum_{\substack{m,n \in Q \\ m < n}} b^2 T^{m+n+1}(1+\varepsilon T)^{(\nu_{im}+\nu_{in})/2},$$

for large T, where
$$c_j = a^2(2^{-(j+\frac{1}{2})} - 2^{-(2j+1)})/(2j+1) > 0.$$

Let $k = \max_{j \in Q} j$. Then $\nu_{ij} - \nu_{ik} = 2(k-j)$ and therefore

$$I(\varepsilon:T) \geqslant T^\nu \varepsilon^{\nu_{ik}} \Big\{ \sum_{j \in Q} c_j \varepsilon^{2(k-j)} - 2\sum_{\substack{m,n \in Q \\ m > n}} b^2 \varepsilon^{2k-m-n}(1+(\varepsilon T)^{-1})^{(\nu_{im}+\nu_{in})/2} \Big\}.$$

This shows that

$$\liminf_{T\to\infty} T^{-\nu} I(\varepsilon:T) \geqslant \varepsilon^{\nu_{ik}} \Big\{ c_k + \sum_{j \in Q'} c_j \varepsilon^{2(k-j)} - 2\sum_{\substack{m,n \in Q \\ m > n}} b^2 \varepsilon^{2k-m-n} \Big\},$$

where Q' is the set of all $j \neq k$ in Q. Now $c_k > 0$ and $2k - m - n \geqslant 1$ since $m < n$. Hence

$$\liminf_{T\to\infty} T^{-\nu} I(\varepsilon:T) > 0$$

if ε is sufficiently small. This proves Lemma 44.

If $l = \dim \mathfrak{c}_\mathfrak{p}$, it is obvious thas $\dim (\mathfrak{c}_\mathfrak{p} \cap \mathfrak{g}_1) = l - 1$. Hence, by induction hypothesis, $\nu_{ij} \geqslant l - 1$ whenever $\phi_{ij} \neq 0$. Therefore

$$\nu = \max_{i,j} (\nu_{ij} + 2j + 1) \geqslant l = \dim \mathfrak{c}_\mathfrak{p}.$$

Now we come to the last assertion of Lemma 42. By the triangle inequality,

$$\{\|\phi\|_{T_2} - \|\phi\|_{T_1}^2\}^{\frac{1}{2}} \leqslant \sum_{1 \leqslant i \leqslant p} \{\|\phi_i\|_{T_2}^2 - \|\phi_i\|_{T_1}^2\}^{\frac{1}{2}}$$

for $T_2 \geqslant T_1 \geqslant 0$. Here

$$\phi_i(\exp tH_0 \cdot y) = \phi_i(t:y) = e^{(-1)^{\frac{1}{2}}\lambda_i t} \sum_{0 \leqslant j \leqslant d} t^j \phi_{ij}(y) \quad (t \in \mathbf{R}, \ y \in G_1)$$

as before. Put $\nu_i = 1 + \max_j (2j + \nu_{ij})$. Then $\nu = \max_i \nu_i$ and so it would clearly be enough to consider the case when $p = 1$. Hence we may assume that

$$\phi(t:y) = e^{(-1)^{\frac{1}{2}}\lambda t} \sum_{0 \leqslant j \leqslant d} t^j \phi_j(y),$$

where λ is real. Then again, by the triangle inequality, we have

$$\{\|\phi\|_{T_2}^2 - \|\phi\|_{T_1}^2\}^{\frac{1}{2}} \leqslant \sum_{0 \leqslant j \leqslant d} \left\{ \int_{T_1^2 \leqslant t^2 + \sigma(y)^2 \leqslant T_2^2} t^{2j} |\phi_j(y)|^2 \, dt \, dy \right\}^{\frac{1}{2}}$$

and so it would be sufficient to consider the case when

$$\phi(t:y) = e^{(-1)^{\frac{1}{2}}\lambda t} t^j \phi_1(y).$$

Then $\nu = 2j + 1 + \nu_1$ where ν_1 is the index of the function ϕ_1 on G_1.

Now suppose $0 \leqslant T_1 \leqslant T_2$. Then

$$\|\phi\|_{T_2}^2 - \|\phi\|_{T_1}^2 = 2 I_1(T_1:T_2) + 2 I_2(T_1:T_2),$$

where
$$I_1(T_1:T_2) = \int_0^{T_1} t^{2j} \, dt \int_{(T_1^2 - t^2) \leqslant \sigma(y)^2 \leqslant (T_2^2 - t^2)} |\phi_1(y)|^2 \, dy$$

and
$$I_2(T_1:T_2) = \int_{T_1}^{T_2} t^{2j} \, dt \int_{\sigma(y)^2 \leqslant T_2^2 - t^2} |\phi_1(y)|^2 \, dy.$$

Fix $\varepsilon > 0$ and select a small number $\varepsilon_1 > 0$. Then, by induction hypothesis, we can choose $t_0 \geqslant 1$ and $\delta_1 (0 < \delta_1 < 1)$ with the property that

$$\int_{t_1 \leqslant \sigma(y) \leqslant t_2} |\phi_1(y)|^2 \, dy \leqslant \varepsilon_1 t_1^{\nu_1}$$

if $t_0 \leqslant t_1 \leqslant t_2 \leqslant (1 + \delta_1) t_1$. Fix a small positive number $\delta (0 < \delta < 1)$ and put $\gamma = 1 - \delta^{\frac{1}{2}}$. Then if $0 \leqslant t \leqslant \gamma T_1$ and $0 < T_1 \leqslant T_2 \leqslant (1 + \delta) T_1$, we have

$$(T_2^2 - t^2)^{\frac{1}{2}} (T_1^2 - t^2)^{-\frac{1}{2}} - 1 \leqslant (T_2^2 - t^2)(T_1^2 - t^2)^{-1} - 1$$
$$= (T_2^2 - T_1^2)(T_1^2 - t^2)^{-1} \leqslant \{(1 + \delta)^2 - 1\}(1 - \gamma^2)^{-1} \leqslant 3 \delta^{\frac{1}{2}} \leqslant \delta_1,$$

provided δ is sufficiently small. Moreover,

$$(T_1^2 - t^2)^{\frac{1}{2}} \geqslant T_1 (1 - \gamma^2)^{\frac{1}{2}} \geqslant T_1 \delta^{\frac{1}{2}} \geqslant t_0.$$

if $T_1 \geqslant T_0 = \delta^{-\frac{1}{2}} t_0$. Therefore if $T_0 \leqslant T_1 \leqslant T_2 \leqslant (1 + \delta) T_1$ and $0 \leqslant t \leqslant \gamma T_1$, we have

$$\int\limits_{(T_1^2 - t^2) \leqslant \sigma(y)^2 \leqslant (T_2^2 - t^2)} |\phi_1(y)|^2 dy \leqslant \varepsilon_1 (T_1^2 - t^2)^{\nu_1/2}.$$

But then since $\nu = \nu_1 + 2j + 1$, we conclude that

$$\int_0^{\gamma T_1} t^{2j} dt \int\limits_{(T_1^2 - t^2) \leqslant \sigma(y)^2 \leqslant (T_2^2 - t^2)} |\phi_1(y)|^\nu dy \leqslant \varepsilon_1 T_1^\nu.$$

On the other hand, we can select $b > 0$ such that

$$\int_{G_1(t)} |\phi_1(y)|^2 dy \leqslant b(1 + t)^{\nu_1}$$

for $t \geqslant 0$. Therefore if $T_0 \leqslant T_1 \leqslant T_2 \leqslant (1 + \delta) T_1$, we have

$$\int_{\gamma T_1}^{T_1} t^{2j} dt \int\limits_{(T_1^2 - t^2) \leqslant \sigma(y)^2 \leqslant (T_2^2 - t^2)} |\phi_1(y)|^2 dy \leqslant b \int_{\gamma T_1}^{T_1} t^{2j} (1 + T_2)^{\nu_1} dt \leqslant 3^{\nu_1} b T_1^\nu \delta^{\frac{1}{2}}$$

since $1 + T_2 \leqslant 1 + (1 + \delta) T_1 \leqslant 3 T_1$. This proves that

$$I_1(T_1 : T_2) \leqslant T_1^\nu (3^{\nu_1} b \delta^{\frac{1}{2}} + \varepsilon_1)$$

if $T_0 \leqslant T_1 \leqslant T_2 \leqslant (1 + \delta) T_1$. On the other hand, it is obvious that

$$I_2(T_1 : T_2) \leqslant T_2^{2j} (T_2 - T_1) b (1 + T_2)^{\nu_1} \leqslant 3^\nu b T_1^\nu \delta.$$

Therefore

$$\|\phi\|_{T_2}^2 - \|\phi\|_{T_1}^2 = 2 I_1(T_1 : T_2) + 2 I_2(T_1 : T_2) \leqslant 2(\varepsilon_1 + 3^{\nu_1} b \delta^{\frac{1}{2}} + 3^\nu b \delta) T_1^\nu \leqslant \varepsilon T_1^\nu,$$

if ε_1 and δ are chosen sufficiently small. This completes the proof of all the statements of Lemma 42 in case dim $\mathfrak{c}_\mathfrak{p} \geqslant 1$.

Now we come to Lemma 43. Since $\nu \geqslant \dim \mathfrak{c}_\mathfrak{p} \geqslant 1$, condition 1) of Lemma 43 cannot be fulfilled in the present case. Since it is obvious that 3) implies 2), it would be enough to verify that 2) is never satisfied in our case. But since $\varrho(H_0) = 0$, it is clear from [4 (j), § 15] that 2) implies that $\phi = 0$, giving a contradiction. This proves Lemma 43 when $\dim \mathfrak{c}_\mathfrak{p} \geqslant 1$.

We state the following result, which has been proved above, as a lemma for future reference.

LEMMA 45. *Given $\varepsilon > 0$, we can choose $\delta > 0$ and $T_0 > 0$ such that*

$$\int_{T_1^2 \leqslant t^2 + \sigma(y)^2 \leqslant T_2^2} |\phi(t:y)|^2 \, dt \, dy \leqslant \varepsilon T_1^\nu$$

for $T_0 \leqslant T_1 \leqslant T_2 \leqslant (1+\delta) T_1$.

§ 27. The differential equation for Φ

In order to complete the proofs of Lemmas 42 and 43, we may now assume that $\mathfrak{c}_\mathfrak{p} = \{0\}$ and $\mathfrak{p} \neq \{0\}$. Fix an element $H_0 \in S^+$ and let $\mathfrak{m}$ be the centralizer of H_0 in $\mathfrak{g}$ and $\mathfrak{l}$ the centralizer of $\mathfrak{m}$ in $\mathfrak{a}_\mathfrak{p}$. Since $\mathfrak{a}_\mathfrak{p} \subset \mathfrak{m}$ and $\mathfrak{a}_\mathfrak{p}$ is maximal abelian in $\mathfrak{p}$, $\mathfrak{l}$ is also the centralizer of $\mathfrak{m}$ in $\mathfrak{p}$. Moreover, $\dim \mathfrak{m} < \dim \mathfrak{g}$.

Put $\mathfrak{M} = \mathfrak{S}(\mathfrak{m}_c)$ and let $\mathfrak{Z}_\mathfrak{m}$ denote the center of $\mathfrak{M}$ and μ_0 the homomorphism (see [4 (o), § 12]) of $\mathfrak{Z}$ into $\mathfrak{Z}_1 = \mathfrak{Z}_\mathfrak{m}$. Define $\mathfrak{U}_1 = \mathfrak{Z}_1 \mu_0(\mathfrak{U})$ where $\mathfrak{U}$ is the ideal consisting of all $u \in \mathfrak{Z}$ such that $u\phi = 0$. Then it follows from condition 1) of Lemma 42 and [4 (o), Lemma 21] that $\dim \mathfrak{Z}/\mathfrak{U}$ and $\dim \mathfrak{Z}_1/\mathfrak{U}_1$ are both finite. Let $\zeta \to \zeta^*$ denote the natural projection of $\mathfrak{Z}_1$ on $\mathfrak{Z}_1{}^* = \mathfrak{Z}_1/\mathfrak{U}_1$. We regard $\mathfrak{Z}_1{}^*$ as a $\mathfrak{Z}_1$-module in the usual way so that $z\zeta^* = (z\zeta)^*$ $(z, \zeta \in \mathfrak{Z}_1)$. Let $\mathfrak{Z}_1{}^{**}$ be the vector space dual to $\mathfrak{Z}_1{}^*$. Then since $\mathfrak{Z}_1$ is abelian, $\mathfrak{Z}_1{}^{**}$ is also a left $\mathfrak{Z}_1$-module by duality. Put $\mathbf{V} = V \otimes \mathfrak{Z}_1{}^{**}$ and let $\boldsymbol{\Gamma}$ denote the representation of $\mathfrak{Z}_1$ on $\mathbf{V}$ defined by

$$\boldsymbol{\Gamma}(z)(v \otimes \zeta^{**}) = v \otimes z\zeta^{**} \quad (z \in \mathfrak{Z}_1, \, v \in V, \, \zeta^{**} \in \mathfrak{Z}_1{}^{**}).$$

Moreover, by making K act trivially on $\mathfrak{Z}_1{}^{**}$, we can regard $\mathbf{V}$ as a double K-module. Note that $\boldsymbol{\Gamma}(z)$ $(z \in \mathfrak{Z}_1)$ commutes with the operations of K on $\mathbf{V}$.

Fix elements $\eta_1 = 1, \eta_2, ..., \eta_p$ in $\mathfrak{Z}_1$ such that $\eta_1{}^*, ..., \eta_p{}^*$ is a base for $\mathfrak{Z}_1{}^*$. Let $\eta_i{}^{**}$ $(1 \leqslant i \leqslant p)$ be the dual base for $\mathfrak{Z}_1{}^{**}$. We regard $\mathfrak{Z}_1{}^{**}$ as a Hilbert space with $(\eta_1{}^{**}, ..., \eta_p{}^{**})$ as an orthonormal base. Then $\mathbf{V}$ also becomes a Hilbert space.

Let Σ be the set of all positive roots of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$, Σ_1 the subset of those $\alpha \in \Sigma$ which vanish identically on $\mathfrak{l}$ and Σ_2 the complement of Σ_1 in Σ. For any $\alpha \in \Sigma$, let $\mathfrak{g}_\alpha$ denote the set of all $X \in \mathfrak{g}$ such that $[H, X] = \alpha(H)X$ for $H \in \mathfrak{a}_\mathfrak{p}$. Put

$$\mathfrak{n}_i = \sum_{\alpha \in \Sigma_i} \mathfrak{g}_\alpha \quad (i = 1, 2)$$

and $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$. Let M be the analytic subgroup of G corresponding to $\mathfrak{m}$. Then M normalizes $\mathfrak{n}_2$. Put

$$d(m) = |\det (\mathrm{Ad}\,(m))_{\mathfrak{n}_2}|^{\frac{1}{2}} \quad (m \in M)$$

and $\zeta' = d^{-1}\zeta \circ d$ for $\zeta \in \mathfrak{Z}_1$. It is easy to verify (see the Appendix, § 45) that $\zeta \to \zeta'$ is an automorphism of $\mathfrak{Z}_1$. Now define

$$\Phi(m) = \sum_{1 \leqslant i \leqslant p} \phi_i(m) \otimes \eta_i^{**} \quad (m \in M),$$

where $$\phi_i(m) = d(m)\phi(m; \eta_i').$$

Then it is obvious that Φ is a spherical function from M to $\mathbf{V}$ corresponding to the double representation of $K_1 = M \cap K$ on $\mathbf{V}$ which we denote by $\mu = (\mu_1, \mu_2)$.

For any $\zeta \in \mathfrak{Z}_1$, there exist unique complex numbers c_{ij} such that

$$u_i(\zeta) = \zeta\eta_i - \sum_{1 \leqslant j \leqslant p} c_{ij}\eta_j \in \mathfrak{U}_1 \quad (1 \leqslant i \leqslant p).$$

LEMMA 46. *Let $\zeta \in \mathfrak{Z}_1$. Then*

$$\Phi(m; \zeta) - \mathbf{\Gamma}(\zeta)\,\Phi(m) = \sum_{1 \leqslant i \leqslant p} d(m)\,\phi(m; u_i(\zeta)') \otimes \eta_i^{**}$$

for $m \in M$.

Fix ζ and define c_{ij} as above. Then

$$\Phi(m; \zeta) = \sum_i \phi_i(m; \zeta) \otimes \eta_i^{**} \quad \text{and} \quad \zeta'\eta_i' = \sum_j c_{ij}\eta_j' + u_i',$$

where $u_i = u_i(\zeta)$. Therefore since

$$\phi_i(m; \zeta) = d(m)\,\phi(m; \zeta'\eta_i'),$$

our assertion is obvious.

Put $$\Psi_\zeta(m) = \sum_{1 \leqslant i \leqslant p} d(m)\,\phi(m; u_i(\zeta)') \otimes \eta_i^{**} \quad (\zeta \in \mathfrak{Z}_1, m \in M).$$

Then the following corollary is merely a restatement of the above lemma.

COROLLARY 1. $\Phi(m; \zeta) = \Gamma(\zeta)\,\Phi(m) + \Psi_\zeta(m)$ *for $m \in M$ and $\zeta \in \mathfrak{Z}_1$.*

Now $\mathfrak{l}$ lies in the center of $\mathfrak{m}$. Hence if $H \in \mathfrak{l}$, we conclude from the above corollary that

$$d(e^{-t\Gamma(H)}\Phi(m \exp tH))/dt = e^{-t\Gamma(H)}\Psi_H(m \exp tH)$$

for $t \in \mathbf{R}$ and $m \in M$. Therefore the following result is now obvious.

COROLLARY 2. $\Phi(m \exp TH) = e^{T\Gamma(H)}\Phi(m) + \int_0^T e^{(T-t)\Gamma(H)}\Psi_H(m \exp tH)\,dt$ for $m \in M$, $H \in \mathfrak{l}$ and $T \in \mathbf{R}$.

§ 28. Some estimates for Φ and Ψ_ζ

Let $\Xi_M = \Xi_1$ be the function on M which corresponds to Ξ when (G, K) is replaced by (M, K_1).

LEMMA 47. *There exist numbers c_0, $r_0 \geqslant 0$ such that*

$$d(\exp H)\Xi(\exp H) \leqslant c_0 \Xi_1(\exp H)(1 + \|H\|)^{r_0}$$

for $H \in \mathfrak{a}_\mathfrak{p}^+$.

We shall give a proof of this lemma in § 45.

Put $M^+ = K_1 A_\mathfrak{p}^+ K_1$. Since K_1 lies in the kernel of the homomorphism d, the following corollary is obvious.

COROLLARY. $d(m)\Xi(m) \leqslant c_0 \Xi_1(m)(1 + \sigma(m))^{r_0}$ $(m \in M^+)$.

Fix $r \geqslant 0$ as in condition 2) of Lemma 42.

LEMMA 48. *Given $g_1, g_2 \in \mathfrak{G}$, we can choose a number $c(g_1, g_2) \geqslant 0$ such that*

$$|\phi(g_1; x; g_2)| \leqslant c(g_1, g_2)\Xi(x)(1 + \sigma(x))^r \quad (x \in G).$$

In view of Lemma 17, it is enough to consider the case $g_1 = 1$. By Theorem 1, we can choose $\alpha \in C_c^\infty(G)$ such that $\phi * \alpha = \phi$. Then $g\phi = \phi * (g\alpha)$ $(g \in \mathfrak{G})$ and so our assertion follows immediately from Lemma 10 and [4 (q), Lemma 32].

Put $r_1 = r + r_0$.

COROLLARY. *For any $v \in \mathfrak{M}$, we can choose a number $c(v) \geqslant 0$ such that*

$$|\Phi(m; v)| \leqslant c(v)\Xi_1(m)(1 + \sigma(m))^{r_1} \quad (m \in M^+).$$

This is obvious from Lemma 48 and the corollary of Lemma 47.

For $\alpha \in \Sigma$, define $\mathfrak{g}_\alpha$ as in § 27.

LEMMA 49. *Fix $g \in \mathfrak{G}$, $\alpha \in \Sigma$ and $X \in \mathfrak{g}_\alpha$. Then we can choose $c_1 \geqslant 0$ such that*

$$|d(h)\phi(h; \theta(X)g)| \leqslant c_1 e^{-\alpha(\log h)}\Xi_1(h)(1 + \sigma(h))^{r_1}$$

for $h \in A_\mathfrak{p}^+$.

Since $\phi(h; \theta(X)g) = e^{-\alpha(\log h)}\phi(\theta(X); h; g)$, this follows from Lemmas 47 and 48.

COROLLARY. *Fix $\zeta \in \mathfrak{Z}_1$ and $v \in \mathfrak{M}$. Then we can choose $c \geqslant 0$ such that*

$$|\Psi_\zeta(m \exp H; v)| \leqslant c e^{-\beta(H)}\Xi_1(m)(1 + \sigma(m))^{r_1}(1 + \|H\|)^{r_1}$$

for $H \in \mathfrak{l}^+ = \mathfrak{l} \cap \mathfrak{a}_\mathfrak{p}^+$ *and* $m \in M^+$. *Here*

$$\beta(H) = \min_{\alpha \in \Sigma_1} \alpha(H).$$

We know that (see the Appendix, § 45)

$$z - \mu_0(z)' \in \theta(\mathfrak{n}_2)\, \mathfrak{G}.$$

Fix $g \in \mathfrak{G}$, $u \in \mathfrak{U}$ and put $g_1 = -(u - \mu_0(u)')g \in \theta(\mathfrak{n}_2)\,\mathfrak{G}$. Since $u\phi = 0$, it is clear that

$$\phi(m; \mu_0(u)'g) = \phi(m; g_1).$$

Define the automorphism $v \to v' = d^{-1}v \circ d$ on $\mathfrak{M}$ as in the Appendix, § 45. Fix v and ζ as above. Then

$$\Psi_\zeta(m; v) = \sum_{1 \leqslant i \leqslant p} d(m)\,\phi(m; v'\,u_i(\zeta)') \otimes \eta_i^{**}.$$

Since $u_i(\zeta) \in \mathfrak{U}_1 \subset \mathfrak{Z}_1$, v commutes with $u_i(\zeta)$ and therefore

$$vu_i(\zeta) = u_i(\zeta)\,v \in \mu_0(\mathfrak{U})\,\mathfrak{M}.$$

Hence, in view of the remark above, we can choose $g_i \in \theta(\mathfrak{n}_2)\,\mathfrak{G}$ such that

$$\phi(m; v'u_i(\zeta)') = \phi(m; g_i) \quad (m \in M).$$

Since $M^+ = K_1 A_\mathfrak{p}^+ K_1$, our assertion now follows easily from Lemma 49.

