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Discrete superpositions of coherent states and phase properties of elliptically polarized light propagating in a Kerr medium

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Abstract. The problem of the formation of discrete superpositions of coherent states when elliptically polarized light is propagating through a nonlinear Kerr medium is considered. It is shown that superpositions with any number of components can be obtained if the evolution time is taken as a fraction M/N of the period, where M and N are mutually prime integers. Exact analytical formulae for finding the superposition coefficients are given. It is shown that the coupling between the two circularly polarized components of the ellpitically polarized light caused by the asymmetry of the nonlinear properties of the medium can suppress the number of components in the superposition from N^2 to N, if the asymmetry parameter takes appropriate values. The phase distribution function $P(\theta_+, \theta_-)$ for the two-mode field is obtained according to the new Pegg–Barnett phase formalism. This function exhibits a well resolved, multi-peak structure, clearly indicating the formation of the discrete superpositions of coherent states. Examples of the phase distribution function for several superposition states are illustrated graphically, showing in a very spectacular way the formation of such superpositions.

1. Introduction

It is well known that the polarization of strong elliptically polarized light propagating through an isotropic nonlinear medium varies. This effect of self-induced rotation of the polarization ellipse was first reported by Maker *et al* [1] in 1964. Since then propagation of light through a Kerr medium has been very extensively studied, and it has become a standard topic in textbooks on nonlinear optics [2, 3]. Those classical results assumed the field as being classical. Quantum properties of the field propagating through a Kerr medium were taken into consideration by Ritze and Bandilla [4], Tanaś and Kielich [5], and Ritze [6] from the point of view of photon statistics and antibunching. Propagation of light in a Kerr medium appeared to be even more

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attractive, and promising, from the point of view of squeezing, which is a phase-sensitive effect and the nonlinear changes in phase of the field were expected to be an effective mechanism for producing squeezed states of light. Tanaś and Kielich [7] have shown that intense light propagating through a nonlinear Kerr medium can squeeze its own quantum fluctuations. They referred to this effect as self-squeezing, and have proved the possibility of as much as 98 per cent of squeezing in this process. The description of the field in [7] was the two-mode quantized-field description of the elliptically polarized light propagating in the medium. The one-mode version of the self-squeezing effect applicable for circularly polarized light propagating in an isotropic Kerr medium has been considered by Tanaś [8] in terms of an anharmonic oscillator model. He has shown that the same amount of self-squeezing as in the two-mode case is attainable. This very simple, exactly solvable model of the anharmonic oscillator appeared to be very attractive and many properties of the quantum states generated from this model have been discussed recently [9–26].

Milburn [9] has discussed the evolution of the quasiprobability distribution function $Q(\alpha, \alpha^*, t)$ for the anharmonic oscillator showing periodic recurrences of its initial form. Milburn and Holmes [10] have shown that dissipation in the model rapidly destroys the quantum recurrence effects. Kitagawa and Yamamoto [11], who also considered the quasiprobability distribution $Q(\alpha, \alpha^*, t)$ for the states obtained in the model, referred to squeezing in this case as 'crescent' squeezing (in contrast to 'elliptic' squeezing) because of the crescent shape of the quasiprobability distribution contours.

Yurke and Stoler [12] have shown that the states produced in the anharmonic oscillator model become a superposition of a finite number of coherent states under a proper choice of the evolution time. Tombesi and Mecozzi [13] have obtained the superposition states for the two-mode case and an arbitrary initial state of the field. Agarwal and Puri [24] have reexamined the problem of propagation of elliptically polarized light through a Kerr medium using the two-mode quantum description of the field. They have discussed the properties of the field states produced in the process and they also indicate the possibility of obtaining, for a special choice of the evolution time, a superposition of two coherent states. Miranowicz *et al* [26] have recently shown that, in the one-mode case, superpositions can be obtained with not only even but also odd numbers of components. They have also shown that the maximum number of well distinguished states is proportional to the field amplitude $|\alpha|$ and that the quasiprobability distribution $Q(\alpha, \alpha^*, t)$ indicates such superpositions in a very spectactular fashion.

In this paper we shall discuss the problem of producing discrete superpositions of coherent states in the propagation of elliptically polarized light through a nonlinear Kerr medium. We shall give the general conditions for obtaining the discrete superpositions of coherent states as well as the exact analytical formulae describing such superpositions for an arbitrary number of components. The possibility of superpositions with both even and odd number of components is shown. It is also shown that the coupling between the two modes propagating in the nonlinear medium can considerably suppress the number of components in the superposition. To illustrate graphically the superpositions obtained in the two-mode description of the field we shall use the two-dimensional phase distribution function $P(\theta_+, \theta_-)$, which is obtained within the framework of the new Pegg–Barnett [27–29] phase formalism. This distribution function in the case of a superposition of well distinguished coherent states splits into separate peaks clearly indicating the superposition.

2. Quantum evolution of elliptically polarized light propagating in a Kerr medium

The classical description of light propagating through a nonlinear Kerr medium is related to the third-order nonlinear polarization of the medium. A monochromatic light field of frequency ω induces the third-order polarization of the medium at frequency ω, which can be written as follows [2, 3]

$$P_i(\omega)=\sum_{jkl}\chi_{ijkl}(-\omega,-\omega,\omega,\omega)E_j^{(-)}(\omega)E_k^{(+)}(\omega)E_l^{(+)}(\omega). \tag{1}$$

Here $\chi_{ijkl}(-\omega,-\omega,\omega,\omega)$ is the third-order nonlinear susceptibility tensor of the medium and the electromagnetic field is decomposed into the positive- and negative-frequency parts

$$E_i(\boldsymbol{r},t)=E_i^{(+)}(\boldsymbol{r},t)+E_i^{(-)}(\boldsymbol{r},t) \tag{2}$$

with

$$E_i^{(+)}(\boldsymbol{r},t)=E_i^{(+)}(\omega)\exp[-\mathrm{i}(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})]. \tag{3}$$

The above definition of the field gives the following relation between the field amplitudes and the intensity of the beam

$$I(\omega)=\frac{cn(\omega)}{2\pi}\sum_i E_i^{(-)}(\omega)E_i^{(+)}(\omega) \tag{4}$$

where $n(\omega)$ is the linear refractive index of the medium for frequency ω.

