DISCRETE SPECTRA AND DAMPED WAVES IN

QUASILINEAR THEORY*

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ABSTRACT

The consequences of the quasilinear equations are explored. Particular attention is paid to the differences between the one dimensional and the two and three dimensional cases, and to the differences between the cases of discrete and continuous wave number spectra. The possibilities of and problems associated with including damped waves are treated. The relation between conservation laws and the "resonance approximation", in which the limit of zero growth rate for the unstable waves is taken at finite times, is clarified. Numerical solutions for the one dimensional case with finite growth rate are presented.

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I. INTRODUCTION

This article addresses itself to the question of the $t \to \infty$ state of the quasilinear theory of weakly unstable plasma oscillations. This is a subject to which numerous authors have devoted attention since 1961 (Drummond and Pines, 1962; Vedenov, Velikhov, and Sagdeev, 1962; Frieman and Rutherford, 1964; Bernstein and Englemann, 1966; Montgomery and Vahala, 1969), but is still characterized by disagreements as to exactly what the theory does in fact say. However, the subject occupies an essentially unique position in continuum mechanics: it is the only example of an unstable continuum system which apparently leads to a simple set of dynamical equations from which a non-trivial final turbulent state can be deduced. For this reason alone, quasi-linear theory deserves to be examined carefully, even though some physical processes such as particle trapping (see, e.g., Armstrong and Montgomery, 1969) we know are not treated by it correctly. (It can always be imagined that the difference between the initial and final electric field energies is small enough that no appreciable trapping occurs, even if this is not the most exciting case.) Numerous consequences of quasilinear theory have been asserted, and one purpose of this paper is to try to separate those which can be proved (convincingly if not rigorously), those which it would be desirable to prove, and those which are not true. Particular attention will be paid to the subjects of discrete, rather than continuous, spectra, and to the possible inclusion in the formalism of Landau-damped waves.

By quasi-linear theory, in this context, we shall mean a theory which deals only with unbounded electron plasmas which obey the Vlasov-Poisson system. We shall include no first-order terms in the discreteness parameter (as do, for example, Harris (1967) or Rogister and Oberman (1969)), and thus obtain no

$t \to \infty$ approach to thermal equilibrium. We shall likewise not include any of the higher-order corrections commonly called "mode coupling" terms. Some considerable attention will be devoted to the differences between the two and three dimensional cases and the one dimensional case. The observation that these are fundamentally different is due to Bernstein and Englemann (1966), but we cannot confirm all of their conclusions. Another (to our knowledge new) consideration that is introduced is that of the differences between the cases of continuous and discrete spectra in wave number space. It appears that the conclusions may be substantially different in the two situations, and whereas laboratory experiments and computer simulations generally deal with the case of discrete spectra, the theoretical treatments have dealt (in two and three dimensions) invariably with the continuous $\vec{K}$ case.

We have some observations to make concerning conservation laws and the so-called "resonance approximation," wherein the growth rate of the unstable waves is allowed to go to zero at finite times. We note that some of the more easily accessible of the conventional conclusions about the $t \to \infty$ state can no longer be retained simultaneously with the laws of conservation of momentum and energy.

Finally, we present some numerical solutions of the one-dimensional, discrete k, quasilinear equations in which the resonance approximation is not made. The results are similar in some respects, and different in others, from the explicit final state which is usually given in terms of the "plateau" construction and the equal-area rule.

II. SUMMARY OF THE DERIVATION OF THE QUASI-LINEAR SYSTEM

We consider an electron plasma with uniform positive background. We shall write the expressions, as a rule, in notation appropriate to the three-dimensional case, with the allowed wave numbers $\vec{k}$ taking on a discrete set of values. Except for numerical factors, the same expressions are appropriate to the two-dimensional case. Where needed, we shall give the corresponding expressions for the one dimensional case, and indicate explicitly how the transition may be made to continuous $\vec{k}$.

The electron distribution function will be written $f(\vec{x}, \vec{v}, t) = f_0(\vec{v}, t) + f^{(1)}(\vec{x}, \vec{v}, t)$, and it will be assumed that

$$| f^{(1)} | \ll | f_0 | \tag{1}$$

for all t. For $f^{(1)}$ we shall assume an expression $f^{(1)}(\vec{x}, \vec{v}, t) = \sum_{\vec{k}\neq 0} f_{\vec{k}}(\vec{v}, t)e^{i\vec{k}\cdot\vec{x}}$ with $f_{-\vec{k}} = f_{\vec{k}}^*$. Similarly, the electric field $\vec{E}(\vec{x},t)$ will be written as $\vec{E}(\vec{x},t) = \sum_{\vec{k}\neq 0} \vec{E}_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}}$,

where $\vec{E}_{\vec{k}}(t) = \hat{k}\, E_{\vec{k}}(t) = \vec{E}^*_{-\vec{k}}(t)$, and $\hat{k} \equiv \vec{k}/k$. $\vec{E}$ is assumed to be of first order in the same small parameter which measures $| f^{(1)}/ f_0 |$, so that from the exact equation for $f_0(\vec{v}, t)$ (obtained by spatially averaging Vlasov's Equation),

$$\frac{\partial f_0}{\partial t} = \frac{e}{m}\frac{\partial}{\partial \vec{v}} \cdot \left\{ \sum_{\vec{k}\neq 0} \vec{E}_{-\vec{k}} f_{\vec{k}} \right\} , \tag{2}$$

we may conclude that f_0 is a slowly-varying function of time, since its time derivative is of second order in this small parameter. (e/m is the electronic charge-to-mass ratio.)

For its part, $f_{\vec{k}}(\vec{v}, t)$ will be assumed to obey the linearized Vlasov equation,

$$\left(\frac{\partial}{\partial t} + i\vec{k} \cdot \vec{v}\right) f_{\vec{k}} - \frac{e}{m} \vec{E}_{\vec{k}} \cdot \frac{\partial f_0}{\partial \vec{v}} = 0. \tag{3}$$

The situation studied is that which results when f_0 is initially a sum of some stable distribution such as the Maxwellian, plus a uniform tenuous stream of suprathermal electrons which pass through the plasma and render it weakly unstable to growing electron plasma oscillations, according to linear theory.

The essential features of quasilinear theory are that: (i) Equation (3) and Poisson's Equation are solved by using the linear Landau (1946) solution with f_0 "frozen," or treated as slowly-varying compared to $f_{\vec{k}}$; and (ii) contributions to $\vec{E}_{\vec{k}}$ and $f_{\vec{k}}$ from the rightmost Landau pole (in the complex Laplace transform plane) only are retained. The location of this growing pole is determined instantaneously by the slowly-changing value of f_0.

Equivalently, we assume for $\vec{E}_{\vec{k}}(t)$ an expression of the form

$$\vec{E}_{\vec{k}}(t) = \vec{E}_{\vec{k}}(0) \exp \left\{ -i \int_0^t \Omega(\vec{k})\, d\tau \right\} \tag{4}$$

where $\Omega(\vec{k}) = \Omega(\vec{k},\tau) = \omega(\vec{k}) + i\gamma(\vec{k})$ determines the location of the pole in the complex plane at time τ.