§ 29. The function Θ

Let Q be the set of all eigenvalues of $\boldsymbol{\Gamma}(H_0)$. Then

$$\mathbf{V} = \sum_{\lambda \in Q} \mathbf{V}_\lambda,$$

where the sum is direct and $\mathbf{V}_\lambda$ is the subspace consisting of all $v \in \mathbf{V}$ such that $(\boldsymbol{\Gamma}(H_0) - \lambda)^m v = 0$ for some integer $m \geqslant 0$. Let E_λ denote the projection of $\mathbf{V}$ on $\mathbf{V}_\lambda$ corresponding to the above sum. We divide Q into three disjoint sets Q^+, Q^0 and Q^-. An element $\lambda \in Q$ lies in Q^+, Q^0 or Q^- according as [1] $\mathfrak{R}\lambda > 0$, $\mathfrak{R}\lambda = 0$ or $\mathfrak{R}\lambda < 0$. Put

$$E^\pm = \sum_{\lambda \in Q^\pm} E_\lambda, \quad E^0 = \sum_{\lambda \in Q^0} E_\lambda$$

and

$$\mathbf{V}^\pm = E^\pm \mathbf{V}, \quad \mathbf{V}^0 = E^0 \mathbf{V}.$$

Since $\mathfrak{Z}_1$ is abelian, $\boldsymbol{\Gamma}(\zeta)$ $(\zeta \in \mathfrak{Z}_1)$ commutes with the projections E_λ.

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[1] $\mathfrak{R}c$ denotes the real part of a complex number c.

Put $Q' = Q^+ \cup Q^-$ and let [1]

$$4\,\varepsilon_0 = \min_{\lambda \in Q'} |\Re\,\lambda|.$$

Then $\varepsilon_0 > 0$. Put $$\beta(H) = \min_{\alpha \in \Sigma_2} \alpha(H) \quad (H \in \mathfrak{l})$$

as before, so that $\beta(H_0) > 0$. Fix a number $\varepsilon(0 < \varepsilon \leqslant \varepsilon_0)$ and an open and relatively compact neighborhood Ω of H_0 in $\mathfrak{l}$. By selecting them sufficiently small, we can arrange that $\beta(H_0) \geqslant 5\,\varepsilon$ and [2]

$$\beta(H) \geqslant 4\,\varepsilon, \quad |\mathbf{\Gamma}(H) - \mathbf{\Gamma}(H_0)| < \varepsilon/2$$

for $H \in \Omega$.

LEMMA 50. *We can choose a number $c \geqslant 0$ such that*

$$\left| e^{-t\mathbf{\Gamma}(H)} E^+ \right| + \left| e^{t\mathbf{\Gamma}(H)} E^- \right| \leqslant c e^{-2\,\varepsilon_0 t} \quad (t \geqslant 0$$

and $$\left| e^{t\mathbf{\Gamma}(H)} E^0 \right| \leqslant c e^{\varepsilon|t|} \quad (t \in \mathbf{R})$$

for $H \in \Omega$.

Fix $\lambda \in Q$. Then

$$\mathbf{\Gamma}(H) E_\lambda = \{\lambda + (\mathbf{\Gamma}(H) - \mathbf{\Gamma}(H_0)) + (\mathbf{\Gamma}(H_0) - \lambda)\} E_\lambda$$

and therefore $$\left| e^{t\mathbf{\Gamma}(H)} E_\lambda \right| \leqslant \exp\{t\Re\lambda + \varepsilon|t|/2\} \left| e^{t(\mathbf{\Gamma}(H_0) - \lambda)} E_\lambda \right|$$

for $H \in \Omega$ and $t \in \mathbf{R}$. Since $(\mathbf{\Gamma}(H_0) - \lambda) E_\lambda$ is nilpotent, our assertions follow without difficulty.

Put $$\Phi^\pm(\mathrm{m}) = E^\pm \Phi(m), \ \Phi^0(m) = E^0 \Phi(m) \quad (m \in M).$$

Then by Corollary 2 of Lemma 41,

$$\Phi^-(m \exp TH) = e^{T\mathbf{\Gamma}(H)} \Phi^-(m) + \int_0^T e^{(T-t)\mathbf{\Gamma}(H)} E^- \Psi_H(m \exp tH)\,dt$$

for $H \in \Omega$ and $T \geqslant 0$. Therefore if $v \in \mathfrak{M}$, we conclude from Lemma 50 that

$$\left| \Phi^-(m \exp TH; v) \right| \leqslant c e^{-2\,\varepsilon_0 T} \left| \Phi^-(m; v) \right| + c \int_0^T e^{-2(T-t)\varepsilon_0} \left| \Psi_H(m \exp tH; v) \right| dt.$$

Put $f(t) = \Psi_H(m \exp tH; v)$ for fixed m, H, v. Then

$$\int_0^T e^{-2(T-t)\varepsilon_0} |f(t)| \, dt \leqslant e^{-T\varepsilon_0} \int_0^{T/2} |f(t)| \, dt + \int_{T/2}^T |f(t)| \, dt$$

$$\leqslant e^{-T\varepsilon_0} \int_0^\infty |f(t)| \, dt + \int_{T/2}^\infty |f(t)| \, dt.$$

On the other hand, since Ψ_H depends linearly on H and Ω is relatively compact, we have the following result from the corollary of Lemma 49.

LEMMA 51. *For a given $v \in \mathfrak{M}$, there exists a number $c(v) \geqslant 0$ such that*

$$\left| \Psi_H(m \exp tH; v) \right| \leqslant c(v) \, e^{-3\varepsilon t} \Xi_1(m) \, (1 + \sigma(m))^{r_1}$$

for $m \in M^+$, $H \in \Omega$ and $t \geqslant 0$.

Therefore in view of the corollary of Lemma 48, we obtain the following lemma.

LEMMA 52. *For any $v \in \mathfrak{M}$, we can select a number $c^-(v) \geqslant 0$ such that*

$$\left| \Phi^-(m \exp TH; v) \right| \leqslant c^-(v) \, e^{-\varepsilon T} \Xi_1(m) \, (1 + \sigma(m))^{r_1}$$

for $m \in M^+$, $H \in \Omega$ and $T \geqslant 0$.

Now we come to Φ^+. Again, by Corollary 2 of Lemma 46, we have

$$\Phi^+(m; v) = e^{-T\Gamma(H)} \Phi^+(m \exp TH; v) - \int_0^T e^{-t\Gamma(H)} E^+ \Psi_H(m \exp tH; v) \, dt.$$

Fix $H \in \Omega$, $m \in M^+$ and let T tend to $+\infty$. Then it is clear from Lemmas 50 and 51 and the corollary of Lemma 48, that

$$\Phi^+(m; v) = - \int_0^\infty e^{-t\Gamma(H)} E^+ \Psi_H(m \exp tH; v) \, dt$$

and therefore

$$\Phi^+(m \exp TH; v) = - \int_T^\infty e^{-(t-T)\Gamma(H)} E^+ \Psi_H(m \exp tH; v) \, dt.$$

In view of Lemmas 50 and 51, this gives the following result.

LEMMA 53. *For any $v \in \mathfrak{M}$, there exists a number $c^+(v) \geqslant 0$ such that*

$$\left| \Phi^+(m \exp TH; v) \right| \leqslant c^+(v) e^{-\varepsilon T} \Xi_1(m) (1 + \sigma(m))^{r_1}$$

for $m \in M^+$, $H \in \Omega$ and $T \geqslant 0$.

We shall now consider Φ^0. But first we need some preparation.

LEMMA 54. *Let C be a compact subset of M. Then we can choose $T_0 \geqslant 0$ such that $m \exp TH \in M^+$ for $m \in C$, $H \in \Omega$ and $T \geqslant T_0$.*

We may obviously assume that $K_1 C K_1 = C$. Let C_1 be the set of all $h \in C \cap A_\mathfrak{p}$ such that $\alpha(\log h) \geqslant 0$ for all $\alpha \in \Sigma_1$. Then C_1 is also compact and $C = K_1 C_1 K_1$. Now choose $T_0 \geqslant 0$ such that

$$\alpha(\log h) + 4\varepsilon T_0 \geqslant 0$$

for all $h \in C_1$ and $\alpha \in \Sigma_2$. Then it is clear that $C_1 \exp TH \subset A_\mathfrak{p}^+$ and therefore $C \exp TH \subset M^+$ for $T \geqslant T_0$ and $H \in \Omega$.

Now fix $v \in \mathfrak{M}$ and $H \in \Omega$. Then we conclude from Lemmas 50, 51 and 54 that the integral

$$\int_0^\infty \left| e^{-t\Gamma(H)} E^0 \Psi_H(m \exp tH; v) \right| dt$$

converges uniformly for $m \in C$. Put

$$\Theta_H(m) = \Phi^0(m) + \int_0^\infty e^{-t\Gamma(H)} E^0 \Psi_H(m \exp tH) \, dt \quad (m \in M).$$

Then it is clear that Θ_H is a C^∞ function on M and

$$\Theta_H(m; v) = \Phi^0(m; v) + \int_0^\infty e^{-t\Gamma(H)} E^0 \Psi_H(m \exp tH; v) \, dt.$$

Moreover, it follows from Corollary 2 of Lemma 46 that

$$\Theta_H(m; v) = \lim_{T \to +\infty} e^{-T\Gamma(H)} \Phi^0(m \exp TH; v).$$

So, in particular, the following result is obvious.

LEMMA 55. $\Theta_H(m \exp tH) = e^{t\Gamma(H)} \Theta_H(m)$ $(m \in M, H \in \Omega, t \in \mathbf{R})$.

We now claim that Θ_H is actually independent of H. Fix $H_1, H_2 \in \Omega$ and $m \in M$ and choose $T_0 \geqslant 0$ such that $m \exp tH \in M^+$ for $t \geqslant T_0$ and $H \in \Omega$. Put $m_2 = m \exp T_2 H_2$ $(T_2 \geqslant T_0)$. Then by Corollary 2 of Lemma 46,

$$e^{-\Gamma(T_1 H_1)} \Phi^0(m_2 \exp T_1 H_1) = \Phi^0(m_2) + \int_0^{T_1} e^{-t\Gamma(H_1)} E^0 \Psi_{H_1}(m_2 \exp tH_1) \, dt,$$

and therefore

$$e^{-\Gamma(T_1 H_1 + T_2 H_2)} \Phi^0(m \exp (T_1 H_1 + T_2 H_2)) - e^{-\Gamma(T_2 H_2)} \Phi^0(m \exp T_2 H_2)$$

$$= \int_0^{T_1} e^{-\Gamma(t_1 H_1 + T_2 H_2)} E^0 \Psi_{H_1}(m \exp (t_1 H_1 + T_2 H_2)) \, dt_1$$

for $T_1 \geqslant 0$. But it follows from Lemma 50 and the corollary of Lemma 49 that there exists a number $c \geqslant 0$ such that

$$\left| e^{-\mathbf{\Gamma}(t_1 H_1 + T_2 H_2)} E^0 \Psi_{H_1}(m \exp (t_1 H_1 + T_2 H_2)) \right| \leqslant c e^{-\varepsilon(t_1 + T_2 - T_0)}$$

for $t_1 \geqslant 0$ and $T_2 \geqslant T_0$. Therefore by making T_1, T_2 tend to $+\infty$, we get

$$\Theta_{H_2}(m) = \lim_{T_1, T_2 \to +\infty} e^{-\mathbf{\Gamma}(T_1 H_1 + T_2 H_2)} \Phi^0(m \exp (T_1 H_2 + T_2 H_2)).$$

Since the right side is symmetrical in H_1, H_2, we conclude that $\Theta_{H_1}(m) = \Theta_{H_2}(m)$.

Hence we may now write Θ instead of Θ_H.

LEMMA 56. $\Theta(m \exp H) = e^{\mathbf{\Gamma}(H)} \Theta(m)$ $\quad (m \in M, H \in \mathfrak{l})$.

Since Ω is open in $\mathfrak{l}$, every $H \in \mathfrak{l}$ can be written in the form $H = \Sigma_{1 \leqslant i \leqslant q} t_i H_i$ $(t_i \in \mathbf{R}, H_i \in \Omega)$. Our assertion therefore follows from Lemma 55.

LEMMA 57. *For any $v \in \mathfrak{M}$, we can choose a number $c^0(v) \geqslant 0$ such that*

$$\left| e^{-t\mathbf{\Gamma}(H)} \Phi^0(m \exp tH; v) - \Theta(m; v) \right| \leqslant c^0(v) \Xi_1(m) (1 + \sigma(m))^{r_1} e^{-2\varepsilon t}$$

for $m \in M^+$, $t \geqslant 0$ and $H \in \Omega$.

Since $\qquad \Theta(m; v) - e^{-T\mathbf{\Gamma}(H)} \Phi^0(m \exp TH; v) = \int_T^\infty e^{-t\mathbf{\Gamma}(H)} E^0 \Psi_H(m \exp tH; v) \, dt,$

our assertion follows immediately from Lemmas 50 and 51.

COROLLARY. *For any $v \in \mathfrak{M}$, there exists a number $c(v) \geqslant 0$ such that*

$$\left| \Phi(m \exp tH; v) - \Theta(m \exp tH; v) \right| \leqslant c(v) \Xi_1(m)(1 + \sigma(m))^{r_1} e^{-\varepsilon t}$$

for $m \in M^+$, $H \in \Omega$ and $t \geqslant 0$.

For,

$\left| \Phi(m \exp tH; v) - \Theta(m \exp tH; v) \right|$

$\quad \leqslant \left| \Phi^+(m \exp tH; v) \right| + \left| \Phi^-(m \exp tH; v) \right| + \left| e^{t\mathbf{\Gamma}(H)} E^0 \right| \left| e^{-t\mathbf{\Gamma}(H)} \Phi^0(m \exp tH; v) - \Theta(m; v) \right|,$

and so our assertion follows immediately from Lemmas 50, 52, 53 and 57.

LEMMA 58. *Let k_1, $k_2 \in K_1$, $m \in M$ and $\zeta \in \mathfrak{Z}_1$. Then*

$$\Theta(k_1 m k_2) = \mu_1(k_1) \Theta(m) \mu_2(k_2), \ \Theta(m; \zeta) = \mathbf{\Gamma}(\zeta) \Theta(m).$$

Moreover, there exist numbers c_2, $r_2 \geqslant 0$ such that

$$|\Theta(m)| \leqslant c_2 \Xi_1(m)(1+\sigma(m))^{r_2} \quad (m \in M).$$

Fix $H \in \Omega$. Since $\boldsymbol{\Gamma}(H)$ commutes with the operations of K on $\mathbf{V}$ (see § 27), the first assertion follows from the relation

$$\Theta(m) = \lim_{t \to +\infty} e^{-t\boldsymbol{\Gamma}(H)} \Phi^0(m \exp tH).$$

Moreover, E^0, $\boldsymbol{\Gamma}(H)$ and $\boldsymbol{\Gamma}(\zeta)$ commute with each other and

$$\Phi(m \exp tH; \zeta) = \boldsymbol{\Gamma}(\zeta)\Phi(m \exp tH) + \Psi_\zeta(m \exp tH)$$

from Corollary 1 of Lemma 46. Hence

$$\Theta(m; \zeta) = \lim_{t \to +\infty} e^{-t\boldsymbol{\Gamma}(H)} \Phi^0(m \exp tH; \zeta) = \boldsymbol{\Gamma}(\zeta)\Theta(m) + \lim_{t \to +\infty} e^{-t\boldsymbol{\Gamma}(H)} E^0 \Psi_\zeta(m \exp tH).$$

But from Lemma 50 and the corollary of Lemma 49, the limit on the right is zero. Therefor $\zeta\Theta = \boldsymbol{\Gamma}(\zeta)\Theta$.

Now put $v = 1$ and $t = 0$ in Lemma 57. Then we conclude from the corollary of Lemma 48 that there exists a number $c_1 \geqslant 0$ such that

$$|\Theta(m)| \leqslant c_1 \Xi_1(m)(1+\sigma(m))^{r_1} \quad (m \in M^+).$$

On the other hand, we can obviously choose a number $\nu_0 \geqslant 0$ such that

$$|\alpha(\log h)| \leqslant \nu_0 \sigma(h) \quad (\alpha \in \Sigma_2,\ h \in A_\mathfrak{p}).$$

Put $\nu = \max(1, \nu_0/4\varepsilon)$. Then it is clear (see the proof of Lemma 54) that m $\exp tH \in M^+$ $(m \in M,\ H \in \Omega)$ provided $t \geqslant \nu\sigma(m)$. Now fix $m \in M$ and put $t_0 = \nu\sigma(m)$ and $m_0 = m \exp t_0 H_0$. Then $m_0 \in M^+$ and

$$\Theta(m) = e^{-t_0 \boldsymbol{\Gamma}(H_0)} \Theta(m_0).$$

Therefore
$$|\Theta(m)| \leqslant c_1 |e^{-t_0 \boldsymbol{\Gamma}(H_0)} E^0| \Xi_1(m_0)(1+\sigma(m_0))^{r_1}.$$

But $\Xi_1(m_0) = \Xi_1(m)$, $\sigma(m_0) \leqslant \sigma(m) + t_0 = (\nu+1)\sigma(m)$ and $\boldsymbol{\Gamma}(H_0) E^0$ has only pure imaginary eigenvalues. Therefore (see the proof of Lemma 50), the last statement of the lemma follows immediately.

For any linear function λ on $\mathfrak{l}_c$, let $\mathbf{V}(\lambda)$ denote the subspace of all $v \in \mathbf{V}$ such that

$$(\boldsymbol{\Gamma}(H) - \lambda(H))^N v = 0 \quad (H \in \mathfrak{l}_c),$$

where $N = \dim \mathbf{V}$. Let $^0\mathbf{V}$ denote the sum

$$\sum_{\Re\lambda=0} \mathbf{V}(\lambda),$$

where λ runs over those linear functions which take only pure imaginary values on $\mathfrak{l}$.

LEMMA 59. $\Theta(m)\in{}^0\mathbf{V}$ $(m\in M)$.

Since $\Theta(m \exp tH)=e^{t\Gamma(H)}\Theta(m)$ $(H\in\Omega,\ t\in\mathbf{R})$, this is obvious from the last statement of Lemma 58.

§ 30. Application of the induction hypothesis to θ

We recall (see § 7) that $\mathfrak{p}$ is a Hilbert space under the norm $\|X\|$ $(X\in\mathfrak{p})$. Put $\mathfrak{a}_2=\mathfrak{l}$ and let $\mathfrak{a}_1$ be the orthogonal complement of $\mathfrak{a}_2$ in $\mathfrak{a}_\mathfrak{p}$. It is clear that Σ_1 is the set of all positive roots of $(\mathfrak{m},\ \mathfrak{a}_\mathfrak{p})$ and $\mathfrak{l}$ is the set of those $H\in\mathfrak{a}_\mathfrak{p}$ where $\alpha(H)=0$ $(\alpha\in\Sigma_1)$. Hence $\mathfrak{a}_1\subset[\mathfrak{m},\ \mathfrak{m}]$. Let $\mathfrak{a}_1{}^+$ be the set of all $H\in\mathfrak{a}_1$ where $\alpha(H)\geqslant0$ $(\alpha\in\Sigma_1)$. Put $A_1{}^+=\exp\mathfrak{a}_1{}^+$ and fix a number $N\geqslant1$ such that if $h_1\in A_1{}^+$ and $t\geqslant N\sigma(h_1)$, then $h_1\exp tH\in A_\mathfrak{p}{}^+$ for $H\in\Omega$. This is possible (see the proof of Lemma 58).

LEMMA 60. *We can choose a number $c'\geqslant0$ such that*

$$\big|\Phi(h_1\exp tH)-\Theta(h_1\exp tH)\big| \leqslant c'e^{-\varepsilon(t-N\sigma(h_1))}\Xi_1(h_1)(1+\sigma(h_1))^{r_1}$$

for $h_1\in A_1{}^+$, $H\in\Omega$ and $t\geqslant N\sigma(h_1)$.

Put $h_0=h_1\exp t_0 H$ where $t_0=N\sigma(h_1)$. Then $\Xi_1(h_1)=\Xi_1(h_0)$ and $\sigma(h_0)\leqslant\sigma(h_1)+N\sigma(h_1)\|H\|$ Since Ω is relatively compact, our assertion follows from the corollary of Lemma 57.

Define $\theta_j(m)\in V$ by

$$\Theta(m)=\sum_{1\leqslant j\leqslant p}\theta_j(m)\otimes\eta_j{}^{**} \quad (m\in M)$$

and put $\theta=\theta_1$.

COROLLARY. *Under the conditions of Lemma 60,*

$$\big|d(h_1\exp tH)\phi(h_1\exp tH)-\theta(h_1\exp tH)\big| \leqslant c'e^{-\varepsilon(t-N\sigma(h_1))}\Xi_1(h_1)(1+\sigma(h_1))^{r_1}.$$

This is obvious since $\eta_1=1$.

Let $\mathfrak{b}$ be the orthogonal complement of H_0 in $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{b}^+$ the set of all $H\in\mathfrak{b}$ where $\alpha(H)\geqslant0$ for every $\alpha\in\Sigma_1$. Let E and E_i denote the orthogonal projections of $\mathfrak{a}_\mathfrak{p}$ on $\mathfrak{b}$ and $\mathfrak{a}_i$ $(i=1,2)$ respectively. Fix a number $\delta>0$ and let U be the set of all $H\in S^+$ such that

$$\|EH\|\leqslant\delta\langle H_0,\ H\rangle,$$

the scalar product being defined as in § 7. We assume that δ is so small that $\delta N\leqslant\tfrac{1}{2}$,

$E_2 U \subset \Omega$ and $\alpha(H) \geqslant 3\varepsilon$ for $\alpha \in \Sigma_2$ and $H \in U$. Clearly this is possible. Then U consists of all elements of the form $cH_0 + H_1$ where $H_1 \in \mathfrak{b}^+$, $c^2 + \|H_1\|^2 = 1$ and $\|H_1\| \leqslant \delta c$ $(c > 0)$, provided δ is chosen sufficiently small. Therefore, in particular, $(1 + \delta^2)^{-\frac{1}{2}} \leqslant c \leqslant 1$. For any $T \geqslant 0$, let $U(T)$ denote the set of all elements tH with $H \in U$ and $0 \leqslant t \leqslant T$.

