2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Discrete-Time Adaptive Control Using Multiple Models

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***Abstract*— In a recent paper [1] the authors proposed a new methodology for the adaptive control of a linear time-invariant plant using multiple models which is significantly different from the "switching" and "switching and tuning" methods which have been in use for over a decade. Extensive simulation studies have also revealed that the performance using the new method is far superior to earlier methods.**

In this paper an attempt is made to extend the same concepts to the adaptive control of discrete-time systems. It is well known that the control of discrete-time systems is simpler than the control of their continuous-time counterparts, that they find wider application in practice, and that the proofs of stability are substantially simpler. Also, in many cases (e.g. periodic systems) discrete-time control may be possible when even the formulation of tractable problems in continuous-time is impossible. The objective of this paper is to examine how the methodology proposed differs in the two cases with regard to transparency of the principal concepts, and effectiveness in practical applications.

I. Introduction

In a recent paper [1] a new methodology for adaptively controlling a linear time-invariant continuous-time system using multiple models was proposed. Assuming that the compact region $\mathcal{S}_\theta$ in which the unknown parameter vector $\theta_p \in \mathbb{R}^m$ of the plant lies is specified, N identification models are chosen so that θ_p belongs to the convex hull $\mathcal{K}(t_0)$ of $\theta_i(t_0)$ $(i \in \Omega = \{1, 2, \ldots, N\})$ at the initial time t_0. Under certain conditions it was shown in [1] that θ_p also belongs to the convex hull $\mathcal{K}(t)$ of $\theta_i(t)$ $i \in \Omega$, and that it is the only element in $\mathcal{S}_\theta$ which satisfies the condition for all t as $t \to \infty$. For more general cases the result was shown to be asymptotically valid. Using this general property, and the parametric trajectories of the N adaptive models, a new adaptive procedure was proposed (referred to as second level adaptation) which was demonstrated to be stable and robust under bounded perturbations. Simulation studies were carried out to compare the above scheme with well established "switching" [2], and "switching and tuning" [3] schemes. The performance of the former was shown to be far superior to those of the latter. As a further step it was shown that similar results could also be obtained by carrying out second level adaptation using N fixed models.

In this paper an attempt is made to extend the same concepts for the adaptive control of discrete-time systems. It is well known that in most practical applications discrete-time control is used. Since the latter involves algebraic equations, they are considerably easier to analyze than continuous-time systems. Also, it has been shown recently [4] in the context of periodic systems, that when mathematically tractable problems are impossible to formulate in the continuous-time case, elegant adaptive solutions can be obtained while dealing with their discrete-time counterparts. In this paper, the study of discrete-time adaptive control using multiple models is undertaken to determine the insights that they provide.

The research was supported by the National Science Foundation under Grant No. ECS-0824118.

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II. Discrete-Time Adaptive Control

Three main reasons have been given for considering discrete-time adaptive control using a single identification model. An obvious reason is that complex systems are generally controlled by computers which result in discrete-time systems. A second and significantly more important reason is the fact that random noise or disturbances can be dealt with more easily in the theoretical analysis in the discrete case. Finally if the methods developed for continuous-time systems prove successful, and are to be extended to nonlinear systems with artificial neural networks as components, a discrete-time framework is preferable.

Discrete-time adaptive control of linear time-invariant systems has been investigated for four decades. In [5], similar methods using multiple models, were proposed and explored in detail for both deterministic and stochastic systems, and during the past decade the results have been used in numerous applications where adaptation has to be carried out in the presence of large uncertainty. In the following sections we develop a general framework for extending the adaptive methods introduced in [1] to discrete-time systems.

It is well known that the judicious choice of the parameter estimation algorithm used in adaptive control plays an important role in the proof of stability, as well as in the performance of the adaptive methods in practical applications. We start our investigations with a plant described by the equation

$$\phi^T[k-1]\theta_p = y[k] \tag{1}$$

since most problems can be conveniently reduced to this form. In equation (1), $\phi[k] \in \mathbb{R}^n$ is a regression vector at time k, composed of past values of the outputs and the inputs of the plant, $\theta_p \in \mathbb{R}^n$ is a constant unknown plant parameter vector that has to be estimated, and $y[k] \in \mathbb{R}$ is the output of the plant.