The solution of Equation (3) associated with (4) is

$$f_{\vec{k}}(\vec{v}, t) = \frac{e}{m} \frac{\vec{E}_{\vec{k}}(t) \cdot \partial f_0 / \partial \vec{v}}{i(\vec{k} \cdot \vec{v} - \Omega(\vec{k}))} . \tag{5}$$

Added to (5) should be other terms, if it is desired to match arbitrary initial values of $f_{\vec{k}}(\vec{v}, 0)$. These would be of two types: (i) other, Landau-damped, contributions; and (ii) terms of fixed amplitude whose time dependence is of the form $\exp(-i\vec{k} \cdot \vec{v}t)$. Both are omitted by the following reasoning. It is assumed that $f_{\vec{k}}(\vec{v},0)$ is very small; both the additional classes of terms remain of this same order or smaller. We include only waves which grow initially and e-fold several times, so that $E_{\vec{k}}(t)$ contains a factor $\exp\left(\int_0^t \gamma(\vec{k})d\tau\right) \gg 1$. This does not prohibit a wave which has grown to an eventually non-neglible amplitude, according to this criterion, from eventually becoming damped ($\gamma(\vec{k}) < 0$). It only asserts that the validity of the approximation requires $\int_0^t \gamma(\vec{k})d\tau \gg 1$.

Closing the loop, the expression (5) is substituted into Poisson's equation, $i\vec{k} \cdot \vec{E}_{\vec{k}} = -4\pi\, en \int f_{\vec{k}}\, d\vec{v}$, where n is the number density of electrons averaged over space. If $E_{\vec{k}} \neq 0$, this requires

$$1 = \frac{\omega_p^2}{k^2}\, \vec{k} \cdot \int \frac{\partial f_0/\partial \vec{v}}{\vec{k} \cdot \vec{v} - \Omega(\vec{k})}\, d\vec{v} \qquad (6)$$

where $\omega_p^2 = 4\pi\, ne^2/m$.

Equation (6) is the Landau dispersion relation. As long as $\gamma(\vec{k}) > 0$, the contours of integration are just along the real $\vec{v}$-axes. If $\gamma(\vec{k})$ went negative, then one would loop around the pole $\vec{k} \cdot \vec{v} = \Omega(\vec{k})$ in the usual way. If in addition, $\exp \int_0^t \gamma(\vec{k})d\tau$ ceased to be $\gg 1$, one would pick up the $\exp(-i\vec{k} \cdot \vec{v}t)$ terms omitted from Equation (5). We shall assume for the present that even if some $\gamma(\vec{k})$ were to go negative, $\int_0^t \gamma(\vec{k})d\tau$ remains $\gg 1$. [Even then, it will not be totally clear that negative $\gamma(\vec{k})$ can be fit into the theory.]

In the situation of interest, we know that the solutions to (6) are, to a good approximation,

$$\omega^2(\vec{k}) \cong \omega_p{}^2 + 3\, k^2 V_e{}^2 + \ldots \tag{7a}$$

$$\gamma(\vec{k}) \cong \frac{\pi}{2} \frac{\omega_p{}^3}{k^2} F_0{}'(\omega/k) \tag{7b}$$

where $F_0(\mu) \equiv \int d\vec{v}\ \delta(\mu-\hat{k}\cdot\vec{v})\, f_0(\vec{v}, t)$, and $V_e{}^2 \equiv \int_{-\infty}^{\infty} \mu^2 F_0(\mu)\, d\mu$. Note that $\omega(\vec{k})$ depends only upon the bulk properties of the plasma, and if the growth of the instability ceases before $\Sigma_{\vec{k}} \mid \vec{E}_{\vec{k}} \mid^2$ becomes comparable to $mV_e{}^2$, $\omega(\vec{k})$ will satisfy (7a) always and be time-independent. Equation (7b), however, shows that $\gamma(\vec{k})$ <u>will</u> vary with time; anything which suffices to change $F_0(\mu)$ locally near $\mu \cong \omega/k$ will cause $\gamma(\vec{k})$ to vary without regard to the moments. On Equations (7), we also have the symmetry conditions

$$\omega(-\vec{k}) = -\ \omega(\vec{k})$$

$$\gamma(-\vec{k}) = +\ \gamma(\vec{k}). \tag{8}$$

(Both $\vec{k}$ and $-\vec{k}$ must be present for reality, and the growing contribution from $-\vec{k}$ must have the same phase velocity direction as $+\vec{k}$.)

Substituting Equation (5) into Equation (2) gives, using (8),

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \vec{v}} \cdot \left(\overleftrightarrow{D} \cdot \frac{\partial f_0}{\partial \vec{v}} \right) \tag{9}$$

where the <u>diffusion</u> <u>tensor</u> $\overleftrightarrow{D}$ is defined by

$$\overleftrightarrow{D} = \frac{e^2}{m^2} \Sigma_{\vec{k}\neq 0} \frac{\hat{k}\,\hat{k}\,\mathcal{E}(\vec{k})\ \gamma(\vec{k})}{(\omega(\vec{k})-\vec{k}\cdot\vec{v})^2+\gamma^2(\vec{k})} \tag{10}$$

with $\mathcal{E}(\vec{k}) \equiv \mid \vec{E}_{\vec{k}}(t) \mid^2$. $\mathcal{E}(\vec{k})$ clearly evolves according to

$$\frac{\partial \mathcal{E}(\vec{k})}{\partial t} = 2\,\gamma(\vec{k})\,\mathcal{E}(\vec{k}). \tag{11}$$

Both $\omega(\vec{k})$ and $\gamma(\vec{k})$ are given by Equations (7), and it is to be emphasized that in Equation (10), we have a discrete sum over $\vec{k}$.

Equations (7) through (11) are what we shall be calling the quasilinear equations. It is their consequences, rather than the validity of the underlying assumptions in their derivation, which is of primary concern to us here.

If $\vec{k}$ is a continuously distributed variable, the same set of equations results, but with the modification that in Equation (10), the $\sum_{\vec{k}} \mathcal{E}(\vec{k})$ goes over into $\int d\vec{k}\, \mathcal{E}_c(\vec{k})$ where $\mathcal{E}_c(\vec{k}) \equiv \lim_{L\to\infty} \left(\frac{2\pi}{L}\right)^3 \mathcal{E}(\vec{k})$; L is the periodicity length which, when finite, had defined the allowed discrete $\vec{k}$ values in $\mathcal{E}(\vec{k})$.

The same expressions apply to the two dimensional (2D) case, up to numerical factors.