Put $\varrho(H) = \frac{1}{2} \operatorname{tr} (\operatorname{ad} H)_\mathfrak{n}$ and $\varrho_i(H) = \frac{1}{2} \operatorname{tr} (\operatorname{ad} H)_{\mathfrak{n}_i}$ $(H \in \mathfrak{a}_\mathfrak{p}, i = 1, 2)$ in the notation of § 27. Then $\varrho = \varrho_1 + \varrho_2$ and we claim that $\varrho_i(H) = \varrho(E_i H)$ $(H \in \mathfrak{a}_\mathfrak{p})$. Since $\mathfrak{l}$ centralizes $\mathfrak{n}_1 = \mathfrak{m} \cap \mathfrak{n}$, $\varrho_1 = 0$ on $\mathfrak{a}_2 = \mathfrak{l}$. Moreover $\mathfrak{m}$ normalizes $\mathfrak{n}_2$ and $\mathfrak{a}_1 \subset [\mathfrak{m}, \mathfrak{m}]$. Therefore $\varrho_2 = 0$ on $\mathfrak{a}_1$ and clearly this implies our assertion. We also note that $d(\exp H) = e^{\varrho_2(H)}$ $(H \in \mathfrak{a}_\mathfrak{p})$.

LEMMA 61. *There exists a number $c \geqslant 0$ with the following property. Suppose $\|H_1\| \leqslant \delta t$ $(H_1 \in \mathfrak{b}^+, t \geqslant 0)$. Then*

$$\left| e^{\varrho(tH_0 + E_2 H_1)} \phi(\exp (tH_0 + H_1)) - \theta(\exp (tH_0 + H_1)) \right| \leqslant c e^{-\varepsilon t/3} e^{-\varrho_1(H_1)}.$$

If $\alpha \in \Sigma_1$, it is clear that $\alpha(E_1 H_1) = \alpha(H_1) \geqslant 0$. Hence $E_1 H_1 \in \mathfrak{a}_1^+$. Moreover $N\|E_1 H_1\| \leqslant N\|H_1\| \leqslant N\delta t \leqslant t/2$. Therefore it follows without difficulty from the corollary of Lemma 60 that the left side is majorized by

$$c' e^{-\varepsilon t/2} \Xi_1(h_1)(1 + \sigma(h_1))^{r_1},$$

where $h_1 = \exp E_1 H_1$. Therefore our assertion follows by applying [4 (j), Theorem 3] to Ξ_1 and observing that $\sigma(h_1) = \|E_1 H_1\| \leqslant \delta t$.

As in § 7, put

$$D(h) = \prod_{\alpha \in \Sigma} \{e^{\alpha(\log h)} - e^{-\alpha(\log h)}\}^{m_\alpha} \quad (h \in A_\mathfrak{p}),$$

where m_α is the multiplicity of α. Then $D(h) \geqslant 0$ for $h \in A_\mathfrak{p}^+$. Put

$$F_U(T) = \int_{\log h \in U(T)} |\phi(h)|^2 D(h) \, dh \quad (T \geqslant 0),$$

where dh is the Haar measure on A. Also let us recall that $\phi \neq 0$ and it is analytic (see [4 (q), Lemma 33]).

LEMMA 62. *There exists a unique integer $\nu \geqslant 0$ with the following property. We can select numbers $a, b(0 < a \leqslant b < \infty)$ such that*

$$aT^\nu \leqslant F_U(T) \leqslant bT^\nu$$

for all $T \geqslant 1$. $\nu = 0$ if and only if $\theta = 0$.

The uniqueness of ν is obvious from its definition. So we have only to verify its existence.

Let $U_0(T)$ denote the subset of all $H \in \mathfrak{a}_\mathfrak{p}$ of the form $H = tH_0 + H_1$ where $0 \leqslant t \leqslant T$, $H_1 \in \mathfrak{b}^+$ and $\|H_1\| \leqslant \delta t$. Then

$$U(T) \subset U_0(T) \subset U(T(1 + \delta^2)^{\frac{1}{2}}).$$

Put
$$F_0(T) = \int_{\log h \in U_0(T)} |\phi(h)|^2 D(h)\, dh.$$

Then
$$F_0(T(1 + \delta^2)^{-\frac{1}{2}}) \leqslant F(T) \leqslant F_0(T),$$

where $F = F_U$. Therefore it would be enough to prove the existence of an integer $\nu \geqslant 0$ such that
$$0 < \liminf_{T \to \infty} T^{-\nu} F_0(T) \leqslant \limsup_{T \to \infty} T^{-\nu} F_0(T) < \infty.$$

For any $c \geqslant 0$, let $\mathfrak{b}^+(c)$ denote the set of all $H \in \mathfrak{b}^+$ with $\|H\| \leqslant c$. Then if $\psi(H) = \phi(\exp H)$ $(H \in \mathfrak{a}_\mathfrak{p})$, we have

$$F_0(T) = \int_0^T dt \int_{\mathfrak{b}^+(\delta t)} |e^{\varrho(tH_0 + E_2 H_1)} \psi(tH_0 + H_1)|^2 D_1(H_1) \prod_{\alpha \in \Sigma_2} (1 - e^{-2\alpha(tH_0 + H_1)})^{m_\alpha}\, dH_1,$$

where
$$D_1(H_1) = \prod_{\alpha \in \Sigma_1} (e^{\alpha(H_1)} - e^{-\alpha(H_1)})^{m_\alpha} \quad (H_1 \in \mathfrak{b})$$

and dH_1 is the (suitably normalized) Euclidean measure on $\mathfrak{b}$. Choose $T_0 \geqslant 0$ so large that $2\alpha(tH_0 + H_1) \geqslant \log 2$ for $\alpha \in \Sigma_2$, $H_1 \in \mathfrak{b}^+(\delta t)$ and $t \geqslant T_0$, and put

$$J(T) = \int_{T_0}^T dt \int_{\mathfrak{b}^+(\delta t)} |e^{\varrho(tH_0 + E_2 H_1)} \psi(tH_0 + H_1)|^2 D_1(H_1)\, dH_1$$

for $T \geqslant T_0$. Then if $q = \dim \mathfrak{n}_2$, it is clear that

$$2^{-q} J(T) \leqslant F_0(T) - F_0(T_0) \leqslant J(T) \quad (T \geqslant T_0).$$

Also put $\psi_\infty(H) = \theta(\exp H)$ $(H \in \mathfrak{a}_\mathfrak{p})$ and

$$J_\infty(T) = \int_{T_0}^T dt \int_{\mathfrak{b}^+(\delta t)} |\psi_\infty(tH_0 + H_1)|^2 D_1(H_1)\, dH_1.$$

Then, by the triangle inequality, we have

$$|J(T)^{\frac{1}{2}} - J_\infty(T)^{\frac{1}{2}}|^2 \leqslant \int_{T_0}^T dt \int_{\mathfrak{b}^+(\delta t)} |e^{\varrho(tH_0 + E_2 H_1)} \psi(tH_0 + H_1) - \psi_\infty(tH_0 + H_1)|^2 D_1(H_1)\, dH_1.$$

Now apply Lemma 61 and observe that

$$e^{-2\varrho_1(H_1)}D_1(H_1) \leqslant 1 \quad (H_1 \in \mathfrak{b}^+).$$

Then it follows easily that

$$\left| J(T)^{\frac{1}{2}} - J_\infty(T)^{\frac{1}{2}} \right| \leqslant c \quad (T \geqslant T_0),$$

where c is a fixed positive number.

Now first suppose $\theta = 0$. Then $\psi_\infty = 0$ and therefore $J_\infty(T) = 0$. Hence $J(T() \leqslant c^2$ for all $T \geqslant T_0$ and it is clear that we can take $\nu = 0$.

So let us now suppose that $\theta \neq 0$. Then it is obvious from Lemma 58 that all the assumptions of Lemma 42 are fulfilled if we replace (G, K, ϕ) by (M, K_1, θ). Since dim $\mathfrak{m} < \dim \mathfrak{g}$ and $H_0 \in \mathfrak{l} \subset \mathfrak{m} \cap \mathfrak{p}$, it follows from the induction hypothesis that the index ν_∞ of θ (on M) is positive. Hence it follows from Lemma 44 and [4 (d), Lemma 38] that

$$0 < \liminf_{T \to \infty} T^{-\nu_\infty}J_\infty(T) \leqslant \limsup_{T \to \infty} T^{-\nu_\infty}J_\infty(T) < \infty.$$

Since $\nu_\infty \geqslant 1$, it is clear that similar inequalities hold if we replace $J_\infty(T)$ by $J(T)$. Hence we can take $\nu = \nu_\infty$.

LEMMA 63. *Define ν as in Lemma 62. Then for any $\varepsilon_1 > 0$, we can choose $T_0 > 0$ and $\delta_1 > 0$ such that*

$$T_1^{-\nu}\{F_U(T_2) - F_U(T_1)\} \leqslant \varepsilon_1$$

for $T_0 \leqslant T_1 \leqslant T_2 \leqslant (1 + \delta_1)T_1$.

Put $F = F_U$ and let $0 \leqslant T_1 \leqslant T_2$. Then

$$F(T_2) - F(T_1) \leqslant \int_{\mathfrak{a}(T_1, T_2)} \left| e^{\varrho(tH_0 + E_2 H_1)}\psi(tH_0 + H_1) \right|^2 D_1(H_1)\,dt\,dH_1,$$

where $\mathfrak{a}(T_1, T_2)$ is the set of all points in $\mathfrak{a}_\mathfrak{p}$ of the form $tH_0 + H_1$ with $t \geqslant 0$, $H_1 \in \mathfrak{b}^+(\delta t)$ and $T_1^2 \leqslant t^2 + \|H_1\|^2 \leqslant T_2^2$. Therefore

$$\{F(T_2) - F(T_1)\}^{\frac{1}{2}} \leqslant I_1(T_1, T_2)^{\frac{1}{2}} + I_2(T_1, T_2)^{\frac{1}{2}},$$

where

$$I_1(T_1, T_2) = \int_{\mathfrak{a}(T_1, T_2)} |\psi_\infty(tH_0 + H_1)|^2 D_1(H_1)\,dt\,dH_1$$

and

$$I_2(T_1, T_2) = \int_{\mathfrak{a}(T_1, T_2)} \left| e^{\varrho(tH_0 + E_2 H_1)}\psi(tH_0 + H_1) - \psi_\infty(tH_0 + H_1) \right|^2 D_1(H_1)\,dt\,dH_1.$$

If we apply Lemma 61 and observe, as before, that $e^{-2\varrho_1(H_1)}D_1(H_1) \leqslant 1$ for $H_1 \in \mathfrak{b}^+$, it follows easily that

$$I_2(T_1, T_2) \leqslant c_2 e^{-\varepsilon' T_1} \quad (T_2 \geqslant T_1 \geqslant 0,$$

where c_2 is a fixed positive number and $\varepsilon' = (1 + \delta^2)^{-\frac{1}{2}} \cdot \varepsilon/2$.

Now first suppose $\theta = 0$. Then $\nu = 0$ and $I_1(T_1, T_2) = 0$. Hence

$$F(T_2) - F(T_1) \leqslant I_2(T_1, T_2) \leqslant c_2 e^{-\varepsilon' T_1}$$

and the statement of Lemma 63 follows immediately. So let us assume that $\theta \neq 0$. Then $\nu = \nu_\infty \geqslant 1$, as we have seen above, and therefore the required assertion is a consequence of Lemma 45.

§ 31. Completion of the proofs of Lemmas 42 and 43

We shall now finish the proofs of Lemmas 42 and 43. For any open subset U of S^+ define

$$F_U(T) = \int_{\log h \in U(T)} |\phi(h)|^2 D(h) \, dh \quad (T \geqslant 0),$$

where $U(T)$ is the set of all elements in $\mathfrak{a}_\mathfrak{p}^+$ of the form tH $(0 \leqslant t \leqslant T, H \in U)$. Since S^+ is compact, we can choose open sets U_i in S^+ and integers $\nu_i \geqslant 0$ $(1 \leqslant i \leqslant q)$ such that $S^+ = \bigcup_{1 \leqslant i \leqslant q} U_i$ and the statements of Lemmas 62 and 63 hold for (U_i, ν_i) in place of (U, ν). Put $F_i = F_{U_i}$ and $\nu = \max_i \nu_i$. It is clear (see [4 (d), Lemma 38] that

$$\max_i F_i(T) \leqslant \|\phi\|_T^2 \leqslant \sum_{1 \leqslant i \leqslant q} F_i(T),$$

if the measure dh is suitably normalized. So it is obvious that

$$0 < \liminf_{T \to \infty} T^{-\nu/2} \|\phi\|_T \leqslant \limsup_{T \to \infty} T^{-\nu/2} \|\phi\|_T < \infty.$$

Moreover, $\qquad \|\phi\|_{T_2}^2 - \|\phi\|_{T_1}^2 \leqslant \sum_{1 \leqslant i \leqslant q} \{F_i(T_2) - F_i(T_1)\} \quad (0 \leqslant T_1 \leqslant T_2).$

Therefore the last assertion of Lemma 42 follows immediately from Lemma 63.

Now we come to the proof of Lemma 43. First assume that $\nu = 0$ and fix $H_0 \in S^+$. Then $\nu_i = 0$ $(1 \leqslant i \leqslant q)$ in the above proof. We may suppose that $H_0 \in U_1$. Define U as in § 30 for H_0 and let $A_\mathfrak{p}(U)$ denote the set of all $h \in A_\mathfrak{p}$ of the form $h = \exp tH$ $(t \geqslant 0, H \in U)$. We may obviously assume that $U \subset U_1$. Then it follows from Lemma 62 that $\theta = 0$ and therefore

$$\left| e^{\varrho(tH)} \phi(\exp tH) \right| \leqslant c e^{-\varepsilon' t} \quad (H \in U, t \geqslant 0)$$

from Lemma 61 where $\varepsilon' = (1 + \delta^2)^{-\frac{1}{2}} \cdot \varepsilon/3$. Hence it is clear (see [4 (j), Lemma 36]) that

$$\sup_{h \in A_\mathfrak{p}^+(U)} |\phi(h)| \, \Xi(h)^{-1} (1 + \sigma(h))^r < \infty$$

for every $r \geqslant 0$. Since $G = K A_\mathfrak{p}^+ K$ and S^+ is compact, this means that

$$\sup_{x \in G} |\phi(x)| \, \Xi(x)^{-1} (1 + \sigma(x))^r < \infty$$

for any $r \geqslant 0$. But then it follows easily (see the proof of Lemma 48) that $\phi \in C(G) \otimes V$. Thus 1) implies 3) in Lemma 43. In view of Lemma 11 and [4 (j), Theorem 3], it is obvious that 3) implies both 1) and 2). Hence it remains to prove that 2) implies 1).

So suppose 2) holds. Fix $H_0 \in S^+$ and use the notation of § 30. Then it follows from Lemma 61 that

$$\lim_{t \to +\infty} \theta(\exp(tH_0 + H_1)) = 0 \quad (H_1 \in \mathfrak{b}^+).$$

Now fix $H_1 \in \mathfrak{b}^+$ and put $f(t) = \theta(\exp(tH_0 + H_1))$ $(t \in \mathbf{R}.)$. We have seen in § 29 that

$$\Theta(\exp(tH_0 + H_1)) = e^{t\Gamma(H_0)} \Theta(\exp H_1)$$

and all eigenvalues of $\boldsymbol{\Gamma}(H_0) E^0$ are pure imaginary. Hence it is clear that

$$f(t) = \sum_{1 \leqslant i \leqslant r} p_i(t) \, e^{(-1)^{\frac{1}{2}} \lambda_i t} \quad (t \in \mathbf{R}),$$

where $\lambda_1, \ldots, \lambda_r$ are distinct real numbers and p_i are polynomial functions from $\mathbf{C}$ to V. Since $f(t) \to 0$ as $t \to +\infty$, we conclude (see [4 (j), § 15]) that $f = 0$. Since $\mathfrak{b}^+ + \mathbf{R} H_0$ is open in $\mathfrak{a}_\mathfrak{p}$ and θ is analytic (see [4 (q), Lemma 33]), it follows that $\theta = 0$ and therefore $\nu = 0$ in Lemma 62. This being true for every $H_0 \in S^+$, we conclude (see the proof of Lemma 42 given above) that the index of ϕ is zero. This shows that 2) implies 1) and so the proof of Lemma 43 is now complete.

Part III. Applications to harmonic analysis on G

§ 32. Lemma 64 and its consequences

Let $\mathfrak{a} = \theta(\mathfrak{a})$ be a Cartan subalgebra of $\mathfrak{g}$ and A the corresponding Cartan subgroup of G. For $f \in C(G)$, define $F_f \in C(A'(I))$ as in § 18. Let J be the algebra of all invariants of $W(\mathfrak{g}/\mathfrak{a})$ in $\mathfrak{A} = \mathfrak{S}(\mathfrak{a}_c)$. Then we have a canonical isomorphism γ of $\mathfrak{Z}$ onto J (see [4 (e), Lemma 19]). Moreover, since $C_c^\infty(G)$ is dense in $C(G)$, it follows (see § 18 and [4 (o), § 22]) that

$$F_{zf} = \gamma(z) F_f \quad (z \in \mathfrak{Z}, \, f \in C(G)).$$

Let T be a distribution on G. We recall that T is said to be $\mathfrak{Z}$-finite, if the space of distributions of the form zT $(z \in \mathfrak{Z})$ has finite dimension. In particular, we can speak of a locally summable function being $\mathfrak{Z}$-finite.

LEMMA 64. *Let f be a $\mathfrak{Z}$-finite function in $\mathcal{C}(G)$. Then $F_f = 0$ unless $\mathfrak{a} \cap \mathfrak{p} = \{0\}$.*

Put $\mathfrak{a}_1 = \mathfrak{a} \cap \mathfrak{k}$, $\mathfrak{a}_2 = \mathfrak{a} \cap \mathfrak{p}$, $A_1 = A \cap K$ and $A_2 = \exp \mathfrak{a}_2$ and suppose that $\mathfrak{a}_2 \neq \{0\}$. Fix a point $a \in A'(I)$ and let $a = a_1 a_2$ $(a_i \in A_i,\ i = 1, 2)$. Select an open and connected neighborhood $\mathfrak{a}_1{}^0$ of zero in $\mathfrak{a}_1$ such that

$$a_1 \exp \mathfrak{a}_1{}^0 \cdot A_2 \subset A'(I).$$

This is clearly possible. Put $\mathfrak{a}^0 = \mathfrak{a}_1{}^0 + \mathfrak{a}_2$ and $A^0 = a_1 \exp \mathfrak{a}^0$.

Let $\mathfrak{U}$ be the set of all $u \in \mathfrak{Z}$ such that $uf = 0$. Then $\mathfrak{U}$ is an ideal in $\mathfrak{Z}$ of finite codimension. Since $\mathfrak{A}$ is a finite module over J, it follows that $\mathfrak{B} = \mathfrak{A}\gamma(\mathfrak{U})$ has finite codimension in $\mathfrak{A}$. Moreover $\gamma(u) F_f = F_{uf} = 0$ $(u \in \mathfrak{U})$ and therefore $vF_f = 0$ for $v \in \mathfrak{B}$. Since $\mathfrak{a}^0$ is connected, we conclude from [4 (e), p. 131] that

$$F_f(a_1 \exp H) = \sum_{1 \leqslant i \leqslant r} p_i(H) e^{\lambda_i(H)} \quad (H \in \mathfrak{a}^0),$$

where λ_i are linear functions and p_i plynomial functions on $\mathfrak{a}_c$ $(1 \leqslant i \leqslant r)$. Put

$$g(H) = F_f(a_1 \exp H) \quad (H \in \mathfrak{a}_2).$$

Since $F_f \in \mathcal{C}(A'(I))$, it is clear that

$$\sup_{\mathfrak{a}_2} |g(H)| (1 + \|H\|)^m < \infty$$

for any $m \geqslant 0$. Since $\mathfrak{a}_2 \neq \{0\}$, we conclude from [4 (j), § 15] that $g = 0$. This shows that $F_f = 0$ on $a_1 A_2$ and therefore $F_f(a) = 0$.

COROLLARY 1. *Suppose $f \neq 0$ in the above lemma. Then* rank $G =$ rank K.

For let us otherwise assume that rank $G >$ rank K. Choose $\mathfrak{a}$ so that it is fundamental in $\mathfrak{g}$. Then $\mathfrak{a} \cap \mathfrak{p} \neq \{0\}$ and therefore $F_f = 0$. But then it follows from Lemma 38 that $f(1) = 0$. Now fix $x \in G$ and put $f_x = r(x)f$ in the notation of § 10. Then f_x is also a $\mathfrak{Z}$-finite function in $\mathcal{C}(G)$ and $f(x) = f_x(1) = 0$, from the above proof. This shows that $f = 0$, giving a contradiction.

COROLLARY 2. *Suppose* rank $G =$ rank K *and f is a $\mathfrak{Z}$-finite function in $\mathcal{C}(G)$. Then*

$$cf(1) = (-1)^q \sum_{\lambda \in L} \varpi(\lambda) \Theta_\lambda(f)$$

in the notation of Theorem 8.

It is clear from Lemma 64 that $\phi_{f,i}=0$ $(2 \leqslant i \leqslant r)$ in Theorem 8. Hence our assertion is obvious.

An element γ of G is called elliptic if it is contained in some compact Cartan subgroup.

COROLLARY 3. *Suppose γ is a semisimple element of G which is not elliptic. Then if f is a $\mathfrak{Z}$-finite function in $C(G)$, we can conclude that*

$$\int_{G/G_\gamma} f(\gamma^{\bar{x}})\, d\ddot{x}=0$$

in the notation of Lemma 28.

Define $\mathfrak{z}$ and $\mathfrak{h}$ as in Theorem 6. By replacing γ by γ^y for some $y \in G$, we may assume that $\theta(\mathfrak{h})=\mathfrak{h}$. Let A be the Cartan subgroup of G corresponding to $\mathfrak{h}$. Then $\gamma \in A$. Since γ is not elliptic, we conclude that A is not compact and therefore $\mathfrak{h} \cap \mathfrak{p} \neq \{0\}$. Our assertion now follows from Lemmas 28 and 64.

§ 33. Proof of a conjecture of Selberg

Let f be a measurable function on G. We say that[1] f is K-finite, if the left and right translates of f, under K, span a finite-dimensional space.

LEMMA 65. *Let f be a function in $C^\infty(G) \cap L_2(G)$ which is K-finite as well as $\mathfrak{Z}$-finite. Then there exist numbers c, $r \geqslant 0$ such that*

$$|f(x)| \leqslant c\Xi(x)(1+\sigma(x))^r \quad (x \in G).$$

We shall give a proof of this lemma in § 38.

