INTERNATIONAL JOURNAL OF ADAPTIVE CONTROL AND SIGNAL PROCESSING, VOL. 11, 501–517 (1997)

DISCRETE-TIME COMBINED MODEL REFERENCE ADAPTIVE CONTROL

MANUEL A. DUARTE* AND RODRIGO F. PONCE

Department of Electrical Engineering, University of Chile, Casilla 412-3, Santiago, Chile

SUMMARY

The discrete-time version of continuous-time combined model reference adaptive control (CMRAC) is presented in this paper. A global stability proof of the overall adaptive scheme is given using arguments similar to those used in discrete-time direct model reference adaptive control (DMRAC) but properly modified to account for the different structure of CMRAC with respect to DMRAC. © 1997 by John Wiley & Sons, Ltd.

Int. J. Adapt. Control Signal Process., **11**, 501–517 (1997)
No. of Figures: 4 No. of Tables: 0 No. of References: 21

Key words: discrete-time adaptive control; adaptive systems; model reference adaptive control; direct and indirect adaptive control; combined adaptive control

1. INTRODUCTION

The main objective in model reference adaptive control is to reduce the error between the plant output and the model reference output by adjusting the controller parameters. This can be accomplished either using plant parameter estimates, i.e. by an indirect technique, or using the tracking error to directly adjust the controller parameters, i.e. by a direct technique. However, it is possible to improve the transient behaviour in terms of speed, accuracy and robustness if both techniques are used together. The difficulty in this case arises from the fact that information acquired from the two methods has to be suitably combined to guarantee global stability of the resulting method. This problem has already been resolved by Duarte and Narendra[1,2] for the continuous-time case under ideal conditions (constant parameters, absence of external perturbations and unmodelled dynamics).

In CMRAC the plant parameter estimates and controller parameters are continuously adjusted through difference (differential) equations. This avoids algebraic problems arising from the

This article was recommended for publication by editor R. Ortega

* Correspondence to: M. A. Duarte, Department of Electrical Engineering, University of Chile, Casilla 412-3, Santiago, Chile.

Contract grant sponsor: CONICYT.
Contract grant number: FONDECYT 1950502.

CCC 0890–6327/97/060501–17$17.50

Received 12 April 1994
Revised 1 December 1996

certainty equivalence principle (Bezout or Diophantine equation) which require the inversion of matrices containing parameter estimates.

Furthermore, at any instant of time the controller and identifier parameters are assumed to be constant, so that a closed-loop transfer function (involving both plant and controller) can be determined. Deviations of these coefficients from the desired coefficients given by the model reference transfer function are used to define the closed-loop estimation errors. These errors provide the connection between the direct and indirect methods and are additionally used in the adaptive laws.

Some of the previous ideas have also been used in other contexts. In this sense it is important to mention the works by Kreisselmeier,[3,4] where the adaptive control of an unknown continuous-time plant is discussed and the synthesis of a feedback matrix is realized asymptotically. The proof of convergence of such a system depends upon the persistent excitation of an external command signal. In an other context, Kreisselmeier and Smith[5] state an interesting method of adaptive regulation for nth-order plants based on the so-called 'identification mismatch error'. Recently, the idea of a dynamical certainty equivalence principle has been discussed by Morse[6] and Ortega.[7]

The global stability of continuous-time CMRAC has been completely studied.[1,2] Owing to the importance of discrete-time algorithms in digital control a study of the discrete-time version of continuous-time CMRAC is attempted in this paper. Several conditions for global stability of the overall scheme are derived under ideal conditions. Some remarks on the robustness of discrete-time CMRAC are made towards the end.

In order to prove global stability of the overall discrete-time scheme proposed here, some modifications with respect to continuous-time CMRAC and discrete-time DMRAC have to be introduced in the design process. For example, an auxiliary signal denoted by $G(\cdot)$ has to be introduced in the design of the identifier.

To update the controller and identifier parameters, it is necessary to compute at every instant of time the maximum eigenvalue of a matrix involving some of the controller parameters. Because of the structure of such a matrix, the computation turns out to be very easy.

Discrete-time versions of adaptive control schemes are more suitable for implementation in practice than are the corresponding continuous-time algorithms. However, it is not always true that discretized versions of continuous-time algorithms work properly using a small sampling period and approximating time derivatives by differences. Therefore the analysis of the discrete-time version of CMRAC is done in this paper using some of the ideas contained in continuous-time CMRAC but also introducing new concepts to provide the scheme with global stability properties under ideal conditions.

