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# Discrete strip-concave functions, Gelfand–Tsetlin patterns, and related polyhedra

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## Abstract

Discrete strip-concave functions considered in this paper are, in fact, equivalent to an extension of Gelfand–Tsetlin patterns to the case when the pattern has a not necessarily triangular but convex configuration. They arise by releasing one of the three types of rhombus inequalities for discrete concave functions (or “hives”) on a “convex part” of a triangular grid. The paper is devoted to a combinatorial study of certain polyhedra related to such functions or patterns, and results on faces, integer points and volumes of these polyhedra are presented. Also some relationships and applications are discussed.

In particular, we characterize, in terms of valid inequalities, the polyhedral cone formed by the boundary values of discrete strip-concave functions on a grid having trapezoidal configuration. As a consequence of this result, necessary and sufficient conditions on a pair of vectors to be the shape and content of a semi-standard skew Young tableau are obtained.

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*Keywords:* Triangular grid; Gelfand–Tsetlin pattern; Discrete concave function; Young tableau

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**1. Introduction**

Let  $n \in \mathbb{N}$  Consider a two-dimensional array  $X = (x_{ij})_{0 \leq i \leq n, a_i \leq j \leq b_i}$  of reals, where the index bounds  $a_i, b_i$  (depending on rows) are integers satisfying  $a_i \leq b_i$  and

$$a_0 = 0, \quad 0 \leq a_1 - a_0 \leq a_2 - a_1 \leq \dots \leq a_n - a_{n-1} \leq 1, \\ \text{and} \quad 1 \geq b_1 - b_0 \geq b_2 - b_1 \geq \dots \geq b_n - b_{n-1} \geq 0. \tag{1}$$

We denote the set of pairs  $ij$  of indices in  $X$  by  $V$  and say that  $X$  has *convex configuration*. (This term is justified by the fact that  $V$  can be identified with the set of nodes of a convex triangular grid; see Remark 1. We visualize  $X$  so that  $(x_{00}, \dots, x_{0b_0})$  is the topmost row and each triple  $x_{ij}, x_{i+1,j}, x_{i+1,j+1}$  or  $x_{ij}, x_{i,j+1}, x_{i+1,j+1}$  is disposed so as to form an equilateral triangle. Then the array is shaped like a convex polygon, with 3–6 sides.) Two examples of such arrays are depicted in Fig. 1.

Depending on the shape of the corresponding convex polygon, we may speak of hexagonal configuration, pentagonal configuration, etc. Although main results in this paper will be applicable to any of these, three special cases with  $a_1 = \dots = a_n = 0$  are of most interest for us: (a)  $b_i = i$  for each  $i$  (giving a  $\Delta$ -array); (b)  $b_i = i + m$  for each  $i$  (a  $\square$ -array), see Fig. 1b; (c)  $b_i = m$  for each  $i$  (a  $\square$ -array), where  $m \in \mathbb{N}$ . In these cases we will also refer to an array as having *triangular, trapezoidal, or parallelogram-wise configuration*, respectively (usually ignoring other possible dispositions of triangle, trapezoid, or parallelogram). We say that  $X$  has *size*  $n$  in case (a), and  $(n, m)$  in cases (b),(c). Sometimes we will admit  $m = 0$  in case (b), regarding  $\Delta$ -arrays as a degenerate case of  $\square$ -arrays.

Let us associate with  $X$  the array  $\partial X = (\partial x_{ij})_{0 \leq i \leq n, a_i+1 \leq j \leq b_i}$  of local differences  $\partial x_{ij} := x_{ij} - x_{i,j-1}$ , referring to  $\partial X$  as the *row derivative* of  $X$ . We deal with arrays  $X$  satisfying the following condition: for  $i = 1, \dots, n$  and  $j = a_i + 1, \dots, b_i$ ,

$$\partial x_{ij} \geq \partial x_{i-1,j} \quad (\text{when } j \leq b_{i-1}) \quad \text{and} \quad \partial x_{i-1,j} \geq \partial x_{i,j+1} \quad (\text{when } j < b_i). \tag{2}$$

The array  $\partial X$  obeying (2) and having triangular configuration is said to be a *Gelfand–Tsetlin pattern*, and in this paper we apply the same name to  $\partial X$  with such a property when  $X$  has an arbitrary convex configuration as well. In this case we call  $X$  a *strip-concave* array, using an analogy with the corresponding functions explained in Remark 1. For example, both arrays in Fig. 1 are strip-concave; their row derivatives are shown in Fig. 2.

One can identify the set of all arrays for  $V$  with the Euclidean space  $\mathbb{R}^V$  whose unit base vectors are indexed by the pairs  $ij \in V$ . Let  $\mathcal{SC}_V$  denote the set of arrays  $X \in \mathbb{R}^V$  that

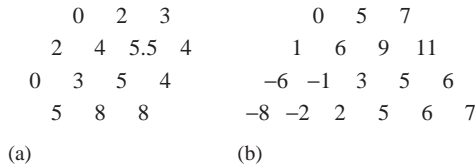


Fig. 1. (a) A hexagonal array with  $n = 3, a = (0, 0, 0, 1), b = (2, 3, 3, 3)$ ; (b) a trapezoidal array with  $n = 3, a = (0, 0, 0, 0), b = (2, 3, 4, 5)$ .

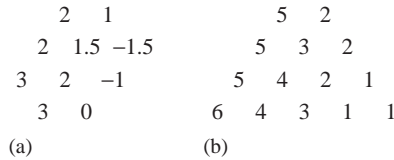


Fig. 2. Gelfand–Tsetlin pattern examples: (a)  $\partial X$  for  $X$  in Fig. 1a; (b)  $\partial X$  for  $X$  in Fig. 1b.

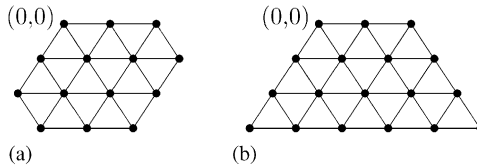


Fig. 3. (a) The grid for the array in Fig. 1a; (b) the grid for the array in Fig. 1b.

satisfy property (2) and the normalization condition  $x_{00} = 0$ ; imposing this condition leads to no loss of generality in what follows. Then  $\mathcal{SC}_V$  is a polyhedral cone in  $\mathbb{R}^V$ .

**Remark 1.** Let  $\alpha, \beta$  be linearly independent vectors in  $\mathbb{R}^2$ . By a *convex (triangular) grid* we mean a finite planar graph  $G = (V, E)$  embedded in the plane so that each node of  $G$  is a point with integer coordinates  $(i, j)$  in the basis  $(\alpha, \beta)$ , each edge is the straight-line segment connecting a pair  $u, v$  of nodes with  $u - v \in \{\alpha, \beta, \alpha + \beta\}$ , each bounded face is a triangle with three edges (a *little triangle* of  $G$ ), and the union  $\mathcal{R}$  of bounded faces covers all nodes and forms a convex polygon in the plane. A convex grid can be considered up to an affine transformation, and to agree with the above visualization of arrays, one should take the generating vectors as, e.g.,  $\alpha := (-1/2, -\sqrt{3}/2)$  and  $\beta := (1, 0)$  and assume that  $(0, 0) \in V$  and  $(i, j) \geq (0, 0)$  for all  $(i, j) \in V$ . (The convex grids behind the arrays in Fig. 1 are exposed in Fig. 3). A function  $x : V \rightarrow \mathbb{R}$  determines an array  $X$  of convex configuration in a natural way:  $x_{ij} := x(i, j)$ . The arrays in  $\mathcal{SC}_V$  (considering  $V$  as the index set) are determined by the functions  $x$  having the following property: if  $f$  is the extension of  $x$  to  $\mathcal{R}$  which is affinely linear on each bounded face of  $G$ , then  $f$  is a concave function within each region (strip) confined by the boundary of  $G$  and lines  $i\alpha + \mathbb{R}\beta$  and  $(i - 1)\alpha + \mathbb{R}\beta$ ,  $i = 1, 2, \dots$ . We call such a function  $x$  *discrete strip-concave* (by an analogy with discrete concave functions; see Remark 2 in the end of this section), and accordingly apply the adjective “strip-concave” to the arrays with property (2).

Local differences on the “boundary” of  $X$  will be of most interest for us in this paper. These are represented by four tuples  $\lambda^X, \bar{\lambda}^X, \mu^X, \nu^X$  (concerning the lower, upper, left and right boundaries, respectively) defined by

$$\begin{aligned} \lambda_j^X &:= \partial x_{nj}, \quad j = 1, \dots, b_n; & \bar{\lambda}_{j'}^X &:= \partial x_{0j'}, \quad j' = 1, \dots, b_0; \\ \mu_i^X &:= x_{ia_i} - x_{i-1, a_{i-1}} & \text{and} & \quad \nu_i^X := x_{ib_i} - x_{i-1, b_{i-1}}, \quad i = 1, \dots, n. \end{aligned}$$

( $\bar{\lambda}^X$  vanishes when  $b_0 = 0$ .) For example, the array  $X$  in Fig. 1a has  $\lambda^X = (3, 0)$ ,  $\bar{\lambda}^X = (2, 1)$ ,  $\mu^X = (2, -2, 5)$  and  $\nu^X = (1, 0, 4)$ , and the array  $X$  in Fig. 1b has  $\lambda^X = (6, 4, 3, 1, 1)$ ,  $\bar{\lambda}^X = (5, 2)$ ,  $\mu^X = (1, -7, -2)$  and  $\nu^X = (4, -5, 1)$ .

Given  $\lambda = (\lambda_{a_n+1}, \dots, \lambda_{b_n})$ ,  $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_{b_0})$  and  $\mu, \nu \in \mathbb{R}^n$ , define

$$SC(\lambda \setminus \bar{\lambda}, \mu, \nu) := \{X \in SC_V : (\lambda^X, \bar{\lambda}^X, \mu^X, \nu^X) = (\lambda, \bar{\lambda}, \mu, \nu)\}.$$

This set, if nonempty, forms a bounded polyhedron (a polytope) in  $\mathbb{R}^V$  in case of  $\triangleleft$ - and  $\square$ -arrays. Indeed, (2) and  $x_{00} = 0$  imply

$$x_{ij} \leq \mu_1 + \dots + \mu_i + \lambda_1 + \dots + \lambda_j \quad \text{and} \quad x_{ij} \geq \mu_1 + \dots + \mu_i + q, \tag{3}$$

where  $q := \lambda_{n-i+1} + \dots + \lambda_{n-i+j}$  for  $\triangleleft$ -arrays, and  $q := \bar{\lambda}_1 + \dots + \bar{\lambda}_j$  for  $\square$ -arrays. (On the other hand, such a polyhedron  $\mathcal{P}$  is unbounded when there is at least one interior entry and both left and right boundaries make a bend, i.e.,  $0 < a_n < n$  and  $0 < b_n - b_0 < n$ ; in particular, if the hexagonal configuration takes place. One can check that adding any positive constant to all interior entries of an array  $X \in \mathcal{P}$  gives a point in  $\mathcal{P}$  as well.)