For an isotropic medium with a centre of inversion, the nonlinear susceptibility tensor, $\chi_{ijkl}(\omega)=\chi_{ijkl}(-\omega,-\omega,\omega,\omega)$, can be written as [2, 3]

$$\chi_{ijkl}(\omega)=\chi_{xxyy}(\omega)\delta_{ij}\delta_{kl}+\chi_{xyxy}(\omega)\delta_{ik}\delta_{jl}+\chi_{xyyx}(\omega)\delta_{il}\delta_{jk} \tag{5}$$

with the additional relation

$$\chi_{xxxx}(\omega)=\chi_{yyyy}(\omega)=\chi_{xxyy}(\omega)+\chi_{xyxy}(\omega)+\chi_{xyyx}(\omega). \tag{6}$$

Assuming that the beam propagates along the z-axis of the laboratory reference frame, inserting equation (5) into equation (1) and introducing the circular basis

$$E_\pm^{(+)}(\omega)=2^{-1/2}[E_x^{(+)}(\omega)\mp \mathrm{i}E_y^{(+)}(\omega)\} \tag{7}$$

one obtains for the circular components of the nonlinear polarization of the medium the expression

$$\begin{aligned}P_\pm^{(+)}(\omega)&=[\chi_{xyxy}(\omega)+\chi_{xyyx}(\omega)]|E_\pm^{(+)}(\omega)|^2E_\pm^{(-)}(\omega)\\&\quad+[2\chi_{xxyy}(\omega)+\chi_{xyxy}(\omega)+\chi_{xyyx}(\omega)]|E_\mp^{(+)}(\omega)|^2E_\pm^{(+)}(\omega).\end{aligned} \tag{8}$$

Taking into account the permutation symmetry of the tensor χ with respect to the first and the second pairs of indices, we have $\chi_{xyxy}(\omega)=\chi_{xyyx}(\omega)$, and formula (8) can be further simplified.

Inserting equation (8) as a source term into the Maxwell equations, and using the slowly varying amplitude approximation, one obtains the following equation for the amplitudes of the circular components of the field:

$$\frac{\mathrm{d}E_\pm^{(+)}(\omega)}{\mathrm{d}z}=\frac{\mathrm{i}2\pi\omega}{n(\omega)c}\{2\chi_{xyxy}(\omega)|E_\pm^{(+)}(\omega)|^2+2[\chi_{xxyy}(\omega)+\chi_{xyxy}(\omega)]|E_\mp^{(+)}(\omega)|^2\}E_\pm^{(+)}(\omega) \tag{9}$$

where the amplitudes $E_{\pm}^{(+)}(\omega)$ are assumed to be dependent on z. Equation (9) immediately shows an advantage of the circular basis used here. One can easily check that $|E_{\pm}^{(+)}(\omega)|^2$ does not depend on z, that is $(\mathrm{d}/\mathrm{d}z)|E_{\pm}^{(+)}(\omega)|^2=0$, and then equation (9) has simple exponential solutions [30].

To describe the quantum evolution of the field, we need quantum equations of motion for the field. Such equations for the field operators, which are the Heisenberg equations of motion, can be obtained from the following effective interaction Hamiltonian [7]:

$$H_{\mathrm{I}}=\tfrac{1}{2}\hbar\kappa(a_{+}^{\dagger 2}a_{+}^{2}+a_{-}^{\dagger 2}a_{-}^{2}+4da_{+}^{\dagger}a_{-}^{\dagger}a_{-}a_{+}). \tag{10}$$

Here the nonlinear coupling constant κ is real and is given by

$$\kappa=\frac{V}{\hbar}\left(\frac{2\pi\hbar\omega}{n^2(\omega)V}\right)^2 2\chi_{xyxy}(\omega) \tag{11}$$

with V denoting the quantization volume, and we have introduced a nonlinear asymmetry parameter d defined as

$$2d=1+\frac{\chi_{xxyy}(\omega)}{\chi_{xyxy}(\omega)}. \tag{12}$$

If the nonlinear susceptibility tensor χ is symmetric with respect to all its indices, the asymmetry parameter d is equal to unity. Otherwise $d\neq 1$ and describes the asymmetry of the nonlinear properties of the medium. When the medium is composed of identical molecules the asymmetry parameter d is related to the nonlinear polarizability of individual molecules [7]. Ritze [6] has calculated this asymmetry parameter for atoms with a degenerate one-photon transition obtaining the results

$$d=\begin{cases}(2J-1)(2J+3)/[2(2J^2+2J+1)]\\(2J^2+3)/[2(6J^2-1)]\end{cases} \tag{13}$$

for $J\leftrightarrow J$ and $J\leftrightarrow J-1$ transitions, respectively.

The operators $a_{\pm}$ in the Hamiltonian (10) are the annihilation operators for the circularly right- and left-polarized modes, satisfying the commutation relations

$$[a_i,a_j^{\dagger}]=\delta_{ij} \qquad (i,j=+\mathrm{or}-). \tag{14}$$

The relation between the annihilation operators, which are dimensionless, and the corresponding field operators is given by

$$E_{\pm}^{(+)}(\omega)=\mathrm{i}\left(\frac{2\pi\hbar\omega}{n^2(\omega)V}\right)^{1/2}a_{\pm}. \tag{15}$$

Using the interaction Hamiltonian (10) and the commutation rules (14), one can easily write the Heisenberg equations of motion describing the time evolution of the field operators. In the travelling wave case the time t is replaced by $-n(\omega)z/c$, and we obtain the following equation:

$$\frac{\mathrm{d}a_{\pm}(z)}{\mathrm{d}z}=\mathrm{i}\frac{n(\omega)}{c}\kappa[a_{\pm}^{\dagger}(z)a_{\pm}(z)+2da_{\mp}^{\dagger}(z)a_{\mp}(z)]a_{\pm}(z). \tag{16}$$

When the relation (15) is applied, equation (16) reverts to the form of equation (9), which makes the quantum–classical correspondence quite transparent.