As in the continuous-time case, the persistent excitation conditions needed to provide parameter convergence of discrete-time CMRAC are no more restrictive than those obtained for discrete-time DMRAC. This means that in spite of the increasing number of difference equations in the overall adaptive scheme, the persistent excitation of CMRAC is similar to that derived for DMRAC.

2. DISCRETE-TIME COMBINED MRAC

Let us consider an nth-order, discrete-time, linear and time-invariant plant defined as

$$W_{\mathrm{p}}(z) \triangleq \frac{Y_{\mathrm{p}}(z)}{U(z)} = K_{\mathrm{p}} \frac{Z_{\mathrm{p}}(z)}{R_{\mathrm{p}}(z)} = K_{\mathrm{p}} \frac{z^m + \cdots + b_1 z + b_0}{z^n + \cdots + a_1 z + a_0} \tag{1}$$

where K_p is the high-frequency gain, $Z_p(z)$ and $R_p(z)$ are monic, coprime polynomials of degree m and $n > m$ respectively and b_j, $j = 0, 1, \dots, m-1$, and a_i, $i = 0, 1, \dots, n-1$, are real but unknown constants. We denote the vectors $a = [a_0, a_1, \dots, a_{n-1}]^T \in \mathbb{R}^n$ and $b = [b_0, b_1, \dots, b_{m-1}]^T \in \mathbb{R}^m$. It will be assumed that $|K_p| \leqslant h$, with h known to the designer. $Y_p(k)$ and $U(k)$ are the plant input and output respectively.

Let $W_M(z)$ be the model reference transfer function given by

$$W_M(z) \triangleq \frac{Y_M(z)}{R(z)} = K_M \frac{Z_M(z)}{R_M(z)} = K_M \frac{z^m + \cdots + b_1^m z + b_0^m}{z^n + \cdots + a_1^m z + a_0^m} \tag{2}$$

where b_j^m, $j = 0, 1, \dots, m-1$, a_i^m, $i = 0, 1, \dots, n-1$, and K_M are real known constants, $Z_M(z)$ is a monic, Hurwitz polynomial of degree m and $R_M(z)$ is a monic, Hurwitz polynomial of degree $n > m$. $Y_M(k)$ and $R(k)$ are the model reference input and output respectively.

For this study it has been found that the most suitable parametrization of the plant is of the form[2]

$$Y_p(k) = \frac{K_p}{K_M}(\bar{U}(k) + \beta^T \bar{V}_1(k)) + \alpha^T \bar{V}_2(k) \tag{3}$$

where $K_p \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}^n$ are the new plant parameters. For notation purposes the vectors α and β are defined as $\alpha = [\alpha_0, \alpha_1, \dots, \alpha_{n-1}]^T \in \mathbb{R}^n$ and $\beta = [\beta_0, \beta_1, \dots, \beta_{n-1}]^T \in \mathbb{R}^n$. $\bar{U}(k) \in \mathbb{R}$ and $\bar{V}_1(k)$, $\bar{V}_2(k) \in \mathbb{R}^n$ are filtered versions of the signals $U(k) \in \mathbb{R}$ and $V_1(k)$, $V_2(k) \in \mathbb{R}^n$ defined as

$$\bar{V}_1(k) = W_M(z) I_n V_1(k), \qquad \bar{V}_2(k) = W_M(z) I_n V_2(k), \qquad \bar{U}(k) = W_M(z) U(k) \tag{4}$$

The signals $V_1(k)$, $V_2(k) \in \mathbb{R}^n$ are filtered versions of the plant input and output defined as

$$V_1(k+1) = \Lambda V_1(k) + lU(k), \qquad V_2(k+1) = \Lambda V_2(k) + lY_p(k) \tag{5}$$

where the pair (Λ, l) is any controllable pair with $\Lambda \in \mathbb{R}^{n \times n}$ an asymptotically stable matrix.[8] In order to make the relationship between the parameters $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and α, $\beta \in \mathbb{R}^n$ simple enough, (Λ, l) is chosen in the controllable canonical form[9]

$$\Lambda = \begin{bmatrix} 0 & | & & & & \\ \cdot & | & & I_{n-1} & & \\ \cdot & | & & & & \\ 0 & | & & & & \\ - & - & - & - & - & - \\ -\lambda_0 & -\lambda_1 & \cdot & \cdot & \cdot & -\lambda_{n-1} \end{bmatrix}, \qquad l = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \tag{6}$$

where I_{n-1} denotes the identity matrix of order $n-1$. With this choice the relationship between the parameters a, b, α and β is given by the polynomial identity[10]

$$R_p(z)(\beta(z) + \Lambda(z)) + K_M Z_p(z)\alpha(z) = Z_p(z) R_M(z) \Lambda_1(z)$$

where

$$\beta(z) = z^n + \beta_{n-1} z^{n-1} + \cdots + \beta_1 z + \beta_0$$
$$\alpha(z) = z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0$$
$$\Lambda(z) = z^n + \lambda_{n-1} z^{n-1} + \cdots + \lambda_1 z + \lambda_0$$