COROLLARY 1. *Let f be a function in $L_2(G)$ which is both K-finite and $\mathfrak{Z}$-finite. Then $f \in C(G)$.*

We regard $\mathfrak{H}=L_2(K \times K)$ as a Hilbert space in the usual way and define a unitary double representation $\mu^0=(\mu_1,\mu_2)$ of K on $\mathfrak{H}$ as follows (cf. § 12). If $u \in \mathfrak{H}$ and $k \in K$, then the functions $u_1=\mu_1(k)u$ and $u_2=u\mu_2(k)$ are given by

$$u_1(k_1,k_2)=u_1(k^{-1}k,k_2), \quad u_2(k_1,k_2)=u(k_1,k_2k^{-1}) \quad (k_1,k_2 \in K).$$

It follows from [4 (q), Lemma 33] that f is analytic. For any $x \in G$, let $\phi(x)$ denote the function

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[1] We do not distinguish between two measurable functions which differ only on a set of measure zero.

$$(k_1, k_2) \to f(k_1^{-1}xk_2^{-1}) \quad (k_1, k_2 \in K)$$

in $\mathfrak{H}$. Then it is clear that $\phi(k_1 x k_2) = \mu_1(k_1)\phi(x)\mu_2(k_2)$ $(k_1, k_2 \in K)$. Let V be the subspace of $\mathfrak{H}$ spanned by $\phi(x)$ for all $x \in G$. Then V is stable under μ^0 and dim $V < \infty$ since f is K-finite. Let μ denote the restriction of μ^0 on V. Then ϕ is a C^∞ μ-spherical function from G to V and it is clear from Lemma 65 that Lemmas 42 and 43 are applicable to ϕ, provided $f \neq 0$. Since $f \in L_2(G)$, we conclude that the index of ϕ is zero and therefore $\phi \in C(G) \otimes V$ from Lemma 43. Obviously this implies that $f \in C(G)$.

COROLLARY 2. *Suppose* $f \neq 0$ *in Corollary* 1. *Then* rank $G =$ rank K.

This follows immediately from Corollary 1 of Lemma 64.

If we combine Corollary 3 of Lemma 64 with Corollary 1 of Lemma 65, we get the following theorem.

THEOREM 11. *Suppose* γ *is a semisimple element of* G, *which is not elliptic, and* f *a function in* $L_2(G)$, *which is both* K-*finite and* $\mathfrak{Z}$-*finite. Then* $f \in C(G)$ *and, in the notation of Lemma* 28, *the integral*

$$\int_{G/G_\gamma} f(\gamma^{\bar{x}}) \, d\bar{x}$$

exists and its value is zero.

This theorem represents, essentially, a conjecture of Selberg [9, p. 70]. I understand that R. P. Langlands had obtained a similar but somewhat weaker result, a few years ago.

§ 34. The behaviour of certain eigenfunctions at infinity

We now return to the notation of § 27. Extend $\mathfrak{a}_\mathfrak{p}$ to a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. Define $\varpi = \varpi^\mathfrak{a}$, $W = W(\mathfrak{g}/\mathfrak{a})$ and $W_1 = W(\mathfrak{m}/\mathfrak{a})$ as usual (see [4 (p), § 12]) and, for a given linear function λ on $\mathfrak{a}_c$, put

$$\chi_\lambda(z) = \chi_\lambda^\mathfrak{a}(p_z) \quad (z \in \mathfrak{Z})$$

in the notation of [4 (p), § 12] and [4 (o), § 14]. Let $\mathfrak{U}_\lambda$ denote the kernel of χ_λ in $\mathfrak{Z}$ and put $\mathfrak{U}_{1\lambda} = \mathfrak{Z}_1\mu_0(\mathfrak{U}_\lambda)$, $\mathfrak{Z}_{1\lambda}^* = \mathfrak{Z}_1/\mathfrak{U}_{1\lambda}$. (Here $\mu_0 = \mu_{\mathfrak{g}/\mathfrak{m}}$ as in § 27.) Let σ_λ denote the natural representation of $\mathfrak{Z}_1$ on $\mathfrak{Z}_{1\lambda}^*$.

Let $r = [W : W_1]$ and select elements $s_1 = 1, s_2, ..., s_r$ in W such that $W = \bigcup_{1 \leq i \leq r} W_1 s_i$. Consider the subalgebras J and J_1 of all invariants of W and W_1 respectively in $\mathfrak{S}(\mathfrak{a}_c)$. Then we have the canonical isomorphisms $\gamma : \mathfrak{Z} \to J$ and $\gamma_1 : \mathfrak{Z}_1 \to J_1$ (see [4(o), § 12]) and $\gamma = \gamma_1 \circ \mu_0$. We identify $\mathfrak{S}(\mathfrak{a}_c)$ with $S = S(\mathfrak{a}_c)$ as usual and denote by $\gamma_1(\zeta : \lambda)$ $(\zeta \in \mathfrak{Z}_1)$ the value at λ of the element $\gamma_1(\zeta) \in S$.

LEMMA 66. $\dim \mathcal{Z}_{1\lambda}^* = r$ and, if $\varpi(\lambda) \neq 0$, we can choose a base $(v_1, ..., v_r)$ for $\mathcal{Z}_{1\lambda}^*$ such that

$$\sigma_\lambda(\zeta) v_i = \gamma_1(\zeta : s_i\lambda) v_i \quad (\zeta \in \mathcal{Z}_1, \ 1 \leqslant i \leqslant r).$$

Since W and W_1 are both generated by reflexions, the results of [4 (j), § 3] are applicable. Therefore by taking into account the isomorphisms γ and γ_1, our assertions follow immediately from Lemmas 13 and 15 of [4 (j)].

Let μ and V have the same meaning as in Lemma 42.

LEMMA 67. *Let λ be a linear function on $\mathfrak{a}_c$ and ϕ a C^∞ μ-spherical function from G to V. Suppose the following conditions are fulfilled:*

1)   rank $G = $ rank K.
2) λ takes only real values on $\mathfrak{a} \cap \mathfrak{p} + (-1)^{\frac{1}{2}} \mathfrak{a} \cap \mathfrak{k}$ and $\varpi(\lambda) \neq 0$.
3) $z\phi = \chi_\lambda(z)\phi \quad (z \in \mathcal{Z})$.
4)   *There exist numbers $c, s \geqslant 0$ such that*

$$|\phi(x)| \leqslant c\Xi(x)(1 + \sigma(x))^s \quad (x \in G).$$

Then $\phi \in C(G) \otimes V$.

We may obviously assume that $\phi \neq 0$ and G is not compact so that $\mathfrak{a}_\mathfrak{p} \neq \{0\}$. Then, in view of Lemma 43, it would be enough to verify that

$$\lim_{t \to +\infty} e^{t\varrho(H)} \phi(h \exp tH) = 0$$

for $H \in S^+$ and $h \in A_\mathfrak{p}^+$. For any $H_0 \in S^+$, let $\mathfrak{m}_{H_0}$ denote the centralizer of H_0 in $\mathfrak{g}$. Suppose the above condition does not hold. Then we can choose $H_0 \in S^+$ such that:

1)   For some $h \in A_\mathfrak{p}^+$, $e^{t\varrho(H_0)}\phi(h \exp tH_0)$ does not tend to zero as $t \to +\infty$.
2) $\dim \mathfrak{m}_{H_0}$ is minimum possible consistent with condition 1).

Put $\mathfrak{m} = \mathfrak{m}_{H_0}$, $\mathfrak{m}_1 = \mathfrak{m} \cap \mathfrak{k} + [\mathfrak{m}, \mathfrak{m}] \cap \mathfrak{p}$ and let $\mathfrak{l}$ be the centralizer of $\mathfrak{m}$ in $\mathfrak{p}$. Then $\mathfrak{m}_1 \cap \mathfrak{p}$ is the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{m} \cap \mathfrak{p}$. Let M_1 be the analytic subgroup of G corresponding to $\mathfrak{m}_1$. We now use the notation of §§ 27–30 for this particular H_0. (Note that $\mathfrak{c}_\mathfrak{p} = \{0\}$ in the present case since rank $\mathfrak{g} = $ rank $\mathfrak{k}$.) Define $\Theta(m)$ and $\theta(m)$ $(m \in M)$ as in §§ 29, 30. We know from Lemma 66 that the representation Γ of $\mathcal{Z}_1$ is semisimple. Moreover it is clear from condition 2) of Lemma 67 that $s_i\lambda$ takes only real values on $\mathfrak{a}_\mathfrak{p}$ and therefore also on $\mathfrak{l}$. Hence we conclude from Lemma 59 that

$$\Theta(m \exp H) = \Theta(m) \quad (m \in M, H \in \mathfrak{l}).$$

This implies, in particular, that

$$\theta(m \exp H) = \theta(m) \quad (m \in M, \ H \in \mathfrak{l}).$$

Hence it follows from Lemma 61 and the definition of H_0, that $\theta \neq 0$.

Fix an element $H \neq 0$ in $\mathfrak{a}_1^+$. Then if $h \in A_1^+$, we claim that

$$e^{t\varrho_1(H)}\theta(h \exp tH) \to 0$$

as $t \to +\infty$. Put $H_1 = c_1(H_0 + cH)$, where c is a small positive number and $c_1 = \|H_0 + cH\|^{-1}$. Then $H_1 \in S^+$ and it is obvious that

$$\dim \mathfrak{m}_{H_1} < \dim \mathfrak{m}_{H_0}.$$

Hence we conclude from definition of H_0 that

$$e^{t\varrho(H_1)}\phi(h_0 \exp tH_1) \to 0 \quad (h_0 \in A_{\mathfrak{p}}^+)$$

as $t \to +\infty$. Define U as in § 30 and let U_0 denote the interior of U in S^+. Then, by choosing c sufficiently small, we can assume that $H_1 \in U_0$ and therefore

$$\lim_{t \to +\infty} \left| e^{t\varrho(H_1)}\phi(h \exp tH_1) - e^{\varrho_1(tH_1)}\theta(h \exp tH_1) \right| = 0 \quad (h \in A_1^+)$$

from Lemma 61. Fix $h \in A_1^+$. Since $\alpha(H_1) > 0$ $(\alpha \in \Sigma_2)$, we can choose $t_0 \geqslant 0$ such that $h_0 = h \exp t_0 H_1 \in A_{\mathfrak{p}}^+$. Hence it follows from what we have seen above that

$$e^{t\varrho(H_1)}\phi(h \exp tH_1) \to 0$$

as $t \to +\infty$. Put $c_2 = c_1 c$. Then $\varrho_1(H_1) = c_2\varrho_1(H)$ and $H_1 = c_1 H_0 + c_2 H$. Therefore since $H_0 \in \mathfrak{l}$, we conclude that

$$e^{t\varrho_1(H)}\theta(h \exp tH) \to 0$$

and this proves our assertion.

Let θ_1 denote the restriction of θ on M_1. It is clear that $\theta_1 \neq 0$ and we conclude from Lemmas 43 and 58 that $\theta_1 \in C(M_1) \otimes V$. But then rank $\mathfrak{m}_1 = $ rank $(\mathfrak{m}_1 \cap \mathfrak{k})$ from Corollary 1 of Lemma 64.

Fix a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{m}_1 \cap \mathfrak{k}$. Then $\mathfrak{h} = \mathfrak{l} + \mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{m}$ and therefore also of $\mathfrak{g}$. Since rank $\mathfrak{g} = $ rank $\mathfrak{k}$ and $H_0 \in \mathfrak{l}$, $\mathfrak{h}$ cannot be fundamental in $\mathfrak{g}$. Hence there exists a root β of $(\mathfrak{g}, \mathfrak{h})$ such that [1] $H_\beta \in \mathfrak{h} \cap \mathfrak{p} = \mathfrak{l}$ (see [4 (g)], Lemma 33]). Let M_c denote the (connected) complex adjoint group of $\mathfrak{m}_c$. We can choose $y \in M_c$ such that $\mathfrak{h}_c^y = \mathfrak{a}_c$. Then $\alpha = \beta^y$ is a root of $(\mathfrak{g}, \mathfrak{a})$ and $H_\alpha = (H_\beta)^y = H_\beta \in \mathfrak{l}$.

Now we know that $\Theta \neq 0$ and by Lemma 59 $\Theta(m) \in {}^0V$ $(m \in M)$. Therefore it follows from Lemma 66 and the definition of Γ, that there exists an element $s \in W$ such that $s\lambda = 0$ on $\mathfrak{l}$. But then $s\lambda(H_\alpha) = 0$ and therefore $\varpi(\lambda) = 0$, contrary to our hypothesis. This proves the lemma.

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[1] Here H_β has the usual meaning (see [4 (n)], § 4]).

§ 35. Eigenfunctions of $\mathfrak{Z}$ in $C(G)$

Let us now assume that rank G = rank K and use the notation of § 20. Let L' be the set of all $\lambda \in L$ where $\varpi(\lambda) \neq 0$. We denote by χ_λ ($\lambda \in L'$) the corresponding homomorphism (see [4 (p), § 29]) of $\mathfrak{Z}$ into C so that $z\Theta_\lambda = \chi_\lambda(z)\Theta_\lambda$ ($z \in \mathfrak{Z}$). Consider the space $C_\lambda(G)$ of all functions $f \in C(G)$ such that $zf = \chi_\lambda(z)f$ ($z \in \mathfrak{Z}$). Let $\mathfrak{H}_\lambda$ denote the closure of $C_\lambda(G)$ in $L_2(G)$ and $\mathfrak{H}$ the smallest closed subspace of $L_2(G)$ containing $\bigcup_{\lambda \in L'} \mathfrak{H}_\lambda$.

It is obvious from the definition of Θ_λ (see [4 (p), Theorem 3]) that

$$\Theta_\lambda(x^{-1}) = \operatorname{conj} \Theta_\lambda(x) = (-1)^m \Theta_{-\lambda}(x) \quad (\lambda \in L', \, x \in G'),$$

where $m = \frac{1}{2}$ (dim $\mathfrak{g}$ − rank $\mathfrak{g}$). Hence it follows that

$$\chi_\lambda(z^*) = \operatorname{conj} \chi_\lambda(\eta(z)) = \chi_{-\lambda}(z) \quad (z \in \mathfrak{Z}),$$

where z^* denotes the adjoint of the differential operator z and η the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$.

LEMMA 68. *Let f be any eigenfunction of $\mathfrak{Z}$ in $C(G)$. Then $f \in C_\lambda(G)$ for some $\lambda \in L'$.*

We may obviously suppose that $f \neq 0$. Let χ be the homomorphism of $\mathfrak{Z}$ into C such that $zf = \chi(z)f$ ($z \in \mathfrak{Z}$). We have to show that $\chi = \chi_\lambda$ for some $\lambda \in L'$. Suppose this is false. Fix $\lambda \in L$ and consider $\Theta_\lambda(f)$. Then

$$\chi(z)\Theta_\lambda(f) = \Theta_\lambda(zf) = \chi_\lambda(z^*)\Theta_\lambda(f) = \chi_{-\lambda}(z)\Theta_\lambda(f) \quad (z \in \mathfrak{Z}).$$

Since $\chi \neq \chi_{-\lambda}$, we conclude that $\Theta_\lambda(f) = 0$. In view of Corollary 2 of Lemma 64, this implies that $f(1) = 0$.

Now fix $x \in G$ and put $f_x = r(x)f$ in the notation of § 10. Then the above proof is applicable to f_x and therefore $f(x) = f_x(1) = 0$. This shows that $f = 0$ and so we get a contradiction. Hence the lemma.

COROLLARY. *Let ϕ be an element in $L_2(G)$ which is an eigendistribution of $\mathfrak{Z}$. Then $\phi \in \mathfrak{H}_\lambda$ for some $\lambda \in L'$.*

We may again assume that $\phi \neq 0$. Let l and r respectively denote the left- and right-regular representations of G on $L_2(G)$ and ν the usual norm on $L_2(G)$. For $\alpha, \beta \in C(K)$, define

$$\alpha * \phi * \beta = \int_{K \times K} \alpha(k_1)\beta(k_2)\,l(k_1)\,r(k_2^{-1})\,\phi\,dk_1\,dk_2$$

as usual. Let $\mathcal{L}$ denote the space of all K-finite functions in $C(K)$. Since $\mathcal{L}$ is dense in $C(K)$

in the norm $|\alpha|_\infty = \sup|\alpha|$ $(\alpha \in C(K))$, it follows easily (see § 3) that, for any $\varepsilon > 0$, we can choose $\alpha, \beta \in \mathcal{L}$ such that

$$\nu(\phi - \alpha \divideontimes \phi \divideontimes \beta) < \varepsilon.$$

Put $\psi = \alpha \divideontimes \phi \divideontimes \beta$ and suppose $\varepsilon < \nu(\phi)$. Then it is clear that $\psi \neq 0$ and $z\psi = \chi(z)\psi$ $(z \in \mathcal{Z})$. Hence we conclude from Lemma 68 and Corollary 1 of Lemma 65 that $\psi \in C_\lambda(G)$ for some $\lambda \in L'$. Therefore, in particular, $\chi = \chi_\lambda$. Since the space $C_\lambda(G)$ depends only on χ_λ (and not on λ), this shows that $\phi \in \mathrm{Cl}(C_\lambda(G)) = \mathfrak{H}_\lambda$.

§ 36. The role of the distributions Θ_λ in the harmonic analysis on G

For any $\mathfrak{b} \in \mathcal{E}_K$, let $\Theta_{\lambda,\mathfrak{b}}$ $(\lambda \in L')$ denote the corresponding Fourier component of Θ_λ (see [4 (q), § 17]).

THEOREM 12. $\Theta_{\lambda,\mathfrak{b}} \in C_\lambda(G)$ for $\lambda \in L'$ and $\mathfrak{b} \in \mathcal{E}_K$.

This follows from Theorem 9 and Lemma 67 (see also the proof of Corollary 1 of Lemma 65).

COROLLARY 1. $C_\lambda(G) \neq \{0\}$ for $\lambda \in L'$.

Since $\Theta_\lambda \neq 0$, we conclude from Lemma 9 that $\Theta_{\lambda,\mathfrak{b}} \neq 0$ for some $\mathfrak{b} \in \mathcal{E}_K$. This implies our assertion.

Fix $\lambda_0 \in L'$ and let $L(\lambda_0)$ denote the set of all $\lambda \in L$ of the form $\lambda = s\lambda_0$ $(s \in W = W(\mathfrak{g}/\mathfrak{b})$ in the notation of [4 (p)]). Let E_{λ_0} denote the orthogonal projection of $L_2(G)$ on $\mathfrak{H}_{\lambda_0}$ and define

$$(f, g) = \int_G (\mathrm{conj}\, f)g\, dx \quad (f, g \in L_2(G))$$

as usual.

Let $\mathcal{L}(G)$ denote the space of all K-finite functions in $C_c^\infty(G)$.

COROLLARY 2. Let $\gamma \in \mathcal{L}(G)$ and $\lambda_0 \in L'$. Then $E_{\lambda_0}\gamma \in C_{\lambda_0}(G)$ and

$$\Theta_\lambda(E_{\lambda_0}\gamma) = \begin{cases} \Theta_\lambda(\gamma), & \text{if } \lambda \in L(-\lambda_0), \\ 0 & \text{otherwise,} \end{cases}$$

for $\lambda \in L'$.

It is obvious that E_{λ_0} commutes with the translations of G and therefore $f = E_{\lambda_0}\gamma$ is K-finite. Hence we conclude from Corollary 1 of Lemma 65 that $f \in C_{\lambda_0}(G)$. Therefore (see the proof of Lemma 68), $\Theta_\lambda(f) = 0$ $(\lambda \in L')$ unless $\lambda \in L(-\lambda_0)$. Now fix $\lambda \in L(-\lambda_0)$. Then conj $\Theta_{\lambda,\mathfrak{b}} \in C_{\lambda_0}(G)$ from Theorem 12 and therefore

$$\Theta_{\lambda,\mathfrak{b}}(\gamma) = (\text{conj } \Theta_{\lambda,\mathfrak{b}}, \gamma) = (\text{conj } \Theta_{\lambda,\mathfrak{b}}, E_{\lambda_0}\gamma) = \Theta_{\lambda,\mathfrak{b}}(f) \quad (\mathfrak{b} \in \mathcal{E}_K)$$

from the corollary of Theorem 9. Therefore, since γ and f are both K-finite, we have

$$\Theta_{\lambda}(\gamma) = \sum_{\mathfrak{b}} \Theta_{\lambda,\mathfrak{b}}(\gamma) = \sum_{\mathfrak{b}} \Theta_{\lambda,\mathfrak{b}}(f) = \Theta_{\lambda}(f).$$

LEMMA 69. *Fix* $\lambda_0 \in L'$ *and define* c *and* q *as in Theorem 8. Then*

$$(-1)^q \sum_{\lambda \in L(-\lambda_0)} \varpi(\lambda) \Theta_{\lambda}(\check{\alpha} \divideontimes f) = c(\alpha, E_{\lambda_0} f)$$

for $\alpha \in C_c^{\infty}(G)$ *and* $f \in \mathcal{C}(G)$. *Here* $\check{\alpha}(x) = \text{conj } \alpha(x^{-1})$ $(x \in G)$.

Since $\check{\alpha} \divideontimes f \in \mathcal{C}(G)$ (see § 10), the left side is defined. Fix $\alpha_0, \beta \in \mathcal{L}(G)$ and put $g = E_{\lambda_0}\beta$. Then $\check{\alpha}_0 \divideontimes g = E_{\lambda_0}(\check{\alpha}_0 \divideontimes \beta)$. Now apply Corollary 2 of Lemma 64 to $\check{\alpha}_0 \divideontimes g$, taking into account Corollary 2 of Theorem 12 with $\gamma = \check{\alpha}_0 \divideontimes \beta$. Then we get

$$c(\alpha_0, E_{\lambda_0}\beta) = (-1)^q \sum_{\lambda \in L(-\lambda_0)} \varpi(\lambda) \Theta_{\lambda}(\check{\alpha}_0 \divideontimes \beta).$$

On the other hand, $\mathcal{L}(G)$ is dense both in $C_c^{\infty}(G)$ and $\mathcal{C}(G)$, by Lemmas 9, 16 and 19. Moreover, convergence in either one of these spaces implies convergence in $L_2(G)$ (see Lemma 11). Finally, if α_0 and β are two variable elements of $\mathcal{L}(G)$, which converge to α and f in $C_c^{\infty}(G)$ and $\mathcal{C}(G)$ respectively, then it is obvious from § 10 that $\check{\alpha}_0 \divideontimes \beta$ tends to $\check{\alpha} \divideontimes f$ in $\mathcal{C}(G)$. Therefore the statement of Lemma 69 now follows immediately.

Define the representation r of G on $\mathcal{C}(G)$ as in § 10.