Since the number of photons in the two modes $a^\dagger_\pm a_\pm$ are constants of motion (they commute with the Hamiltonian (10)), equation (16) has the simple exponential solution [6, 7]

$$a_\pm(\tau) = \exp\{i\tau[a^\dagger_\pm(0)a_\pm(0) + 2da^\dagger_\mp(0)a_\mp(0)]\}a_\pm(0) \tag{17}$$

where we have introduced the notation

$$\tau = n(\omega)\kappa z/c. \tag{18}$$

The solutions (17) are exact operator solutions for the field operators of light propagating through a nonlinear, isotropic Kerr medium. These equations were used for calculations of quantum effects such as photon antibunching [6] and squeezing [7].

To describe the evolution of the field states we can use the evolution operator $U(\tau)$, which according to equations (10) and (18), and after the replacement $t = -n(\omega)z/c$, has the form

$$\begin{aligned} U(\tau) &= \exp[\tfrac{1}{2}\tau i(a^{\dagger 2}_+ a^2_+ + a^{\dagger 2}_- a^2_- + 4da^\dagger_+ a^\dagger_- a_- a_+)] \\ &= \exp[\tfrac{1}{2}\tau i(\hat{n}_+(\hat{n}_+ - 1) + \hat{n}_-(\hat{n}_- - 1) + 4d\hat{n}_+\hat{n}_-)] \end{aligned} \tag{19}$$

where we have introduced the number operators $\hat{n}_\pm = a^\dagger_\pm a_\pm$ for the two circularly polarized modes. The resulting state of the field is thus given by

$$|\psi(\tau)\rangle = U(\tau)|\psi(0)\rangle \tag{20}$$

where $|\psi(0)\rangle$ is the initial state of the field. If the initial state of the field is a coherent state of elliptically polarized light one obtains [24]

$$\begin{aligned} |\psi(\tau)\rangle &= U(\tau)|\alpha_+, \alpha_-\rangle \\ &= \sum_{n_+, n_-} b_{n_+} b_{n_-} \exp\{i(n_+\varphi_+ + n_-\varphi_-) \\ &\quad + \tfrac{1}{2}\tau i[n_+(n_+ - 1) + n_-(n_- - 1) + 4dn_+n_-]\}|n_+, n_-\rangle \end{aligned} \tag{21}$$

where

$$b_{n_\pm} = \exp(-\tfrac{1}{2}|\alpha_\pm|^2)|\alpha_\pm|^{n_\pm}/\sqrt{n_\pm!} \tag{22}$$

and the state $|n_+, n_-\rangle = |n_+\rangle|n_-\rangle$ is the Fock state. We have used $\alpha_\pm = |\alpha_\pm|\exp(i\varphi_\pm)$ here.

The properties of the states (21) have recently been discussed by Agarwal and Puri [24], who have shown that, using our notation, for $\tau=\pi$ and $d=\frac{3}{2}$, the resulting state is a superposition of two coherent states. We are going to study the problem of generating discrete superpositions of coherent states in the model in more detail in the next section.

3. Discrete superpositions of coherent states

To make our analysis simpler, we start with the one-mode problem. If light propagating through a Kerr medium is circularly polarized, say right, then the state of the left-polarized mode is the vacuum and it remains vacuum all the time. This means that propagation of circularly polarized light in a Kerr medium can be described in terms of the anharmonic oscillator model. In this case the state of the field is given by [11]

$$|\psi(\tau)\rangle=\sum_n b_n \exp\{\mathrm{i}[n\varphi+\tfrac{1}{2}\tau n(n-1)]\}|n\rangle. \tag{23}$$

The problem of generating discrete superpositions of coherent states in such a model has been discussed by Miranowicz *et al* [26], who have shown that superpositions with both even and odd numbers of components are possible. Here, we shall give exact anaytical formulae for the superpositions with any number, N, of components.

It was shown by Białynicka-Birula [31] that under periodic conditions the generalized coherent states, like (23), become a discrete superposition of N coherent states, and that the superposition coefficients can be found by solving a system of N algebraic equations. Such a system of equations has been solved for several N-values by Miranowicz *et al* [26], and analytical formulae for the superposition states have been obtained. Averbukh and Perelman [32] have considered the problem of the evolution of wave packets formed by highly excited states of quantum systems that shows a possibility of 'fractional revivals' of the initial wave packet. Their calculations effectively lead to the anharmonic oscillator model, similar to our model. They have shown that because of the periodicity the superposition coefficients can be obtained for arbitrary N. We shall follow their approach here.

First of all, it is easy to note that $|\psi(\tau+T)\rangle=|\psi(\tau)\rangle$, for $T=2\pi$, because $n(n-1)$ is an even number. This means that the evolution is periodic in time (or length of the medium) with the period $T=2\pi$. Moreover, we have

$$\exp[\tfrac{1}{2}\tau\mathrm{i}(n+2N)(n+2N-1)]=\exp[\tfrac{1}{2}\tau\mathrm{i}n(n-1)]\exp[\mathrm{i}\tau N(2N+2n-1)] \tag{24}$$

which means that for

$$\tau=\frac{M}{N}2\pi=\frac{M}{N}T \tag{25}$$

the exponential becomes periodic with the period $2N$. We assume that M and N are mutually prime integers. If τ is taken according to equation (25), as a fraction of the period, the state (23) becomes a superposition of coherent states [31]

$$|\psi(\tau=MN^{-1}T)\rangle=\sum_{k=0}^{2N-1} c_k|\exp(\mathrm{i}\varphi_k)\alpha\rangle \tag{26}$$

where $|\alpha\rangle$ is the initial coherent state. The phases φ_k are given by

$$\varphi_k=\frac{\pi}{N}k \qquad k=0,1,\ldots,2N-1 \tag{27}$$

and the coefficients c_k are given by the set of $2N$ equations

$$\sum_{k=0}^{2N-1} c_k\exp(\mathrm{i}n\varphi_k)=\exp\left(\mathrm{i}\pi\frac{M}{N}n(n-1)\right) \tag{28}$$

where $n=0,1,\ldots,2N-1$. Equation (28) can be rewritten as

$$\sum_{k=0}^{2N-1} c_k\exp\left(\mathrm{i}\frac{\pi}{N}[nk-Mn(n-1)]\right)=1 \tag{29}$$

which after summing over n and a minor rearrangement gives

$$\sum_{k=0}^{2N-1} c_k\frac{1}{2N}\sum_{n=0}^{2N-1}\exp\left(\mathrm{i}\frac{\pi}{N}[nk-Mn(n-1)]\right)=1. \tag{30}$$