The first problem we deal with in this paper is to characterize the set  $\mathcal{B}_V$  of all quadruples  $(\lambda, \bar{\lambda}, \mu, \nu)$  (depending on  $V$ ) such that  $SC(\lambda \setminus \bar{\lambda}, \mu, \nu)$  is nonempty. Two conditions on such quadruples are trivial. The first one comes up from the fact that (2) implies that  $\lambda^X$  is weakly decreasing, i.e.,  $\lambda_{a_n+1}^X \geq \dots \geq \lambda_{b_n}^X$ , and similarly for  $\bar{\lambda}$ . The second one comes up by observing that

$$|\lambda^X| - |\bar{\lambda}^X| + |\mu^X| - |\nu^X| = (x_{nb_n} - x_{na_n}) - (x_{0b_0} - x_{00}) + (x_{na_n} - x_{00}) - (x_{nb_n} - x_{0b_0}) = 0,$$

where for a tuple (vector)  $d = (d_p, \dots, d_q)$ ,  $|d|$  stands for  $\sum(d_i : i = p, \dots, q)$ .

To obtain the desired characterization, we need to introduce certain values depending on  $\lambda, \bar{\lambda}$ . For  $k \in \mathbb{Z}_+$ , define

$$\delta_k(j) := \max\{0, \bar{\lambda}_{j-k} - \lambda_j\}, \quad j = a_n + 1, \dots, b_n, \quad \text{and} \\ \Delta_k := \delta_k(a_n + 1) + \dots + \delta_k(b_n),$$

letting by definition  $\delta_k(j) := 0$  if  $j - k \leq 0$  or  $j - k > b_0$ . We refer to  $\Delta_k$  as the  $k$ th deficit of  $\lambda \setminus \bar{\lambda}$ .

We shall explain later that the above problem is reduced to the case of trapezoidal configuration. Necessary and sufficient conditions on the corresponding quadruples for  $\triangleleft$ -arrays are given in the following theorem. Hereinafter, for  $d = (d_p, \dots, d_q)$  and  $I \subseteq \{p, \dots, q\}$ ,  $d(I)$  denotes  $\sum(d_i : i \in I)$ , and for  $p \leq k \leq k' \leq q$ ,  $d[k, k']$  denotes  $d_k + \dots + d_{k'}$ .

**Theorem 1.** For  $n \in \mathbb{N}$  and  $m \in \mathbb{Z}_+$ , let  $\lambda = (\lambda_1, \dots, \lambda_{n+m})$  and  $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_m)$  be weakly decreasing, and let  $\mu, \nu \in \mathbb{R}^n$  be such that  $|\lambda| - |\bar{\lambda}| + |\mu| - |\nu| = 0$ . Then a strip-concave  $\triangleleft$ -array  $X$  with  $(\lambda^X, \bar{\lambda}^X, \mu^X, \nu^X) = (\lambda, \bar{\lambda}, \mu, \nu)$  exists if and only if the inequality

$$\lambda[1, |I|] + \mu(I) - \nu(I) - \Delta_{|I|} \geq 0 \tag{4}$$

holds for each (including empty) subset  $I \subseteq \{1, \dots, n\}$ . Furthermore, if  $\lambda, \bar{\lambda}, \mu, v$  are integer and the polytope  $SC(\lambda \setminus \bar{\lambda}, \mu, v)$  is nonempty, then it contains an integer point.

In particular,  $\mathcal{B}_V$  forms a polyhedral cone (in  $\mathbb{R}^{n+m} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n$ ) for  $V$  in question. Also (4) implies evident relations  $\lambda_j \geq \bar{\lambda}_j$  ( $j = 1, \dots, m$ ) and  $\lambda_j \leq \bar{\lambda}_{j-n}$  ( $j = n+1, \dots, n+m$ ), where the former is easily obtained by taking  $I = \emptyset$ , and the latter by comparing  $|\lambda| - |\bar{\lambda}| + |\mu| - |v| = 0$  with (4) for  $I = \{1, \dots, n\}$ .

Note that relation (4) involves a piece-wise linear term, namely,  $\Delta_{|I|}$ . One can replace each instance of (4) by a collection of  $2^m$  linear inequalities, yielding an equivalent version of Theorem 1. This version, giving rise to a description of the facets of the cone  $\mathcal{B}_V$ , is discussed in Section 4. (It turns out that the number of facets of  $\mathcal{B}_V$  grows exponentially in  $n, m$ . On the other hand, to verify that a given quadruple  $(\lambda, \bar{\lambda}, \mu, v)$  belongs to  $\mathcal{B}_V$ , it suffices to check validity of (4) only for  $n + 1$  sets  $I$ : for  $k = 0, \dots, n$ , take  $I$  with  $|I| = k$  maximizing  $(v - \mu)(I)$ .)

For an arbitrary convex configuration, the problem with prescribed local differences  $\lambda, \bar{\lambda}, \mu, v$  is reduced to the trapezoidal case as follows. Since the polyhedron  $\mathcal{P} := SC(\lambda \setminus \bar{\lambda}, \mu, v)$  is described by a linear system formed by the inequalities in (2) and the corresponding equalities involving  $\lambda, \bar{\lambda}, \mu, v$ , one can efficiently compute a number  $c \in \mathbb{R}_+$  such that if  $\mathcal{P}$  is nonempty, then there exists  $X \in \mathcal{P}$  with  $|x_{ij}| < c/2$  for all entries  $x_{ij}$ . (For example, one can roughly take  $c$  equal to  $|V|^{|V|}$  times the maximum absolute value  $\alpha$  of the entries in  $\lambda, \bar{\lambda}, \mu, v$ , taking into account that the constraint matrix of the system has entries 0, 1, -1. In fact, there is a bound  $c$  linear in  $\alpha|V|$ ; cf. (3) for  $\square$ -arrays.) Suppose  $a_n \neq 0$  and take the maximum  $p$  with  $a_p = 0$  (then  $a_i = i - p$  for  $p < i \leq n$ ). Add to  $V$  the set  $A$  of pairs  $ij$  with  $0 \leq j < i - p \leq n - p$ , define  $\lambda'_j := c$  for  $j = 1, \dots, n - p$ , and define  $\mu'_i := \mu_i - c$  for  $i = p + 1, \dots, n$ . Symmetrically, if  $b_n < b_0 + n$ , we take the maximum  $q$  with  $b_q = b_0 + q$ , add the set  $B$  of pairs  $ij$  with  $1 \leq j - b_n \leq i - q \leq n - q$ , define  $\lambda'_j := -c$  for  $j = b_n + 1, \dots, b_n + n - q$ , and define  $v'_i := v_i - c$  for  $i = q + 1, \dots, n$ . Let  $\lambda'$  coincide with  $\lambda$  for the remaining entries, and similarly for  $\mu', v'$ . The resulting  $V' := V \cup A \cup B$  gives a trapezoid (of size  $(n, b_0)$ ), and it is straightforward to verify that the set  $\mathcal{P}' := SC(\lambda'/\bar{\lambda}, \mu', v')$  (concerning  $V'$ ) is nonempty if and only if  $\mathcal{P}$  is so, that the restriction of any  $X' \in \mathcal{P}'$  to  $V$  belongs to  $\mathcal{P}$ , and that  $X$  as above is extended in a natural way to an array in  $\mathcal{P}'$ .

Applying this reduction to the parallelogram-wise configuration of size  $(n, m)$ , one can derive the following corollary from Theorem 1.

**Corollary 1.** *Let  $n, m \in \mathbb{N}$ , and let  $\mu, v \in \mathbb{R}^n$  and weakly decreasing  $\lambda, \bar{\lambda} \in \mathbb{R}^m$  satisfy  $|\lambda| - |\bar{\lambda}| + |\mu| - |v| = 0$ . Then a strip-concave  $\square$ -array  $X$  with  $(\lambda^X, \bar{\lambda}^X, \mu^X, v^X) = (\lambda, \bar{\lambda}, \mu, v)$  exists if and only if for each subset  $I \subseteq \{1, \dots, n\}$ , the inequality*

$$\lambda[1, |I|] - \bar{\lambda}[m - |I| + 1, m] + \mu(I) - v(I) - \Delta_{|I|} \geq 0$$

holds for  $|I| \leq m$ , and the inequality

$$|\lambda| - |\bar{\lambda}| + \mu(I) - v(I) \geq 0$$

holds for  $|I| > m$ . Furthermore, if  $\lambda, \bar{\lambda}, \mu, \nu$  are integer and the polytope  $SC(\lambda \setminus \bar{\lambda}, \mu, \nu)$  is nonempty, then it contains an integer point.

(To see this, observe that each entry  $v'_i$  for the new right boundary tuple is equal to  $v_i - c$ , that  $\mu' = \mu$ , and that  $\lambda'_j = -c$  for  $j = m + 1, \dots, m + n$ . The fact that  $\bar{\lambda}$  has all entries greater than  $-c$  implies that for  $k = 0, \dots, n$ , each  $j$  with  $\max\{m, k\} < j \leq m + k$  contributes  $\bar{\lambda}_{j-k} + c$  units to the new  $k$ -deficit  $\Delta'_k$  (whereas  $\delta'_k(j) = \delta_k(j)$  for  $j = 1, \dots, m$  and  $\delta'_k(j) = 0$  for the remaining  $j$ 's). Therefore, given  $I \subseteq \{1, \dots, n\}$ , the new  $|I|$ -deficit becomes  $\Delta_{|I|} + \bar{\lambda}[m + 1 - |I|, m] + |I|c$  whenever  $|I| \leq m$ , and  $|\bar{\lambda}| + mc$  whenever  $|I| > m$ . Also  $\lambda'[1, |I|] = \lambda[1, |I|]$  if  $|I| \leq m$ , and  $\lambda'[1, |I|] = |\lambda| - (|I| - m)c$  if  $|I| > m$ . Now Corollary 1 is obtained from Theorem 1 by substituting these relations, together with  $\mu'(I) = \mu(I)$  and  $\nu'(I) = \nu(I) - |I|c$ , into relation (4) (taken with primes).)