The analogous one dimensional (1D) equations are

$$\frac{\partial F_0}{\partial t}(\mu,t) = \frac{\partial}{\partial \mu}\left(D(\mu,t)\,\frac{\partial F_0(\mu,t)}{\partial \mu}\right) , \tag{9I}$$

$$D = \frac{e^2}{m^2} \sum_{k\neq 0} \frac{\mathcal{E}(k)\,\gamma(k)}{(\omega(k)-k\mu\,)^2 + \gamma^2(k)} , \tag{10I}$$

$$\frac{\partial \mathcal{E}(k)}{\partial t} = 2\gamma\,(k)\,\mathcal{E}(k). \tag{11I}$$

Both $\omega(k)$ and $\gamma(k)$ are still given by Equations (7), and the symmetry conditions analogous to (8) are $\omega(-k) = -\omega(k)$, $\gamma(-k) = +\,\gamma(k)$. The transition to the continuum limit is straightforward. It should be noted that in 2D and 3D, $k = |\,\vec{k}\,| > 0$, but that in 1D, k can be either positive or negative.

It is readily proved, by use of the dispersion relation, that these quasilinear equations conserve both momentum and energy. For example, in 3D, it is readily shown that as a consequence of Equations (9)-(11), $d\,W_{Tot.}/dt = 0$ and $d\,\vec{P}_{Tot.}/dt = 0$, where

$$W_{Tot.} \equiv n\int d\vec{x}\, d\vec{v}\, \tfrac{1}{2}\, m\, f_0\, \vec{v}^{\,2} + \int d\vec{x}\, \frac{\vec{E}^{\,2}}{8\pi}$$

$$\vec{P}_{Tot.} \equiv n\int d\vec{x}\, d\vec{v}\, m\vec{v}\, f_0 \, .$$

Both these expressions are infinite, but can be referred to unit volume and thereby made finite. Similar expressions can readily be proved constant in 2D and 1D. Obviously, all systems also conserve particle number.

It can also be readily shown that Equations (10), (10I) do not conserve $W_{Tot.}$ and $\vec{P}_{Tot}$. in the large t limit, if the limit $\gamma/\omega \to 0$ is taken at <u>finite</u> times. We will return to this point later.

III. IMPLICATIONS OF THE BERNSTEIN-ENGELMANN "H-LIKE" THEOREM

A. The 2D and 3D Cases

Throughout this Section, we shall be assuming that the solution to the quasilinear system can be obtained assuming no $\gamma(\vec{K})$ in the $\sum_{\vec{K}}$ ever becomes < 0. As long as we remain in the regime of discrete $\vec{K}$, we shall encounter no contradictions. Only in the limit of continuous $\vec{K}$ does the contradiction noted by Bernstein and Engelmann (1966) arise. We shall show that all $\gamma(\vec{K})$ must eventually approach zero; whether this happens in a finite or an infinite time has everything to do with whether the quasilinear system has a well-behaved solution with only non-negative $\gamma(\vec{K})$. We return in Section IV to the possibility of negative γ.

If we first restrict the sum in Equation (10) to growing waves, $\gamma(\vec{K}) > 0$, it is clear that for any vector $\vec{a}$, $\vec{a} \cdot \overset{\Rightarrow}{D} \cdot \vec{a} \geq 0$. The equality sign, if more than two linearly independent values of $\vec{K}$ are present, can only hold if $\vec{a} = 0$. Using Equation (9) and integrating by parts,

$$\frac{d}{dt} \int \tfrac{1}{2} f_0{}^2 \, d\vec{v} = - \int \frac{\partial f_0}{\partial \vec{v}} \cdot \overset{\Rightarrow}{D} \cdot \frac{\partial f_0}{\partial \vec{v}} \, d\vec{v} \leq 0, \qquad (12)$$

with the equality sign holding only if

$$\frac{\partial f_0}{\partial \vec{v}} \cdot \overset{\Rightarrow}{D} \cdot \frac{\partial f_0}{\partial \vec{v}} = 0, \qquad (13)$$

almost everywhere (i.e., except for a possible set of measure zero) in $\vec{v}$. Again let us assume that more than two linearly independent values of $\vec{K}$ are present (otherwise the problem would not be three dimensional). Then Equation (13) could not be satisfied unless $\partial f_0/\partial \vec{v} = 0$ for almost all $\vec{v}$, or equivalently, $f_0(\vec{v})$ = const., almost all $\vec{v}$. Since the equations conserve particles, this constant could only be zero. However, f_0 cannot approach zero in such a way as to conserve $\int f_0 \, d\vec{v}$ and keep $\int \vec{v}^{\,2} f_0 \, d\vec{v}$ finite, and both these quantities are bounded for all t (see Section II). Therefore we have proved that as long as $\gamma(\vec{K}) > 0$ for three or more linearly independent values of $\vec{K}$, $\frac{1}{2}\int f_0{}^2 \, d\vec{v}$ must decrease. However, since this integral is positive, this derivative of $\frac{1}{2}\int f_0{}^2 \, d\vec{v}$ must approach zero. Clearly, the only way to reconcile these two statements is to have the $\gamma(\vec{K}) \longrightarrow 0_+$ as $t \to \infty$, as long as we are not allowing negative $\gamma(\vec{K})$.

In fact, <u>all</u> the $\gamma(\vec{K})$ must go to zero, for the requirement (13) is equivalent to $\vec{K} \cdot \partial f_0/\partial \vec{v} = 0$ if the corresponding $\gamma(\vec{K}) \neq 0$. But f_0 must $\longrightarrow 0$ as $|\vec{v}| \longrightarrow \infty$. By tracing $\hat{k} \cdot \partial f_0/\partial \vec{v} = 0$ in from infinity parallel to the $\vec{K}$

direction, we could prove $f_0 = 0$ at any value of $\vec{v}$ otherwise, and we have already observed this cannot happen. Therefore, all $\gamma(\vec{k}) \to 0_+$ as $t \to \infty$ must characterize any acceptable solution.

The content of Equation (13) is then

$$\overset{\Rightarrow}{D}(\vec{v}, \infty) = \Sigma_{\vec{k}} \ \hat{k}\hat{k}\, \mathcal{E}(\vec{k}) \ \pi\delta \ (\omega(\vec{k}) - \vec{k} \cdot \vec{v}) \tag{14}$$

which then leads to the much weaker statement that

$$\vec{k} \cdot \frac{\partial f_0(\vec{v}, \infty)}{\partial \vec{v}} = 0 \text{ for } \vec{k} \cdot \vec{v} = \omega(\vec{k}), \tag{15}$$

again with the possible exception of a set of measure zero. Equation (15) is simply the statement that the unit normal to the surface $f_0(\vec{v})$ = const. is perpendicular to $\vec{k}$ when $\vec{k} \cdot \vec{v} = \omega(\vec{k})$.

A similar statement applies in two dimensions, and Equations (14) and (15) hold for the two-dimensional case as well.

The loci of points $\vec{v}$ satisfying $\vec{k} \cdot \vec{v} = \omega(\vec{k})$ are a discrete set of planes in three dimensions and a discrete set of lines in two dimensions. It is shown in Figure 1 how the surfaces f_0 = const. might arrange themselves to satisfy Equation (15) in the $t \to \infty$ state.