COROLLARY 1. *Let* $f \in \mathcal{C}(G)$. *Then* $E_{\lambda_0} f$ *is a continuous function on* G *given by*

$$E_{\lambda_0} f(x) = c^{-1}(-1)^q \sum_{\lambda \in L(-\lambda_0)} \varpi(\lambda) \Theta_{\lambda}(r(x)f) \quad (x \in G).$$

It is obvious that the right side is continuous in x and the equality follows from Lemma 69, if we observe that $\Theta_{\lambda}(\check{\alpha} \divideontimes f) = \Theta_{\lambda}(f \divideontimes \check{\alpha})$ $(\alpha \in C_c^{\infty}(G))$, in view of the invariance of Θ_{λ}.

Let E denote the orthogonal projection of $L_2(G)$ on $\mathfrak{H}$.

COROLLARY 2. $c(\alpha, Ef) = (-1)^q \sum_{\lambda \in L} \varpi(\lambda) \Theta_{\lambda}(\check{\alpha} \divideontimes f)$ *for* $\alpha \in C_c^{\infty}(G)$ *and* $f \in \mathcal{C}(G)$.

This is obvious from Lemma 69.

COROLLARY 3. *For* $f \in \mathcal{C}(G)$, *put*

$$f^{\natural}(x) = c^{-1}(-1)^q \sum_{\lambda \in L} \varpi(\lambda) \Theta_{\lambda}(r(x)f) \quad (x \in G).$$

Then $f^{\natural}$ *is a continuous function on* G *and* $f^{\natural} = Ef$.

Put $$T(f) = c^{-1}(-1)^q \sum_{\lambda \in L} \varpi(\lambda) \Theta_\lambda(f) \quad (f \in \mathcal{C}(G)).$$

We have seen in § 20 that T is a tempered distribution. Hence it is obvious that $f^\natural$ is continuous. The rest follows from Corollary 2, if we take into account the fact that

$$T(\tilde{\alpha} * f) = T(f * \tilde{\alpha}) = \int f^\natural(x) \operatorname{conj} \alpha(x) \cdot dx.$$

Let $W(\lambda)$ $(\lambda \in L')$ be the set of all $s \in W$ such that $s\lambda \in L$. Put ([1])

$$\Theta_\lambda^* = [W(\lambda)]^{-1} \sum_{s \in W(\lambda)} \varepsilon(s) \Theta_{s\lambda} \quad (\lambda \in L').$$

COROLLARY 4. *For any $\lambda \in L'$, the distribution $(-1)^q \varpi(\lambda) \Theta_\lambda^*$ is of positive type.*

This is obvious from Lemma 69 since $c(\alpha, E_\lambda \alpha) \geqslant 0$ for $\alpha \in C_c^\infty(G)$.

LEMMA 70. *Fix $\mathfrak{d} \in \mathcal{E}_K$. Then there exist only a finite number of $\lambda \in L'$ such that $\Theta_{\lambda, \mathfrak{d}} \neq 0$.*

Let $\mathfrak{c}$ be the center and $\mathfrak{g}_1$ the derived algebra of $\mathfrak{g}$. Fix a quadratic form Q on $\mathfrak{g}$ such that 1) Q is negative-definite on $\mathfrak{c}$, 2) $\mathfrak{c}$ and $\mathfrak{g}_1$ are mutually orthogonal under Q and 3) $Q(X) = \operatorname{tr}(\operatorname{ad} X)^2$ for $X \in \mathfrak{g}_1$. Then Q is negative-definite on $\mathfrak{k}$, positive-definite on $\mathfrak{p}$ and it is invariant under G. Moreover, $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal under Q. Fix bases $(Y_1, ..., Y_p)$ and $(Z_1, ..., Z_q)$ for $\mathfrak{p}$ and $\mathfrak{k}$ which are orthonormal with respect to Q and $-Q$ respectively and put

$$\omega_\mathfrak{p} = Y_1^2 + ... + Y_p^2, \quad \omega_\mathfrak{k} = -(Z_1^2 + ... + Z_q^2)$$

in $\mathfrak{G}$. Then $$\omega = \omega_\mathfrak{p} + \omega_\mathfrak{k} \in \mathfrak{Z}$$

Let γ denote the canonical isomorphism of $\mathfrak{Z}$ into ([2]) $S(\mathfrak{b}_c) = \mathfrak{S}(\mathfrak{b}_c)$ (see [4 (e)], Lemma 19) and $\gamma(z:\mu)$ $(z \in \mathfrak{Z}, \mu \in \mathfrak{F})$ the value of the polynomial function $\gamma(z)$ at μ. Put

$$\varrho = \tfrac{1}{2} \sum_{\alpha \in P} \alpha,$$

where P is the set of all positive roots of $(\mathfrak{g}, \mathfrak{b})$. Then (see [4 (e)], p. 144])

$$|\mu|^2 = \gamma(\omega:\mu) + \tfrac{1}{2} \sum_{\alpha \in P} \varrho(H_\alpha) \quad (\mu \in \mathfrak{F})$$

is a positive-definite quadratic form on $\mathfrak{F}$ and $\gamma(\omega:\varrho) = 0$, so that

$$\gamma(\omega:\mu) = |\mu|^2 - |\varrho|^2.$$

Moreover, $\chi_\lambda(\omega) = \gamma(\omega:\lambda) = |\lambda|^2 - |\varrho|^2$ for $\lambda \in L'$.

([1]) See the foot-note on p. 3.

([2]) Here the notation is the same as in [4 (p)].

Let $\mathfrak{K}$ be the subalgebra of $\mathfrak{G}$ generated by $(1, \mathfrak{k}_c)$ and $\mathfrak{Z}_K$ the center of $\mathfrak{K}$. Then $\omega_\mathfrak{k} \in \mathfrak{Z}_K$ and

$$\omega_\mathfrak{k} \, \alpha_\mathfrak{b} = \chi_\mathfrak{b}(\omega_\mathfrak{k}) \, \alpha_\mathfrak{b},$$

where $\chi_\mathfrak{b}(\omega_\mathfrak{k})$ is a number $\geqslant 0$ (see § 3). Therefore since L is a lattice in $\mathfrak{F}$, it would be sufficient to prove the following lemma.

LEMMA 71. *Suppose λ and $\mathfrak{b}$ are two elements in L' and $\mathcal{E}_K$ respectively such that $\Theta_{\lambda, \mathfrak{b}} \neq 0$. Then*

$$|\lambda|^2 \leqslant \chi_\mathfrak{b}(\omega_\mathfrak{k}) + |\varrho|^2.$$

We know from Theorem 12 that $f = \Theta_{\lambda, \mathfrak{b}} \in \mathcal{C}_\lambda(G) \subset L_2(G)$. Let r denote the right-regular representation of G on $L_2(G)$, V the smallest closed subspace of $L_2(G)$ containing f which is invariant under r, and π the restriction of r on V. Since convergence in $\mathcal{C}(G)$ implies convergence in $L_2(G)$, it follows from Lemma 15 that f is differentiable under π and

$$\pi(\omega)f = \chi_\lambda(\omega)f, \qquad \pi(\omega_\mathfrak{k})f = \chi_\mathfrak{b}(\omega_\mathfrak{k})f.$$

Hence $\qquad \{|\lambda|^2 - |\varrho|^2\} \|f\|^2 = (f, \pi(\omega)f) = \chi_\mathfrak{b}(\omega_\mathfrak{k}) \|f\|^2 + (f, \pi(\omega_\mathfrak{p})f),$

where $\| \ \|$ denotes the usual norm in $L_2(G)$. But since π is unitary and f is differentiable under π, it is obvious that

$$(f, \pi(\omega_\mathfrak{p})f) = - \sum_{1 \leqslant i \leqslant p} \|\pi(Y_i)f\|^2 \leqslant 0.$$

Hence our assertion follows immediately from the fact that $\|f\| > 0$.

LEMMA 72. *Let f be a K-finite function in $\mathcal{C}(G)$. Then $E_\lambda f \in \mathcal{C}_\lambda(G)$ $(\lambda \in L')$ and $E_\lambda f = 0$ for all $\lambda \in L'$ except a finite number. Hence $Ef \in \mathcal{C}(G)$ and it is both K-finite and $\mathfrak{Z}$-finite.*

Since E_λ commutes with the translations of G, it is clear that $E_\lambda f$ is K-finite and therefore, by Corollary 1 of Lemma 65, it lies in $\mathcal{C}_\lambda(G)$. Now select a finite subset F of $\mathcal{E}_K$ such that $f = \alpha_F \ast f \ast \alpha_F$ in the notation of § 12. Replacing F by $F \cup F^*$, we may assume that $F = F^*$. (Recall that $\mathfrak{b}^*$ is the class in $\mathcal{E}_K$ contragradient to $\mathfrak{b}$.) Put

$$f_\lambda(x) = \Theta_\lambda(r(x)f) \quad (x \in G)$$

for $\lambda \in L'$. Then it is obvious that

$$f_\lambda(x) = \Theta_{\lambda, F}(r(x)f),$$

where $\qquad \Theta_{\lambda, F} = \sum_{\mathfrak{b} \in F} \Theta_{\lambda, \mathfrak{b}}.$

Hence we conclude from Lemma 70 that $f_\lambda = 0$ for all $\lambda \in L'$ except a finite number, and therefore the assertions of the lemma follow from Corollaries 1 and 3 of Lemma 69.

Define $B' = B \cap G'$ as in § 19.

LEMMA 73. *Fix* $\lambda \in L'$ *and let* Θ *be a tempered and invariant distribution on* G *such that* $z^* = \chi_\lambda(z)\Theta$ $(z \in \mathfrak{Z})$. *Then* $\Theta_\mathfrak{b} \in C_\lambda(G)$ *for* $\mathfrak{b} \in \mathcal{E}_K$. *Moreover, in order to show that* $\Theta = 0$, *it is sufficient to verify either one of the following two conditions.*

1) $\Theta(f) = 0$ *for every K-finite function* $f \in C_{-\lambda}(G)$.
2) $\Theta = 0$ *pointwise*(1) *on* B'.

It follows from Theorem 9 and Lemma 67 that $\Theta_\mathfrak{b} \in C_\lambda(G)$ $(\mathfrak{b} \in \mathcal{E}_K)$ and therefore $f = \text{conj } \Theta_\mathfrak{b} \in C_{-\lambda}(G)$ (see § 35). Then

$$\Theta_\mathfrak{b}(f) = \int_G |\Theta_\mathfrak{b}(x)|^2 \, dx$$

from the corollary of Theorem 9. Therefore $\Theta_\mathfrak{b} = 0$ under condition 1). In view of Lemma 9, this implies that $\Theta = 0$.

On the other hand, by Lemma 64 and the corollary of Theorem 7, 2) implies 1) and so the lemma is proved.

Define W_G as in [4 (p), Theorem 3]. Then L is stable under W_G and $\Theta_{s\lambda} = \varepsilon(s)\Theta_\lambda$ $(s \in W_G, \lambda \in L')$.

LEMMA 74. *Let* Θ *be an invariant eigendistribution of* $\mathfrak{Z}$ *on* G, *which is tempered. Suppose there exists an element* $\mathfrak{b} \in \mathcal{E}_K$ *such that* $\Theta_\mathfrak{b} \neq 0$ *and* $\Theta_\mathfrak{b} \in L_2(G)$. *Then we can choose* $\lambda \in L'$ *such that* $z\Theta = \chi_\lambda(z)\Theta$ $(z \in \mathfrak{Z})$. *Moreover, for any such* λ, *there exist unique complex numbers* c_s $(s \in W(\lambda))$ *such that* $c_{ts} = c_s$ $(t \in W_G)$ *and*

$$\Theta = [W_G]^{-1} \sum_{s \in W(\lambda)} \varepsilon(s) c_s \Theta_{s\lambda}.$$

The first statement follows from the corollary of Lemma 68. Now put

$$\Phi(b) = \Delta(b)\Theta(b) \quad (b \in B'),$$

where Δ has the usual meaning (see [4 (p), Theorem 3]). Then it follows from [4 (o), Lemma 31] that Φ extends to an analytic function on B. Moreover, we know from [4 (e), Theorem 2] that

$$\gamma(z)\Phi = \chi_\lambda(z)\Phi \quad (z \in \mathfrak{Z}),$$

---

(1) Here we have to make use of Theorem 2 of [4 (o)].

where $\gamma(z) \in \mathfrak{S}(\mathfrak{b}_c)$ is to be regarded as a differential operator on B. Finally it is obvious that $\Phi(b^s) = \varepsilon(s)\Phi(b)$ for $s \in W_G$ and $b \in B$. Therefore (see [4 (f), p. 102] and [4 (p), Theorem 3]), we can choose unique complex numbers c_s $(s \in W(\lambda))$ such that $c_{ts} = c_s$ $(t \in W_G)$ and the distribution

$$\Theta' = \Theta - [W_G]^{-1} \sum_{s \in W(\lambda)} \varepsilon(s) c_s \Theta_{s\lambda}$$

vanishes pointwise on B'. It is clear (see § 20) that Θ' is tempered and $z\Theta' = \chi_\lambda(z) \Theta'$ $(z \in \mathfrak{Z})$. Therefore $\Theta' = 0$ from Lemma 73.

LEMMA 75. *Let* $\mathfrak{d}_0$ *denote the class of the trivial representation of* K. *Then* $\Theta_{\lambda, \mathfrak{d}_0} = 0$ *for* $\lambda \in L'$.

Fix $\lambda \in L'$ and put $\Phi_\lambda = \Theta_{\lambda, \mathfrak{d}_0}$. Then from Theorem 9, there exist numbers $c, m \geqslant 0$ such that

$$|\Phi_\lambda| \leqslant c\Xi(1+\sigma)^m.$$

This is the analogue of [4 (q), Lemma 43]. By making use of the corollary of [4 (q), Lemma 47], we prove in the same way as in [4 (q), § 20] that $\Phi_\lambda = 0$.

§ 37. The discrete series for G

Let G be a locally compact unimodular group satisfying the second axiom of countability. Fix a Haar measure dx on G. By a unitary representation of G, we mean a representation of G on a Hilbert space, which is unitary. Let $\mathcal{E}$ be the set of all equivalence classes of irreducible unitary representations of G.

Let π be an irreducible unitary representation of G on a Hilbert space $\mathfrak{H}$. We say that π is *square-integrable* if any one of the following two mutually equivalent conditions holds (see [7, p. 640]).

(1)   There exist nonzero elements ϕ, ψ in $\mathfrak{H}$ such that

$$\int_G |(\phi, \pi(x) \psi)|^2 dx < \infty.$$

(2)   There exists a closed subspace V of $L_2(G)$ stable under the right-regular representation r of G on $L_2(G)$, such that π is equivalent to the restriction of r on V.

It is known (see [7, p. 640]) that, if π is square-integrable, there exists a number $d(\pi) > 0$ such that

$$\int_G |(\phi, \pi(x) \psi)|^2 dx = d(\pi)^{-1} \|\phi\|^2 \|\psi\|^2 \quad (\phi, \psi \in \mathfrak{H}),$$

where $\|\phi\|$ is the norm of ϕ in $\mathfrak{H}$. We shall call $d(\pi)$ the *formal degree* of π (see [4 (d), § 3]). It is obvious that square-integrability, as well as the formal degree, are invariant under equivalence. We call a class $\omega \in \mathcal{E}$ *discrete* if every representation $\pi \in \omega$ is square-integrable and put $d(\omega) = d(\pi)$.

Let $\mathcal{E}_d$ denote the set of all discrete classes in $\mathcal{E}$. Then $\mathcal{E}_d$ is called the discrete series of G.

Now let us return to the case when G and K are defined as in § 7. For any $w \in \mathcal{E}$, let Θ_ω denote the character (see [4 (b), § 5]) and χ_ω the infinitesimal character of ω so that $z\Theta_\omega = \chi_\omega(z)\Theta_\omega$ $(z \in \mathfrak{Z})$.

LEMMA 76. *Let* $\omega \in \mathcal{E}_d$. *Then* Θ_ω *is tempered and* $\Theta_{\omega,\mathfrak{b}} \in L_2(G)$ *for* $\mathfrak{b} \in \mathcal{E}_K$.

Fix $\pi \in \omega$ and let $\mathfrak{H}$ be the representation space of π. We now use the notation of § 25 and put
$$\phi_i(x) = (\psi_i, \pi(x)\psi_i) \quad (x \in G, i \in J).$$

Then
$$\Theta_\omega(f) = \sum_{i \in J} \int f\phi_i\, dx \quad (f \in C_c^\infty(G)),$$

the series being absolutely convergent (see [4 (b), p. 243]). Moreover, ϕ_i is analytic from [4 (q), Lemma 33]. Fix an integer $m \geqslant 0$ as in Lemma 7. Then
$$\int f\phi_i\, dx = c(\mathfrak{b})^{-m} \int f \cdot \Omega^m \phi_i\, dx = c(\mathfrak{b})^{-m} \int \Omega^m f \cdot \phi_i\, dx \quad (i \in J_\mathfrak{b}),$$

in the notation of Lemma 6. Hence, by the Schwartz inequality, we get
$$\left| \int f\phi_i\, dx \right| \leqslant c(\mathfrak{b})^{-m} \|\Omega^m f\| \, d(\pi)^{-\frac{1}{2}} \quad (i \in J_\mathfrak{b}),$$

where $\| \ \|$ denotes the usual norm in $L_2(G)$. This shows that
$$|\Theta_\omega(f)| \leqslant d(\omega)^{-\frac{1}{2}} \|\Omega^m f\| \sum_{\mathfrak{b} \in \mathcal{E}_K} c(\mathfrak{b})^{-m} \dim \mathfrak{H}_\mathfrak{b} \quad (f \in C_c^\infty(G)).$$

But
$$\sum_\mathfrak{b} c(\mathfrak{b})^{-m} \dim \mathfrak{H}_\mathfrak{b} \leqslant N \sum_\mathfrak{b} c(\mathfrak{b})^{-m} d(\mathfrak{b})^2 < \infty$$

from Lemma 7. Since
$$\nu(g) = \|\Omega^m g\| \quad (g \in C(G))$$

is a continuous seminorm on $C(G)$ (see Lemma 11), we conclude that Θ_ω is tempered. Moreover, (see § 25),
$$\Theta_{\omega,\mathfrak{b}} = \sum_{i \in J_\mathfrak{b}} \phi_i \quad (\mathfrak{b} \in \mathcal{E}_K)$$

and therefore $\Theta_{\omega,\mathfrak{b}} \in L_2(G)$.

§ 38. Proof of Lemma 65

As before, let r denote the right-regular representation of G on $L_2(G)$.

LEMMA 77. *Suppose $f \neq 0$ is a $\mathfrak{Z}$-finite function in $L_2(G)$ such that the right translates of f, under K, span a finite-dimensional space. Let V be the smallest closed subspace of $L_2(G)$ containing f, which is stable under r. Then*

$$V = \sum_{1 \leqslant i \leqslant p} U_i,$$

where U_i are mutually orthogonal closed subspaces of V, which are invariant and irreducible under r.

Let π denote the restriction of r on V. For any finite subset F of $\mathcal{E}_K$, define E_F as in § 25 and put $V_F = E_F V$. It is clear that F can be so chosen that $f \in V_F$. Define V^∞ as in Lemma 4. We know from [4 (q), Lemma 33] that f is analytic and therefore, by Theorem 1, $f = f * \alpha$ for some $\alpha \in C_c^\infty(G)$. This shows (Lemma 2) that $f \in V^\infty$. Moreover, a simple argument (see § 8) shows that $W = \pi(\mathfrak{G})f$ is dense in V. Finally we conclude from [4 (a), Theorem 1] that

$$W = \sum_{\mathfrak{b} \in \mathcal{E}_K} E_\mathfrak{b} W$$

and $W_F = E_F W$ has finite dimension. Since W_F is dense in V_F, it follows that $W_F = V_F$.

Let $U \neq \{0\}$ be any closed subspace of V stable under π. We claim that $E_F U \neq \{0\}$. For otherwise suppose $E_F U = \{0\}$. Let U' denote the orthogonal complement of U in V. Since π is unitary, U' is also stable under π and $f \in V_F \subset U'$. But this implies that $V \subset U'$ and therefore $U = \{0\}$, contradicting our hypothesis.

Let U_i ($1 \leqslant i \leqslant p$) be a finite set of mutually orthogonal, closed, nonzero subspaces of V, which are stable under π. Then since $U_i \cap V_F \neq \{0\}$, we conclude that $p \leqslant \dim V_F < \infty$. Therefore the required result follows immediately by assuming that p has the largest possible value.

Now we come to the proof of Lemma 65. We may assume that $f \neq 0$. Define V and U_i ($1 \leqslant i \leqslant p$) as in Lemma 77 and put $f_i = E_i f$, where E_i is the orthogonal projection of V on U_i. It is obvious that f_i is $\mathfrak{Z}$-finite as well as K-finite. Hence by [4 (q), Lemma 33], it is analytic. Moreover, $f = f_1 + \ldots + f_p$. Therefore it would be sufficient to prove Lemma 65 for each f_i. Thus we may assume that V is irreducible under π so that π is square-integrable. By Theorem 1, there exists an element $\alpha \in C_c^\infty(G)$ such that $f = \alpha * f$. Since f is K-finite, we can obviously assume that α is also K-finite. But then

$$f(x) = (\check{\alpha}, \pi(x)f) = (E\check{\alpha}, \pi(x)f) \quad (x \in G)$$

where $\check{\alpha}$ is defined as in Lemma 69 and E is the orthogonal projection of $L_2(G)$ on V. The required inequality is now an immediate consequence of Theorem 10 and Lemma 76.

§ 39. The existence of the discrete series

Henceforward we assume, for convenience, that G is acceptable.

THEOREM 13. *G has a discrete series if and only if* rank G = rank K.

Suppose $\mathcal{E}_d \neq \emptyset$. Fix $\omega \in \mathcal{E}_d$ and choose $\mathfrak{b} \in \mathcal{E}_K$ such that $f = \Theta_{\omega, \mathfrak{b}} \neq 0$. Then it follows from Lemma 76 and Corollary 2 of Lemma 65 that rank G = rank K.

Conversely suppose rank G = rank K. Fix $\lambda \in L'$ and choose $\mathfrak{b} \in \mathcal{E}_K$ such that $\Theta_{\lambda, \mathfrak{b}} \neq 0$. Then by Theorem 12, the function $f = \Theta_{\lambda, \mathfrak{b}}$ satisfies the hypotheses of Lemma 77. Let π_i denote the restriction of r on U_i $(1 \leqslant i \leqslant p)$, in the notation of Lemma 77. Then π_i is square-integrable and therefore $\mathcal{E}_d \neq \emptyset$.