A converse reduction, from  $\triangleleft$ - to  $\square$ -case, is easily constructed as well, and Theorem 1 follows from Corollary 1. In contrast, we cannot point out a “simple” reduction of Theorem 1 to its special case with  $m = 0$  concerning  $\Delta$ -arrays. (Nevertheless, a more intricate, though constructive, way of reducing does exist, as we explain in part D of Section 4. In fact, this sort of reduction is behind our method of proof of Theorem 1 where the case  $m = 0$  is used as a base.)

Another object of our study is the set of vertices of the polyhedron formed by strip-concave arrays  $X$  with convex configuration whose entries are fixed only on the lower, upper and left boundaries. More precisely, for  $\lambda = (\lambda_{a_n+1}, \dots, \lambda_{b_n}), \bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_{b_0})$  and  $\mu \in \mathbb{R}^n$ , define

$$SC(\lambda \setminus \bar{\lambda}, \mu) := \{X \in SC_V : (\lambda^X, \bar{\lambda}^X, \mu^X) = (\lambda, \bar{\lambda}, \mu)\}.$$

(This polyhedron is bounded in case of  $\Delta$ -,  $\triangleleft$ -, or  $\square$ -configuration since the bounds on  $x_{ij}$  indicated in (3) remain valid in this case too.) We show the following.

**Theorem 2.** For an arbitrary convex configuration and integer  $\lambda, \bar{\lambda}, \mu$ , the polyhedron  $SC(\lambda \setminus \bar{\lambda}, \mu)$  is integral, i.e., each face of this polyhedron contains an integer point.

Note that for arbitrary reals  $q_1, \dots, q_n$ , the transformation of an array  $X$  into the array  $X'$  with entries  $x'_{ij} := x_{ij} + q_i$  preserves the row derivative. Such a transformation shifts a polyhedron  $SC(\lambda \setminus \bar{\lambda}, \mu, \nu)$  into  $SC(\lambda \setminus \bar{\lambda}, \mu', \nu')$  with  $\mu'_i := \mu_i + q_i - q_{i-1}$  and  $\nu'_i := \nu_i + q_i - q_{i-1}$  (letting  $q_0 := 0$ ) and it maintains relation (4). This implies that, without loss of generality, in Theorem 1 one can consider only the quadruples of the form  $(\lambda, \bar{\lambda}, 0^n, \nu)$  (where  $0^n$  is the zero  $n$ -tuple). Similarly, one can restrict  $\mu$  to be  $0^n$  in Theorem 2 as well.

When dealing with  $\Delta$ -configuration, for a triple  $(\lambda, 0^n, \nu)$ , inequality (4) turns into the majorization condition  $\lambda[1, |I|] \geq \nu(I)$ . Therefore, for a fixed  $\lambda$ , the set  $\{\nu : (\lambda, 0^n, \nu) \in \mathcal{B}_V\}$  forms a *permutohedron*, a polytope  $\mathcal{P}$  formed by all vectors  $z \in \mathbb{R}^n$  with the same value  $|z|$  such that for  $k = 1, \dots, n - 1$ , the sum of any  $k$  entries of  $z$  does not exceed a constant depending only on  $k$ . (The vertices of  $\mathcal{P}$  are obtained by permuting entries of a fixed  $n$ -vector  $h$ ; in our case,  $h = \lambda$ .) It is known that for nonnegative integer  $\lambda, \nu$ , the majorization condition is necessary and sufficient for the existence of a *semi-standard Young tableau with shape  $\lambda$  and content  $\nu$* , and that these tableaux one-to-one correspond to the

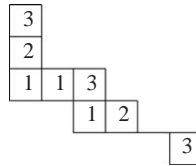


Fig. 4. The semi-standard skew Young tableau corresponding to the pattern in Fig. 2b (here  $\lambda = (6, 4, 3, 1, 1)$ ,  $\bar{\lambda} = (5, 2)$  and  $v = (3, 2, 3)$ ).

integer Gelfand–Tsetlin patterns respecting  $\lambda, v$ ; for a definition and a survey, see [11]. Theorem 1 (Corollary 1) shows that in case of  $\triangleleft$ -arrays (resp.  $\square$ -arrays) and  $\lambda, \bar{\lambda}$  fixed, the analogous set  $\{v : (\lambda, \bar{\lambda}, 0^n, v) \in \mathcal{B}_v\}$  forms a permutohedron in  $\mathbb{R}^n$  as well (but now the corresponding vertex generating vector  $h$  becomes less trivial to write down; it will be indicated in Section 4). Each integer (generalized) Gelfand–Tsetlin pattern for nonnegative integer  $\lambda, \bar{\lambda}, v$  determines a so-called semi-standard skew Young tableau with shape  $\lambda \setminus \bar{\lambda}$  and content  $v$  (cf. [11]), and our theorem (corollary) yields necessary and sufficient conditions for the existence of such tableaux. Fig. 4 illustrates an instance of semi-standard skew Young tableau.

It should be noted that in case of  $\Delta$ -configuration one can obtain the claim of Theorem 2 by using a description for the generators of the Gelfand–Tsetlin patterns cone given in [1].

Our method of proof of Theorem 2 is based on attracting a certain equivalent flow model and showing that the integer points in  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n)$  one-to-one correspond to the integer flows in a certain directed graph. In addition, we explain how to use the flow approach to easily show that Kostka coefficient  $K(\lambda, v)$  (or  $K(\lambda \setminus \bar{\lambda}, v)$ ), as well as the intrinsic volume of  $\mathcal{SC}(\lambda, 0^n, v)$  (resp.  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n, v)$ ) in the nondegenerate case, preserves under a permutation of the entries of  $v$ . Here  $K(\lambda, v)$  is the number of semi-standard Young tableaux with shape  $\lambda$  and content  $v$  (which is equal to the number of integer points in  $\mathcal{SC}(\lambda, 0^n, v)$ ), while  $K(\lambda \setminus \bar{\lambda}, v)$  concerns the corresponding skew tableaux.

This paper is organized as follows. Theorems 1 and 2 are proved in Sections 2 and 3, respectively. The concluding Section 4 discusses some additional aspects related to these theorems and demonstrates consequences from the proving method of Theorem 2: a combinatorial characterization of the vertices of polyhedra  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, \mu)$ , the above-mentioned facts on integer points and volumes, and others.

We conclude this section with two more remarks.

**Remark 2.** Let us say that an array  $X$  (as in (1)) is (fully) concave if it satisfies (2) and

$$x_{ij} - x_{i+1,j} \geq x_{i-1,j-1} - x_{i,j-1} \quad \text{for all } 1 \leq i < n \text{ and } a_i < j \leq b_i. \tag{5}$$

This is equivalent to saying that the extension  $f$  of the function  $x$  on the nodes of the corresponding grid  $G$  (cf. Remark 1) is concave in the entire region  $\mathcal{R}$ . The functions  $x$  with such a property are often called discrete concave ones, and a series of interesting results on these have been obtained. Knutson et al. [9] pointed out the precise list of facets of the cone  $\text{BNDR}_n$  formed by all possible triples  $(\lambda, \mu, v)$  of  $n$ -tuples whose entries are the differences  $x(v) - x(u)$  on boundary edges  $uv$  for a discrete concave function  $x$  on the triangular grid

of size  $n$ , or a *hive* (equivalently:  $\lambda, \mu, \nu$  are the spectra of three Hermitian  $n \times n$  matrices with zero sum). Also it is shown in [8] that for each integer  $(\lambda, \mu, \nu) \in \text{BNDR}_n$  there exists an integer discrete concave function  $x$  as required for this triple. (A history of studying this cone and related topics are reviewed in [5], see also [3]). Nontrivial constraints for  $\text{BNDR}_n$  are expressed by Horn’s inequalities. These are generalized to an arbitrary convex grid (see [7]), and relation (4) in Theorem 1 is, in essence, equivalent to a special case of Horn’s inequalities. We will briefly explain in Section 4 that Theorem 1 can be derived from the above-mentioned results on discrete concave functions. At the same time, our direct proof of Theorem 1 is much simpler compared with the proofs of the corresponding theorems in [8,9].

**Remark 3.** The polyhedron integrality claimed in Theorem 2 need not hold when the array entries are fixed on the whole boundary. More precisely, by a result due to De Loera and McAllister [4], for any  $k \in \mathbb{N}$ , there exist  $\lambda, \mu, \nu \in \mathbb{Z}^n$  and a triangular array  $X$  of size  $n$ , with  $n = O(k)$ , such that  $X$  is a vertex of the polytope  $\mathcal{SC}(\lambda, \mu, \nu)$  and some entry of  $X$  has denominator  $k$ . (Some ingredient from a construction in [4] is used in [6] to obtain an analogous result for fully concave triangular arrays in the case when the values are fixed only on two “sides”.) Nevertheless, for  $\triangleleft$ -,  $\square$ - or  $\Delta$ -configuration, at least one integer vertex in each nonempty polytope  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, \mu, \nu)$  with  $\lambda, \bar{\lambda}, \mu, \nu$  integer does exist, as explained in the end of Section 4.

**2. Proof of Theorem 1**

As explained in the Introduction, it suffices to consider the case  $\mu = 0^n$ .

To show part “only if” in the theorem, we use induction on  $n$ . Case  $n = 1$  is trivial, so assume  $n > 1$ . Let  $(\lambda, \bar{\lambda}, 0^n, \nu) \in \mathcal{B}_V$  (for  $V$  determined by  $n, m$ ) and consider an array  $X \in \mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n, \nu)$  and a set  $I = \{i(1), \dots, i(k)\}$  with  $1 \leq i(1) < \dots < i(k) \leq n$ .

Define  $I' := I \cap \{1, \dots, n - 1\}$  and  $\lambda'_j := \partial_{x_{n-1,j}}$  for  $j = 1, \dots, n + m - 1$ . Then  $\lambda_j \geq \lambda'_j \geq \lambda_{j+1}$  (by (2)). By induction,

$$\lambda'[1, |I'|] - \nu(I') - \Delta'_{|I'|} \geq 0, \tag{6}$$

where  $\Delta'_{k'}$  stands for the  $k'$ ’th deficit for  $\lambda', \bar{\lambda}$ , i.e.,  $\Delta'_{k'} := \delta'_{k'}(1) + \dots + \delta'_{k'}(n + m - 1)$ , where  $\delta'_{k'}(j) := \max\{0, \bar{\lambda}_{j-k'} - \lambda'_j\}$ . Two cases are possible.