It is here that the first significant difference between the case of discrete and continuous $\vec{k}$ has shown up. We see that in the former it is not unthinkable that an internally consistent final state might exist where all the $\gamma(\vec{k}) \longrightarrow 0_+$, but do not pass through zero. Bernstein and Englemann concluded that this was impossible for the continuous $\vec{k}$ case, for the following reason. If $\vec{k}$ is continuously distributed, (15) has to be replaced by the much stronger condition that $\partial f_0 / \partial \vec{v} = 0$ over an infinite, continuous subregion of the $\vec{v}$ space, so that f_0 = a constant there. Since particles and energy are (again) conserved, this constant can only be zero. Thus, the only imaginable way we could find a consistent $t \to \infty$ state with all $\gamma(\vec{k}) \geq 0$ would be for the particles to evacuate completely an infinite subregion of the $\vec{v}$ space. This seems physically very unlikely, though no thoroughly satisfactory proof exists that it is impossible.

Bernstein and Englemann were led to the conclusion that an internally consistent final state would require negative $\gamma(\vec{k})$, and that all $\gamma(\vec{k})$ would eventually become negative. We shall return to discuss this possibility in Section IV. For the present, we note that no such necessity arises in the discrete $\vec{k}$ case, although it remains an open question whether or not the solutions of Equations (9)-(11) do have in fact the $t \to \infty$ form we have shown is possible for them.

B. The One Dimensional Case

In one dimension, the condition analogous to $\vec{a} \cdot \overleftrightarrow{D} \cdot \vec{a} > 0$ is $D(\mu,t) > 0$, which holds if we have any $\gamma(k) > 0$ and we restrict γ to be non-negative. Analogous to Equation (12), we have

$$\frac{d}{dt}\int \tfrac{1}{2} F_0^{\,2}\, d\mu = -\int \left(\frac{\partial F_0}{\partial \mu} \right)^2 D\, d\mu \leq 0, \tag{12I}$$

with the equality holding only if

$$\left(\frac{\partial F_0}{\partial \mu} \right)^2 \quad D(\mu,t) = 0 \tag{13I}$$

for all μ. The 1D case is uncomplicated by the geometrical considerations of the 2D and 3D cases. Since if any $\gamma(k)$ is > 0, $D > 0$ everywhere, Equation (13I) would only be satisfied (for any $\gamma(k)>0$) by F_0 = const. This constant would have to be zero by conservation of particles, but F_0 must have a finite second moment, so this cannot be.

We again have a positive definite integral, $\int \frac{1}{2} F_0^{\,2}\, d\mu$, whose time derivative is negative; it must therefore approach a constant as $t \to \infty$. By the remarks of the last paragraph, this can only occur if the final state is characterized by all $\gamma(k) \to 0_+$.

For this reason, the relation analogous to (14) is

$$D(\mu,\infty) = \Sigma_k\, \mathcal{E}(k)\; \pi\delta\; (\omega(k)-k\mu); \tag{14I}$$

note that (14I) is asserted only at $t \to \infty$, a very different matter from asserting it at finite times. The equality (13I) now just requires the vanishing of $F_0'(\mu)$ at a discrete set of points:

$$\frac{\partial\, F_0(\omega(k)/k,\, \infty)}{\partial\, \mu} = 0. \tag{15I}$$

At other values of μ, no information on $F_0(\mu,\infty)$ is explicitly provided.

There is no doubt, then, that such a consistent picture of the 1D final state as $t \to \infty$ can be given, and some numerical details of the approach to this state appear in Section V. There is no reason a priori to include negative $\gamma(k)$, though this possibility is discussed in Section IV.

Passage to the continuum limit is straightforward in 1D, and unlike 2D and 3D, no qualitative conclusions are changed by doing so. Unfortunately, the explicit solution for $\mathcal{E}(k)$ at $t = \infty$ given in the original papers can only be given if the relation (14I) is asserted at <u>finite</u> t. We have already observed the violation of conservation laws this introduces, so there seems to be little practical incentive for considering the continuous k case.

IV. DAMPED WAVES AND NEGATIVE DIFFUSION COEFFICIENTS

Some obscurity has surrounded the subject of the inclusion of damped waves ($\gamma(\vec{k}) < 0$) in the formalism. They have been conventionally held to be important for two and three dimensions, and though not essential in one, desirable. It is our purpose in this section to point out some difficulties associated with the inclusion of $\gamma(\vec{k}) < 0$ waves.

Several preliminaries are in order. First notice that the "H-like" theorem discussed in Section III depends on the result $\vec{a} \cdot \overset{\Rightarrow}{D} \cdot \vec{a} \geq 0$, all $\vec{v}$, in 2D and 3D and on $D \geq 0$ in 1D. From Equations (10) and (10I), we see that these statements no longer hold if some $\gamma(\vec{k})$ can be negative. In particular, if some γ goes from positive to negative, there will always be negative values of $\vec{a} \cdot \overset{\Rightarrow}{D} \cdot \vec{a}$ (or D) near enough to $\omega(\vec{k}) = \vec{k} \cdot \vec{v}$ (or $\omega(k) = k\mu$) immediately after γ goes negative. Since we have considered that only the perturbations which grow out of the noise by e-folding several times are being considered, it makes sense to treat $\int_0^t \gamma(\vec{k})d\tau$ as $\gg 1$, at least until damped waves have been present for several e-folding times. Therefore if damped waves are to be present, they must arise from the transition of some γ from positive to negative. It likewise follows that the presence of any damped waves implies at least a temporary loss of the positive-definite character of the diffusion coefficient over a finite region of its arguments.

Second, it is well known that serious difficulties arise in diffusion-like equations when the diffusion coefficients become negative. This can be illustrated by the following one dimensional example. Consider the equation

$$\frac{\partial F(\mu,t)}{\partial t} = \frac{\partial}{\partial \mu} \left\{ d(\mu) \frac{\partial F(\mu,t)}{\partial \mu} \right\} \qquad (16)$$

where $d(\mu)$ is some given function of μ which is negative in some interval. In that interval, consider a perturbation on the solution of the initial form $F_\alpha(\mu;\ t=0) = F_\alpha(\mu)\ e^{i\alpha\mu}$, where the μ-space wavelength, $2\pi/\alpha$, is so much less than the characteristic size of the interval over which $d(\mu)$ varies that $d(\mu)$ can be treated as essentially constant:

$$\alpha \gg \left| \frac{d'(\mu)}{d(\mu)} \right| . \tag{17}$$

$F_\alpha(\mu,t)$ then obeys the equation

$$\frac{\partial F_\alpha(\mu,t)}{\partial t} \cong -\alpha^2 d F_\alpha(\mu,t), \tag{18}$$

to which the solution is

$$F_\alpha(\mu,t) = F_\alpha(\mu)\ \exp\left\{ -\alpha^2 dt \right\} . \tag{19}$$

Consider $F_\alpha(\mu)$ as given and small, and t as fixed. For $d < 0$, we can make $F_\alpha(\mu,t)$ as large as we like by making the wavelength $2\pi/\alpha$ as small as we like. Said another way, arbitrarily small changes in the initial data make arbitrarily large changes in the answer after fixed, finite t. Such a lack of well-posedness is generally held to render a differential equation useless for describing a physical system. (Note that this is not simply an "instability", where, to get answers arbitrarily far apart for small initial data differences, one must go to <u>infinite</u> t.)