In view of the condition

$$\sum_{k=0}^{2N-1} c_k c_k^*=1 \tag{31}$$

we immediately obtain

$$c_k=\frac{1}{2N}\sum_{n=0}^{2N-1}\exp\left(-\mathrm{i}\frac{\pi}{N}[nk-Mn(n-1)]\right). \tag{32}$$

Formula (32) gives the coefficients c_k of the superposition (26) for any M and N. Because of the symmetry of the system, only one half of the coefficients c_k is different from zero [26], and the superposition (26) has only N components although the summation contains $2N$ terms. Anticipating this, we have extended the summations twice in order to keep the number of components equal to N. Thus, the denominator of the fraction M/N in equation (25) determines the number of components that appear in the superposition (26), which will contain the components with even (or odd) index only. Examples of such states are given in [26].

Proceeding along the same lines, we can find the superposition states for the two-mode case. Taking the evolution time as in equation (25) and assuming $2d$ to be an integer, the state (21) can be written as the superposition

$$|\psi(\tau=MN^{-1}T)\rangle=\sum_{n=0}^{2N-1}\sum_{k=0}^{2N-1}c_{nk}|\exp(\mathrm{i}\varphi_n)\alpha_+\rangle|\exp(\mathrm{i}\varphi_k)\alpha_-\rangle \tag{33}$$

where φ_n and φ_k are given by equation (27), and the coefficients c_{nk} are given by

$$c_{nk}=\frac{1}{(2N)^2}\sum_{n_+=0}^{2N-1}\sum_{n_-=0}^{2N-1}$$
$$\times\exp\left(-\mathrm{i}\frac{\pi}{N}\{nn_++kn_--M[n_+(n_+-1)+n_-(n_--1)+4dn_+n_-]\}\right) \tag{34}$$

It is obvious from equation (34) that the coefficients c_{nk} are symmetric,

$$c_{nk}=c_{kn}. \tag{35}$$

If $d=0$, which can happen according to equation (13) for the $\frac{1}{2}\leftrightarrow\frac{1}{2}$ transitions, there is no coupling between the two modes, and the coefficients c_{nk} factorize into the product of the coefficients for individual modes. In this case the number of components in the superposition (33) is equal to N^2. Generally $d\neq 0$, and the two modes propagating in the medium are coupled. This coupling can suppress the number of components in the superposition (33) considerably. It must be kept in mind, however, that in order to obtain the superposition it is necessary to have $2d$ integer, which restricts the possible values of d to integers or half-integers. We have, according to equation (13), $d=0$ for $\frac{1}{2}\to\frac{1}{2}$ transitions and $d=\frac{1}{2}$ for $1\to 0$ transitions, and according to equation (12), $d=1$ for the completely symmetric susceptibility tensor.

Starting with $d=0$ and adding one half in a subsequent step, we arrive for $d=N/2M$ at the initial state with $d=0$ because of the periodicity in d of equation (34).

There are some more symmetry properties of the coefficients c_{nk}, but instead of discussing them in detail, we shall rather give several examples of the superposition states.

For $N=2$, $M=1$ and $d=0$, we have

$$c_{11}=-c_{33}=-\tfrac{1}{2}\mathrm{i} \qquad c_{13}=c_{31}=\tfrac{1}{2}$$

and the resulting state is

$$|\psi(\tau=\pi)\rangle_{d=0}=\tfrac{1}{2}(-\mathrm{i}|\mathrm{i}\alpha_+,\mathrm{i}\alpha_-\rangle+\mathrm{i}|-\mathrm{i}\alpha_+,-\mathrm{i}\alpha_-\rangle+|\mathrm{i}\alpha_+,-\mathrm{i}\alpha_-\rangle+|-\mathrm{i}\alpha_+,\mathrm{i}\alpha_-\rangle) \tag{36}$$

where $|\alpha_+,\alpha_-\rangle$ stands for $|\alpha_+\rangle|\alpha_-\rangle$.

For $N=2$, $M=1$ and $d=\frac{1}{2}$, we have

$$c_{11}=c_{33}^*=2^{-1/2}\,\mathrm{e}^{-\mathrm{i}\pi/4} \qquad c_{13}=c_{31}=0$$

with the resulting state

$$|\psi(\tau=\pi)\rangle_{d=1/2}=2^{-1/2}(\mathrm{e}^{-\mathrm{i}\pi/4}|\mathrm{i}\alpha_+,\mathrm{i}\alpha_-\rangle+\mathrm{e}^{\mathrm{i}\pi/4}|-\mathrm{i}\alpha_+,-\mathrm{i}\alpha_-\rangle). \tag{37}$$

This is the superposition obtained by Agarwal and Puri [24], although they have taken $d=\frac{3}{2}$, but because of periodicity it gives the same state. Two of the states of the superposition (36) have been eliminated in the superposition (37) owing to the coupling between the two modes.