*Case 1.* Let  $n \notin I$ , i.e.,  $I' = I$ . Since  $\delta_k(j) = \max\{0, \bar{\lambda}_{j-k} - \lambda_j\}$ ,  $\delta'_k(j) = \max\{0, \bar{\lambda}_{j-k} - \lambda'_j\}$  and  $\lambda_j \geq \lambda'_j$ , we have  $\delta_k(j) \leq \delta'_k(j)$ , implying  $\Delta_k \leq \Delta'_k$ . Now, using (6),

$$\lambda[1, k] - \nu(I) - \Delta_k \geq \lambda'[1, k] - \nu(I) - \Delta'_k \geq 0,$$

and (4) follows (with  $\mu = 0^n$ ).

*Case 2.* Let  $n \in I$ . Then  $|I'| = k - 1$ . Summing up (6) and the evident equality  $|\lambda| - |\lambda'| - \nu_n = 0$ , we obtain

$$\lambda[1, k] + \sum (\lambda_j - \lambda'_{j-1} : j = k + 1, \dots, n + m) - \nu(I) - \Delta'_{k-1} \geq 0. \tag{7}$$



Note also that  $\lambda_j + \delta_k(j) = \max\{\lambda_j, \bar{\lambda}_{j-k}\}$  (in view of  $\delta_k(j) = \max\{0, \bar{\lambda}_{j-k} - \lambda_j\}$ ), and similarly  $\lambda'_{j-1} + \delta'_{k-1}(j-1) = \max\{\lambda'_{j-1}, \bar{\lambda}_{j-k}\}$ . Since  $\lambda_j \leq \lambda'_{j-1}$ , we have  $\lambda_j + \delta_k(j) \leq \lambda'_{j-1} + \delta'_{k-1}(j-1)$ . Therefore,  $\sum(\lambda_j - \lambda'_{j-1} : j = k+1, \dots, n+m) - \Delta'_{k-1} \leq -\Delta_k$ . This together with (7) implies (4).

Next we show part “if” in the theorem. We first consider case  $m = 0$  (i.e.,  $\Delta$ -configuration); in this case all deficits  $\Delta_k$  are zeros, which simplifies the consideration. We use induction on  $n$ ; case  $n = 1$  is trivial. Let  $n > 1$  and let (4) hold for all  $I$ . In particular,  $\lambda_1 - v_n \geq 0$  (by taking  $I := \{n\}$ ). Also, subtracting inequality (4) with  $I = \{1, \dots, n-1\}$  from the equality  $|\lambda| - |v| = 0$ , we obtain  $\lambda_n - v_n \leq 0$ . Therefore, as  $\lambda$  is weakly decreasing, there exists  $p \in \{1, \dots, n-1\}$  such that

$$\lambda_p \geq v_n \quad \text{and} \quad \lambda_{p+1} \leq v_n. \tag{8}$$

Assign the  $(n-1)$ -tuple  $\lambda'$  by the following rule:

$$\begin{aligned} \lambda'_j &:= \lambda_j \quad \text{for } j = 1, \dots, p-1; & \lambda'_j &:= \lambda_{j+1} \quad \text{for } j = p+1, \dots, n-1; \\ \text{and } \lambda'_p &:= \lambda_p + \lambda_{p+1} - v_n. \end{aligned} \tag{9}$$

Consider the triple  $(\lambda', 0^{n-1}, v')$ , where  $v' := (v_1, \dots, v_{n-1})$ . We assert that

$$\lambda'[1, |I'|] \geq v(I') \tag{10}$$

holds for each  $I' \subseteq \{1, \dots, n-1\}$ . Consider two cases, letting  $k := |I'|$ .

- (i) Let  $k < p$ . Then  $\lambda'[1, k] = \lambda[1, k]$ , and (10) follows from (4) for  $I := I'$ .
- (ii) Let  $k \geq p$ . Define  $I := I' \cup \{n\}$ . Then  $\lambda'[1, k] = \lambda[1, k+1] - v_n$  (by (9)), and we have (using (4))

$$\lambda'[1, k] - v(I') = \lambda[1, k+1] - v_n - v(I') = \lambda[1, |I|] - v(I) \geq 0.$$

Thus, (10) holds for each  $I'$ . Also (8) and (9) imply  $\lambda_j \geq \lambda'_j \geq \lambda_{j+1}$  for  $j = 1, \dots, n-1$  (in particular,  $\lambda'$  is weakly decreasing), and (9) together with  $|\lambda| = |v|$  implies  $|\lambda'| = |v'|$ . By induction there exists a strip-concave  $\Delta$ -array  $X'$  of size  $n-1$  with  $(\lambda^{X'}, \mu^{X'}, v^{X'}) = (\lambda', 0^{n-1}, v')$ . Assign  $x_{ij} := x'_{ij}$  for  $0 \leq j \leq i \leq n-1$  and  $x_{nj} := \lambda[1, j]$  for  $1 \leq j \leq n$ . The resulting array  $X$  of size  $n$  satisfies (2) and has the desired local differences on the “sides”, namely,  $(\lambda^X, \mu^X, v^X) = (\lambda, 0^n, v)$ . Hence  $(\lambda, 0^n, v) \in \mathcal{B}_V$ . Also when  $\lambda, v$  are integer, the tuple  $\lambda'$  defined by (9) is integer as well, and the last claim in the theorem (for  $m = 0$ ) follows by induction, as the integrality of  $X'$  implies that for  $X$ .

It remains to prove part “if” when  $m > 0$ . Notice that the triple  $\lambda, \bar{\lambda}, v$  can be considered up to adding a constant to all entries (which matches adding a constant to the array row derivative), so one may assume that  $\lambda$  is nonnegative. Also, by compactness and scaling, w.l.o.g. one may assume that  $\lambda, \bar{\lambda}, v$  are integer (this slightly simplifies technical details).

We proceed by induction on  $m + |\lambda|$ ; case  $|\lambda| = 0$  is trivial. Let (4) hold for all  $I$ . In particular,  $\lambda_j \geq \bar{\lambda}_j \geq \lambda_{j+n}$  for  $j = 1, \dots, m$ . If  $\lambda_{n+m} = \bar{\lambda}_m$ , we make a simple reduction to  $\square$ -configuration of size  $(n, m-1)$  by truncating the tuples  $\lambda, \bar{\lambda}$  to  $\lambda' := (\lambda_1, \dots, \lambda_{n+m-1})$  and  $\bar{\lambda}' := (\bar{\lambda}_1, \dots, \bar{\lambda}_{m-1})$ , respectively. (This maintains (4), and if  $X$  is a required array

of size  $(n, m - 1)$  for  $\lambda', \bar{\lambda}', v$ , then adding to  $X$  the elements  $x_{i,i+m} := x_{i,i+m-1} + \bar{\lambda}_m$  for  $i = 0, \dots, n$  produces a required array of size  $(n, m)$  for  $\lambda, \bar{\lambda}, v$ .) A similar reduction (discarding  $\lambda_1, \bar{\lambda}_1$ ) is applied when  $\lambda_1 = \bar{\lambda}_1$ .

Therefore, one may assume  $\lambda_1 > \bar{\lambda}_1$  and  $\bar{\lambda}_m > \lambda_{n+m}$ . Then there are  $1 \leq r \leq n + m - 1$  and  $1 \leq s \leq m$  such that

$$\lambda_r \geq \bar{\lambda}_1 = \dots = \bar{\lambda}_s > \bar{\lambda}_{s+1} \quad \text{and} \quad \bar{\lambda}_s > \lambda_{r+1}, \tag{11}$$

letting  $\bar{\lambda}_{m+1} := 0$ . Note that  $\lambda_r > \bar{\lambda}_{s+1}$  implies  $r \leq s + n$  and  $\bar{\lambda}_s > \lambda_{r+1}$  implies  $r \geq s$ . Define

$$\lambda'_j := \begin{cases} \lambda_j - 1, & j = r - s + 1, \dots, r, \\ \lambda_j, & j = 1, \dots, r - s, r + 1, \dots, n + m; \end{cases} \tag{12}$$

$$\bar{\lambda}'_j := \begin{cases} \bar{\lambda}_j - 1, & j = 1, \dots, s, \\ \bar{\lambda}_j, & j = s + 1, \dots, m. \end{cases} \tag{13}$$

Then  $\lambda', \bar{\lambda}'$  are weakly decreasing and  $|\lambda'| - |\bar{\lambda}'| - |v| = 0$ . We assert that for any  $I \subseteq \{1, \dots, n\}$  and  $k := |I|$ :

$$\lambda'[1, k] - v(I) - \Delta'_k \geq 0, \tag{14}$$

denoting by  $\Delta'_k$  the  $k$ -deficit for  $\lambda', \bar{\lambda}'$ , i.e., the sum of numbers  $\delta'_k(j) := \max\{0, \bar{\lambda}'_{j-k} - \lambda'_j\}$  over  $j$ . To see this, first of all observe that  $\delta'_k(j) = \delta_k(j) = 0$  if  $1 \leq j \leq r$  (since  $\lambda_j \geq \bar{\lambda}_1$  and  $\lambda'_j \geq \bar{\lambda}'_1$ , by (11)–(13)). Consider three cases.

(a) Let  $k \leq r - s$ . Then for  $j = r + 1, \dots, n + m$ , we have  $\lambda'_j = \lambda_j$  and  $\lambda'_{j-k} = \bar{\lambda}_{j-k}$  (in view of  $j - k > s$ ). Hence  $\Delta'_k = \Delta_k$ . Also  $\lambda'[1, k] = \lambda[1, k]$ . Then (14) follows from (4).

(b) Let  $r - s < k \leq r$ . Then  $\delta'_k(j) = \delta_k(j) - 1$  for  $j = r + 1, \dots, k + s$  (as  $1 \leq j - k \leq s$  implies  $\bar{\lambda}'_{j-k} = \bar{\lambda}_{j-k} - 1 \geq \lambda_j = \lambda'_j$ ), and  $\delta'_k(j) = \delta_k(j)$  for  $j = k + s + 1, \dots, n + m$ . So  $\Delta'_k = \Delta_k - (k + s - r)$ . Also  $\lambda'[1, k] = \lambda[1, k] - (k + s - r)$ , and (14) follows.

(c) Let  $r < k \leq n$ . Then  $\delta'_k(j) = \delta_k(j) - 1$  for  $j = k + 1, \dots, k + s$ , and  $\delta'_k(j) = \delta_k(j)$  for  $j = k + s + 1, \dots, n + m$ . So  $\Delta'_k = \Delta_k - s$ . Also  $\lambda'[1, k] = \lambda[1, k] - s$ , and (14) follows.