This is an indication, though not a proof, that damped waves may not be possible to include within quasi-linear theory. [In Equation (9I), for instance, D implicitly involves the unknown F_o through $\gamma(k)$, whereas in the example, $d(\mu)$ was a given time-independent function.] The numerical results of Section V, however, strongly indicate that difficulties do arise, since numerical integration routines for solving Equations (9I)-(11I), which function smoothly for $\gamma(k) \geq 0$, become wildly unstable if any $\gamma(k) < 0$.

Parallel arguments are readily constructed for the 2D and 3D cases. We are left with strong indications, but no proof, that damped waves cannot be included in quasi-linear theory in a consistent way.

One way out of the impasse suggests itself, but unfortunately does not seem to help: namely, the inclusion of contributions from additional poles in Equation (5). The reason this does not help is that all these have time dependences $\lesssim \exp[-i\vec{k} \cdot \vec{v}t]$, and contain as a multiplicative factor the initial small noise level from which the unstable waves are assumed to have grown. Since by assumption $\int_0^t \gamma(\vec{k})\, d\tau \gg 1$ when $\gamma(\vec{k})$ first goes negative, the $\exp[-i\vec{k} \cdot \vec{v}t]$ terms are too small to compensate for those retained in Equation (9).

V. NUMERICAL RESULTS IN ONE DIMENSION

The discussion up to this point has centered around possible final states for the quasilinear equations. In the absence of rigorous existence proofs for the solutions, the main avenue to determining whether the equations do in fact lead to such final states would seem to be by solving them numerically. Drummond and Pines (1962) in their original paper considered the numerical problem of solving Equations (9I)-(11I) in the resonance approximation $\gamma/[\gamma^2+(\omega-k\mu)^2] \rightarrow \pi\delta(\omega-k\mu)$. We have already observed, among other things, that conservation of momentum and energy are no longer preserved under this approximation; it is therefore of interest to give a numerical solution of Equations (9I)-(11I) without making the resonance approximation.

For numerical purposes, it is convenient to refer all quantities to dimensionless units, which may be thought of as lengths being measured in units of the Debye length V_e/ω_p, times in units of the inverse plasma frequency ω_p^{-1}, and velocities in units of the thermal speed V_e.

The initial velocity distribution is a Maxwellian, plus a Maxwellian "bump" on the tail:

$$F_o(\mu,0) = (2\pi)^{-\frac{1}{2}} \Big[0.93 \exp(-\mu^2/2) + 0.07 \exp(-(\mu-4)^2/2) \Big] , \qquad (20)$$

in the dimensionless units. Since we are interested only in the qualitative features of the solutions to (9I)-(11I), having once derived the equations, we can relax somewhat the various inequalities used to derive them. [Thus, for example, the "bump" in Equation (20) is located at about four thermal velocities, and it is largely a matter of taste as to whether 4 can be counted as $\gg$ 1.] The boundary conditions $F_o(7,t) = F_o(7,0)$, $F_o(-7,t) = F_o(-7,0)$ are imposed for convenience; all the significant development of F_o occurs at velocities less than that.

In Figure 2, the development of $F_o(\mu,t)$ in the neighborhood of the unstable waves is shown. There are ten values of k present, and their phase velocities, 1/k, are located between μ=3.0 and μ=3.9. The initial values of the $\mathcal{E}(k)$ are all 0.003. Considerably finer detail in $F_o(\mu,t)$ and $D(\mu,t)$ needs to be retained in the neighborhood of the phase velocities than elsewhere. An appropriately matched <u>explicit</u> finite difference scheme for $-7 \leq \mu \leq 2.1$ and an <u>implicit</u> finite difference scheme for $2 \leq \mu \leq 7$ are used. The γ's decrease, and as they get closer to zero, $D(\mu,t)$ of course becomes more sharply peaked. This necessitates a much smaller step-size ($\Delta\mu=0.025$, $\Delta t=0.005$) in the resonant (implicit) region than elsewhere, where $\Delta\mu$ can be chosen as 0.1, and Δt as 0.01.

It can be seen from Figure 2 that in the course of the development of F_o, regions of positive $F_o'(\mu,t)$ can result at values of μ where $F_o'(\mu,0)$ was negative. It is interesting to observe the different consequences of allowing phase velocities which lie only in the region of initially positive $F_o'(\mu,0)$, and allowing those which also lie in regions which would initially correspond to negative γ. These are the dased and solid lines, respectively, in Figures 2. A stable program can be applied to either case if we simply instruct the computer to set $\partial\mathcal{E}(k)/\partial t = 0$ at a given time step if γ is negative, but to obey Equation (11I) if $\gamma > 0$. (This is the equivalent of only counting growing waves, analytically.)

In Figure 3, a logarithmic plot of $D(\mu,t)$ at an early time and at a late time is shown. In Figure 4, a plot of the entire $F_o(\mu,t)$ is shown; note the modification of $F_o(\mu,\infty)$ far from the resonant region.

Figure 5 is for a three wave case, wherein one of the phase velocities has an associated $\gamma < 0$ initially. By $t = 2.0$, appreciable unstable oscillations have developed, and by a later time, they are off scale. This is characteristic of attempts to include negative γ.

VI. SUMMARY

It has been argued that: (1) in the case of discrete $\vec{k}$-spectra, a consistent version of the quasilinear theory can be given without damped waves in one, two, and three dimensions; (2) the inclusion of damped waves by any known prescription renders the theory ill-posed in one dimension, and very probably in two and three; (3) qualitative differences in the theory, the origin of which is not fully understood, appear in the transition to continuous $\vec{k}$ in two and three dimensions; (4) for an internally self-consistent theory satisfying the conservation laws, one must not use the "resonance approximation" $\gamma/\omega \to 0$ at finite times.

In (4), we should remark on a recent proof by G. Knorr (1969) that if an expansion in γ/ω is done correctly, new terms in the diffusion coefficient appear which recover the conservation laws. However, the usual $t \to \infty$ calculation of the one dimensional $\mathcal{E}(k)$ can no longer be salvaged.

In closing, we should remark on recent numerical simulations for 2D and 3D plasmas by Morse and Nielson (1969). Because the $\mathcal{E}(\vec{k}) \xrightarrow[t \to \infty]{} 0$, they imply confirmation of the Bernstein-Englemann continuous $\vec{k}$ version of the theory, which also has this as a prediction. It appears that no more detailed comparison than that has been attempted, however, and the simulation at this point stands as equally compelling evidence for any other theory that predicts the vanishing of the electrostatic field energy at long times.

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FIGURE CAPTIONS

Figure 1. Possible contours of $f_o(\vec{v})$ = const. in two dimensional case. The smooth curves are the t = 0 values and the distorted ones are for $t = \infty$. Note that for discrete $\vec{K}$, $\vec{K} \cdot \partial f_o/\partial\vec{v} = 0$ at $t = \infty$ (except for a set of measure zero) on the lines $\omega = \vec{k} \cdot \vec{v}$, leaving the integral expression on the right of Equation (12) zero.