A model described by the equation

$$\phi^T[k-1]\hat{\theta}[k-1] = \hat{y}[k] \tag{2}$$

is set up to estimate θ_p, where $\hat{\theta}[k]$ is the estimate at time k, and $\hat{y}[k]$ is the corresponding output of the model. From equation (1) and (2) the following error equation can be derived

$$\phi^T[k-1]\tilde{\theta}[k-1] = e[k] \tag{3}$$

where $\hat{\theta}[k-1] - \theta_p = \tilde{\theta}[k-1]$ is the parametric error and $\hat{y}[k] - y[k] = e[k]$ is the output error.

Numerous adaptive algorithms have been proposed in the past for updating the parameter estimates. A simple algorithm has the form

$$\hat{\theta}[k] = \hat{\theta}[k-1] - \frac{a\phi[k-1]e[k]}{c + \phi^T[k-1]\phi[k-1]} \tag{4}$$

where $0 < a < 2$ and $c > 0$, which is convenient while discussing proof of concept. A more complex but well tested algorithms is the Recursive Least-Squares (RLS) Algorithm in which

$$\hat{\theta}[k] = \hat{\theta}[k-1] - P[k-1]\phi[k-1]e[k] \tag{5}$$

where

$$\begin{aligned} P[k-1] &= P[k-2] \\ &\quad - \frac{P[k-2]\phi[k-1]\phi^T[k-1]P[k-2]}{1 + \phi^T[k-1]P[k-2]\phi[k-1]} \end{aligned} \tag{6}$$

with $P(-1)$ any symmetric positive definite matrix P_0.

In view of its speed of convergence and robustness in the presence of disturbances, this method is commonly used in all discrete-time adaptive control problems. It has been shown that if the algorithms (4) and (5) are used, the following results follow

$$a)\quad \|\hat{\theta}[k] - \theta_0\| \leq k_1\|\hat{\theta}[0] - \theta_0\|^2 \quad k \geq 1 \tag{7}$$

$$b)\quad \lim_{k\to\infty} \frac{e[k]}{1 + k_2\phi^T[k-1]\phi[k-1]} = 0 \tag{8}$$

$$c)\quad \lim_{k\to\infty} \|\hat{\theta}[k] - \hat{\theta}[k-\ell]\| = 0 \text{ for any finite } \ell. \tag{9}$$

Equations (7) and (9) assure that the parameter estimates are bounded and tend to constant values, and (8) assures that the estimation error can grow only at a slower rate than the regression vector. Using (a)-(c), the stability of an adaptive system has been demonstrated in the past [5].

III. Multiple Models And The Convex Hull Property

Let $\mathcal{S}_\theta$, the domain of uncertainty, be a compact set in $\mathbb{R}^m$ and let $\theta_p \in \mathcal{S}_\theta$. The need for multiple models arises because $\mathcal{S}_\theta$ is relatively so large that the adaptive control algorithms suggested earlier cannot assure sufficiently fast convergence of the output error and the parameter error vector to zero. To cope with the large uncertainty N models are set up as shown below:

$$\Sigma_i: \ \phi^T[k-1]\theta_i[k-1] = y_i[k] \ \ i \in \Omega = \{1, 2, \ldots, N\} \tag{10}$$

with the initial parameter estimates $\theta_i[0]$ assuming different independent values.

Comment: If $\theta_p \in \mathbb{R}^n$, only $N = n + 1$ models are needed so that $\mathcal{S}_\theta \subset \mathcal{K}[0]$, where $\mathcal{K}[0]$ is the convex hull in parameter space of the N model $\theta_i[0]$ $i \in \Omega$. The number of models required here is consequently much smaller than that needed in conventional (multiple-model based) adaptive control methods available at the present time, as discussed later in this paper. This is one of the major advantages of the method proposed here.

The parameter estimates are adjusted using the adaptive laws

$$\begin{aligned} \theta_i[k] &= \theta_i[k-1] + P[k-1]\phi[k-1]e_i[k] \\ &= \theta_i[k-1] + \frac{P[k-2]\phi[k-1]e_i[k]}{1 + \phi^T[k-1]P[k-2]\phi[k-1]} \end{aligned} \tag{11}$$

where $P[-1] = P_0$.