Since  $|\lambda'| < |\lambda|$ , by induction the set  $\mathcal{SC}(\lambda' \setminus \bar{\lambda}', 0^n, v)$  is nonempty and contains an integer member  $X'$ . We transform  $X'$  into the desired array  $X$  for  $\lambda, \bar{\lambda}, v$  as follows. Let  $\alpha := \lambda'_{r-s+1} (= \lambda_{r-s+1} - 1)$ . For  $i = 0, \dots, n$ , define  $p(i)$  to be the maximum  $j$  such that  $\partial x'_{ij} > \alpha$ , letting by definition  $\partial x'_{i0} := \infty$ . Then  $p(0) = 0, p(n) = r - s$  and  $p(i) \leq i$  for each  $i$  (as  $\lambda'_{r-s} > \lambda'_{r-s+1} \geq \bar{\lambda}'_1 \geq \partial x'_{i,i+1}$ ). For  $i = 0, \dots, n$ , define

$$x_{ij} := \begin{cases} x'_{ij}, & j = 0, \dots, p(i), \\ x'_{ij} + j - p(i), & j = p(i) + 1, \dots, p(i) + s, \\ x'_{ij} + s, & j = p(i) + s + 1, \dots, i + m. \end{cases} \tag{15}$$

Observe that  $\lambda^X = \lambda, \bar{\lambda}^X = \bar{\lambda}$  and  $v^X = v$  (since  $x_{i,i+m} = x'_{i,i+m} + s$  for each  $i$ ). Also  $X$  satisfies (2). To see the latter, let  $\varepsilon_{ij} := \partial x_{ij} - \partial x'_{ij}$  for all corresponding  $i, j$ ; then  $\varepsilon_{ij} \in \{0, 1\}$ .

Using the definition of  $\alpha$ ,  $p(0), \dots, p(n)$ , relation (15) and the fact that  $X'$  is strip-concave, it is not difficult to conclude that  $\varepsilon_{ij} < \varepsilon_{i-1,j}$  is possible only if  $j = p(i) = p(i - 1) + 1$ . In this case we have  $\partial x'_{ij} \geq \alpha + 1 > \partial x'_{i-1,j}$ , whence  $\partial x_{ij} \geq \partial x_{i-1,j}$ . Similarly, one can see that if  $\varepsilon_{i-1,j} < \varepsilon_{i,j+1}$ , then  $j = p(i - 1) = p(i)$ ; in this case  $\partial x_{i-1,j} \geq \partial x_{i,j+1}$  follows from  $\partial x'_{i-1,j} \geq \alpha + 1 > \partial x'_{i,j+1}$ . This implies that  $X$  is strip-concave.

This completes the proof of Theorem 1.

**Remark 4.** The  $\Delta$ -array  $X$  recursively constructed in the second part of the proof is, in fact, a vertex of the polytope  $\mathcal{SC}(\lambda, 0^n, v)$ . This can be seen as follows. Given  $X' \in \mathcal{SC}(\lambda, 0^n, v)$ , let  $Q(X')$  be the set of all equalities of the form  $\partial x'_{ij} = \partial x'_{i-1,j}$  or  $\partial x'_{i-1,j} = \partial x'_{i,j+1}$ . A trivial observation is that  $X'$  is a vertex of  $\mathcal{SC}(\lambda, 0^n, v)$  if and only if  $X'$  is determined by  $Q(X')$ , i.e., there is no other point  $X''$  in this polytope such that  $Q(X'') \supseteq Q(X')$ . In our case, the equalities as in (9) (in the recursive process) give the corresponding equalities for  $\partial X$ ; clearly the latter equalities determine  $X$  uniquely, so  $X$  is a vertex of  $\mathcal{SC}(\lambda, 0^n, v)$ . Moreover, if  $\lambda, v$  are integer, then  $X$  is integer as well. This strengthens the last claim in the theorem for case  $m = 0$ . On the other hand, the construction of  $\square$ -array  $X$  in the third part of the proof does not guarantee that this  $X$  is a vertex of  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n, v)$ . (Although an integer vertex in this polytope with  $\lambda, \bar{\lambda}, v$  integer does exist, as explained in Section 4.)

**Remark 5.** One can accelerate the process of constructing a required  $\square$ -array  $X$  in the third part of the proof. Given (not necessary integer)  $\lambda, \bar{\lambda}, v$ , define  $\rho := \bar{\lambda}_1 - \max\{\lambda_{r+1}, \bar{\lambda}_{s+1}\}$ , for  $r, s$  as in (11). When  $\lambda_1 > \bar{\lambda}_1$  and  $\bar{\lambda}_m > \lambda_{n+m}$ , we can reduce the corresponding entries of  $\lambda, \bar{\lambda}$  just by  $\rho$  (rather than by one), by setting  $\lambda'_j := \lambda_j - \rho$  and  $\bar{\lambda}'_j := \bar{\lambda}_j - \rho$  in the first lines of (12) and (13), respectively (one shows that (4) is maintained). Given an array  $X'$  for  $\lambda', \bar{\lambda}', v$ , we iteratively transform  $X'$  into an array for  $\lambda, \bar{\lambda}, v$ . More precisely, at the first iteration, for  $\alpha, p(0), \dots, p(n)$  defined as above, we increase the entries  $x'_{ij}$  for  $ij$  as in the second and third lines of (15) by  $\varepsilon(j - p(i))$  and by  $\varepsilon s$ , respectively, where  $\varepsilon$  is the maximum value not exceeding  $\rho$  and such that the resulting array is still strip-concave ( $\varepsilon$  is computed efficiently). If  $\varepsilon < \rho$ , we apply a similar procedure (at the second iteration) to the updated  $X'$  and  $\rho := \rho - \varepsilon$ , and so on. One shows that after  $O(n^2)$  iterations we get  $\rho = 0$ , and that the final  $X'$  is the desired array  $X$  for  $\lambda, \bar{\lambda}, v$ . Hence the number of operations in the whole process of finding a member of  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n, v)$  is polynomial in  $n$ . Such a transformation  $X' \rightarrow X$  is closely related to a rearrangement of flows (associated with strip-concave arrays) explained in part D of Section 4.

### 3. Proof of Theorem 2

First of all we observe that the generic case of convex configuration in this theorem is reduced to the case of  $\square$ -configuration. Indeed, given  $\lambda, \bar{\lambda}, \mu$  for  $V$  as in (1), there exists a (sufficiently large) positive integer  $c$  such that each face of  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, \mu)$  contains a face of the polyhedron  $\mathcal{P}$  formed by the arrays  $X \in \mathcal{SC}(\lambda \setminus \bar{\lambda}, \mu)$  with  $|\partial x_{ij}| \leq c$  for all entries  $\partial x_{ij}$  of  $\partial X$ . Let  $m := b_0$  and extend  $\lambda$  to  $(n + m)$ -tuple  $\lambda'$  by setting  $\lambda'_1 := \dots := \lambda'_{a_n} := c$ ,  $\lambda'_{b(n)+1} := \dots := \lambda'_{n+m} := -c$  and  $\lambda'_j := \lambda_j$  for  $j = a_n + 1, \dots, b_n$ . Accordingly, set

$\mu'_i := \mu_i$  if  $a_i = 0$ , and  $\mu'_i := \mu_i - c$  if  $a_i > 0$ . Then the restriction map  $X' \rightarrow X'|_V$  gives a bijection between the  $\triangleleft$ -arrays  $X'$  with  $\lambda^{X'} = \lambda'$ ,  $\bar{\lambda}^{X'} = \bar{\lambda}$ ,  $\mu^{X'} = \mu'$  and the arrays in  $\mathcal{P}$  (cf. explanations in the Introduction). This implies that  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, \mu)$  is integral if  $\mathcal{SC}(\lambda' \setminus \bar{\lambda}, \mu')$  is such.

In the rest of the proof we deal with  $\triangleleft$ -configuration of size  $(n, m)$ . As before, we may assume  $\mu = 0^n$ . Also one may assume that  $\lambda$  is nonnegative (cf. reasonings in the previous section). For brevity we denote the polytope  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n)$  by  $\mathcal{SC}(\lambda \setminus \bar{\lambda})$ . Theorem 2 will be proved by constructing a bijection between the vertices of  $\mathcal{SC}(\lambda \setminus \bar{\lambda})$  and certain forests in the grid  $G$  (defined in Remark 1 in the Introduction). Establishing this correspondence, we admit  $\lambda$  and  $\bar{\lambda}$  to be real-valued.

The node set  $V$  of  $G$  is naturally partitioned into subsets (*horizontal layers*)  $L_i = \{(i, 0), \dots, (i, i + m)\}$ ,  $i = 0, \dots, n$ . Extract the edges connecting neighbouring layers and orient them from the top to the bottom. Formally, let  $A$  be the set of pairs  $e^0_{ij} := ((i, j), (i + 1, j))$  and  $e^1_{ij} := ((i, j), (i + 1, j + 1))$  of nodes of  $G$ , for  $i = 0, \dots, n - 1$ ,  $j = 0, \dots, i + m$ . Then  $H_{n,m} := H := (V, A)$  is an acyclic digraph in which any maximal (directed) path begins at a node of the “topmost” layer  $L_0$  and ends at a node of the “bottommost” layer  $L_n$ .

We say that a function  $g : A \rightarrow \mathbb{R}_+$  is a  $(\lambda, \bar{\lambda})$ -admissible flow in  $H$  if

$$\operatorname{div}_g(i, j) = \begin{cases} 0, & i = 1, \dots, n - 1, j = 0, \dots, i + m, \\ \lambda_j - \lambda_{j+1}, & i = n, j = 0, \dots, n + m, \\ \bar{\lambda}_{j+1} - \bar{\lambda}_j, & i = 0, j = 0, \dots, m. \end{cases} \tag{16}$$

Here  $\operatorname{div}_g(v)$  ( $v \in V$ ) stands for the value  $\sum_{e=(u,v) \in A} g(e) - \sum_{e=(v,u) \in A} g(e)$ , and we formally extend  $\lambda$  and  $\bar{\lambda}$  by setting  $\lambda_0 := \bar{\lambda}_0 := \lambda_1$  and  $\lambda_{n+m+1} := \bar{\lambda}_{m+1} := 0$ . In particular,  $g(e^0_{n-1,0}) = 0$  and  $g(e^1_{n-1,n+m-1}) = \lambda_{n+m}$ . The set  $\mathcal{F}(\lambda \setminus \bar{\lambda})$  of  $(\lambda, \bar{\lambda})$ -admissible flows forms a polytope in  $\mathbb{R}^{|A|}$ .