For $N=3$, $M=1$ and $d=0$, the non-zero coefficients are

$$c_{00}=c_{04}=c_{40}=c_{44}=\tfrac{1}{3}e^{i\pi/3}$$

$$c_{02}=c_{20}=c_{24}=c_{42}=\tfrac{1}{3}e^{-i\pi/3}$$

$$c_{22}=-\tfrac{1}{3}$$

and the number of states in the superposition is equal to $N^2=9$. The state is given by

$$\begin{aligned}|\psi(\tau=2\pi/3)\rangle_{d=0}=\tfrac{1}{3}e^{i\pi/3}(&|\alpha_+,\alpha_-\rangle+|\alpha_+,e^{-i2\pi/3}\alpha_-\rangle\\&+|e^{-i2\pi/3}\alpha_+,\alpha_-\rangle+|e^{-i2\pi/3}\alpha_+,e^{-i2\pi/3}\alpha_-\rangle)\\&+\tfrac{1}{3}e^{-i\pi/3}(|\alpha_+,e^{i2\pi/3}\alpha_-\rangle+|e^{i2\pi/3}\alpha_+,\alpha_-\rangle\\&+|e^{i2\pi/3}\alpha_+,e^{-i2\pi/3}\alpha_-\rangle+|e^{-i2\pi/3}\alpha_+,e^{i2\pi/3}\alpha_-\rangle)\\&-\tfrac{1}{3}|e^{i2\pi/3}\alpha_+,e^{i2\pi/3}\alpha_-\rangle.\end{aligned} \tag{38}$$

For $N=3$, $M=1$ and $d=\tfrac{1}{2}$, we obtain

$$c_{00}=c_{44}=3^{-1/2}\,e^{i\pi/6} \qquad c_{22}=-3^{-1/2}i$$

which gives the state

$$\begin{aligned}|\psi(\tau=2\pi/3)\rangle_{d=1/2}=3^{-1/2}\,e^{i\pi/6}[&|\alpha_+,\alpha_-\rangle+|e^{-i2\pi/3}\alpha_+,e^{-i2\pi/3}\alpha_-\rangle]\\&-3^{-1/2}i|e^{i2\pi/3}\alpha_+,e^{i2\pi/3}\alpha_-\rangle\end{aligned} \tag{39}$$

with the number of components suppressed to $N=3$. The results for $d=1$ are

$$c_{04}=c_{40}=3^{-1/2}\,e^{i\pi/6} \qquad c_{22}=-3^{-1/2}i$$

and

$$\begin{aligned}|\psi(\tau=2\pi/3)\rangle_{d=1}=3^{-1/2}\,e^{i\pi/6}(&|\alpha_+,e^{-i2\pi/3}\alpha_-\rangle+|e^{-i2\pi/3}\alpha_+,\alpha_-\rangle)\\&-3^{-1/2}i|e^{i2\pi/3}\alpha_+,e^{i2\pi/3}\alpha_-\rangle.\end{aligned} \tag{40}$$

Again we obtain the superposition of $N=3$ states, but this time the components are different from those in equation (39).

In a similar manner we can, using equations (33) and (34), obtain the superposition of coherent states for any values N, M and d. Generally such a superposition can have N^2 components that, however, can be reduced to N components when the asymmetry parameter d takes an appropriate value. The number N can take any integer value. The formation of such superpositions will be illustrated graphically by plotting the phase distribution function of the resulting field.

4. Phase distribution function and superposition states of the field

Recently, Pegg and Barnett [27–29] have introduced a new phase formalism, which allows for the construction of the Hermitian phase operator as well as for defining the phase distribution function. All necessary quantum mechanical phase expectation values can be calculated in a classical manner with this phase distribution function.

The phase distribution function is thus a good representation of the field state, and we use it here to illustrate the formation of the superposition states discussed in section 3.

For the one-mode case, Miranowicz *et al* [26] have used the quasiprobability distribution $Q(\alpha, \alpha^*, t)$ to illustrate the discrete superpositions of coherent states that are formed during the evolution of such a system. The quasiprobability distribution $Q(\alpha, \alpha^*, t)$ can be plotted as a function of two variables (Reα and Imα) in the complex α-plane, giving a three-dimensional picture representing the states of the field. When a superposition of coherent states with well distinguished components occurs the Q-function splits into separate peaks representing coherent states of the superposition.

In the two-mode case considered in this paper, the superposition states are composed of the products of coherent states of the two modes. To use the Q-function to illustrate the superposition states appearing in this case, it would be necessary to superimpose the two α of both components into an effective α describing the one mode of elliptically polarized light that corresponds to these components. Otherwise, we should deal with a function of four variables, which is difficult to illustrate in the three-dimensional space. This problem disappears if we, instead of the Q-function, use the phase probability distribution $P(\theta_+, \theta_-)$ as a representation of the two-mode field state. We deal with the function of two variables θ_+ and θ_-, which describes the phases of the two modes, and this function can be directly plotted.

The new Pegg–Barnett [27–29] phase formalism is based on introducing a finite $(s+1)$-dimensional subspace Ψ spanned by the number states $|0\rangle$, $|1\rangle, \ldots, |s\rangle$. The Hermitian phase operator operates on this finite subspace, and after all necessary expectation values have been calculated in Ψ, the value of s is allowed to tend to infinity. A complete orthonormal basis of $s+1$ states, for a one-mode field, is defined on Ψ as

$$|\theta_m\rangle \equiv (s+1)^{-1/2} \sum_{n=0}^{s} \exp(in\theta_m)|n\rangle \tag{41}$$

where

$$\theta_m \equiv \theta_0 + 2\pi m/(s+1) \qquad m=0, 1, \ldots, s. \tag{42}$$

The value of θ_0 is arbitrary and defines a particular basis set of $s+1$ mutually orthogonal phase states. The Hermitian phase operator is defined as

$$\hat{\phi}_\theta \equiv \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m|. \tag{43}$$

The phase states (41) are eigenstates of the phase operator (43) with the eigenvalues θ_m restricted to lie within a phase window from θ_0 to $\theta_0 + 2\pi$. The expectation value of the phase operator in a state $|\psi\rangle$ is given by

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle = \sum_{m=0}^{s} \theta_m |\langle\theta_m|\psi\rangle|^2 \tag{44}$$

where $|\langle\theta_m|\psi\rangle|^2$ gives the probability of being found in the phase state $|\theta_m\rangle$. The density of phase states is $(s+1)/2\pi$, so in the continuum limit as s tends to infinity, we can write equation (44) as

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle=\int_{\theta_0}^{\theta_0+2\pi}\theta P(\theta)\,\mathrm{d}\theta \tag{45}$$

where the continuum phase distribution function $P(\theta)$ is introduced by

$$P(\theta)=\lim_{s\to\infty}\frac{s+1}{2\pi}|\langle\theta|\psi\rangle|^2 \tag{46}$$

where θ_m is replaced by the continuous variable θ. As the phase distribution function $P(\theta)$ is found for a given state $|\psi\rangle$, all the quantum mechanical phase expectation values can be calculated with this function in a classical-like manner. The choice of the value of θ_0 defines the 2π-range window of the phase values.