Since it is known a priori that the unknown plant parameter vector θ_p lies in $\mathcal{K}[0]$, at time $k = 0$, it follows that non-negative constants α_i exist such that

$$\sum_{i=1}^{N} \alpha_i = 1, \quad \sum_{i=1}^{N} \alpha_i\theta_i[0] = \theta_p. \tag{12}$$

We then have the following theorem which is the discrete-time counterpart of Theorem 1 in [6].

Theorem 1: If N adaptive identification models described in (10) are adjusted using adaptive laws (11) with initial conditions $\theta_i[0]$, and if the plant parameter vector θ_p lies in the convex hull $\mathcal{K}[0]$ of $\theta_i[0]$ $(i \in \Omega)$, then θ_p lies in the convex hull $\mathcal{K}[k]$ of $\theta_i[k]$ $(i \in \Omega)$ for all integers $k > 0$.

Proof: We prove Theorem 1 using induction. From equation (12) it follows that

$$\sum_{i=1}^{N} \alpha_i\theta_i[k] = \theta_p \tag{13}$$

for $k = 0$. Let (13) be valid for $k = k_0$, i.e. $\sum_{i=1}^{N} \alpha_i\theta_i[k_0] = \theta_p$. When $k = k_0 + 1$, it follows from equation (11) and (3) that $\theta_i[k_0 + 1] = \theta_i[k_0] + P[k_0]\phi[k_0]e_i[k_0 + 1] = \theta_i[k_0] + P[k_0]\phi[k_0]\phi^T[k_0]\tilde{\theta}_i[k_0]$. Therefore, $\sum_{i=1}^{N} \alpha_i\theta_i[k_0 + 1] = \sum_{i=1}^{N} \alpha_i\theta_i[k_0] + P[k_0]\phi[k_0]\phi^T[k_0]\sum_{i=1}^{N} \alpha_i\tilde{\theta}_i[k_0] = \theta_p$. (Since by definition the last term is zero). This concludes the proof. ■

Example 1: This example is included here to illustrate Theorem 1. The input and output of a plant are related by the equation $y[k+1] = 0.8y[k] + 0.7u[k]$ where the coefficients are assumed to be unknown. The uncertainty region $\mathcal{S}_\theta$ is assumed to be $\mathcal{S}_\theta = [0.5, 1.5] \times [0.5, 1.5]$. Four adaptive models Σ_i $(i = 1, 2, 3, 4)$ are used to identify the plant. The initial values of the parameters $\theta_i[0]$ are chosen as

$\theta_1[0] = [1.5, 1.5]^T$, $\theta_2[0] = [0.5, 1.5]^T$, $\theta_3[0] = [0.5, 0.5]^T$, and $\theta_4[0] = [1.5, 0.5]^T$, whose convex hull $\mathcal{K}[0]$ is the same as $\mathcal{S}_\theta$. In Figure 1, the evolution of the convex hull as a function of time is indicated, and it is seen that the parameter vector θ_p is contained in the convex hull at every instant.

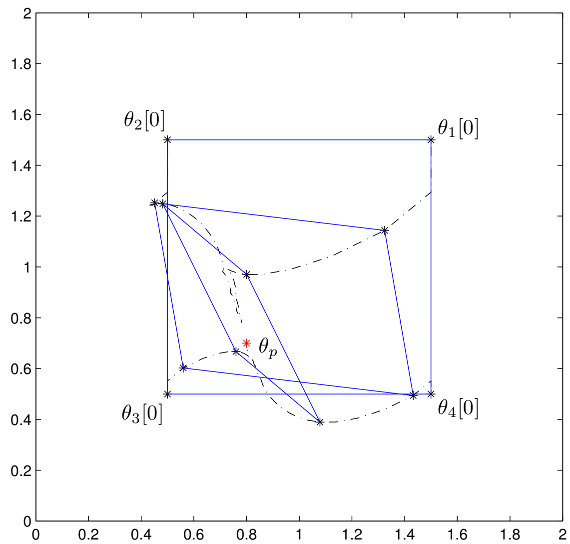


Fig. 1: Convex Hull Property of Multiple Adaptive Models

Comment: As illustrated in Example 1, the plant remains in the convex hull of the adaptive models for all time instants $k \geq 0$. However, it is not necessarily true that the convex hulls $\mathcal{K}[k]$ are nested, i.e. $\mathcal{K}[k_2]$ is not contained in $\mathcal{K}[k_1]$ if $k_2 > k_1$.