§ 40. The characters of the discrete series

In view of Theorem 13, we shall now assume that rank G = rank K and use the notation of §§ 36, 37.

For any $\lambda \in L'$, let $\mathcal{E}_d(\lambda)$ denote the set of all $\omega \in \mathcal{E}_d$ such that $\chi_\omega = \chi_\lambda$. The following result is an immediate consequence of Lemmas 74 and 76.

LEMMA 78. $\mathcal{E}_d = \bigcup_{\lambda \in L'} \mathcal{E}_d(\lambda)$. *Moreover, for any $\omega \in \mathcal{E}_d(\lambda)$ $(\lambda \in L')$, there exist unique complex numbers $c_s(\omega)$ $(s \in W(\lambda))$ such that $c_{ts}(\omega) = c_s(\omega)$ $(t \in W_G)$ and*

$$\Theta_\omega = [W_G]^{-1} \sum_{s \in W(\lambda)} \varepsilon(s) c_s(\omega) \Theta_{s\lambda}.$$

For any $\omega \in \mathcal{E}$, define the analytic function Φ_ω on B (see § 36) by

$$\Phi_\omega(b) = \Delta(b) \Theta_\omega(b) \quad (b \in B')$$

and put

$$F_f(b) = \Delta(b) \int_G f(b^x) dx \quad (b \in B')$$

for $f \in C(G)$ as in § 18. Put $G_B = (B')^G$ and let db denote the normalized Haar measure on B. Then G_B is open in G and (see Lemma 91)

$$\int_G \alpha(x) dx = (-1)^m [W_G]^{-1} \int_B \Delta F_\alpha db \quad (\alpha \in C_c^\infty(G_B)),$$

where $m = \tfrac{1}{2} \dim G/B$.

LEMMA 79. *Let f be a $\mathfrak{Z}$-finite function in $C(G)$. Then*

$$\Theta_\omega(f) = (-1)^m [W_G]^{-1} \int_B F_f \Phi_\omega db \quad (\omega \in \mathcal{E}_d).$$

This follows immediately from Lemmas 64 and 76 and the corollary of Theorem 7.

LEMMA 80. *Let π be a square-integrable representation of G on $\mathfrak{H}$ and ω its class in $\mathcal{E}_d$. Fix two K-finite elements ϕ, $\psi \in \mathfrak{H}$ and put*

$$f(x) = (\phi, \pi(x)\,\psi) \quad (x \in G).$$

Then

$$F_f = d(\omega)^{-1}(\phi, \psi)\,\Phi_\omega$$

on B'.

First observe that $f \in \mathcal{C}(G)$ from Corollary 1 of Lemma 65 and therefore F_f is defined. Now fix $\alpha \in C_c^\infty(G)$ and consider the operator

$$\pi(\alpha) = \int \alpha(x)\,\pi(x)\,dx.$$

Then $\Theta_\omega(\alpha) = \operatorname{tr}\pi(\alpha)$ and the argument of [4 (d), 576] shows that

$$\int_G dx \int_G \alpha(y)\,f(y^x)\,dy = \int_G (\phi, \pi(x)\,\pi(\alpha)\,\pi(x^{-1})\,\psi)\,dx = d(\omega)^{-1}(\phi, \psi)\,\Theta_\omega(\alpha).$$

Now suppose $\alpha \in C_c^\infty(G_B)$. Then we claim that

$$\int |\alpha(y)\,f(y^x)|\,dx\,dy < \infty.$$

For we can choose $\alpha_0 \in C_c^\infty(G_B)$ such that $\alpha_0 \geqslant |\alpha|$. Then

$$\int |\alpha(y)\,f(y^x)|\,dx\,dy \leqslant \int \alpha_0(y)\,|f(y^x)|\,dx\,dy = [W_G]^{-1}\int_B |F_{\alpha_0}(b)|\left\{|\Delta(b)|\int_G |f(b^y)|\,dy\right\}db.$$

Since $|F_{\alpha_0}|$ is bounded on B' (Lemma 26), our assertion follows from Theorem 5.

Therefore we conclude from Fubini's theorem that

$$d(\omega)^{-1}(\phi, \psi)\,\Theta_\omega(\alpha) = \int \alpha(y)\,dy \int f(y^x)\,dx \quad (\alpha \in C_c^\infty(G_B)).$$

But it is clear that

$$\int \alpha(y)\,dy \int f(y^x)\,dx = (-1)^m[W_G]^{-1}\int_B F_\alpha F_f\,db$$

and

$$\Theta_\omega(\alpha) = \int \alpha\,\Theta_\omega\,dx = (-1)^m[W_G]^{-1}\int_B F_\alpha\,\Phi_\omega\,db.$$

This shows that

$$\int F_\alpha F_f db = d(\omega)^{-1}(\phi, \psi) \int F_\alpha \Phi_\omega db \quad (\alpha \in C_c^\infty(G_B)).$$

On the other hand, it is easy to verify (see [4 (o), § 20]) that there exists a C^∞ function u on G_B such that

$$\Delta(b) u(b^x) = F_f(b) - d(\omega)^{-1}(\phi, \psi) \Phi_\omega(b) \quad (b \in B', \ x \in G).$$

Then if follows from the above result that

$$\int \alpha u dx = 0 \quad (\alpha \in C_c^\infty(G_B))$$

and therefore $u = 0$. This implies the assertion of the lemma.

For any $\omega \in \mathcal{E}_d$, we define a subspace $\mathfrak{H}_\omega$ of $L_2(G)$ as follows. Fix $\pi \in \omega$ and let U be the representation space of π. Then $\mathfrak{H}_\omega$ is the smallest closed subspace of $L_2(G)$ containing all functions f of the form

$$f(x) = (\phi, \pi(x)\psi) \quad (x \in G),$$

where $\phi, \psi \in U$. It is clear that this definition is independent of the particular choice of π and $\mathfrak{H}_\omega$ is stable under both left and right translations of G. Put $C_\omega(G) = \mathfrak{H}_\omega \cap C(G)$. Then it follows from Lemma 11 that $C_\omega(G)$ is closed in $C(G)$.

THEOREM 14. $C_\omega(G)$ *is dense in* $\mathfrak{H}_\omega$ *and*

$$F_f = d(\omega)^{-1}f(1)\Phi_\omega$$

for $f \in C_\omega(G)$ *and* $\omega \in \mathcal{E}_d$.

Choose an orthonormal base ψ_i $(i \in J)$ for U as in § 25 so that ψ_i $(i \in J_\mathfrak{b})$ is a base for $U_\mathfrak{b}$ $(\mathfrak{b} \in \mathcal{E}_K)$ and put

$$f_{ij}(x) = (\psi_i, \pi(x)\psi_j) \quad (x \in G).$$

Then it follows from Corollary 1 of Lemma 65 that $f_{ij} \in C_\omega(G)$. This shows that $C_\omega(G)$ is dense in $\mathfrak{H}_\omega$.

Let V be the set of all $f \in C_\omega(G)$ such that

$$F_f = d(\omega)^{-1}f(1)\Phi_\omega.$$

Then it is clear that V is a closed subspace[1] of $C_\omega(G)$. Hence it would be enough to show that V is dense in $C_\omega(G)$.

---

[1] The topology of $C_\omega(G)$ is the one inherited from $C(G)$.

Fix $f_0 \in C_\omega(G)$. Then in view of Lemma 16, it would be enough to prove that $f = \alpha_{\mathfrak{d}_1} \times f_0 \times \alpha_{\mathfrak{d}_2} \in V$ for $\mathfrak{d}_1,\ \mathfrak{d}_2 \in \mathcal{E}_K$. Since $J_\mathfrak{d}$ is a finite set for every $\mathfrak{d} \in \mathcal{E}_K$, it is clear that f is a finite linear combination of f_{ij} $(i,\ j \in J)$ and therefore $f \in V$ from Lemma 80. This proves the theorem.

Remark. The above proof shows that f_{ij} $(i,\ j \in J)$ span a dense subspace of $C_\omega(G)$. Moreover, $f_{ii}(1) = 1$ for $i \in J$.

For any $w \in \mathcal{E}$, let ω^* denote the class contragredient to ω. It is clear that $\Theta_{\omega*} = \operatorname{conj} \Theta_\omega$ as functions on G, and ω^* is discrete whenever ω is discrete.

LEMMA 81. *Let $\omega,\ \omega'$ be two elements in $\mathcal{E}_d$. Then*

$$\Theta_\omega(f) = \begin{cases} d(\omega)^{-1} f(1) & \text{if } \omega' = \omega^*, \\ 0 & \text{otherwise,} \end{cases}$$

for $f \in C_{\omega'}(G)$.

We keep to the above notation. Then it follows easily from Lemma 19 that

$$\Theta_\omega(f) = \sum_i \int f f_{ii}\, dx = \sum_i (\operatorname{conj} f_{ii},\ f)$$

for any K-finite function f in $C_{\omega'}(G)$. Now $\operatorname{conj} f_{ii} \in \mathfrak{H}_{\omega*}$ and if $\omega^* \neq \omega'$, we conclude from the Schur orthogonality relations [4 (d), Theorem 1] that $\mathfrak{H}_{\omega*}$ is orthogonal to $\mathfrak{H}_{\omega'}$ and therefore $\Theta_\omega(f) = 0$. Since K-finite functions are dense in $C_{\omega'}(G)$ by Lemma 16, we get the required assertion in this case.

Now suppose $\omega^* = \omega'$ and $f = \operatorname{conj} f_{ij}$ $(i,\ j \in J)$. Then it follows again from the Schur orthogonality relations that

$$\Theta_\omega(f) = d(\omega)^{-1} f(1).$$

But we have seen above that $\operatorname{conj} f_{ij}$ $(i,\ j \in J)$ span a dense subspace of $C_{\omega*}(G)$ and so the assertion of the lemma is now obvious.

As before, let db denote the normalized Haar measure of B.

COROLLARY 1.

$$\int_B (\operatorname{conj} \Phi_\omega) \Phi_{\omega'}\, db = \begin{cases} [W_G] & \text{if } \omega = \omega', \\ 0 & \text{otherwise.} \end{cases}$$

Fix $f \in C_\omega(G)$ such that $f(1) \neq 0$ (see the remark after Theorem 14) and put $g = \operatorname{conj} f$. Then

$$\Theta_{\omega'}(g) = (-1)^m [W_G]^{-1} \int_B F_g \Phi_{\omega'}\, db$$

from Lemma 79. But $F_g = (-1)^m \operatorname{conj} F_f$

and therefore our assertion follows from Theorem 14 and Lemma 81.

Define the number $c > 0$ by the relation (see Lemma 38)

$$F_f(1; \omega) = (-1)^q \, cf(1) \quad (f \in \mathcal{C}(G)).$$

Then c has the same value as in Theorem 8 (see [4 (q), § 15]). Fix $\lambda \in L'$ and for any $\omega \in \mathcal{E}_d(\lambda)$, define $c_s(\omega) \, (s \in W(\lambda))$ as in Lemma 78. Then

$$\Theta_\omega = \sum_{s \in W_G \backslash W(\lambda)} \varepsilon(s) \, c_s(\omega) \, \Theta_{s\lambda},$$

where the sum is over a complete system of representatives.

COROLLARY 2. *Let* $\omega \in \mathcal{E}_d(\lambda)$. *Then*

$$d(\omega) = (-1)^q \, c^{-1} \, \varpi(\lambda) \sum_{s \in W(\lambda)} c_s(\omega)$$

and $$\sum_{s \in W_G \backslash W(\lambda)} |c_s(\omega)|^2 = 1.$$

We know from Theorem 14 that

$$(-1)^q cf(1) = F_f(1; \varpi) = d(\omega)^{-1} f(1) \Phi_\omega(1; \varpi) \quad (f \in \mathcal{C}_\omega(G)),$$

and the first relation is an immediate consequence of this fact. The second follows by putting $\omega' = \omega$ in Corollary 1 above.

Let $\mathfrak{T}(\lambda) \ (\lambda \in L')$ be the space of all tempered and invariant distributions Θ on G such that $z\Theta = \chi_\lambda(z)\Theta (z \in \mathfrak{Z})$.

THEOREM 15. *Fix* $\lambda \in L'$. *Then* $\Theta_\omega \ (\omega \in \mathcal{E}_d(\lambda))$ *form a base for* $\mathfrak{T}(\lambda)$ *over* $\mathbb{C}$ *and*

$$[\mathcal{E}_d(\lambda)] = \dim \mathfrak{T}(\lambda) = [W(\lambda)][W_G]^{-1}.$$

Moreover, $$\sum_{\omega \in \mathcal{E}_d(\lambda)} d(\omega) \Theta_\omega = (-1)^q c^{-1} \sum_{s \in W(\lambda)} \varpi(s\lambda) \Theta_{s\lambda},$$

where c and q have the same meaning as in Theorem 8.

We know from Lemma 76 that $\Theta_\omega \ (\omega \in \mathcal{E}_d(\lambda))$ lie in $\mathfrak{T}(\lambda)$ and from Lemma 81 that they are linearly independent. Now fix $\Theta \in \mathfrak{T}(\lambda)$. We have to show that Θ is a linear combination of $\Theta_\omega \ (\omega \in \mathcal{E}_d(\lambda))$. Define analytic functions Φ and Φ_0 on B as follows.

$$\Phi(b) = \Delta(b) \Theta(b) \quad (b \in B')$$

and
$$\Phi_0 = \Phi - \sum_{\omega \in \mathcal{E}_d(\lambda)} c(\omega) \Phi_\omega,$$

where
$$c(\omega) = [W_G]^{-1} \int_B \Phi \operatorname{conj} \Phi_\omega \, db.$$

Then it is clear from Corollary 1 of Lemma 81 that Φ_0 is orthogonal to Φ_ω ($\omega \in \mathcal{E}_d(\lambda)$) in $L_2(B)$. Put

$$\Theta_0 = \Theta - \sum_{\omega \in \mathcal{E}_d(\lambda)} c(\omega) \Theta_\omega.$$

We claim that $\Theta_0 = 0$. In view of Lemma 73, it would be enough to verify that $\Theta_0(f) = 0$ for any K-finite function $f \in C_{-\lambda}(G)$. We may obviously assume that $f \neq 0$. Define V and U_i ($1 \leqslant i \leqslant p$) as in Lemma 77 and let E_i denote the orthogonal projection of V on U_i. Put $f_i = E_i f$ ($1 \leqslant i \leqslant p$). Then it follows from Corollary 1 of Lemma 65 that $f_i \in C_{-\lambda}(G)$. Therefore since

$$\Theta_0(f) = \sum_{1 \leqslant i \leqslant q} \Theta_0(f_i),$$

it would be enough to consider the case when V is irreducible under r.

Let π denote the restriction of r on V and ω the class in $\mathcal{E}_d$ such that $\pi \in \omega^*$. Then $\omega \in \mathcal{E}_d(\lambda)$ and, as we have seen in § 38, there exists an element $\alpha \in C_c^\infty(G)$ such that

$$f(x) = (\alpha, \pi(x)f) = (E\alpha, \pi(x)f) \quad (x \in G).$$

(Here E denotes the orthogonal projection of $L_2(G)$ on V.) This shows that $f \in C_{\omega^*}(G)$. On the other hand, it follows from Lemma 64 and the corollary of Theorem 7, that

$$\Theta_0(f) = (-1)^m [W_G]^{-1} \int_B F_f \Phi_0 \, db.$$

Therefore we conclude from Theorem 14 and the definition of Φ_0 that $\Theta_0(f) = 0$.

Let $s_1, s_2, \ldots, s_p$ be a complete set of representatives of $W_G \backslash W(\lambda)$ in $W(\lambda)$. Then by Lemma 74, the distributions $\Theta_{s_i\lambda}$ ($1 \leqslant i \leqslant \lambda$) also form a base for $\mathfrak{T}(\lambda)$ and therefore

$$[W(\lambda)][W_G]^{-1} = p = \dim \mathfrak{T}(\lambda) = [\mathcal{E}_d(\lambda)].$$

Now put
$$\Theta = \sum_{\omega \in \mathcal{E}_d(\lambda)} d(\omega) \Theta_\omega - (-1)^q c^{-1} \sum_{s \in W(\lambda)} \varpi(s\lambda) \Theta_{s\lambda}.$$

We have to show that $\Theta = 0$. In view of Lemma 73, it would be enough to verify that $\Theta(f) = 0$ for any K-finite function $f \in C_{-\lambda}(G)$. By the argument given above, we are reduced to the case when $f \in C_{\omega^*}(G)$ for some $\omega \in \mathcal{E}_d(\lambda)$. But then $\Theta(f) = 0$ from Lemma 81 and Corollary 1 of Lemma 69. This completes the proof of Theorem 15.

§ 41. Explicit determination of these characters

Put $\varepsilon(\lambda) = \text{sign } \varpi(\lambda)$ for $\lambda \in L'$.

THEOREM 16.[1] *For any $\lambda \in L'$, there exists a unique element $\omega(\lambda) \in \mathcal{E}_d$ such that* $\Theta_{\omega(\lambda)} = (-1)^q \varepsilon(\lambda) \Theta_\lambda$. *The mapping $\lambda \to \omega(\lambda)$ of L' into $\mathcal{E}_d$ is surjective and*

$$d(\omega(\lambda)) = c^{-1}[W_G] |\varpi(\lambda)|$$

in the notation of Theorem 15. Finally $\omega(\lambda_1) = \omega(\lambda_2) (\lambda_1, \lambda_2 \in L')$, if and only if λ_1, λ_2 are conjugate under W_G.

We begin by proving the surjectivity first. Fix $\omega \in \mathcal{E}_d$. Then by Lemma 78, Φ_ω is a finite linear combination of the characters of B. Introduce an order on $\mathfrak{F}$ and let λ be the highest element in L such that

$$c_0 = \int_B \Phi_\omega \text{ conj } \xi_\lambda \, db \neq 0.$$

(As before, ξ_λ has the same meaning as in [4 (p), § 24].) Then $\lambda \in L'$.

LEMMA 82. $\Theta_\omega = (-1)^q \varepsilon(\lambda) \Theta_\lambda$.

For the proof of this lemma, we may, by going over to a finite covering group of G, assume that K is also acceptable (see [4 (o), § 18]). Let P be the set of all positive roots of $(\mathfrak{g}, \mathfrak{b})$ and P_0, P_+ respectively the sets of all compact and singular roots in P (see [4 (n), § 4]). Put

$$\varrho = \tfrac{1}{2} \sum_{\alpha \in P} \alpha, \qquad \varrho_0 = \tfrac{1}{2} \sum_{\alpha \in P_0} \alpha, \qquad \varrho_+ = \tfrac{1}{2} \sum_{\alpha \in P_+} \alpha.$$

Then ϱ, ϱ_0, ϱ_+ are all in L and $\varrho = \varrho_0 + \varrho_+$. Hence we can define two analytic functions Δ_0 and Δ_+ as follows.

$$\Delta_0(\exp H) = \prod_{\alpha \in P_0} (e^{\alpha(H)/2} - e^{-\alpha(H)/2}), \qquad \Delta_+(\exp H) = \prod_{\alpha \in P_+} (e^{\alpha(H)/2} - e^{-\alpha(H)/2}) \quad (H \in \mathfrak{b}),$$

so that $\Delta = \Delta_0 \Delta_+$. It is clear that

$$\Delta_0(b^s) = \varepsilon(s) \Delta_0(b), \quad \Delta_+(b^s) = \Delta_+(b) \quad (b \in B)$$

for $s \in W_G$.

Let db and dk denote the normalized Haar measures on B and K respectively. Fix a function $\alpha \in C_c^\infty(G)$ such that α is invariant under right translations by K and

$$\int \alpha(x) \, dx = 1.$$

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[1] Cf. [4 (c), p. 40] and [4(d), Theorem 4].

For any $\beta \in C_c^\infty(B')$, we can define $f_\beta \in C_c^\infty(G_B)$ (see [4 (o), § 20]) by

$$f_\beta(b^x) = \alpha(x)\,\Delta(b)^{-1} \sum_{s \in W_G} \varepsilon(s)\,\beta(b^s) \quad (x \in G,\ b \in B').$$

Similarly define $g_\beta \in C^\infty(K)$ by

$$g_\beta(b^k) = \Delta_+(b)\,\Delta_0(b)^{-1} \sum_{s \in W_G} \varepsilon(s)\,\beta(b^s) \quad (k \in K,\ b \in B').$$

Fix $\pi \in \omega$ and let U be the representation space of π. Define $E_\mathfrak{b}$ and $U_\mathfrak{b}\,(\mathfrak{b} \in \mathcal{E}_K)$ as usual and put

$$\phi_\mathfrak{b}(x) = \mathrm{tr}\,(E_\mathfrak{b}\,\pi(x)\,E_\mathfrak{b}) \quad (x \in G).$$

Then $\phi_\mathfrak{b}(k) = n(\mathfrak{b})\,\chi_\mathfrak{b}(k)\ (k \in K)$, where $n(\mathfrak{b})$ is a nonnegative integer and $\chi_\mathfrak{b}$ is the character of $\mathfrak{b}$. Moreover, as we have seen in § 25, there exists an integer $N \geqslant 1$ such that $n(\mathfrak{b}) \leqslant Nd(\mathfrak{b})\ (\mathfrak{b} \in \mathcal{E}_K)$.

Put $m = \frac{1}{2}\dim G/B$, $m_0 = \frac{1}{2}\dim K/B$ and

$$T_\beta = \int_K g_\beta(k)\,\pi(k)\,dk, \quad \pi(f_\beta) = \int_G f_\beta(x)\,\pi(x)\,dx \quad (\beta \in C_c^\infty(B')).$$

Then $m = m_0 + q$ and it is clear that

$$\pi(f_\beta) = (-1)^m[W_G]^{-1} \int_B \Delta(b) \sum_{s \in W_G} \varepsilon(s)\,\beta(b^s)\,db \int_G \alpha(x)\,\pi(b^x)\,dx$$

$$= (-1)^m \int_B \Delta(b)\,\beta(b)\,db \int_G \alpha(x)\,\pi(b^x)\,dx$$

since $\alpha(xk) = \alpha(x)\ (k \in K)$. Similarly

$$T_\beta = (-1)^{m_0}[W_G]^{-1} \int_B \Delta_0(b)^2 g_\beta(b)\,db \int_K \pi(b^k)\,dk = (-1)^{m_0} \int_B \Delta(b)\,\beta(b)\,db \int_K \pi(b^k)\,dk.$$

Therefore if follows that

$$\pi(f_\beta) = (-1)^q \int_G \alpha(x)\,\pi(x)\,T_\beta\,\pi(x^{-1})\,dx.$$

Now, by [4 (c), Lemma 24], the operator T_β is summable and therefore it follows easily that

$$\Theta_\omega(f_\beta) = \mathrm{tr}\,\pi(f_\beta) = (-1)^q\,\mathrm{tr}\,T_\beta = (-1)^q \sum_{\mathfrak{b} \in \mathcal{E}_K} n(\mathfrak{b}) \int_K g_\beta(k)\,\chi_\mathfrak{b}(k)\,dk.$$

On the other hand, $\quad \Theta_\omega(f_\beta) = \int \Theta_\omega f_\beta\,dx = (-1)^m \int_B \Phi_\omega\,\beta\,db.$

Therefore we have obtained the following result.