**Claim.** For any  $X \in \mathcal{SC}(\lambda \setminus \bar{\lambda})$  there exists a  $(\lambda, \bar{\lambda})$ -admissible flow  $g = \gamma(X)$  satisfying

$$\begin{aligned} g(e^0_{ij}) &= \partial x_{i,j} - \partial x_{i+1,j+1}, \\ g(e^1_{ij}) &= \partial x_{i+1,j+1} - \partial x_{i,j+1}, \end{aligned} \quad i = 0, \dots, n - 1, j = 0, \dots, i + m, \tag{17}$$

letting  $\partial x_{i0} := \lambda_1$  and  $\partial x_{i,i+m+1} := 0$ . Moreover,  $\gamma$  is a bijective mapping of  $\mathcal{SC}(\lambda \setminus \bar{\lambda})$  to  $\mathcal{F}(\lambda \setminus \bar{\lambda})$ .

(Fig. 5 illustrates the flow determined by the array  $X$  with  $\partial X$  as in Fig. 2b; here the flow is integer and its value on an edge is indicated by the number of lines connecting the ends of this edge.)

**Proof.** Let  $X \in \mathcal{SC}(\lambda \setminus \bar{\lambda})$  and let  $g$  be defined by (17). Then for each node  $v = (n, j)$  with  $j = 0, \dots, n + m$ ,

$$\begin{aligned} \operatorname{div}_g(v) &= g(e^1_{n-1,j-1}) + g(e^0_{n-1,j}) = (\partial x_{nj} - \partial x_{n-1,j}) + (\partial x_{n-1,j} - \partial x_{n,j+1}) \\ &= \lambda_j - \lambda_{j+1}, \end{aligned}$$

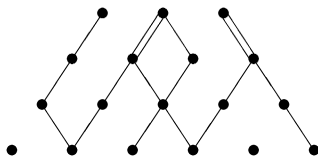


Fig. 5. The flow corresponding to  $\partial X$  in Fig. 2b.

letting  $g(e) := 0$  if the edge  $e$  is void (e.g., for  $e = e_{n-1,-1}^1$ ). Similarly,  $\text{div}_g(v) = \bar{\lambda}_{j+1} - \bar{\lambda}_j$  for each node  $v = (0, j)$ ,  $j = 0, \dots, m$ . And for each node  $v = (i, j)$  with  $1 \leq i \leq n - 1$  and  $0 \leq j \leq i + m$ , one has

$$\begin{aligned} \text{div}_g(v) &= g(e_{i-1,j-1}^1) + g(e_{i-1,j}^0) - g(e_{ij}^0) - g(e_{ij}^1) \\ &= (\partial x_{ij} - \partial x_{i-1,j}) + (\partial x_{i-1,j} - \partial x_{i,j+1}) - (\partial x_{ij} - \partial x_{i+1,j+1}) \\ &\quad - (\partial x_{i+1,j+1} - \partial x_{i,j+1}) = 0. \end{aligned}$$

Also the function  $g$  is nonnegative, as is seen by comparing (17) and (2). Thus,  $g$  is a  $(\lambda, \bar{\lambda})$ -admissible flow.

Conversely, let  $g$  be a  $(\lambda, \bar{\lambda})$ -admissible flow in  $H$ . Assign numbers  $\partial x_{ij}$  recursively by the following rule:

$$\begin{aligned} \partial x_{nj} &:= \lambda_j, & j &= 1, \dots, n + m, \\ \partial x_{ij} &:= \partial x_{i+1,j} - g(e_{i,j-1}^1), & i &= 1, \dots, n - 1, \quad j = 1, \dots, i + m. \end{aligned}$$

This gives the  $\square$ -array  $X$  of size  $(n, m)$ . Reversing the argument above, one can check validity of (17). This and the nonnegativity of  $g$  imply that  $X$  is strip-concave and satisfies  $\lambda^X = \lambda$  and  $\bar{\lambda}^X = \bar{\lambda}$ . Then  $X \in \mathcal{SC}(\lambda \setminus \bar{\lambda})$ , and the claim follows.  $\square$

Thus,  $\gamma$  is a linear operator (in view of (17)) and  $\gamma$  gives a one-to-one correspondence between the points in the polytopes  $\mathcal{SC}(\lambda \setminus \bar{\lambda})$  and  $\mathcal{F}(\lambda \setminus \bar{\lambda})$ . Therefore,  $\gamma$  establishes a one-to-one correspondence between the vertices of these polytopes.

Next we characterize the vertices of  $\mathcal{F}(\lambda \setminus \bar{\lambda})$ . To this aim, we distinguish, in the bottommost layer  $L_n$ , the set  $L(\lambda)$  of nodes  $(n, j)$  ( $1 \leq j \leq n + m$ ) such that  $\lambda_j > \lambda_{j+1}$ , and in the topmost layer  $L_0$ , the subset  $L(\bar{\lambda})$  of nodes  $(0, j)$  ( $0 \leq j \leq m$ ) such that  $\bar{\lambda}_j > \bar{\lambda}_{j+1}$ . Given a flow  $g \in \mathcal{F}(\lambda \setminus \bar{\lambda})$ , let  $H(g)$  denote the subgraph of  $H$  induced by the set of edges  $e$  with  $g(e) > 0$ . From (16) it follows that  $H(g)$  contains  $L(\lambda)$  and  $L(\bar{\lambda})$  and that each node of  $H(g)$  lies on a path from  $L(\bar{\lambda})$  to  $L(\lambda)$ . Suppose there are two different paths  $P, P'$  in  $H(g)$  having the same beginning and the same end. Choose  $\varepsilon > 0$  not exceeding the minimal value of  $g$  on the paths  $P$  and  $P'$ . Then the functions  $g' := g + \varepsilon \chi^P - \varepsilon \chi^{P'}$  and  $g'' := g - \varepsilon \chi^P + \varepsilon \chi^{P'}$  are nonnegative and satisfy (16), where  $\chi^Q \in \{0, 1\}^A$  is the characteristic function of the edge set of a path  $Q$ . So  $g$  is expressed as the half-sum of two different  $(\lambda, \bar{\lambda})$ -admissible flows  $g', g''$ , and therefore,  $g$  cannot be a vertex of  $\mathcal{F}(\lambda \setminus \bar{\lambda})$ .

On the other hand, let for any two nodes  $y$  and  $z$ ,  $H(g)$  contain at most one path from  $y$  to  $z$ , i.e.,  $H(g)$  is a (directed) forest with the set  $L(\bar{\lambda})$  of zero indegree nodes (*roots*) and the set  $L(\lambda)$  of zero outdegree ones (*leaves*). Then  $g$  is the only  $(\lambda, \bar{\lambda})$ -admissible flow taking

zero values on all edges outside  $H(g)$ , i.e.,  $g$  is determined by  $H(g)$ . Indeed, one can see that for each edge  $e = (u, v)$  of  $H(g)$ ,  $g(e)$  is equal to

$$\sum (\lambda_j - \lambda_{j+1} : (n, j) \in V(Q)) - \sum (\bar{\lambda}_j - \bar{\lambda}_{j+1} : (0, j) \in V(Q)), \tag{18}$$

where  $Q$  is the connected component of  $H(g) \setminus \{e\}$  that contains the node  $v$ , denoting by  $V(Q)$  the node set of  $Q$ . This implies that  $g$  is a vertex of  $\mathcal{F}(\lambda \setminus \bar{\lambda})$ . Moreover,  $g$  is integer if  $\lambda, \bar{\lambda}$  are integer, and Theorem 2 follows.  $\square$

Arguing as in the above proof, one can associate the vertices of  $\mathcal{SC}(\lambda \setminus \bar{\lambda})$  with certain subgraphs of  $H$ , as follows.

**Corollary 2.** *In case of  $\square$ -configuration of size  $(n, m)$ , each vertex of  $\mathcal{SC}(\lambda \setminus \bar{\lambda})$  one-to-one corresponds to a forest  $H'$  in  $H_{n,m}$  having  $L(\bar{\lambda})$  as the set of roots and  $L(\lambda)$  as the set of leaves and satisfying the following condition: for each component  $Q$  of  $H'$ , the value in (18) is zero, and for each edge  $e = (u, v)$  and the component  $Q$  of  $H' \setminus \{e\}$  containing  $v$ , the value in (18) is positive. Therefore, in case  $m = 0$ , the vertices of  $\mathcal{SC}(\lambda)$  one-to-one correspond to the rooted trees in  $H_n$  ( $:= H_{n,0}$ ) with root  $(0,0)$  and set of leaves  $L(\lambda)$ .*

**Remark 6.** The flows introduced in the proof of Theorem 2 give an alternative way to represent the Gelfand–Tsetlin patterns (or the strip-concave arrays), and Corollary 2 suggests a way to compute or estimate the number of vertices of the polytope  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, \mu)$  in case of  $\square$ -configuration (or  $\Delta$ -configuration). One can check that the reasonings in the proof of Theorem 2 and the corresponding corollary are applicable to  $\square$ -configuration as well (with  $H_{n,m}$  arising from the corresponding parallelogram-wise grid of size  $(n, m)$ ).

#### 4. Concluding remarks

As mentioned in the Introduction, Theorem 1 admits a reformulation in which the piecewise linear constraints are replaced by linear ones. More precisely, one can see that for each  $I \subseteq \{1, \dots, n\}$ , inequality (4) is equivalent to the set of linear inequalities

$$\lambda[1, |I|] + \lambda(J + |I|) - \bar{\lambda}(J) + \mu(I) - \nu(I) \geq 0, \tag{19}$$

where  $J$  ranges all subsets of  $\{1, \dots, m\}$ , and for  $k \in \mathbb{Z}$ ,  $J + k$  stands for the set  $\{j + k : j \in J\}$ . It turns out that, as a rule, each of the latter inequalities is essential, i.e., determines a facet of the cone  $\mathcal{B}_V$ . More precisely, one can show that for  $\square$ -configuration of size  $(n, m)$ , the set of facets of this cone is described as follows:

- (\*) For  $I, J$  as above, (19) determines a facet of  $\mathcal{B}_V$  if and only if  $|I| + |J| \neq 0$ ,  $n + m$  and either (i)  $|I| \neq 0$ ,  $n$  (and  $J$  is arbitrary), or (ii)  $|I| = 0$  and  $|J| = 1$ , or (iii)  $|I| = n$  and  $|J| = m - 1$ . Furthermore,  $\mathcal{B}_V$  has no other facets if  $n = 1$  or if  $n = 2$  and  $m = 0$ . Otherwise the remaining facets are exactly those determined by the “chamber inequalities”  $\lambda_j \geq \lambda_{j+1}$  ( $j = 1, \dots, n + m - 1$ ) and  $\bar{\lambda}_j \geq \bar{\lambda}_{j+1}$  ( $j = 1, \dots, m - 1$ ).