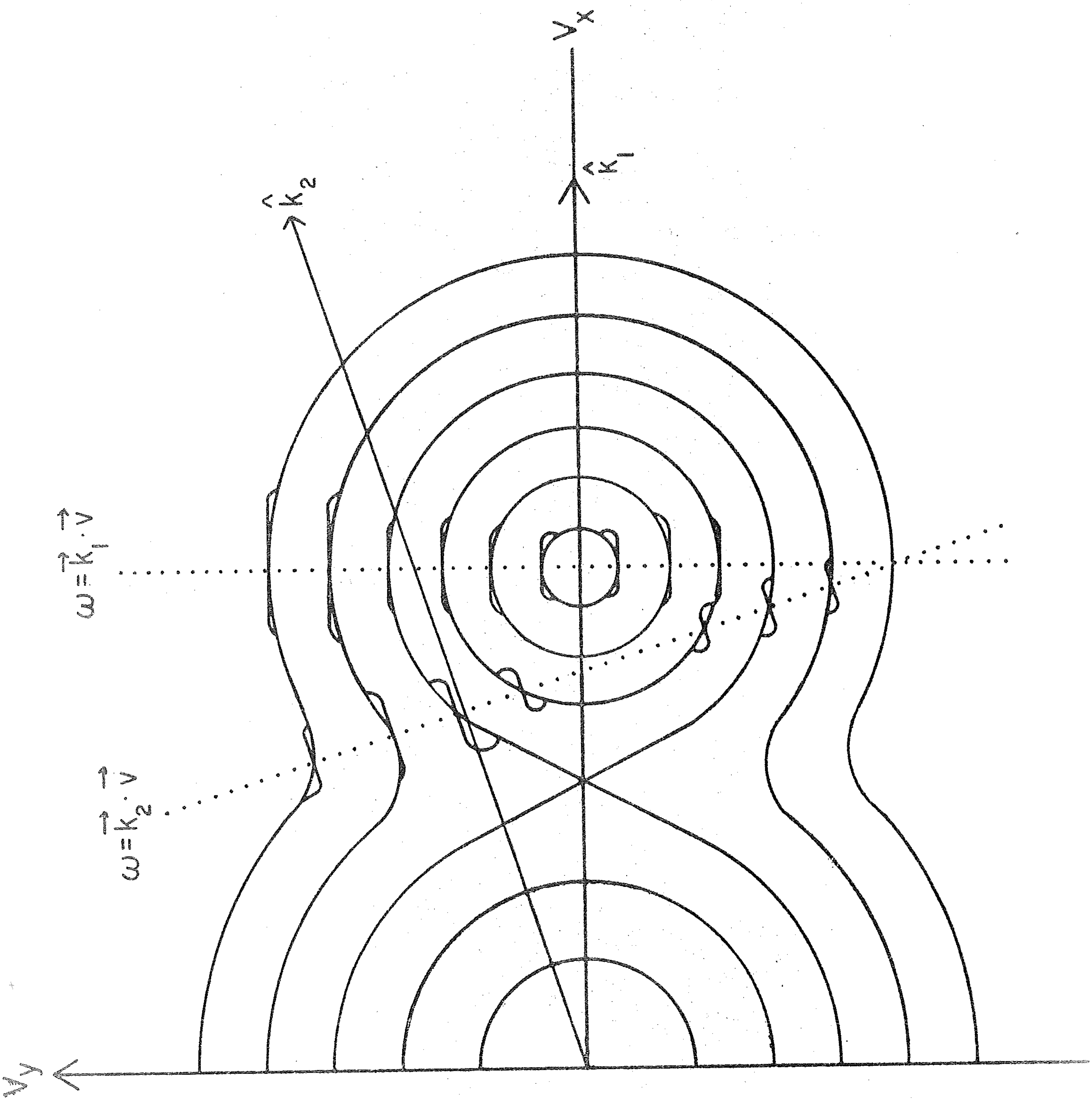
Figure 2. One dimensional $F_0(\mu,t)$ in the neighborhood of the bump on the tail, for the case of ten waves. The dotted lines pertain to the case where only initially positive $\gamma(k)$ are included, whereas the solid curves also include waves which had initially negative γ [and for which $\partial \mathcal{E}(k)/\partial t$ was initially set = 0] which later went positive.

Figure 3. Logarithmic plot of the one dimensional diffusion coefficient $D(\mu,t)$ at two times. The situations correspond to those in Figure 2. Note that by t = 1.66, some of the $\gamma(k)[(\omega-k\mu)^2 + \gamma^2(k)]^{-1}$ have already essentially become delta functions.

Figure 4. Plot of the total distribution in one dimension initially, and what is essentially the $t \to \infty$ state. The region of phase velocities corresponding to waves develops the familiar plateau structure, but the plateau level is higher than that predicted by the resonance approximation. The main body of the distribution is modified down to $\mu = 0$.

Figure 5. The effect of including one negative $\gamma(k)$. In contrast to the cases shown previously, a negative $\gamma(k)$ was included, and the corresponding $\partial \mathcal{E}(k)/\partial t$ was <u>not</u> set equal to zero. There are only three waves present in this case. By t = 2.0, strong numerical instabilities had developed; slightly later, $F_0(\mu,t)$ was off scale and < 0. Such behavior was only observed when it was attempted to compute with negative $\gamma(k)$.

Figure 1



v_x
$\hat{k}_1$
$\hat{k}_2$
$\omega = \vec{k}_1 \cdot \vec{v}$
$\omega = \vec{k}_2 \cdot \vec{v}$
v_y