Generalization of the phase formalism into the two-mode case is straightforward, and for our state (21) we obtain

$$\begin{aligned}\langle\theta_{m_-}|\langle\theta_{m_+}|\psi(\tau)\rangle &= (s_+ +1)^{-1/2}(s_- +1)^{-1/2}\\ &\times\sum_{n_+=0}^{s_+}\sum_{n_-=0}^{s_-} b_{n_+}b_{n_-}\exp\{\mathrm{i}n_+(\varphi_+-\theta_{m_+})+\mathrm{i}n_-(\varphi_--\theta_{m_-})\\ &+\tfrac{1}{2}\tau\mathrm{i}[n_+(n_+-1)+n_-(n_--1)+4dn_+n_-]\}.\end{aligned} \tag{47}$$

Since the initial states of the two modes are coherent states, that is partial phase states, it is convenient to choose the phase value windows to be symmetrical with respect to the phases $\varphi_\pm$ of the coherent states. This means that

$$\theta_0^\pm=\varphi_\pm-\frac{\pi s_\pm}{s_\pm+1} \tag{48}$$

and

$$\varphi_\pm-\theta_{m_\pm}=-\theta_{\mu_\pm} \tag{49}$$

where the new phase labels $\mu_\pm$ run in unit steps between the values $-\frac{1}{2}s_\pm$ and $\frac{1}{2}s_\pm$. By taking the square of the modulus of equation (47), after taking into account equations

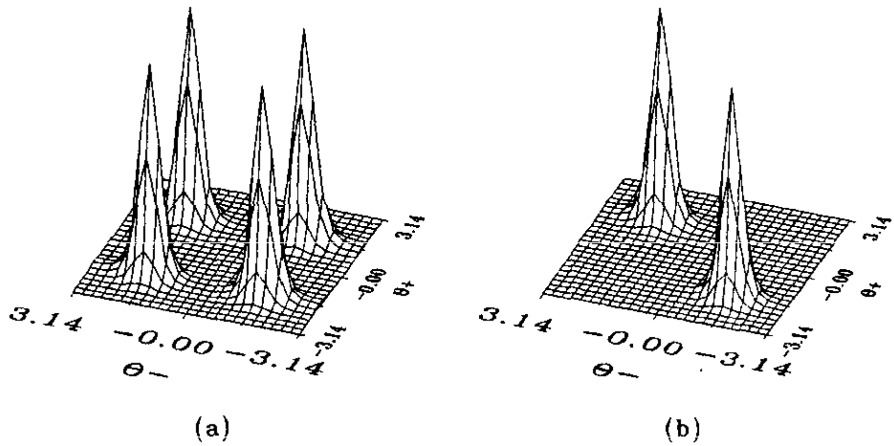

Figure 1. Plot of the phase distribution function $P(\theta_+, \theta_-)$ for the mean number of photons $N_+ = N_- = 4$, $\tau = 2\pi/2 = \pi$, and (a), $d=0$; (b), $d=\frac{1}{2}$.

(48) and (49), and performing the continuum limit transition by making the replacements

$$\sum_{\mu_\pm=-s_\pm/2}^{s_\pm/2} \frac{2\pi}{s_\pm+1} \rightarrow \int_{-\pi}^{\pi} \mathrm{d}\theta_\pm \tag{50}$$

we arrive at the continuous, two-dimensional phase distribution function given by

$$P(\theta_+, \theta_-) = \frac{1}{(2\pi)^2}\left| \sum_{n_+=0}^{\infty} \sum_{n_-=0}^{\infty} b_{n_+} b_{n_-} \right.$$
$$\left. \times \exp\{-\mathrm{i}n_+\theta_+ - \mathrm{i}n_-\theta_- + \tfrac{1}{2}\tau \mathrm{i}[n_+(n_+-1) + n_-(n_--1) + 4dn_+n_-]\} \right|^2 \tag{51}$$

with the normalization

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P(\theta_+, \theta_-)\, \mathrm{d}\theta_+\, \mathrm{d}\theta_- = 1. \tag{52}$$

The phase distribution function $P(\theta_+, \theta_-)$ (equation (51)) describes the phase properties of the field propagating in a Kerr medium. This function depends on τ, and its evolution describes the evolution of the phase properties of a field propagating in the Kerr medium. More detailed discussion of the phase properties of such a field will be given elsewhere [33]. In this paper we use the function (51) to illustrate the formation of the superposition states (33). When the mean numbers of photons in the two modes ($N_+ = |\alpha_+|^2$ and $N_- = |\alpha_-|^2$) are not large, the summation in equation (51) can be performed numerically and the results, for $\tau = 2\pi M/N$ with particular choices of M, N and the asymmetry parameter d, are shown in figures 1 to 4, where the mean numbers of photons in the two modes are assumed to be $N_+ = N_- = 4$.