In particular,  $\mathcal{B}_V$  has  $(2^n - 2)2^m + n + 4m - 2$  facets in case  $n \geq 2$ , and  $2m$  facets in case  $n = 1$ . The proof of (\*) is rather technical and the main part of it consists in showing that for each  $(I, J)$  indicated in the claim, there exist  $3n + 2m - 2$  linearly independent vectors  $(\lambda, \bar{\lambda}, \mu, \nu)$  in  $\mathcal{B}_V$  for which (19) turns into equality. (Note that  $\mathcal{B}_V$  is easily shown to have dimension  $3n + 2m - 1$ .) The details are omitted here and will be given elsewhere.

Next, we outline (in parts A–D) more applications of the flow approach developed in the proof of Theorem 2. Here, unless explicitly said otherwise, we consider the case of  $\triangleleft$ -configuration of size  $(n, m)$ . (Note that the exposed properties remain valid if we deal with  $\square$ -configuration.)

(A) Let  $\mathcal{P} = \mathcal{P}_{n,m}$  be the set of paths in the graph  $H = H_{n,m}$  beginning at a node of the layer  $L_0$  and ending at a node of  $L_n \setminus \{(n, 0)\}$ . Associate with a path  $P \in \mathcal{P}$  the  $\triangleleft$ -array  $Y^P$  with the entries  $y_{i1} = \dots = y_{i,p(i)} = 1$  and  $y_{i,p(i)+1} = \dots = y_{i,i+m} = 0$  for  $i = 0, \dots, n$ , where  $(i, p(i))$  is a node of  $P$ . Considering the case of triangular arrays, Berenstein and Kirillov [1] noticed that the set of arrays  $Y^P$  ( $P \in \mathcal{P}_{n,0}$ ) constitutes a minimal list of generators of the cone of nonnegative Gelfand–Tsetlin patterns of size  $n - 1$ . A similar property takes place for  $\triangleleft$ -patterns (or  $\square$ -patterns) and can be easily shown by use of flows. More precisely, for a strip-concave  $\triangleleft$ -array  $X$  with  $\partial X \geq 0$ , take the flow  $g = \gamma(X)$  defined by (17). Then  $g$  is represented as a nonnegative linear combination  $\alpha_1 \chi^{P_1} + \dots + \alpha_N \chi^{P_N}$ , where  $P_1, \dots, P_N \in \mathcal{P}$ . One can check that  $\partial X = \alpha_1 Y^{P_1} + \dots + \alpha_N Y^{P_N}$ , as required (the minimality of  $\{Y^P : P \in \mathcal{P}\}$  is obvious).

(B) One can establish some invariants for polytopes  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n, \nu)$  when the entries of  $\nu$  are permuted. Consider an array  $X \in \mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n, \nu)$  and the flow  $g = \gamma(X)$  as in (17). For  $i = 1, \dots, n$ , we have  $\sum_{j=1}^{i+m} \partial x_{ij} - \sum_{j=1}^{i+m-1} \partial x_{i-1,j} = x_{i,i+m} - x_{i-1,i+m-1} = \nu_i$ . Also  $\partial x_{ij} - \partial x_{i-1,j} = g(e_{i-1,j-1}^1)$  for  $j = 1, \dots, i + m$  (see Section 3 for the definition of edges  $e_{i',j'}^0$  and  $e_{i',j'}^1$ ; as before,  $\partial x_{i-1,i+m} := 0$ ). Comparing these relations, we conclude that

$$\nu_i = g(e_{i-1,0}^1) + \dots + g(e_{i-1,i+m-1}^1) \quad \text{for } i = 1, \dots, n. \tag{20}$$

Choose  $i \in \{1, \dots, n - 1\}$  and consider the subgraph  $H^i$  of  $H$  induced by the edges connecting the layers  $L_{i-1}, L_i$  or the layers  $L_i, L_{i+1}$ . For  $j = 0, \dots, i + m - 1$ , the nodes  $(i - 1, j)$  and  $(i + 1, j + 1)$  are connected by two paths, namely, by path  $Z_j$  with the edges  $e_{i-1,j}^0, e_{ij}^1$  and by path  $Z'_j$  with the edges  $e_{i-1,j}^1, e_{i,j+1}^0$ . Let us call such a path  $Z$  with edges  $e, e'$  a zigzag and define its capacity to be  $g(Z) := \min\{g(e), g(e')\}$ . The zigzag swapping operation modifies  $g$  within  $H^i$  by swapping the capacities simultaneously for each pair  $Z_j, Z'_j$ . More precisely, for  $j = 0, \dots, i + m - 1$ , assign

$$g'(e) := \begin{cases} g(e) - g(Z_j) + g(Z'_j) & \text{for each edge } e \text{ of } Z_j, \\ g(e) - g(Z'_j) + g(Z_j) & \text{for each edge } e \text{ of } Z'_j, \end{cases}$$

and  $g'(e) := g(e)$  for the remaining edges of  $H$ . Obviously,  $g'$  is again a  $(\lambda, \bar{\lambda})$ -admissible flow. (For example, such an operation applied to the flow in Fig. 5 results in the flow illustrated in Fig. 6.)

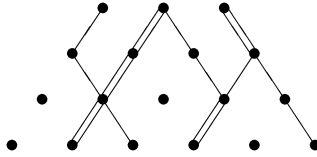


Fig. 6. The flow obtained by the zigzag swapping operation applied to the flow in Fig. 5 at layer 2.

Let  $\sigma_i(X)$  denote the array  $\gamma^{-1}(g')$  and let  $v'$  be the  $n$ -tuple of local differences on the “right boundary” of  $\sigma_i(X)$ . Using (20), one can check that the zigzag swapping operation swaps  $v_i$  and  $v_{i+1}$ , i.e.,  $v'_i = v_{i+1}$ ,  $v'_{i+1} = v_i$  and  $v'_p = v_p$  for  $p \neq i, i + 1$ . Moreover, applying the zigzag swapping operation (with the same  $i$ ) to  $g'$  returns  $g$ .

Thus, for each  $i$ ,  $\sigma_i$  is a continuous bijective mapping of  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n, v)$  to  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n, v')$  (and  $\sigma_i^2$  is the identity on  $\mathcal{SC}(\lambda \setminus \bar{\lambda})$ ).<sup>3</sup> Moreover, for  $k \in \mathbb{N}$ , if  $g$  is  $\frac{1}{k}$ -integer, so is  $g'$ . Therefore,  $\sigma_i$  gives a bijection on the  $\frac{1}{k}$ -integer points in these polytopes for any  $k$ . As a consequence (for  $k = 1$ ), the following known property is obtained: if  $\lambda, \bar{\lambda}, v$  are integer and if  $v'$  is an arbitrary permutation of  $v$ , then Kostka coefficients  $K(\lambda \setminus \bar{\lambda}, v)$  and  $K(\lambda \setminus \bar{\lambda}, v')$  are equal.

(C) Let  $\lambda, \bar{\lambda}, v$  be rational-valued and let  $v'$  be a permutation of  $v$ . Let  $V_0$  denote the set of boundary index pairs in  $V$  (or the boundary nodes in the grid  $G$ ). The fact that each map  $\sigma_i$  is continuous and bijective implies that the polytopes  $\mathcal{SC} := \mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n, v)$  and  $\mathcal{SC}' := \mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n, v')$  have the same dimension (which typically equals  $|V \setminus V_0|$ ). Consider the  $|V \setminus V_0|$ -dimensional affine subspaces  $S$  and  $S'$  containing the polytopes  $\mathcal{SC}$  and  $\mathcal{SC}'$ , respectively, which are obtained by imposing the corresponding equalities on the values on  $V_0$ . Since  $S$  and  $S'$  are parallel, there is  $k' \in \mathbb{N}$  such that for any multiple  $k$  of  $k'$ , the lattice of  $\frac{1}{k}$ -integer points in  $S'$  is obtained by a parallel translation of a similar lattice in  $S$ . So the density of  $\frac{1}{k}$ -integer points in  $S$  and  $S'$  (measured by the number of such points in a unit ball with center at a point of the lattice) is the same. Also the numbers of  $\frac{1}{k}$ -integer points in the polytopes in question are equal. Thus, when  $k$  tends to infinity, we obtain equality for the corresponding volumes and can conclude with the following.

**Proposition 1.** *Given (real-valued)  $\lambda, \bar{\lambda}, v$ , let  $v'$  be a permutation of  $v$ . Then the polytopes  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n, v)$  and  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n, v')$  have the same  $|V \setminus V_0|$ -dimensional volume.*

It should be noted that, although  $\sigma_i$  (being a piece-wise linear operator) brings integer points into integer ones, it need not do so for polytope vertices, even for polytopes  $\mathcal{SC}(\lambda \setminus \bar{\lambda})$ . Indeed, in case  $m = 0$ , take a rooted tree  $T$  in  $H_{n,0}$  (with root  $(0,0)$  and the leaves in  $L_n$ ) such that for some  $i, j$ , the subgraph  $T \cap H^i$  contains zigzags  $Z_j$  and  $Z'_{j+1}$ . Then the zigzag swapping operation (applied to a nowhere zero flow on  $T$ ) transforms the pair  $Z_j, Z'_{j+1}$  into  $Z'_j, Z_{j+1}$ , so the resulting graph  $T'$  is not a tree, as it has two edges entering the node  $(i, j + 1)$ .

<sup>3</sup> Note also that for integer points the zigzag swapping operation produces Bender–Knuth’s involution, cf. [1].



(D) The reduction applied in the proof of part “if” of Theorem 1 in case  $m > 0$  can be described in terms of flows. Moreover, the language of flows is convenient to develop a more general sort of reduction and to demonstrate some additional properties. To explain the idea, consider  $X \in \mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n, v)$  and  $g$  as in (17), assuming that  $\lambda$  is nonnegative. Let  $P$  be a path in  $H$  beginning at a node  $(0, s)$  of the layer  $L_0$ , ending at a node  $(n, t)$  of the layer  $L_n$  and such that the minimum  $\alpha$  of values of  $g$  on the edges of  $P$  is nonzero. Choose  $p \in \mathbb{Z}$  and  $\alpha' \in \mathbb{R}$  satisfying  $0 \leq s + p \leq m$  and  $0 < \alpha' \leq \alpha$  and change  $g$  by moving the path  $P$  with weight  $\alpha'$  at distance  $|p|$ , to the right of left depending on the sign of  $p$ . Formally: define  $P'$  to be the path containing the node  $(i, j + p)$  for each node  $(i, j)$  of  $P$  and transform  $g$  into  $g' := g - \alpha' \chi^P + \alpha' \chi^{P'}$ . This transformation does not change the sum in (20), and therefore, the resulting array  $X' := \gamma^{-1}(g')$  satisfies  $v^{X'} = v$ . When  $p > 0$  ( $p < 0$ ), the row derivative  $\partial X'$  is obtained from  $\partial X$  by increasing (resp. decreasing) by  $\alpha'$  the entries corresponding to the horizontal edges of the grid  $G$  lying between the paths  $P$  and  $P'$ ; the tuples  $\lambda^X$  and  $\bar{\lambda}^X$  are changed accordingly.