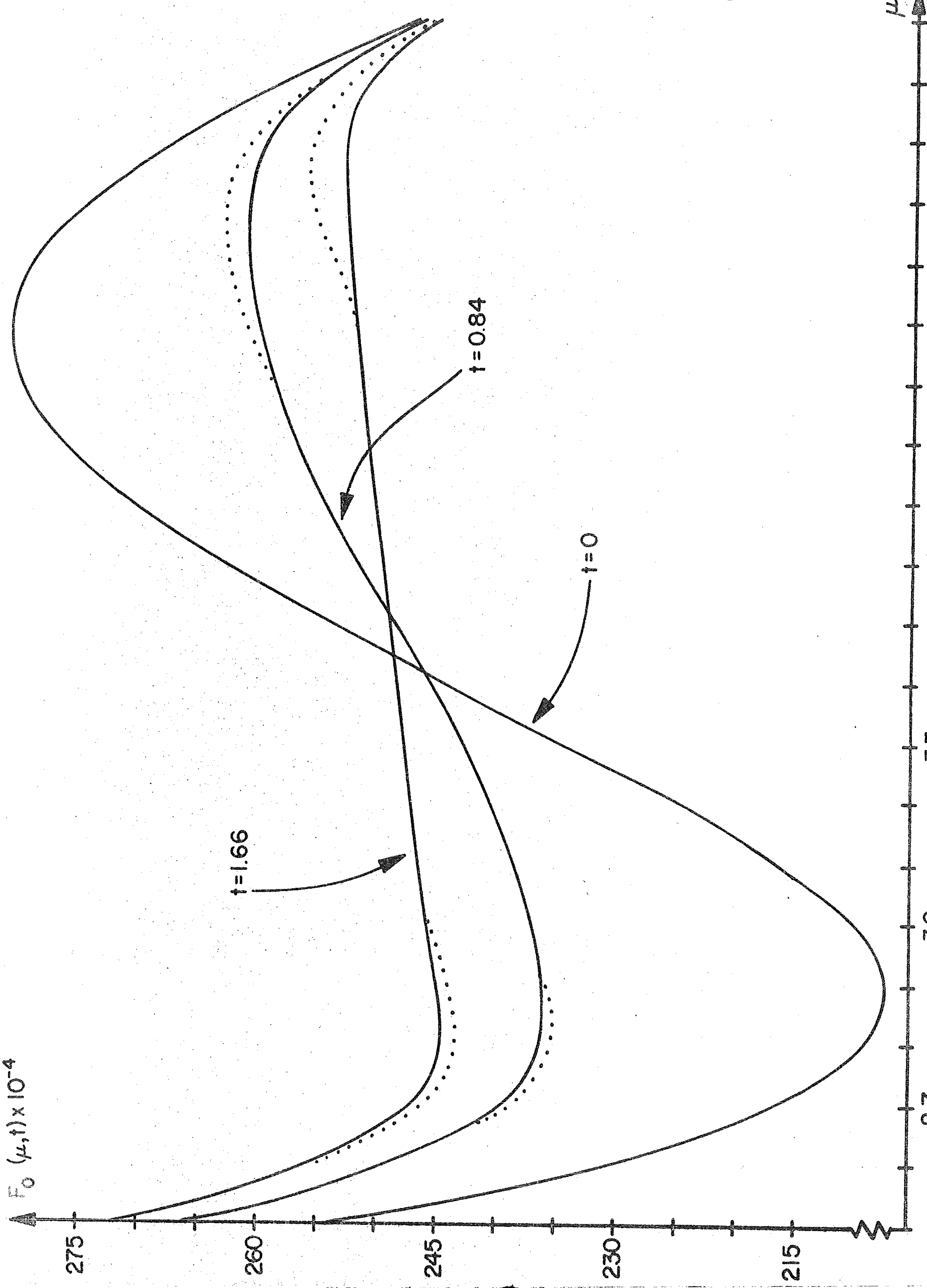


Figure 2

Figure 3

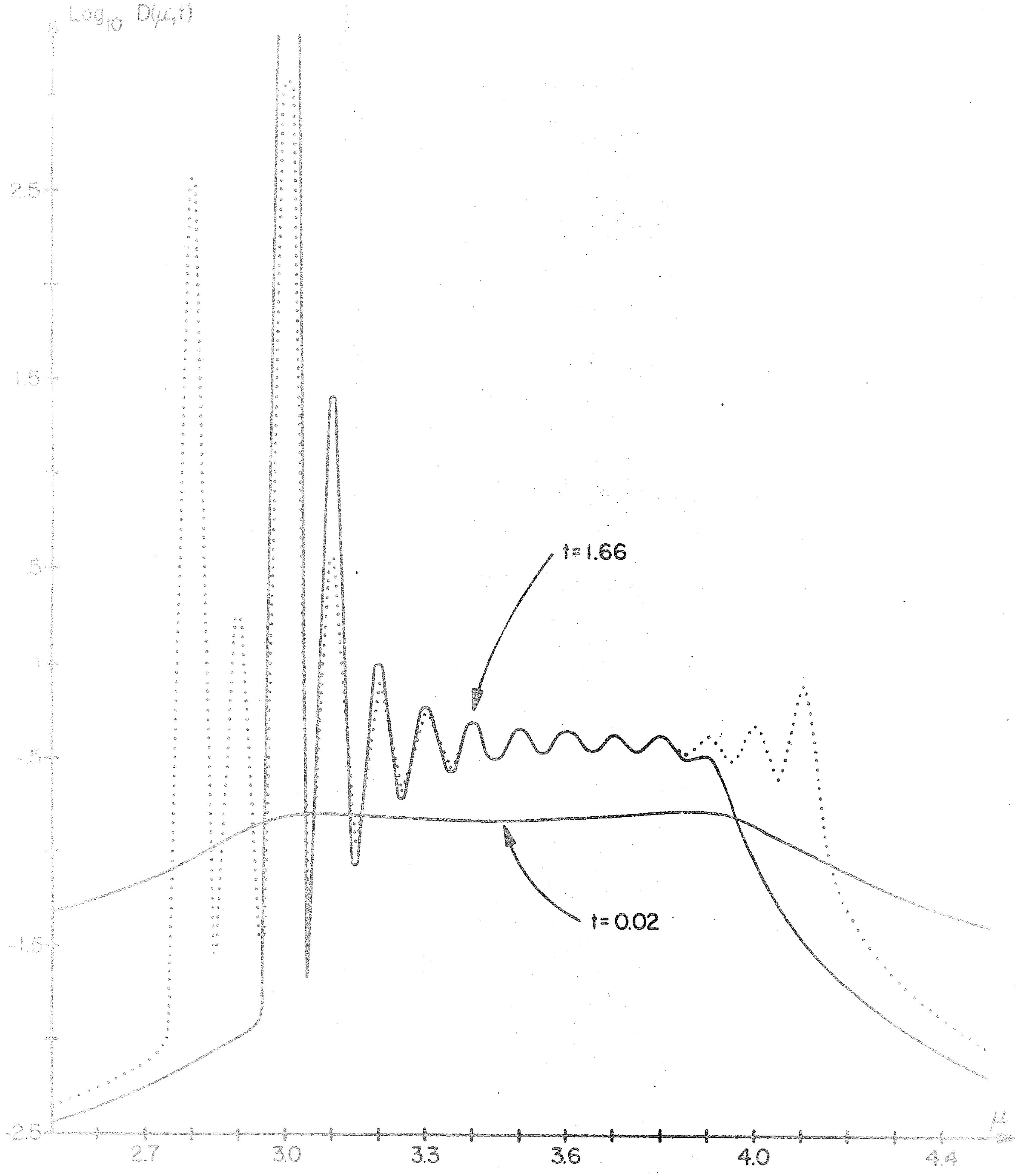
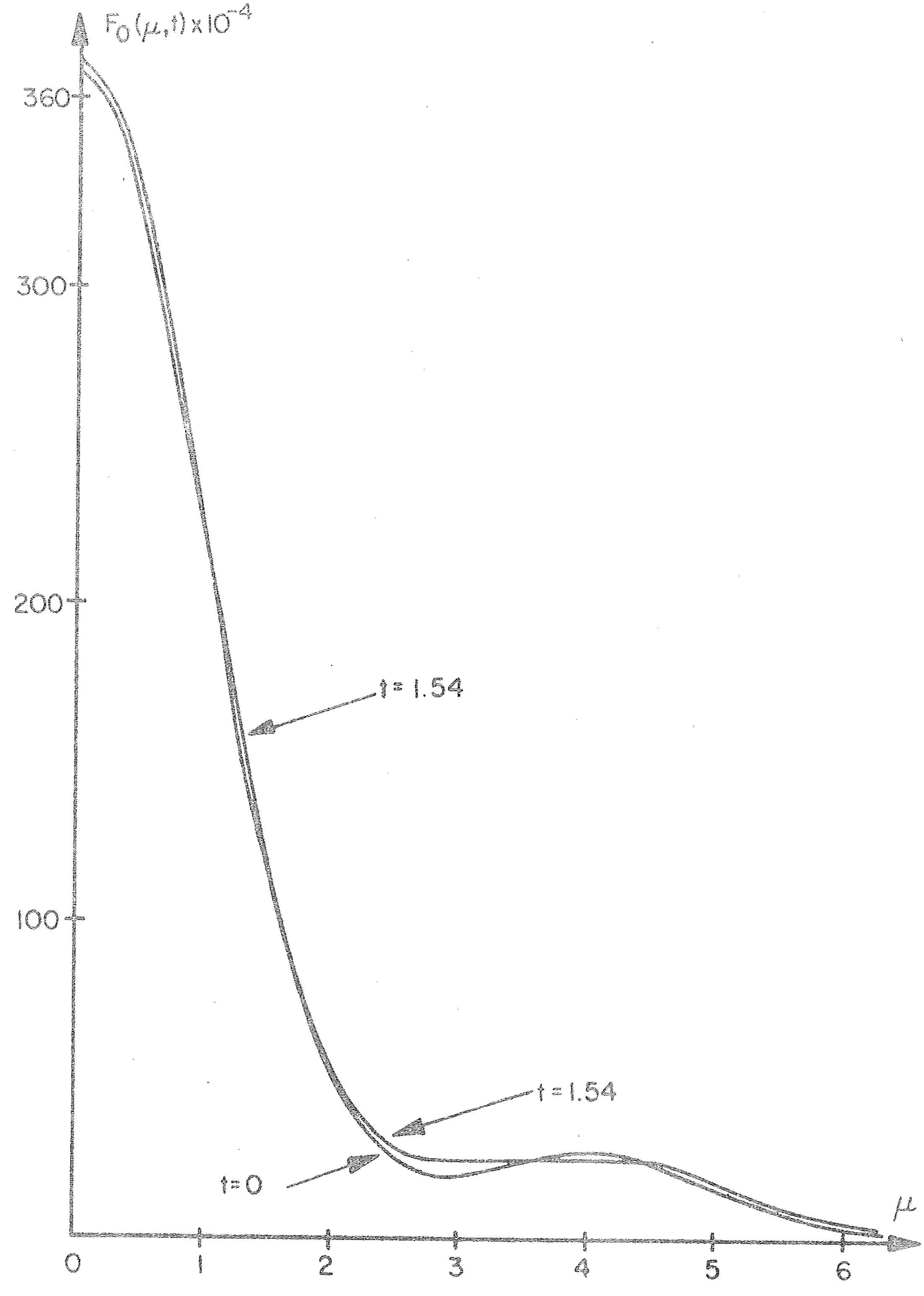