Here $\Lambda(z) = \det(zI - \Lambda)$ is the characteristic polynomial of the matrix Λ. The polynomial $\Lambda(z)$ can always be factorized as $\Lambda(z) = \Lambda_1(z) Z_M(z)$, with $\Lambda_1(z)$ an arbitrary monic, Hurwitz polynomial of degree $n - m$.

The control objective is to find a controller such that the plant output $Y_p(k)$ follows the model reference output $Y_M(k)$ when $k \to \infty$ for any bounded reference $R(k)$.

The control law used to achieve the above objective has the same form as in DMRAC, i.e.

$$U(k) = K_0(k)R(k) + \theta_1^T(k)V_1(k) + \theta_2^T(k)V_2(k) \tag{7}$$

where $K_0(k) \in \mathbb{R}$ and $\theta_1(k)$, $\theta_2(k) \in \mathbb{R}^n$ are the adjustable controller parameters which will be updated in a fashion defined later (see (20)).

It can be shown that with this control law the real plant parameters $K_p \in \mathbb{R}$ and α, $\beta \in \mathbb{R}^n$ and the ideal control parameters $K_0^* \in \mathbb{R}$ and θ_1^*, $\theta_2^* \in \mathbb{R}^n$ (control parameters such that the transfer function of the plant together with the controller exactly matches the model reference transfer function) are related through the equations[10]

$$\theta_1^* + \beta = 0, \qquad K_p\theta_2^* + K_M\alpha = 0, \qquad K_0^* - K_r = 0 \tag{8a}$$

where $K_r = K_M/K_p$. We define the closed-loop estimation errors[1,2] as those obtained from (8a) by replacing the real values by their respective estimates, so that the following relations are obtained:

$$\varepsilon_{\theta_1}(k) = \theta_1(k) + \hat{\beta}(k), \qquad \varepsilon_{\theta_2}(k) = \hat{K}_p(k)\theta_2(k) + K_M\hat{\alpha}(k), \qquad \varepsilon_{K_0}(k) = K_0(k) - \hat{K}_r(k) \tag{8b}$$

where $\varepsilon_{\theta_1}(k)$, $\varepsilon_{\theta_2}(k) \in \mathbb{R}^n$ and $\varepsilon_{K_0}(k) \in \mathbb{R}$.

From (3) the plant output can be expressed as

$$Y_p(k) = \frac{K_p}{K_M}(\bar{U}(k) + \beta^T(k)\bar{V}_1(k)) + \alpha^T(k)\bar{V}_2(k) + \frac{K_p}{K_M N_k}(\bar{V}_1^T(k)\varepsilon_{\theta_1}(k)) - \frac{K_p}{K_M N_k}(\bar{V}_1^T(k)\varepsilon_{\theta_1}(k)) \tag{9a}$$

where $N_K \in \mathbb{R}$ is an arbitrary constant greater than one. Based on this plant representation, an identifier of the following form is proposed:

$$\hat{Y}_p(k) = \frac{\hat{K}_p(k)}{K_M}(\bar{U}(k) + \hat{\beta}^T(k)\bar{V}_1(k)) + \hat{\alpha}^T(k)\bar{V}_2(k) - \frac{\hat{K}_p(k)}{K_M N_k}\bar{V}_1^T(k)\varepsilon_{\theta_1}(k) - \frac{K_M}{N_L(k)}\bar{V}_1^T(k)\varepsilon_{\theta_2}(k) - \frac{1}{K_M N_L(k)}\theta_2^T(k)\varepsilon_{\theta_2}(k)\left[\bar{U}(k) + \left(\hat{\beta}(k) - \frac{\varepsilon_{\theta_1}(k)}{N_k}\right)^T \bar{V}_1(k)\right] \tag{9b}$$

where $\hat{K}_p(k) \in \mathbb{R}$ and $\hat{\alpha}(k)$, $\hat{\beta}(k) \in \mathbb{R}^n$ are the estimates of $K_p \in \mathbb{R}$ and α, $\beta \in \mathbb{R}^n$ respectively. These estimates will be delivered by the identifier. $N_L(k) \in \mathbb{R}$ is such that