IV. Second Level Adaptation

The property discussed in the previous section leads to new and novel ways of viewing the adaptive identification and control problems, and has resulted, as shown in [1] for continuous-time systems, in powerful methods for adaptive control.

As shown in the previous section, if $\theta_p \in \mathcal{K}[0]$, $\theta_p = \sum_{i=1}^{N} \alpha_i \theta_i[0]$. This also assures that $\theta_p \in \mathcal{K}[k]$ $k \geq 0$, with

$$\theta_p = \sum_{i=1}^{N} \alpha_i \theta_i[k] \qquad \sum_{i=1}^{N} \alpha_i = 1 \qquad \alpha_i \geq 0 \tag{14}$$

for $\sum_{i=1}^{N} \alpha_i = 1$ and $\alpha_i \geq 0$.

The fact that the coefficients α_i in equation (14) remain the same for all $k \geq 0$, makes it relatively easy to estimate them. Also, the value θ_p is seen to depend upon the $N = (n+1)$ trajectories of the models chosen. Equation (14) can be expressed in matrix form as

$$[\theta_1[k], \theta_2[k], \ldots, \theta_N[k]]\, \alpha = \Theta[k]\alpha = \theta_p \tag{15}$$

where $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T \in \mathbb{R}^N$, $\Theta[k] \in \mathbb{R}^{n \times n+1}$, and $\theta_i[k]$ is the i^{th} adaptive parameter vector at time k. Since

$$\Theta[k-1]\alpha = \theta_p \tag{16}$$

we have from (15) and (16)

$$\Delta\Theta[k]\alpha = (\Theta[k] - \Theta[k-1])\alpha = 0. \tag{17}$$

It readily follows that the i^{th} column of $\Delta\Theta[k]$ is $\theta_i[k] - \theta_i[k-1]$, (which by the adaptive law for adjusting the parameter θ_i) is given by

$$\theta_i[k] - \theta_i[k-1] = -P[k-1]\phi[k-1]e_i[k]. \tag{18}$$

From equation (18), it follows that equation (17) can be rewritten as $\sum_{i=1}^{N} P[k-1]\phi[k-1]e_i[k]\alpha_i = 0$ or

$$P[k-1]\phi[k-1]E[k]\alpha = 0 \tag{19}$$

where $E[k]$ is the row vector $[e_1[k], e_2[k], \ldots, e_N[k]]$.

Defining

$$M[k] = P[k-1]\phi[k-1]E[k] \tag{20}$$

equation (19) reduces to

$$M[k]\alpha = 0 \tag{21}$$

which represents n equations in $(n+1)$ unknowns where the latter satisfy the constraint $\sum_{i=1}^{N} \alpha_i = 1$. Equation (21) can be expressed as

$$W[k]\alpha = b \tag{22}$$

to take into account the constraint, where

$$W[k] = \left[\begin{array}{c} M[k] \\ \hline \ell^T \end{array} \right], \qquad \ell = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{23}$$

and $b = [0, \ldots, 0, 1]^T$.

Comment: Equation (22) is an algebraic equation in $(n+1)$ unknowns, of which only n are independent. From classical adaptive control theory it is well known that it can be solved either algebraically or adaptively using a model. It is the latter that we refer to as second level adaptation.

Comment: In the discrete case the form of equation (20) defining $M[k]$ provides considerable insight into the adaptive model. $P[k-1]\phi[k-1] \in \mathbb{R}^n$ is a common vector at any instant, and the errors $e_i[k]$ $i \in \Omega$ vary from column to column. Due to space limitations, the advantages of this fact are not described here in detail.