Using such operations, one can transform  $g$  more globally, still preserving  $v$ : decompose  $g$  into the sum of path flows  $\alpha_q \chi^{P_q}$  ( $\alpha_q > 0$ ),  $q = 1, \dots, N$ , and move each path  $P_q$  to the left so that the resulting  $P'_q$  begin at the node  $(0,0)$ . This gives an array  $X'$  with  $\partial x'_{ij} = 0$  for  $i = 0, \dots, n$  and  $j = i + 1, \dots, i + m$ , i.e., in essence,  $X'$  is equivalent to a  $\Delta$ -array. One can deduce that the first  $n$  entries of the tuple  $\lambda' := \lambda^{X'}$  are expressed as follows:

$$\lambda'_k = \sum_{t=k}^{n+m} [|\lambda_t, \lambda_{t+1}] \cap [|\lambda_1, \bar{\lambda}_{t-k+1}] \quad \text{for } k = 1, \dots, n, \tag{21}$$

denoting by  $|[a, b]$  the length  $b - a$  of a segment  $[a, b]$  and letting  $\bar{\lambda}_j := 0$  for  $j > m$ . Conversely, given  $\lambda, \bar{\lambda}, v$ , define the  $n$ -tuple  $\lambda'$  by (21) and consider a  $\Delta$ -array  $X' \in \mathcal{SC}(\lambda', 0^n, v)$ . Then one can determine a special path decomposition for  $\gamma(X')$  and move each path at a due distance to the right so as to obtain a flow determining a  $\square$ -array in  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, 0^n, v)$  (moreover,  $\lambda'$  is integer when  $\lambda, \bar{\lambda}$  are such and one can maintain flow and array integrality under the transformation). This gives a constructive way to reduce the trapezoidal case to the triangular one. The tuple  $\lambda'$  is weakly decreasing and it just represents the vertex generating vector for the permutohedron mentioned in the Introduction.

Next we explain the idea of deriving Theorem 1 from results in [8,9] (mentioned in Remark 2 in Section 1). We use the equivalence between  $\square$ -arrays of size  $(n, m)$  and functions on the node set of the corresponding grid  $G = (V, E)$ . Given tuples  $\lambda, \bar{\lambda}, \mu, v$ , let us choose a positive integer  $c$  and replace  $\mu, v$  by  $\mu', v'$  defined by  $\mu'_i := \mu_i - ic$  and  $v'_i := v_i - ic, i = 1, \dots, n$ . This turns the polytope  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, \mu, v)$  into  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, \mu', v')$  (each array  $X$  in the former polytope corresponds to  $X'$  defined by  $x'_{ij} := x_{ij} - \frac{i(i+1)}{2}c$ ); for brevity, we denote the latter polytope by  $\mathcal{C}$ . When  $c$  is large enough,  $\mathcal{C}$  consists of fully concave arrays, and we can apply results on the corresponding discrete concave functions. The second part of Theorem 1 follows from a result in [8] (in fact, shown there for any convex grid) which in our case reads: if  $\lambda, \bar{\lambda}, \mu', v'$  are integer and if  $\mathcal{C} \neq \emptyset$ , then  $\mathcal{C}$  contains an integer point.

The first part of Theorem 1 follows from a combinatorial characterization for the existence of a discrete concave function under prescribed boundary data (we use its extension to an arbitrary convex grid given in [7]). It uses a notion of *puzzle* (originally introduced for  $\Delta$ -grids in [9]). This is a subdivision  $\Pi$  of the grid into a set of little triangles and little rhombi (the union of two little triangles sharing an edge), along with a 0,1-labeling of the edges of  $G$  occurring in the boundaries of these pieces, satisfying the following properties:

- (i) for each little triangle  $\tau$  in  $\Pi$ , the edges of  $\tau$  are all labeled either by 0 or by 1;
- (ii) for each little rhombus  $\rho$  in  $\Pi$ , a side edge of  $\rho$  is labeled 1 if clockwise of an obtuse angle, and 0 if clockwise of an acute angle.

Then a necessary and sufficient condition on the nonemptiness of  $\mathcal{C}$  (in  $\triangleleft$ -case) is that each puzzle  $\Pi$  satisfies the inequality

$$\lambda(I) - \bar{\lambda}(J) + \mu'(K) - v'(L) \geq 0, \tag{22}$$

where  $I, J, K, L$  are the sets of edges labeled 1 in the lower, upper, left and right sides of  $G$ , respectively. To show the necessity is rather easy, as follows. Let  $\mathcal{C} \neq \emptyset$  and let  $x \in \mathcal{C}$  (considering  $x$  as a function on  $V$ ). The discrete concavity of  $x$  implies that for each little rhombus  $\rho$  with obtuse vertices  $u, u'$  and acute vertices  $v, v'$ , one has  $q(x, \rho) := x(u) + x(u') - x(v) - x(v') \geq 0$ . When summing up these inequalities for all rhombi in  $\Pi$  and the equalities  $(x(v) - x(u)) + (x(w) - x(v)) + (x(u) - x(w)) = 0$  for all little triangles labeled 1, with vertices  $u, v, w$  in the anticlockwise order, the terms  $x(\cdot)$  for interior vertices cancel out and we just obtain (22) with  $I, J, K, L$  to be the sets of edges labeled 1 on the corresponding sides.

When  $c$  tends to  $+\infty$ , the value  $q(x, \rho)$  does so as well (uniformly for all  $x \in \mathcal{C}$ ) for each little rhombus  $\rho$ , if any, whose smaller diagonal is parallel to the bottom side of  $G$ . The grow of  $q(x, \rho)$  must cause a similar behavior for the left-hand side in (22). This implies that the puzzles containing at least one of such rhombi  $\rho$  can be excluded from the consideration, as they become redundant in verification of the nonemptiness of  $\mathcal{C}$ . Now relation (4) in Theorem 1 can be deduced from (22) when the remaining puzzles  $\Pi$  are considered.

In conclusion, it should be noted that, using the above reduction to the fully concave case and an argument in [2] (where an alternative proof of the integrality theorem from [8] is given), one can show the following sharper version of the last claim in Theorem 1.

**Proposition 2.** *For integer  $\lambda, \bar{\lambda}, \mu, v$ , the down hull  $\mathcal{D}$  of  $SC(\lambda \setminus \bar{\lambda}, \mu, v)$  (i.e., the polyhedron  $SC(\lambda \setminus \bar{\lambda}, \mu, v) - \mathbb{R}_+^V$ ) is integral.*

One can give a direct, relatively simple, proof of this proposition. A sketch: Consider a vertex  $X$  of  $\mathcal{D}$ ; then there is no array  $X' \neq X$  in  $\mathcal{D}$  with  $X' \geq X$ . Let  $V_1, \dots, V_N$  be the minimal nonempty sets of index pairs such that for  $q = 1, \dots, N$  and for any  $ij$  and  $i'j'$  with  $i' = i + 1, j' \in \{j, j + 1\}$  and  $\partial x_{ij} = \partial x_{i'j'}$ , the set  $V_q$  contains either both or none of  $ij$  and  $i'j'$ . Let  $c_q := \partial x_{ij}$  for  $ij \in V_q$ . Each  $V_q$  is associated with the corresponding subset of horizontal edges in the grid  $G$ ; let  $R_q$  denote the union of little triangles containing an edge in this subset. Then the interior of each region  $R_q$  is connected, and each maximal horizontal line  $\mathcal{L}_i$  in  $G$  (corresponding to the  $i$ th row in  $\partial X$ ) intersects  $R_q$  by a connected, possibly empty, set. We say that  $R_q$  is an *intermediate* region if it has no edge in the lower

or upper boundary of  $G$ ; let for definiteness  $R_1, \dots, R_\ell$  be the intermediate regions. One shows that if the set  $W_q$  of nodes of  $G$  occurring in the interior of an intermediate region  $R_q$  is nonempty, then one can increase the function  $x$  by a (small) positive constant within the set  $W_q$  so as to preserve the strip-concavity; the boundary tuples  $\lambda^X, \bar{\lambda}^X, \mu^X, \nu^X$  are preserved automatically. (This relies on the observation that if, e.g.,  $\partial x_{ij} = \partial x_{i-1,j}$  and the vertex  $(i, j-1)$  is in  $W_q$ , then  $(i-1, j-1)$  is in  $W_q$  as well, in view of  $\partial x_{ij} = \partial x_{i,j-1} = \partial x_{i-1,j-1}$ .) Therefore,  $W_q = \emptyset$  for all  $q = 1, \dots, \ell$ ; in other words, each horizontal line  $\mathcal{L}_i$  contains at most one edge within  $R_q$ .

Now associate with  $R_q$  ( $1 \leq q \leq \ell$ ) a real variable  $z_q$ . Let  $A = (a_{iq})$  be the  $(n-1) \times \ell$  matrix in which  $a_{iq}$  is the number of edges of the line  $\mathcal{L}_i$  occurring in  $R_q$ . Form the linear system  $Az = b$ , where for  $i = 1, \dots, n-1$ ,  $b_i$  is equal to  $x_{i,i+m} - x_{i0}$  minus the sum of values  $\partial x_{ij}$  over all  $ij$  concerning the edges of nonintermediate regions. Then for the numbers  $c_q$  as above, the tuple  $z := (c_1, \dots, c_\ell)$  is a solution to this system. Note that each  $b_i$  is an integer. (Indeed, each of the above values  $\partial x_{ij}$  is equal to some entry of  $\lambda$  or  $\bar{\lambda}$ , which is an integer;  $x_{i0}$  and  $x_{i,i+m}$  are integers as well.) Also  $A$  is a 0,1-matrix and the ones in each column go in succession, i.e.  $A$  is an *interval* matrix. So  $A$  is totally unimodular (cf. [10, Section 19.4]) and must have full column rank (otherwise  $Az = 0$  has a nonzero solution and we can represent  $X$  as the half-sum of two other points in  $\mathcal{SC}(\lambda \setminus \bar{\lambda}, \mu, \nu)$ ). Then  $c_1, \dots, c_\ell$  are integers, as required.

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