In figure 1 the function $P(\theta_+, \theta_-)$ is shown for $M=1$, $N=2$ and $d=0$ (figure 1(a)), and $d=\frac{1}{2}$ (figure 1(b)). The four-peak structure corresponding to the state (36) appears in figure 1(a), which when $d=\frac{1}{2}$ is suppressed into two peaks that correspond to the superposition state obtained by Agarwal and Puri [24]. Since coherent states belong to a class of partial phase states, their phase distribution function has a well resolved peak structure. This suggests that the appearance of separate peaks in the phase distribution function can be an indication of the fact that the state of the field is a superposition (or close to the superposition) state. We have shown [34] that in the one-mode case of the anharmonic oscillator model the (one-dimensional) phase distribution function $P(\theta)$ splits into N peaks if the evolution time is taken as $t=2\pi/N$. This leads to N^2 peaks in the two-mode case if the two modes propagate independently ($d=0$). The coupling between the two modes ($d\neq0$) can reduce the number of peaks to N. This is convincingly shown in the figures. In figure 2 the case $M=1$, $N=3$ is illustrated for various values of d. For $d=0$ there are nine peaks that are reduced to three peaks when $d=\frac{1}{2}$ and $d=1$. In figure 3 the case $N=3$, but $M=2$, is illustrated. Again for $d=\frac{1}{2}$ and $d=1$ three peaks remain; however, for $d=1$ the structure is different than in figure 2. All these features can be explained when the analytical formulae for the superposition states obtained in the previous section are consulted (see, for example, formulae (38)–(40)). The case $M=1$ and $N=4$ is illustrated in figure 4 showing either sixteen or four peaks, as expected.

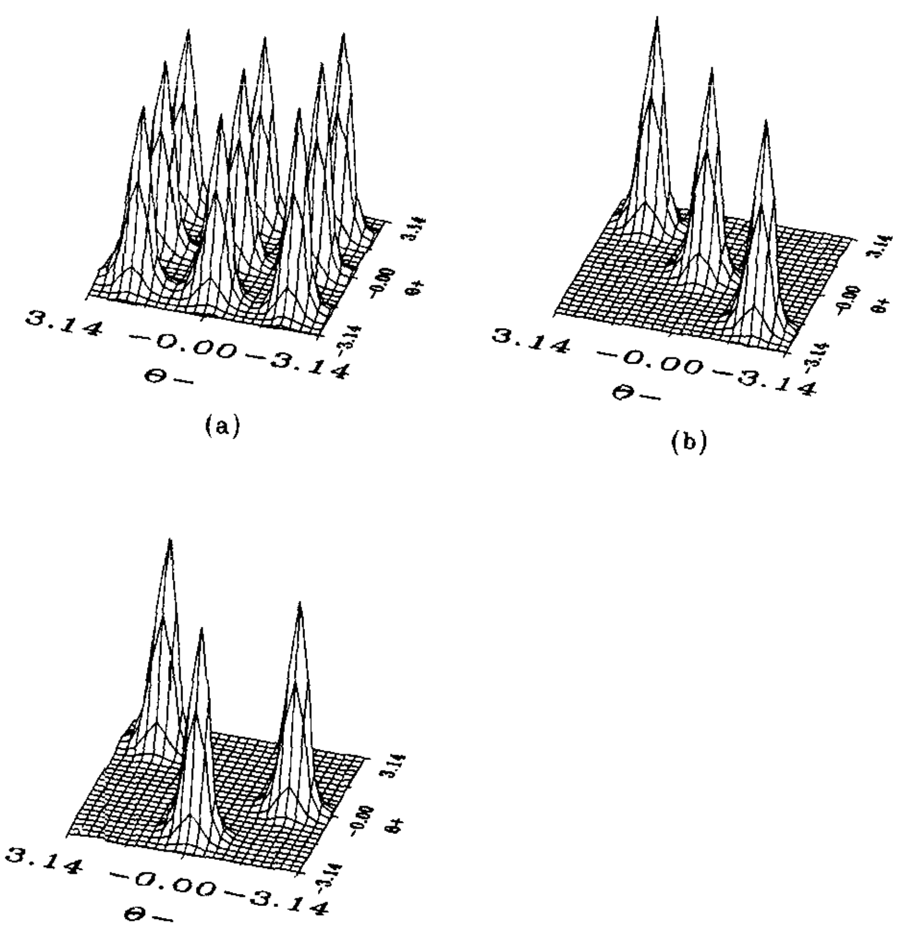

Figure 2. Same as figure 1, but for $\tau=2\pi/3$, and (a), $d=0$; (b), $d=\frac{1}{2}$; (c), $d=1$.

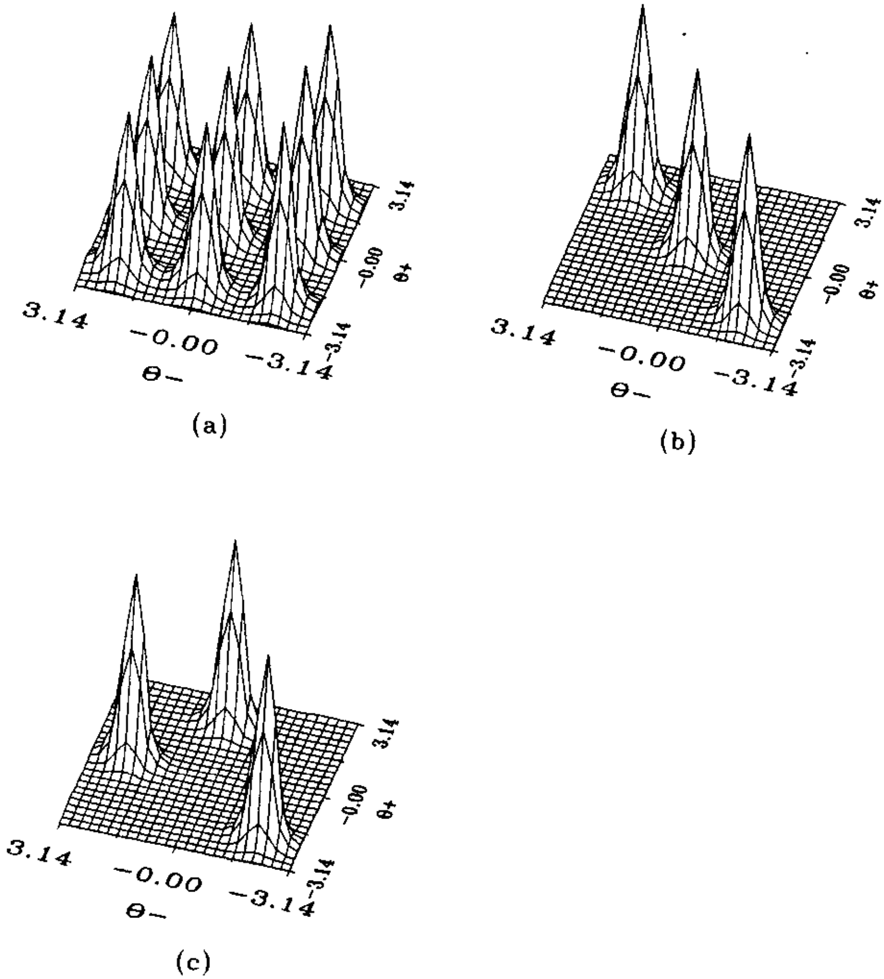

Figure 3. Same as figure 1, but for $\tau=4\pi/3$, and (a), $d=0$; (b), $d=\frac{1}{2}$; (c), $d=1$.