Figure 4



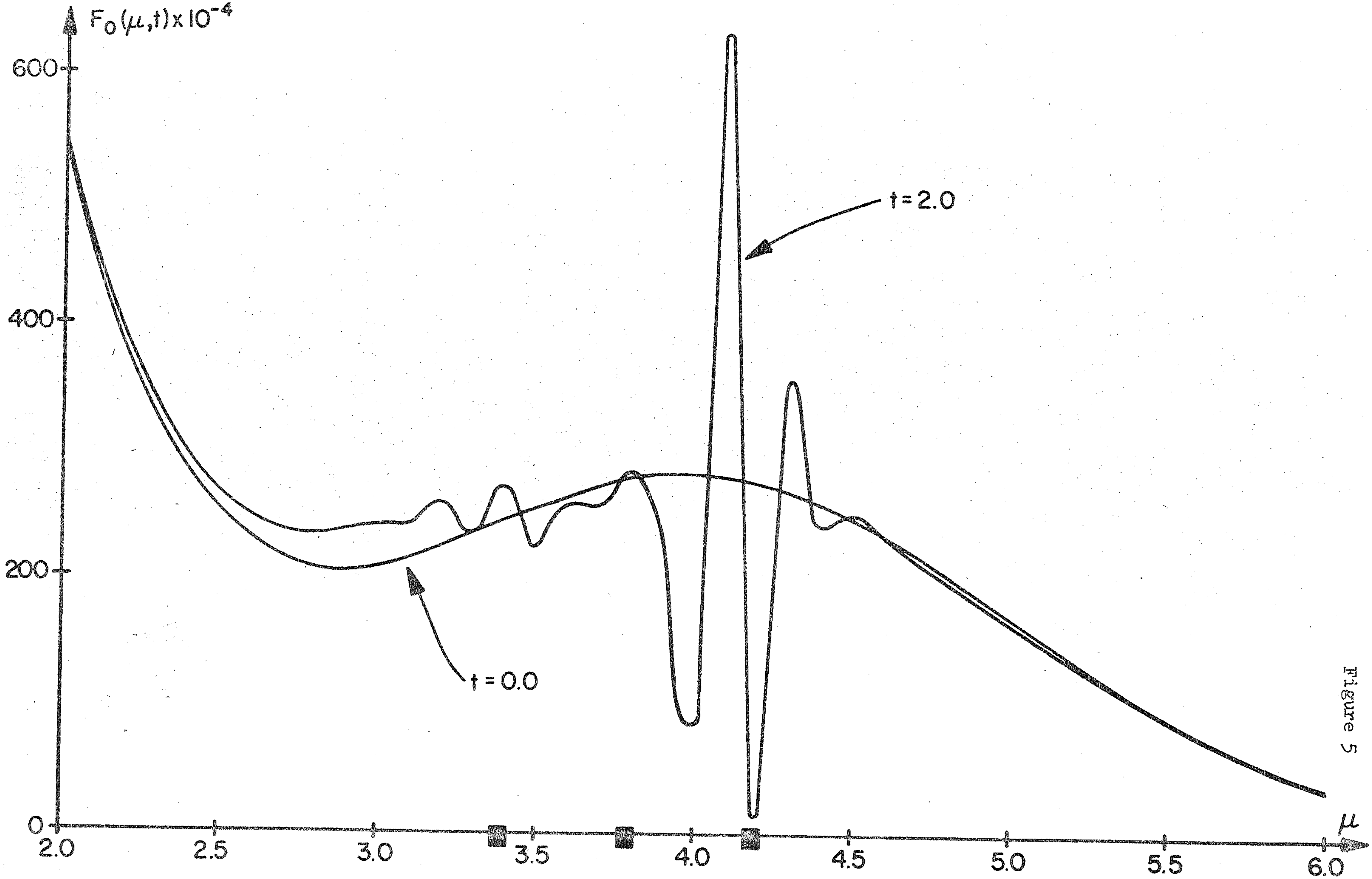


Figure 5