LEMMA 83.   *Put $\eta_{\mathfrak{b}}(b) = \Delta_0(b) \chi_{\mathfrak{b}}(b)$ ($\mathfrak{b} \in \mathcal{E}_K$, $b \in B$).   Then*

$$\int_B \Phi_\omega \beta \, db = \sum_{\mathfrak{b} \in \mathcal{E}_K} n(\mathfrak{b}) \int_B \Delta_+ \beta \, \eta_{\mathfrak{b}} \, db$$

for $\beta \in C_c^\infty(B')$, the series being absolutely convergent.

Define $\mathfrak{K}$ and Ω as in the proof of Lemma 5 and let $\mathcal{Z}_K$ denote the center of $\mathfrak{K}$. Then $\Omega \in \mathcal{Z}_K$. Let u be the image of Ω in $\mathfrak{S}(\mathfrak{b}_c)$ under the canonical isomorphism (see [4 (e), Lemma 19]) of $\mathcal{Z}_K$ into $\mathfrak{S}(\mathfrak{b}_c)$. Then it follows from [4 (e), Theorem 2] that

$$u\eta_{\mathfrak{b}} = c(\mathfrak{b}) \eta_{\mathfrak{b}} \quad (\mathfrak{b} \in \mathcal{E}_K)$$

in the notation of Lemma 6. Hence

$$\int \beta \eta_{\mathfrak{b}} \, db = c(\mathfrak{b})^{-p} \int u^{*p} \beta \cdot \eta_{\mathfrak{b}} \, db \quad (\beta \in C^\infty(B)),$$

where u^* is the adjoint of the differential operator u and p any positive integer. It follows from Weyl's formula for $\chi_{\mathfrak{b}}$ that $|\eta_{\mathfrak{b}}| \leqslant [W_G]$. Therefore

$$\sum_{\mathfrak{b} \in \mathcal{E}_K} n(\mathfrak{b}) \left| \int \beta \eta_{\mathfrak{b}} \, db \right| \leqslant N[W_G] \sup |u^{*p} \beta| \sum_{\mathfrak{b} \in \mathcal{E}_K} c(\mathfrak{b})^{-p} d(\mathfrak{b}),$$

and so we conclude from Lemma 7 that there exists a distribution S_0 on B such that

$$S_0(\beta) = \sum_{\mathfrak{b} \in \mathcal{E}_K} n(\mathfrak{b}) \int_B \beta \eta_{\mathfrak{b}} \, db \quad (\beta \in C^\infty(B)).$$

Put $$S = \Phi_\omega - \Delta_+ S_0.$$

Then it follows from Lemma 83 that $S = 0$ on B'. Therefore since B is compact, we can choose (see [4 (m), Lemma 21]) an integer $p \geqslant 0$ such that $\Delta^p S = 0$. This means that

$$\int \Phi_\omega \Delta^p \beta \, db = \sum_{\mathfrak{b} \in \mathcal{E}_K} n(\mathfrak{b}) \int \beta \Delta^p \Delta_+ \eta_{\mathfrak{b}} \, db$$

for $\beta \in C^\infty(B)$. Now put $\beta = \text{conj } \xi_{\lambda + p\varrho}$. Then it is clear that the left side is equal to c_0. Moreover, we know from Weyl's formula that $\Delta^p \Delta_+ \eta_{\mathfrak{b}}$ is a finite linear combination of characters of B with coefficients in $\mathbf{Z}$. Therefore, since $n(\mathfrak{b})$ is an integer, we conclude that $c_0 \in \mathbf{Z}$. Since $c_0 \neq 0$, this shows that $|c_0| \geqslant 1$.

On the other hand, let $s_1 = 1$, s_2, ..., s_r be a complete set of representatives in $W(\lambda)$ for $W_G \backslash W(\lambda)$. Then, by Lemma 78 there exist unique complex numbers c_i such that

$$\Theta_\omega = \sum_{1 \leqslant i \leqslant r} \varepsilon(s_i) \, c_i \, \Theta_{s_i \lambda}.$$

It is obvious that $c_1 = c_0$ and we know from Corollary 2 of Lemma 81 that

$$\sum_{1 \leqslant i \leqslant r} |c_i|^2 = 1.$$

Therefore $c_1 = c_0 = \pm 1$ and $c_i = 0$ for $i \geqslant 2$. This shows that

$$\Theta_\omega = c_0 \Theta_\lambda.$$

By Theorem 15, there exist exactly r distinct elements $\omega_1 = \omega$, ω_2, ..., ω_r in $\mathcal{E}_d(\lambda)$. For each i, we can, by the above proof, choose $s_i \in W(\lambda)$ ($s_1 = 1$) and a number $c_i = \pm 1$ such that

$$\Theta_{\omega_i} = c_i \Theta_{s_i \lambda} \quad (1 \leqslant i \leqslant r).$$

Then it follows from the linear independence of Θ_{ω_i} (Lemma 81) that $s_1, s_2, ..., s_r$ form a complete system of representatives of $W_G \backslash W(\lambda)$. Therefore

$$\sum_i d(\omega_i) \Theta_{\omega_i} = (-1)^q c^{-1} [W_G] \sum_i c_i \, \varpi(s_i \lambda) \Theta_{\omega_i}$$

from Theorem 15 and this shows that

$$d(\omega_i) = (-1)^q c^{-1} [W_G] |\varpi(\lambda)| c_i \varepsilon(s_i \lambda) \quad (1 \leqslant i \leqslant r).$$

But $d(\omega_i) > 0$ and so we conclude that $c_i = (-1)^q \varepsilon(s_i \lambda)$ and

$$d(\omega_i) = c^{-1} [W_G] |\varpi(\lambda)|.$$

Hence in particular $c_0 = c_1 = (-1)^q \varepsilon(\lambda)$ and this proves Lemma 82.

We now come to Theorem 16. Since a class $\omega \in \mathcal{E}$ is completely determined by its character (see [4 (b), p. 250]), the uniqueness of $\omega(\lambda)$ ($\lambda \in L'$) is obvious. Moreover since $\Theta_{s\lambda} = \varepsilon(s) \Theta_\lambda$ ($s \in W_G$) from the definition of Θ_λ (see [4 (p), Theorem 3]), it follows from the linear independence of the characters, that $\omega(\lambda_1) = \omega(\lambda_2)$ ($\lambda_1, \lambda_2 \in L'$) if and only if λ_1, λ_2 are conjugate under W_G. Now fix $\lambda \in L'$ and let $r = \dim \mathfrak{T}(\lambda)$. Then by Theorem 15, there are exactly r distinct elements $\omega_1, \omega_2, ..., \omega_r$ in $\mathcal{E}_d(\lambda)$. Moreover, from the above proof, we can choose a complete set of representatives $(s_1, s_2, ..., s_r)$ for $W_G \backslash W(\lambda)$ such that

$$\Theta_{\omega_i} = (-1)^q \varepsilon(s_i \lambda) \Theta_{s_i \lambda} \quad (1 \leqslant i \leqslant r).$$

We may assume that $s_1 \in W_G$ and therefore

$$\Theta_{\omega_1} = (-1)^q \varepsilon(s_1 \lambda) \Theta_{s_1 \lambda} = (-1)^q \varepsilon(\lambda) \Theta_\lambda.$$

This shows that $\omega_1 = \omega(\lambda)$. We have already seen that $d(\omega_1) = c^{-1}[W_G] |\omega(\lambda)|$ and therefore the proof of Theorem 16 is now complete.

Theorem 16 shows that

$$c^{-1}(-1)^q \sum_{\lambda \in L} \varpi(\lambda) \Theta_\lambda = \sum_{\omega \in \mathcal{E}_d} d(\omega) \Theta_\omega$$

and we know (see [4 (d), § 5]) that this distribution represents the contribution of the discrete series to the Plancherel formula of G.

Part IV. Some inequalities and their consequences

§ 42. Proof of the inequalities

Let us use the notation of § 14 and put $\mathfrak{a}_2 = \mathfrak{h} \cap \mathfrak{p}$, $\mathfrak{n}_2 = \mathfrak{n}_\mathfrak{h}$, $\mathfrak{a}_1 = \mathfrak{a} \cap [\mathfrak{m}, \mathfrak{m}]$, $\mathfrak{k}_1 = \mathfrak{k} \cap \mathfrak{m}$, $\mathfrak{n}_1 = \mathfrak{n} \cap \mathfrak{m}$ and $\mathfrak{m}_1 = \mathfrak{k}_1 + \mathfrak{a}_1 + \mathfrak{n}_1$. We denote the analytic subgroup of G corresponding to a subalgebra of $\mathfrak{g}$ by the corresponding capital latin letter e.g. A_2 and N correspond to $\mathfrak{a}_2$ and $\mathfrak{n}$ respectively. Then $G = KAN$ and $M_1 = K_1 A_1 N_1$ are the Iwasawa decompositions of G and M_1 respectively. For any $x \in G$, let $\varkappa(x)$ and $H(x)$ denote the unique elements $k \in K$ and $H \in \mathfrak{a}$ respectively, such that $x = k \exp H \cdot n$ $(n \in N)$. Let $H_i(x)$ denote the component of $H(x)$ in $\mathfrak{a}_i$ $(i = 1, 2)$ so that $H(x) = H_1(x) + H_2(x)$.

We fix orders in the duals of the real vector spaces $\mathfrak{a}_2$, $\mathfrak{a}$ and $\mathfrak{h}^* = \mathfrak{h} \cap \mathfrak{p} + (-1)^{\frac{1}{2}} \mathfrak{h} \cap \mathfrak{k}$ and assume that they are compatible for the pairs $(\mathfrak{a}, \mathfrak{a}_2)$ and $(\mathfrak{h}^*, \mathfrak{a}_2)$. Let P denote the set of positive roots of $(\mathfrak{g}, \mathfrak{h})$ and Σ the set of positive roots of $(\mathfrak{g}, \mathfrak{a})$. Let P_1 and Σ_1 be the sets of those elements in P and Σ respectively, whose restrictions on $\mathfrak{a}_2$ are zero. We denote by P_2 and Σ_2 the complements of P_1 and Σ_1 in P and Σ respectively.

Put $M_{1\mathfrak{p}} = \exp(\mathfrak{m}_1 \cap \mathfrak{p})$ so that $G = K M_{1\mathfrak{p}} A_2 N_2$ (see [4 (g), Lemma 11]). Fix $x \in G$ and let $x = kman$ $(k \in K, m \in M_{1\mathfrak{p}}, a \in A_2, n \in N_2)$. Then k, m, a, n are uniquely determined. Put $\mu(x) = m$. Since M_1 and A_2 commute, it is easy to verify that

$$\varkappa(x) = k\varkappa(m), \quad H_2(x) = \log a, \quad H_1(x) = H(m).$$

Define ϱ and $\Xi_1 = \Xi_M$ as in § 14. Since $\mathfrak{m}$ normalizes $\mathfrak{n}_2$, it is clear that $\operatorname{tr} (\operatorname{ad} X)_{\mathfrak{n}_2} = 0$ for $X \in [\mathfrak{m}, \mathfrak{m}]$. Therefore since $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$, we conclude that

$$\varrho(H) = \tfrac{1}{2} \operatorname{tr} (\operatorname{ad} H)_{\mathfrak{n}_1} \quad (H \in \mathfrak{a}_1).$$

LEMMA 84. *Let $d_1 k$ denote the normalized Haar measure on K_1. Then*

$$\int_{K_1} e^{-\varrho(H(xk))} d_1 k = e^{-\varrho(H_2(x))} \Xi_1(\mu(x)) \quad (x \in G).$$

Since K_1 normalizes N_2, it is easy to see that

$$H(xk) = H_1(xk) + H_2(xk) = H(\mu(xk)) + H_2(x) = H(\mu(x) k) + H_2(x) \quad (k \in K_1).$$

Hence our assertion follows from the fact (see [4 (q), Lemma 31]) that

$$\Xi_1(m) = \int_{K_1} e^{-\varrho(H(mk))} d_1 k \quad (m \in M_1).$$

As usual let dk denote the normalized Haar measure on K. Then the following result is an immediate consequence of Lemma 84 and [4 (q), Lemma 31].

COROLLARY.

$$\int_K e^{-\varrho(H_2(xk))} \Xi_1(\mu(xk)) dk = \Xi(x) \quad (x \in G).$$

Put $\bar{N} = \theta(N)$, $\bar{N}_i = \theta(N_i)$ $(i = 1, 2)$ and

$$\beta(H) = \inf_{\alpha \in \Sigma_2} \alpha(H) \quad (H \in \mathfrak{a}).$$

LEMMA 85. $\varrho(H_2(\bar{n})) \geqslant 0$ *and* $\varrho(H(\bar{n})) \geqslant 0$ *for* $\bar{n} \in \bar{N}$. *Moreover, if* $a \in A_2$ *and* $\beta(\log a) \geqslant 0$, *we have*

$$\exp \varrho(H_2(\bar{n}^a)) \leqslant 1 + \exp \left\{ -\tfrac{1}{2}\beta(\log a) + \varrho(H_2(\bar{n})) \right\} \quad (\bar{n} \in \bar{N})$$

and
$$\exp \varrho(H(\bar{n}^a)) \leqslant 1 + \exp \left\{ -\beta(\log a) + \varrho(H(\bar{n})) \right\} \quad (\bar{n} \in \bar{N}_2).$$

Let Z be the center of G. Then, for the purpose of this lemma, we can obviously replace G by G/Z. Hence we may agree to subscribe to the assumptions and conventions of [4 (j), p. 244].

For any linear function λ on $\mathfrak{h}_c$, define $H_\lambda \in \mathfrak{h}_c$ as usual by the condition

$$\text{tr} (\text{ad } H \text{ ad } H_\lambda) = \lambda(H) \quad (H \in \mathfrak{h}_c).$$

Also put $\langle \lambda_1, \lambda_2 \rangle = \lambda_1(H_{\lambda_2})$ for two such functions λ_1, λ_2. Let J_P denote the set of all λ such that $2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle$ is a nonnegative integer for every $\alpha \in P$. Then for every $\lambda \in J_P$, we have an irreducible representation π_λ of G on a finite-dimensional (complex) Hilbert space V_λ with the highest weight λ (with respect to $\mathfrak{h}$). We denote the corresponding representation of $\mathfrak{G}$ also by π_λ. Let v_λ denote a unit vector in V_λ belonging to the highest weight λ.

LEMMA 86. *Fix $\lambda \in J_P$ and let U be the subspace consisting of all $v \in V_\lambda$ such that $\pi_\lambda(H)v = \lambda(H)v$ for all $H \in \mathfrak{a}_2$. Then U is invariant and irreducible under $\pi_\lambda(\mathfrak{m})$. Moreover, $\dim U = 1$ if and only if $\langle \lambda, \alpha \rangle = 0$ for $\alpha \in P_1$.*

We write V and π for V_λ and π_λ respectively. It is clear that

$$\mathfrak{g} = \theta(\mathfrak{n}_2) + \mathfrak{m} + \mathfrak{n}_2$$

and therefore $\mathfrak{G} = \theta(\mathfrak{N}_2)\mathfrak{M}\mathfrak{N}_2,$

where[1] $\mathfrak{M} = \mathfrak{S}(\mathfrak{m}_c)$ and $\mathfrak{N}_2 = \mathfrak{S}((\mathfrak{n}_2)_c).$

For any $\alpha \in P$ define X_α, $X_{-\alpha}$ as in [4 (n), § 4] and put

$$\mathfrak{l}_\alpha = \mathbf{C}H_\alpha + \mathbf{C}X_\alpha + \mathbf{C}X_{-\alpha}.$$

Fix $u \neq 0$ in U. Then if $\alpha \in P_2$, it is clear that

$$\pi(HX_\alpha)u = (\lambda(H) + \alpha(H))\pi(X_\alpha)u \quad (H \in \mathfrak{a}_2).$$

Since λ is the highest weight of π, we conclude from the definition of our order, that $\pi(X_\alpha)u = 0$. Hence

$$V = \pi(\mathfrak{G})u = \pi(\theta(\mathfrak{N}_2)\mathfrak{M})u.$$

But then it is obvious from the definition of U that $U = \pi(\mathfrak{M})u$. This proves that U is invariant and irreducible under $\pi(\mathfrak{m})$.

Now fix $\alpha \in P$ and observe that $\pi(X_\alpha)v_\lambda = 0$. Hence by considering the subalgebra $\mathfrak{l}_\alpha$, it follows (see [4 (m), Lemma 25]) that $\pi(X_{-\alpha})v_\lambda = 0$ if and only if $\langle \lambda, \alpha \rangle = 0$. On the other hand, $\mathfrak{m}' = [\mathfrak{m}, \mathfrak{m}]$ is clearly generated, as a Lie algebra, by $\theta(\mathfrak{n}_1) + \mathfrak{n}_1$. Hence $\pi(\mathfrak{m}')v_\lambda = \{0\}$ if and only if $\langle \lambda, \alpha \rangle = 0$ for all $\alpha \in P_1$. Since U is irreducible under $\pi(\mathfrak{m})$, the second assertion of the lemma is now obvious

LEMMA 87. *Fix $\Lambda \in J_P$. Then $\Lambda(H_2(\bar{n})) \geqslant 0$ and*

$$\exp \Lambda(H_2(\bar{n}^a)) \leqslant 1 + \exp \left\{ -\tfrac{1}{2}\beta(\log a) + \Lambda(H_2(\bar{n})) \right\}$$

for $\bar{n} \in \bar{N}$ and $a \in A_2$ provided $\beta(\log a) \geqslant 0$.

Put $\lambda = \Lambda - \theta\Lambda$ so that

$$\langle \lambda, \alpha \rangle = \langle \Lambda, \alpha \rangle - \langle \Lambda, \theta\alpha \rangle \quad (\alpha \in P).$$

Obviously this is zero if $\alpha \in P_1$. On the other hand, $-\theta\alpha \in P_2$ whenever $\alpha \in P_2$ and therefore,

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[1] We use here the notation of [4 (m)], p. 280].

since [1] $|\theta\alpha|^2 = |\alpha|^2$, it follows that $\lambda \in J_p$. Put $\pi = \pi_\lambda$, $V = V_\lambda$, $v = v_\lambda$ and observe that $\lambda = 2\Lambda$ on $\mathfrak{h} \cap \mathfrak{p} = \mathfrak{a}_2$ and $\lambda = 0$ on $\mathfrak{h} \cap \mathfrak{k}$. Hence dim $U = 1$ and $\pi(\mathfrak{m}') U = \{0\}$ from Lemma 86. On the other hand, it is easy to see that $\mathfrak{m}_1 \cap \mathfrak{p} \subset \mathfrak{m}'$ so that $\pi(m) v = v$ for $m \in M_{1\mathfrak{p}}$. Therefore it is obvious that

$$\left| \pi(x) v \right| = e^{\lambda(H_2(x))} = e^{2\Lambda(H_2(x))} \quad (x \in G).$$

Let E denote the orthogonal projection of V on U. Then if $X \in \theta(\mathfrak{n}_2)$, it is obvious that $E\pi(X^r)v = 0$ for $r \geqslant 1$. Moreover, $\theta(\mathfrak{n}_1) \subset \mathfrak{m}'$ and therefore $\pi(\theta(\mathfrak{n}_1))v = \{0\}$. On the other hand, $\mathfrak{n}_2$ is an ideal in $\mathfrak{n}$ and therefore $\bar{N} = \bar{N}_2 \bar{N}_1$. Hence $E\pi(\bar{n})v = v$ and this shows that $|\pi(\bar{n})v| \geqslant |v| = 1$ $(\bar{n} \in \bar{N})$. Hence

$$\Lambda(H_2(\bar{n})) \geqslant 0 \quad (\bar{n} \in \bar{N}).$$

Put $E' = 1 - E$ and let $\|T\|$ denote the Hilbert–Schmidt norm of a linear transformation T in V. Since $U = \mathbb{C}v$, it is clear that

$$\exp 4\Lambda(H_2(\bar{n}^a)) = \|\pi(\bar{n}^a)E\|^2 = \|E\pi(\bar{n}^a)E\|^2 + \|E'\pi(\bar{n}^a)E\|^2 = 1 + \|E'\pi(\bar{n}^a)E\|^2$$

since $E\pi(\bar{n}^a)E = E$ as we saw above. On the other hand, we have seen during the proof of Lemma 86 that

$$V = \pi(\theta(\mathfrak{N}_2)\mathfrak{M})v = \pi(\theta(\mathfrak{N}_2))v.$$

Therefore every weight of π, other than λ, is of the form $\lambda - \sigma$ with

$$\sigma = \sum_{1 \leqslant i \leqslant r} \alpha_i,$$

$\alpha_i \in P_2$ and $r \geqslant 1$. Let $\lambda = \lambda_0 > \lambda_1 > \dots \lambda_p$ be all the weights of π and V_i the subspace of V consisting of all vectors belonging to the weight λ_i $(0 \leqslant i \leqslant p)$. Since $\mathfrak{h} = \theta(\mathfrak{h})$, V is the orthogonal sum of V_i $(0 \leqslant i \leqslant p)$. Put $\sigma_i = \lambda - \lambda_i$ and let E_i denote the orthogonal projection of V on V_i. Then it is clear that $E' = E_1 + \dots + E_p$ and therefore

$$E'\pi(\bar{n}^a)E = \sum_{1 \leqslant i \leqslant p} e^{-\sigma_i(\log a)} E_i \pi(\bar{n})E.$$

On the other hand, since $\beta(\log a) \geqslant 0$, it is obvious that

$$\sigma_i(\log a) \geqslant \inf_{\alpha \in P_2} \alpha(\log a) = \beta(\log a).$$

Therefore

$$\|E'\pi(\bar{n}^a)E\|^2 \leqslant e^{-2\beta(\log a)} \sum_{1 \leqslant i \leqslant p} \|E_i \pi(\bar{n})E\|^2$$
$$\leqslant e^{-2\beta(\log a)} |\pi(\bar{n})v|^2 = \exp\{-2\beta(\log a) + 4\Lambda(H_2(\bar{n}))\},$$

and the assertion of the lemma is now obvious.