Second Level Adaptation (Method I):

In Method I, the $N(= n+1)$ adaptive models at the first level provide the information to adjust the parameters of an estimation model having the form

$$W[k]\hat{\alpha}[k] = \hat{b}[k] \tag{24}$$

where $\hat{\alpha}[k]$ is the estimate of $\alpha \in \mathbb{R}^{n+1}$ and $\hat{b}[k]$ is the corresponding estimate of b. At this stage $\hat{\alpha}[k]$ can be adjusted using any standard adaptive law. We use the law

$$\hat{\alpha}[k+1] = \hat{\alpha}[k] - \frac{\mu W^T[k]\tilde{b}[k]}{1 + \|W[k]\|^2} \tag{25}$$

where $\tilde{b}[k] = \hat{b}[k] - b$. As shown in [7], [5], the optimal adaptive gain is $\mu_{opt} = 1$ in the ideal case when no observation noise is present.

At every instant the estimate $\hat{\alpha}[k]$ of α provides the estimate of the plant parameter θ_p, using the equation

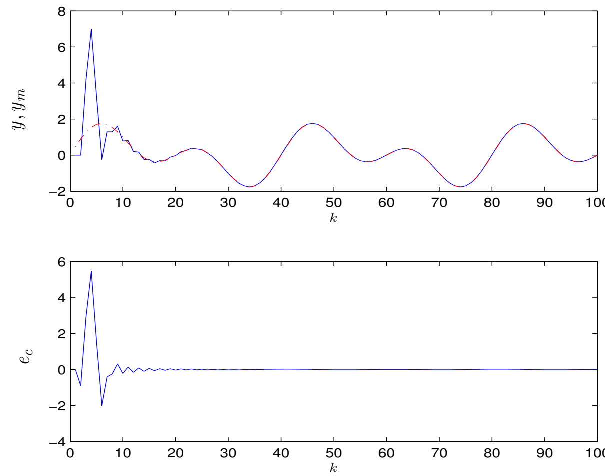
$$\hat{\theta}_p[k] = \sum_{i=1}^{N} \hat{\alpha}_i[k]\theta_i[k], \tag{26}$$

and this, in turn, is used to control the plant.

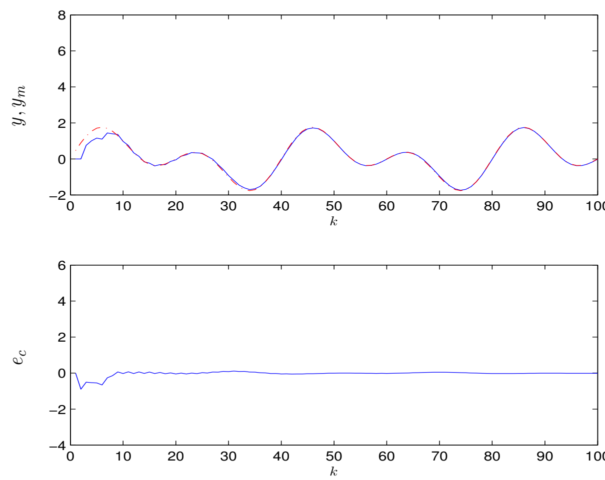
Comment: From the above discussion it is seen that the second level adaptation is carried out where no distinction is made among the N identification models on the first level. As shown in [1], a different approach can also be used where one of the identification models is chosen to be a base and the other $N-1$ models provide information relative to it. These two approaches use essentially the same information provided by the N models and give similar results.

Comment: The main difference between second level adaptation as discussed above and currently existing methods is that the outputs of all the N models are used in computing the estimate of the plant parameter vector.

Example 2: A second order unstable plant described by the equation $y[k+1] = 0.97y[k] + 0.26y[k-1] + 0.66u[k] + 0.528u[k-1]$ is simulated to illustrate the effectiveness of second level adaptation. The coefficients of the plant are assumed to be unknown but lie in the uncertainty region $\mathcal{S}_\theta = [0,1] \times [0,1] \times [0,1] \times [0,1]$. The objective is to adaptively stabilize the plant and control it in such a fashion that the output tracks the reference signal $y_m[k] = \sin[\frac{\pi k}{20}] + \sin[\frac{\pi k}{10}]$. Two simulations are plotted in Figure 2 where the plant output and the reference signal are shown together while the output error is given separately on the same scale. In the first simulation a single adaptive model is used to identify and control the given plant based on the measured input and output. The initial estimation of the coefficients are chosen randomly in the unit cube $\mathcal{S}_\theta$. It is seen from Figure 2(a) that the performance of the system suffers from large and oscillatory initial transients which has a magnitude of about four times that of the reference signal. In the second simulation, five models are used for second level adaptation since the parameter space has a dimensionality of four. The initial values of the adaptive models are chosen such that $\mathcal{K}[0] \supset \mathcal{S}_\theta$. It is seen from Figure 2(b) that the response is improved significantly.