Of course, the peak structure of the phase distribution function will be well resolved when the interference terms are negligible and the distribution can be written as a sum of the phase distributions of individual component states.

Our results show that the phase distribution function is a good alternative with respect to, for example, the quasiprobability distribution function in describing the superpositions of coherent states.

5. Conclusions

In this paper we have considered the problem of the propagation of an elliptically polarized quantum field in a nonlinear Kerr medium from the point of view of the generation of a discrete superposition of coherent states. The two-mode description of elliptically polarized light has been applied. It has been shown that when the evolution time takes the fraction M/N of the period, with M and N being mutually prime integers, the state of the resulting field becomes a superposition of coherent states with the number of components equal to N^2 if the two modes are uncoupled. If the two modes are coupled because of the asymmetry properties of the nonlinear medium the number of components in the superposition can be reduced to N, if the asymmetry parameter d of the medium takes appropriate values. The exact analytical formulae for finding the superposition coefficients are given for any N. Superpositions of coherent states with any even or odd number of components are shown to appear. Examples of such superposition states are given explicitly. The phase distribution function obtained according to the new Pegg–Barnett phase formalism has been used to illustrate graphically the process of formation of the superposition states in the propagation of elliptically polarized light through a Kerr medium. We have shown that the two-dimensional phase distribution function exhibits a clear multi-peak structure when the state of the field becomes a superposition of coherent states. This structure is well resolved when the states forming the superposition are well distinguished. This means that the interference terms are negligible and the phase distribution of the resulting field becomes a sum of the phase distribution functions of

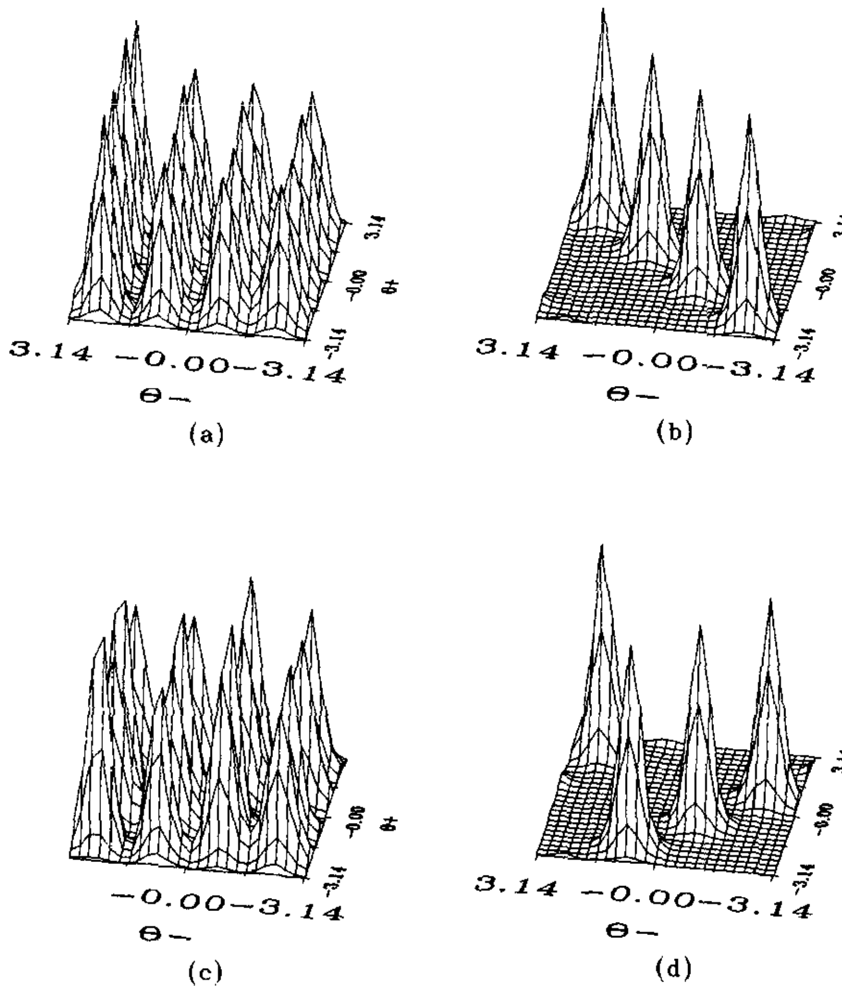

Figure 4. Same as figure 1, but for $\tau = 2\pi/4 = \pi/2$, and (a), $d=0$; (b), $d=\frac{1}{2}$; (c), $d=1$; (d), $d=\frac{3}{2}$.

the component states of the distribution. Thus, the phase distribution function can be treated as a good representation of the field state, indicating the formation of discrete superpositions of coherent states. This approach is an alternative with respect to the quasiprobability distribution function, and its application to describe the two-mode field has clear advantages.

Of course, to produce the superposition states in the process of light propagation in the Kerr medium very long times (or lengths of the medium) are needed to obtain the required τ-values. This means that the possibility of creating a superposition of macroscopically distinguishable quantum states in the process of the propagation of light in Kerr media is rather difficult in practice. Optical fibres with high nonlinearities and very small losses can be considered as possible candidates for experimental testing of the quantum effects discussed in this paper.

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