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[1] As usual, $|\alpha|^2 = \langle \alpha, \alpha \rangle$.

Let π be an irreducible finite-dimensional representation of G on V and Λ the highest weight of π with respect to $\mathfrak{a}$. Let U be the subspace of those vectors $u \in V$ for which $\pi(H)u = \Lambda(H)u$ $(H \in \mathfrak{a}_2)$. We denote by E the orthogonal projection of V on U.

LEMMA 88. *Fix $a \in A_2$ such that $\beta(\log a) \geqslant 0$. Then*

$$\exp \Lambda(H(\bar{n}^a)) \leqslant 1 + \exp\{-\beta(\log a) + \Lambda(H(\bar{n}))\}$$

and
$$\|\pi(\bar{n}^a)E\| \leqslant \|E\| + e^{-\beta(\log a)}\|\pi(\bar{n})E\|$$

for $\bar{n} \in \bar{N}_2$. Moreover, $\Lambda(H(\bar{n})) \geqslant 0$ for $\bar{n} \in \bar{N}$.

Let $\Lambda = \Lambda_1 > \Lambda_2 > \ldots > \Lambda_p$ be all the weights of π with respect to $\mathfrak{a}$. It follows from the definition of our orders that there exists an integer $q \geqslant 1$ such that $\sigma_i = \Lambda - \Lambda_i$ is zero on $\mathfrak{a}_2$ for $i \leqslant q$ while $\sigma_i \neq 0$ on $\mathfrak{a}_2$ for $i > q$. Let E_i denote the orthogonal projection of V on the space V_i consisting of all vectors belonging to the weight Λ_i. Then $E = E_1 + \ldots + E_q$ and

$$E' = 1 - E = \sum_{q < i \leqslant p} E_i.$$

Fix $i > q$. Then it is clear (see the proof of Lemma 86) that

$$\sigma_i = \sum_{\alpha \in \Sigma} r(\alpha)\alpha,$$

where $r(\alpha)$ are nonnegative integers and $r(\alpha) \geqslant 1$ for some $\alpha \in \Sigma_2$. Hence if v is a unit vector in V_1, it is clear that

$$|\pi(\bar{n}^a)v|^2 = 1 + |E'\pi(\bar{n}^a)v|^2 \leqslant 1 + e^{-2\beta(\log a)}|\pi(\bar{n})v|^2 \quad (\bar{n} \in \bar{N}_2)$$

and from this the first inequality follows immediately. Moreover,

$$\|\pi(\bar{n}^a)E\|^2 = \|E\|^2 + \|E'\pi(\bar{n}^a)E\|^2 \leqslant \|E\|^2 + e^{-2\beta(\log a)}\|E'\pi(\bar{n})E\|^2$$
$$\leqslant \|E\|^2 + e^{-2\beta(\log a)}\|\pi(\bar{n})E\|^2 \quad (\bar{n} \in \bar{N}_2)$$

and this gives the second inequality. The last statement of Lemma 88 has already been proved in [4 (j), Lemma 43].

Now if we take $\Lambda = \frac{1}{2}\sum_{\alpha \in P}\alpha$ in Lemma 87, and choose π, in Lemma 88, such that its highest weight, with respect to $\mathfrak{a}$, is ϱ, then we get Lemma 85 immediately.[1]

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[1] The second inequality of Lemma 88 has been proved for later applications.

§ 43. Applications of the above inequalities

Define the number $d \geqslant 0$ as in § 14 and observe that $\varrho(H(\bar{n})) \geqslant 0$ for $\bar{n} \in \bar{N}$ from Lemma 85. Let $d_i \bar{n}$ denote the Haar measure on $\bar{N}_i$ $(i = 1, 2)$.

LEMMA 89.[1] *For any $\varepsilon > 0$,*

$$\int_{\bar{N}_2} e^{-\varrho(H(\bar{n}))} \{1 + \varrho(H(\bar{n}))\}^{-(d+\varepsilon)} d_2 \bar{n} \leqslant \infty .$$

We can choose $c > 0$ (see [4 (j), Theorem 3]) such that

$$e^{\varrho(\log a)} \Xi(a) \leqslant c(1 + \sigma(a))^d \quad (a \in A).$$

On the other hand, by [4 (j), Cor. 2, p. 289],

$$e^{\varrho(\log a)} \Xi(a) = \int_{\bar{N}} \exp \left\{ -\varrho(H(\bar{n}^a)) - \varrho(H(\bar{n})) \right\} d\bar{n} \quad (a \in A) ,$$

where $d\bar{n}$ is the (suitably normalized) Haar measure on $\bar{N}$. We may assume that $d\bar{n} = d_2 \bar{n}_2 \cdot d_1 \bar{n}_1$ for $\bar{n} = \bar{n}_2 \bar{n}_1$ $(\bar{n}_i \in \bar{N}_i, \ i = 1, 2)$. Now

$$\bar{n} = \bar{n}_2 \bar{n}_1 \in \bar{n}_2 k^{-1} \exp H(\bar{n}_1) \cdot N_1,$$

where $k = \varkappa(\bar{n}_1)^{-1} \in K_1$. Since M normalizes $\theta(\mathfrak{n}_2)$, we get

$$H(\bar{n}) = H(\bar{n}_2^{\,k}) + H(\bar{n}_1).$$

We may normalize $d_1 \bar{n}_1$ in such a way (see [4 (j), Lemma 44]) that

$$\int_{\bar{N}_1} e^{-2\varrho(H(\bar{n}_1))} d_1 \bar{n}_1 = 1.$$

Then, since $d_2 \bar{n}_2^{\,k} / d_2 \bar{n}_2 = 1$ $(k \in K_1)$ and A_2 commutes with $\bar{N}_1$, we conclude that

$$e^{\varrho(\log a)} \Xi(a) = \int_{\bar{N}_2} \exp \left\{ -\varrho(H(\bar{n}^a)) - \varrho(H(\bar{n})) \right\} d_2 \bar{n} \leqslant c(1 + \sigma(a))^d \quad (a \in A_2).$$

Put $a_t = \exp tH$ $(t \in \mathbf{R})$ where H is an element in $\mathfrak{a}_2$ such that

$$b = \beta(H) > 0.$$

Then if $a = a_t$ $(t \geqslant 0)$, it follows from Lemma 85 that

$$\exp \varrho(H(\bar{n}^a)) \leqslant 1 + \exp \{\varrho(H(\bar{n})) - bt\} \quad (\bar{n} \in \bar{N}_2)$$

and therefore

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[1] Cf. [4 (j), Lemma 45].

$$\int_{\bar{N}_2} e^{-\varrho(H(\bar{n}))}\{1 + e^{\varrho(H(\bar{n}))-bt}\}^{-1} d_2\bar{n} \leqslant c(1+\sigma(a_t))^d \leqslant c'(1+t)^d$$

for $t \geqslant 0$. Here c' is a positive constant independent of t. Let $\bar{N}_{2,r}$ denote the set of all $\bar{n} \in \bar{N}_2$ with $\varrho(H(\bar{n})) \leqslant 2^r$ and put $t = 2^r b^{-1}$. Then

$$e^{\varrho(H(\bar{n}))-bt} \leqslant 1$$

for $\bar{n} \in \bar{N}_{2,r}$ and therefore

$$\int_{\bar{N}_{2,r}} e^{-\varrho(H(\bar{n}))} d_2\bar{n} \leqslant 2c'(1+b^{-1}2^r)^d \leqslant c_1 2^{rd} \quad (r \geqslant 0),$$

where c_1 is a positive number independent of r. Let $\bar{N}_2(r)$ denote the complement of $\bar{N}_{2,r-1}$ in $\bar{N}_{2,r}$ $(r \geqslant 1)$. Then

$$\int_{\bar{N}_2(r)} e^{-\varrho(H(\bar{n}))}\{1 + \varrho(H(\bar{n}))\}^{-(d+\varepsilon)} d_2\bar{n} \leqslant c_1 2^{rd-(r-1)(d+\varepsilon)} = c_1 2^{d-(r-1)\varepsilon} \quad (r \geqslant 1).$$

Since $\bar{N}_{2,0}$ is compact [4 (j), Lemma 40], we get the required result from the convergence of the series $\sum_{r \geqslant 0} 2^{-r\varepsilon}$.

LEMMA 90. *Put $\bar{n} = \theta(n^{-1})$ for $n \in N$. Then there exists a number $c \geqslant 1$ such that*

$$1 + \max(\sigma(h), \varrho(H(\bar{n}))) \leqslant c(1+\sigma(hn))$$

and $$\Xi(hn) \leqslant c(1+\sigma(hn))^d \exp\{-\varrho(\log h) - \varrho(H(\bar{n}))\}$$

for $h \in A$ and $n \in N$.

It is clear that, for the proof of this lemma, we may assume, as in § 42, that the conditions of [4 (j), p. 244] hold. Fix an irreducible finite-dimensional representation π of G with the highest weight ϱ with respect to $\mathfrak{a}$. Define A^+ as in § 14. Then $G = KA^+K$ and therefore $hn = k_1 h' k_2$ $(k_1, k_2 \in K;\ h' \in A^+)$. Then

$$\bar{n}h = \theta(hn)^{-1} = k_2^{-1} h' k_1^{-1}$$

and therefore $\|\pi(hn)\| = \|\pi(h')\| = \|\pi(\bar{n}h)\|$. Hence if p is the degree of π, we get

$$e^{2\varrho(\log h')} \leqslant \|\pi(h')\|^2 \leqslant p e^{2\varrho(\log h')}.$$

Let $\mathfrak{w}$ be the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{a}$ (see [4 (j), p. 249]). Fix $s \in \mathfrak{w}$ and choose $k \in K$ such that $\mathrm{Ad}(k)H = sH$ for all $H \in \mathfrak{a}$. Then (see [1, § 7.4]) $\bar{n}^k = \bar{n}_1 n_2$ where

$$\bar{n}_1 \in \bar{N} \cap \bar{N}^k, \quad n_2 \in N \cap \bar{N}^k.$$

Hence
$$\|\pi(h')\| = \|\pi(\bar{n}^k h^s)\| = \|\pi(\bar{n}_1 h^s n_2')\|,$$

where $n_2' = (h^s)^{-1} n_2 h^s$. Therefore

$$\|\pi(h')\|^2 \geqslant |\pi(\bar{n}_1 h^s)\psi|^2 \geqslant e^{2\varrho(\log h^s)},$$

where ψ is a unit vector belonging to the highest weight ϱ. This shows that

$$\varrho(\log h') + \tfrac{1}{2}\log p \geqslant \max_{s\in\mathfrak{w}} \varrho(\log h^s) \geqslant |\varrho(\log h)|$$

(see [4 (j), p. 281]). On the other hand, we can obviously choose c_1, $c_2 > 0$ such that

$$c_1\sigma(h_1) \leqslant \varrho(\log h_1) \leqslant c_2\sigma(h_1) \quad (h_1 \in A^+).$$

Then
$$c_2\sigma(h') + \tfrac{1}{2}\log p \geqslant \max_{s\in\mathfrak{w}} \varrho(\log h^s) \geqslant c_1\sigma(h).$$

Since $\sigma(h') = \sigma(hn)$, this shows that we can choose $c_3 > 0$ such that

$$c_3(1 + \sigma(hn)) \geqslant 1 + \sigma(h) \quad (h \in A, \, n \in N).$$

Moreover, we know [4 (k), Lemma 42] that

$$\varrho(\log h') \geqslant \varrho(\log h) + \varrho(H(\bar{n})).$$

Since
$$\varrho(\log h') + \tfrac{1}{2}\log p \geqslant -\varrho(\log h)$$

by our result above, we conclude that

$$2\varrho(\log h') + \tfrac{1}{2}\log p \geqslant \varrho(H(\bar{n})).$$

Hence we can choose $c_4 > 0$ such that

$$1 + \varrho(H(\bar{n})) \leqslant c_4(1 + \sigma(hn)) \quad (h \in A, \, n \in N).$$

Now select $c_5 > 0$ such that

$$\Xi(h_1) \leqslant c_5 \, e^{-\varrho(\log h_1)} (1 + \sigma(h_1))^d \quad (h_1 \in A^+).$$
Then

$$\Xi(hn) = \Xi(h') \leqslant c_5 \, e^{-\varrho(\log h')} (1 + \sigma(h'))^d \leqslant c_5 (1 + \sigma(hn))^d \exp\{-\varrho(\log h) - \varrho(H(\bar{n}))\}.$$

This proves Lemma 90.

Put $\varrho_i(H) = \mathrm{tr}\,(\mathrm{ad}\,H)_{\mathfrak{n}_i}$ $(H \in \mathfrak{a},\, i = 1, 2)$ so that $\varrho = \varrho_1 + \varrho_2$.

COROLLARY 1. *Suppose r_1, r_2 are two numbers $\geqslant 0$ and $r = r_1 + r_2$. Then*

$$e^{\varrho_2(\log h)} \, \Xi\,(hn)\,(1 + \sigma(hn))^{-(r+d)} \leqslant c^{r+1} e^{-\varrho_1(\log h)}\,(1+\sigma(h))^{-r_1} e^{-\varrho(H(\bar{n}))}\,(1+\varrho(H(\bar{n})))^{-r_2}$$

for $h \in A$ and $n \in N$.

For, in the above notation, we have

$$\Xi\,(hn)\,(1+\sigma(hn))^{-(r+d)} = \Xi\,(h')\,(1+\sigma(h'))^{-(r+d)} \leqslant c e^{-\varrho(\log h')}\,(1+\sigma(h'))^{-r}.$$

But $(1+\sigma(h'))^{-r} \leqslant c^r (1+\sigma(h))^{-r_1}\,(1+\varrho(H(\bar{n})))^{-r_2}$

and $\varrho(\log h') \geqslant \varrho(\log h) + \varrho(H(\bar{n})).$

Hence our assertion is obvious.

Let $d_2 n$ denote the Haar measure on N_2.

COROLLARY 2. *Let Ω be a compact set in G. Then if $r > 2d$, the integral*

$$\int_{N_2} \Xi\,(xn)\,(1+\sigma(xn))^{-r}\,d_2 n$$

converges uniformly for $x \in \Omega$.

Let $x = k_0 h n_0$ ($k_0 \in K$, $h \in A$, $n_0 \in N$). Then $\Xi(xn) = \Xi(hn_0 n)$ and $\sigma(xn) = \sigma(hn_0 n)$. Now let $\bar{n}_0 = \bar{n}_2 \bar{n}_1$ where $\bar{n}_i \in \bar{N}_i$ ($i = 1, 2$). Then h and $\bar{n}_i$ remain bounded [1] and

$$H(\theta(n_0 n)^{-1}) = H(\bar{n}\bar{n}_2\bar{n}_1) = H(\bar{n}\bar{n}_2 k^{-1}) + H(\bar{n}_1) = H((\bar{n}\bar{n}_2)^k) + H(\bar{n}_1) \quad (n \in N_2),$$

where $k = \varkappa(\bar{n}_1)^{-1} \in K_1$. Fix a compact set U in $\bar{N}_2$ such that $\bar{n}_2$ stays within U. Since $r - d > d$, we can, by Lemma 89, choose, for a given $\varepsilon > 0$, a compact set V_0 in $\bar{N}_2$ such that

$$\int_{^c V_0} e^{-\varrho(H(\bar{n}))}\,\{1 + \varrho(H(\bar{n}))\}^{-r+d}\,d_2\bar{n} \leqslant \varepsilon,$$

where $^c V_0$ denotes the complement of V_0 in $\bar{N}_2$. Put $V = V_0^{K_1} U^{-1}$. Then if $\bar{n} \in {}^c V$, it is clear that $(\bar{n}\bar{n}_2)^k \in {}^c V_0$. Moreover, since h remains bounded, we can choose c_1 such that $c^{r+1} e^{-\varrho(\log h)} \leqslant c_1$. Then we conclude from Corollary 1 above that

$$\Xi\,(hn)\,(1+\sigma(hn))^{-r} \leqslant c_1 e^{-\varrho(H(\bar{n}))}\,(1+\varrho(H(\bar{n})))^{-r+d} \quad (n \in \bar{N}_2).$$

We may obviously assume that $d_2 n = d_2 \bar{n}$ under the mapping $n \to \bar{n}$. Therefore since $\varrho(H(\bar{n}_1)) \geqslant 0$ by Lemma 85, it is clear that

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[1] This means that they stay within compact sets as x varies in Ω.

$$\int_{cV} \Xi(hn_0 n)\,(1 + \sigma(hn_0 n))^{-r}\,d_2 n \leqslant c_1 \int_{cV_0} e^{-\varrho(H(\bar{n}))}\{1 + \varrho(H(\bar{n}))\}^{-r+d}\,d_2\bar{n} \leqslant c_1\,\varepsilon.$$

This proves our assertion.

§ 44. Proof of Lemma 21

We now come to the proof of Lemma 21. Put $A_1{}^+ = \exp \mathfrak{a}_1{}^+$, $\mathfrak{a}_1{}^+$ being the set of all $H \in \mathfrak{a}_1$ where $\alpha(H) \geqslant 0$ $(\alpha \in \Sigma_1)$. Then $M_1 = K_1 A_1{}^+ K_1$ and $M = M_1 A_2 = K_1 (A_1{}^+ A_2) K_1$. Therefore it is obviously enough to consider the case when $m = h = h_1 h_2$ $(h_1 \in A_1{}^+, h_2 \in A_2)$. Put $r_1 = r'$ and $r_2 = d + r - r'$. Then it follows from Lemma 89 and Corollary 1 of Lemma 90 that

$$e^{\varrho_2(\log h)} \int_{N_2} \Xi(hn)\,(1 + \sigma(hn))^{-(r+2d)}\,d_2 n \leqslant c_1\,e^{-\varrho_1(\log h)}\,(1 + \sigma(h))^{-r'},$$

where c_1 is a positive number independent of h. Since

$$e^{\varrho_1(\log h)}\,\Xi_1(h) = e^{\varrho_1(\log h_1)}\,\Xi_1(h_1) \geqslant 1$$

from [4 (j), Lemma 36], the first statement of Lemma 21 is now obvious. The second is an immediate consequence of Lemma 90 and the relation $M = K_1 A K_1$.

§ 45. Appendix

We now use the notation of §§ 27, 28. Put $\varrho_i(H) = \frac{1}{2}\operatorname{tr}(\operatorname{ad} H)_{\mathfrak{n}_i}$ $(H \in \mathfrak{a}_\mathfrak{p},\ i = 1, 2)$ so that $\varrho = \varrho_1 + \varrho_2$. Let M_1 and A_2 be the analytic subgroups of G corresponding to $\mathfrak{m}_1 = \mathfrak{k} + \mathfrak{p} \cap [\mathfrak{m}, \mathfrak{m}]$ and $\mathfrak{a}_2 = \mathfrak{l}$ respectively. Then $M = M_1 A_2$ and $d(ma) = e^{\varrho(\log a)}$ $(m \in M_1,\ a \in A_2)$. Hence it follows without difficulty that

$$d^{-1} X \circ d = X' \quad (X \in \mathfrak{m}),$$

where $X \to X'$ is the isomorphism of $\mathfrak{m}$ into $\mathfrak{M}$ given by $H' = H + \varrho(H)$, $Y' = Y$ $(H \in \mathfrak{l},\ Y \in \mathfrak{m}_1)$. This gives rise to an automorphism $v \to v'$ of $\mathfrak{M}$ which preserves $\mathfrak{Z}_1$.

Now let $H \in \mathfrak{a}_\mathfrak{p}{}^+$. Then

$$d(\exp H)\,\Xi(\exp H) = e^{\varrho_2(H)}\,\Xi(H) = e^{-\varrho_1(H)}\,e^{\varrho(H)}\,\Xi(H).$$

The assertion of Lemma 47 now follows immediately if we apply [4 (j), Theorem 3] and observe [4 (j), Lemma 36] that

$$1 \leqslant e^{\varrho_1(H)}\,\Xi_1(\exp H).$$

Since $\mathfrak{g} = \theta(\mathfrak{n}_2) + \mathfrak{m} + \mathfrak{n}_2$, it is clear that

$$\mathfrak{G} = \theta(\mathfrak{N}_2)\,\mathfrak{M}\mathfrak{N}_2,$$

where $\mathfrak{N}_2 = \mathfrak{S}(\mathfrak{n}_{2c})$. We know (see the proof of the corollary of Lemma 13 of [4 (q)]) that

$$z - \mu_0(z)' \in \mathfrak{G}\mathfrak{n}_2 \quad (z \in \mathfrak{Z}).$$

Put $u = z - \mu_0(z)'$. Then u commutes with $\mathfrak{l}$ and since

$$\mathfrak{G} = \mathfrak{M}\mathfrak{N}_2 + \theta(\mathfrak{n}_2)\,\mathfrak{G},$$

it is obvious that $u \in \theta(\mathfrak{n}_2)\,\mathfrak{G}\mathfrak{n}_2$. This is the result needed in § 28.

Now suppose $\mathfrak{g}$ and G are defined as in § 7. Let A be a Cartan subgroup of G, A_0 the center of A and $\tilde{A}$ the normalizer of A in G. Put $W_A = \tilde{A}/A_0$. Then A_0 is open in A and W_A is a finite group (see [4 (o), § 20]). We denote by $x \to x^*$ the natural projection of G on $G^* = G/A_0$.

Let $\mathfrak{h}$ be the Lie algebra of A and dx, da the Haar measures on G and A respectively. Put $G_A = (A')^G$ as usual (see [4 (o), § 20]).

LEMMA 91. *Let dx^* be the invariant measure on G^* such that*

$$\int_G f(x)\,dx = \int_{G^*} dx^* \int_{A_0} f(xa)\,da \quad (f \in C_c(G)).$$

Then
$$\int_{G_A} f(x)\,dx = [W_A]^{-1} \int_A v(a)\,da \int_{G^*} f(a^{x^*})\,dx^* \quad (f \in C_c(G_A))$$

in the notation of [4 (o), § 22], where

$$v(a) = \left|\det\,(\mathrm{Ad}\,(a^{-1}) - 1)_{\mathfrak{g}/\mathfrak{h}}\right|.$$

Let ϕ denote the mapping $(x^*, a) \to a^{x^*}$ of $G^* \times A'$ onto G_A. Then we know (see [4 (o), § 20]) that ϕ is regular and $\phi^{-1}(x)$ $(x \in G_A)$ contains exactly $[W_A]$ points in $G^* \times A'$. Hence our result follows from a simple computation which gives the functional determinant of this mapping.

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