(a) Adaptive Control Using One Model



(b) Adaptive Control Using Second Level Adaptation

Fig. 2: Adaptive Control of a Second Order System

V. Second Level Adaptation with Fixed Models (Method II)

As stated earlier, all the N first level adaptive models continue to evolve with k in the previous method. However, from equations (17)-(20) it is seen that only a knowledge of $\Delta\Theta[k] = \Theta[k] - \Theta[k-1]$ is needed to set up the equations for adaptation. The columns of this matrix depend only on the adaptive laws used by the N models. This implies that the information for second level adaptation can be obtained from all N models, even when they are fixed. This gives rise to a second method of adaptation which is distinctly different from the first.

Comment: A brief comparison of Method I and Method II is provided after the details of the latter are described below.

We now take a closer look at equation (19)

$$P[k-1]\phi[k-1]E[k]\alpha = 0. \tag{27}$$

Recall that from equation (3), $e_i[k] = \phi^T[k-1]\tilde{\theta}_i[k-1]$, it follows that (19) can be rewritten as

$$P[k-1]\phi[k-1]\phi^T[k-1]\tilde{\Theta}[k-1]\alpha = 0. \tag{28}$$

where the i^{th} column of $\tilde{\Theta}[k]$ is $\tilde{\theta}_i[k]$.

Since $\tilde{\theta}_i[k] = \theta_i[k] - \theta_p$, it follows that

$$P[k-1]\phi[k-1]\phi^T[k-1](\Theta[k-1] - \Theta_p)\alpha = 0. \quad (29)$$

Since $P[k]$ is always positive definite, it follows that

$$\phi[k-1]\phi^T[k-1](\Theta[k-1] - \Theta_p)\alpha = 0, \quad (30)$$

i.e. $\phi[k-1]\phi^T[k-1]\Theta[k-1]\alpha = \phi[k-1]\phi^T[k-1]\theta_p = \phi[k-1]y[k]$. If fixed models are used, $\Theta[k] = \Theta[0]$ for all $k \geq 0$.

If the reference input is persistently exciting during the interval $0 \leq k \leq T$, we have

$$\sum_{k=0}^{T} \phi[k]\phi^T[k] \triangleq \Phi(0,T) > \delta I \quad (31)$$

for some positive constant δ [8]. It then follows that $\Phi(0,T)\Theta[0]\alpha = \sum_{k=0}^{T} \phi[k]y[k+1]$, and consequently

$$\Theta[0]\alpha = \Phi(0,T)^{-1} \sum_{k=0}^{T} \phi[k]y[k+1]. \quad (32)$$

If the initial convex hull $\mathcal{K}[0]$ is chosen such that

$$\left[\frac{\Theta[0]}{\mathbf{1}}\right] \quad (33)$$

is of full rank, it follows that

$$\alpha = \left[\frac{\Theta[0]}{\mathbf{1}}\right]^{-1} \left[\frac{\Phi(0,T)^{-1}\sum_{k=0}^{T}\phi[k]y[k+1]}{1}\right]. \quad (34)$$

Comparison of Method I and Method II:

Two methods for estimating $\hat{\alpha}[k]$ were described in this section. In Method I, the N models in the first level are adaptive. In the second, the N models in the first level are fixed but the information provided by them is used in determining $\hat{\alpha}[k]$. The latter information is merely the error, and the regression vector which would have been used for adaptation if the models had been adaptive (and were used in Method I).

It is well known that one of the drawbacks of adaptation based on all adaptive models is that they need to be re-initialized, since they all converge to the same point in parameter space i.e. θ_p. However, before this stage is reached the method is very effective in estimating θ_p. In contrast to this, the second method is based only on fixed models located at the boundary of the region of uncertainty and consequently whose effectiveness is less than that of the adaptive models. Methods of combining the advantages of the two have been investigated and will be reported elsewhere. Simulation results comparing a single model (classical adaptive control) with Method I and Method II for a time-invariant plant and a time-varying plant are shown in Examples 3 and 4.

Example 3: The same problem discussed in Example 1 is now simulated for comparison of three different schemes: (i) adaptive control using a single model, (ii) second level adaptation method I, and (iii) second level adaptation method II.

In simulation 1, a single identification model is randomly initialized in the uncertainty region. In simulation 2, five adaptive models are employed whose initial locations form a convex hull containing $\mathcal{S}_\theta$. Second level adaptation is then carried out as described earlier based on the parameter estimates generated by the five adaptive models. In simulation 3, five fixed models are used whose convex hull contains the uncertainty region and provides information for the second level adaptation.

It can be seen from Figure 3 that second level adaptation results in smooth and rapid adaptation, while adaptive control using a single model leads to large oscillation and unsatisfactory transients. It is seen that adaptive control using second level adaptation gives significant improvement in the system response.

Example 4: In this example, the plant to be controlled switched from Σ_1 to Σ_2 at time $k = 100$, where

$$\Sigma_1: \quad y[k+1] = 0.5y[k] + 1.0u[k], \quad (35)$$

$$\Sigma_2: \quad y[k+1] = 1.2y[k] + 0.7u[k]. \quad (36)$$

The system response and output error of second level adaptation method I with and without re-initialization of the first level adaptive models are shown in Figures 4(b) and (a) respectively. The same plots using second level adaptation method II are given in Figure (c). It is seen that if second level adaptation method I is not properly re-initialized, it would degenerate to a single model after an interval of convergence and would not perform satisfactorily to sudden changes in the system parameters. On the other hand, second level adaptation method I with re-initialization and second level adaptation method II respond smoothly and rapidly to a plant parameter switching as seen in Figures 4(b) and (c).

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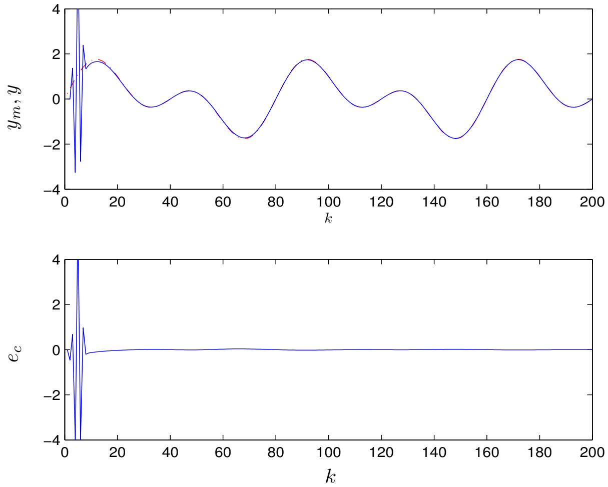
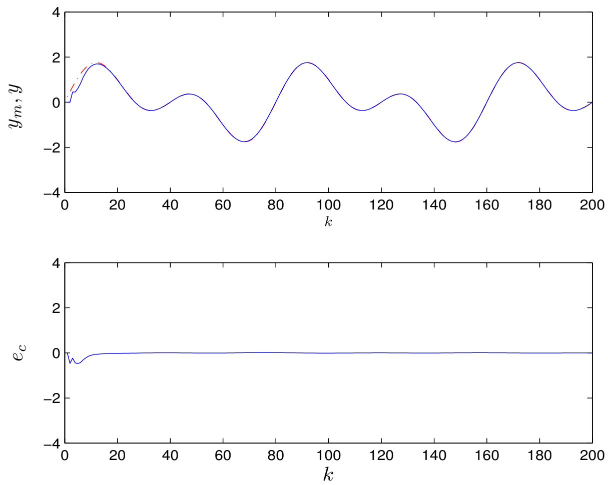
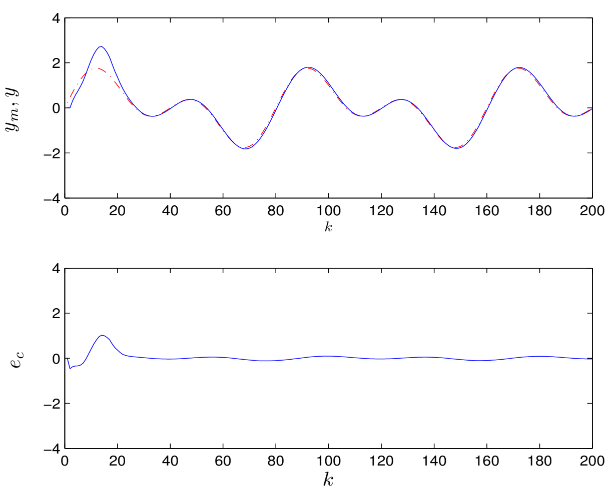


(a) Adaptive Control With A Single Model

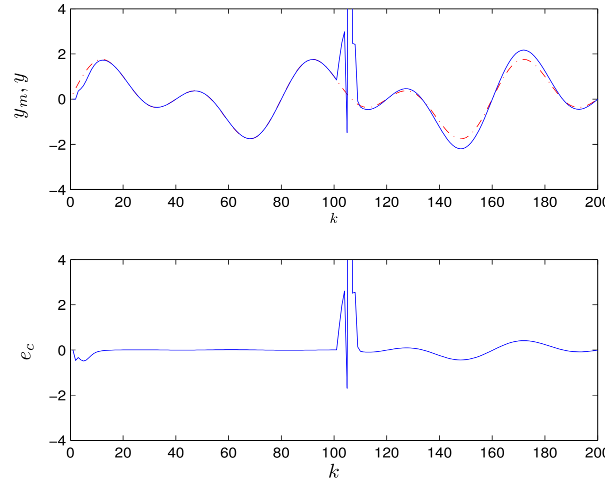


(b) Second Level Adaptation Method I

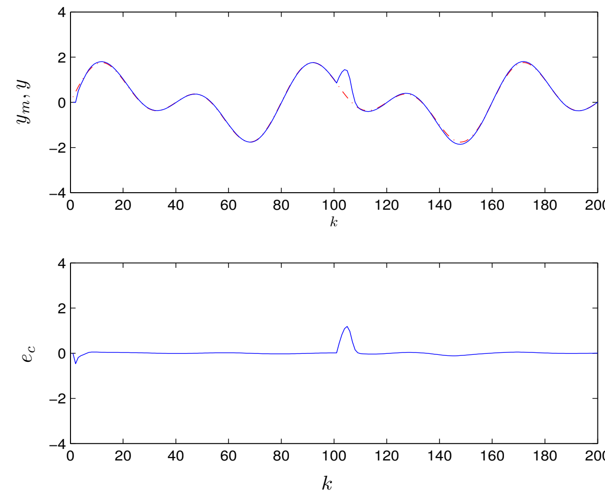


(c) Second Level Adaptation Method II

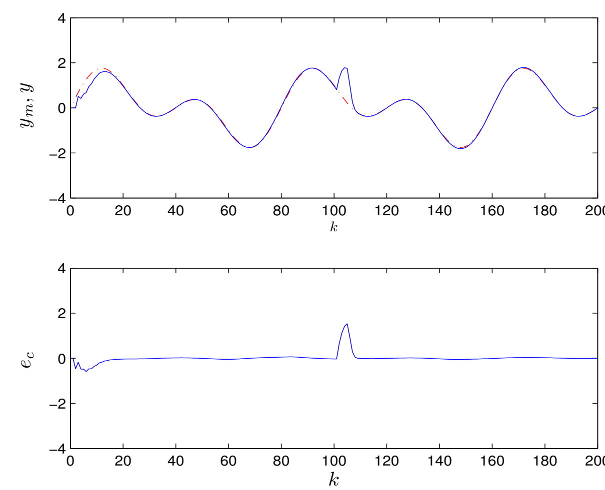
Fig. 3: Adaptive Control of a Time-Invariant Second Order System



(a) Second Level Adaptation Method I: No Re-Initialization



(b) Second Level Adaptation Method I: With Re-Initialization



(c) Second Level Adaptation Method II

Fig. 4: Adaptive Control of a Time-Varying Second Order System