$$N_L(k) \geqslant \tfrac{1}{2}(K_M^2 + h + \lambda_M(k)) \tag{9c}$$

with

$$\lambda_M(k) = \| \theta_2(k) \|^2 \tag{9d}$$

From (8a) and (8b) it is possible to represent the closed-loop estimation errors in terms of the parameter errors (controller and identifier parameter errors), so that

$$\varepsilon_{\theta_1}(k) = \phi_{\theta_1}(k) + \eta_\beta(k), \qquad \varepsilon_{\theta_2}(k) = K_p\phi_{\theta_2}(k) + K_M\eta_\alpha(k) + \theta_2(k)\eta_{K_p}(k),$$
$$\varepsilon_{K_0}(k) = \phi_{K_0}(k) - \eta_{K_r}(k) \tag{10}$$

where the plant parameter errors $\eta_{K_p}(k), \eta_{K_r}(k) \in \mathbb{R}$ and $\eta_\alpha(k), \eta_\beta(k) \in \mathbb{R}^n$ are defined by

$$\eta_{K_p}(k) = \hat{K}_p(k) - K_p, \qquad \eta_\alpha(k) = \hat{\alpha}(k) - \alpha$$
$$\eta_\beta(k) = \hat{\beta}(k) - \beta, \qquad \eta_{K_r}(k) = \hat{K}_r(k) - K_r \tag{11}$$

and the controller parameter errors $\phi_{K_0}(k) \in \mathbb{R}$ and $\phi_{\theta_1}(k), \phi_{\theta_2}(k) \in \mathbb{R}^n$ are defined by

$$\phi_{\theta_1}(k) = \theta_1(k) - \theta_1, \qquad \phi_{\theta_2}(k) = \theta_2(k) - \theta_2, \qquad \phi_{k_0}(k) = K_0(k) - K_0 \tag{12}$$

If we define

$$\bar{U}_0(k) \triangleq \bar{U}(k) + \bar{V}_1^{\mathrm{T}}(k)\left(\hat{\beta}(k) - \frac{\varepsilon_{\theta_1}(k)}{N_k}\right) \tag{13}$$

then from (9a) and (9b) the identification error $e_i(k) = \hat{Y}_p(k) - Y_p(k)$ can be written as

$$e_i(k) = \frac{1}{K_M}\eta_{K_p}(k)\bar{U}_0(k) + \frac{K_p}{K_M}\eta_\beta^{\mathrm{T}}(k)\bar{V}_1(k) + \eta_\alpha^{\mathrm{T}}(k)\bar{V}_2(k) - \frac{K_p}{K_M N_k}\bar{V}_1^{\mathrm{T}}(k)\varepsilon_{\theta_1}(k)$$
$$- \frac{K_M}{N_L(k)}\bar{V}_2^{\mathrm{T}}(k)\varepsilon_{\theta_2}(k) - \frac{1}{K_M N_L(k)}\theta_2^{\mathrm{T}}(k)\varepsilon_{\theta_2}(k)\bar{U}_0(k) \tag{14}$$

Defining the vectors

$$\omega(k) = (R(k), V_1^{\mathrm{T}}(k), V_2^{\mathrm{T}}(k))^{\mathrm{T}} \in \mathbb{R}^{2n+1}, \qquad \bar{\omega}(k) = W_M(z)I_{2n+1}\omega(k) \in \mathbb{R}^{2n+1}$$
$$\theta(k) = (K_0(k), \theta_1^{\mathrm{T}}(k), \theta_2^{\mathrm{T}}(k))^{\mathrm{T}} \in \mathbb{R}^{2n+1}, \qquad \theta^* = (K_0^*, \theta_1^{*\mathrm{T}}, \theta_2^{*\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{2n+1}$$
$$\phi(k) = \theta(k) - \theta = (\phi_{K_0}(k), \phi_{\theta_1}^{\mathrm{T}}(k), \phi_{\theta_2}^{\mathrm{T}}(k))^{\mathrm{T}} \in \mathbb{R}^{2n+1}$$

we can define the auxiliary errors $e_2(k)$ and $e_3(k)$ as

$$e_2(k) \triangleq (\theta^{\mathrm{T}}(k)L^{-1}(z) - L^{-1}(z)\theta^{\mathrm{T}}(k))\omega(k)$$
$$e_3(k) \triangleq -\left(\frac{\bar{R}(k)\varepsilon_{K_0}}{N_k} + \frac{\bar{V}_2^{\mathrm{T}}(k)\varepsilon_{\theta_2}(k)}{\mathrm{sgn}(K_p)N_L(k)} + \frac{\bar{V}_1^{\mathrm{T}}(k)\varepsilon_{\theta_1}(k)}{N_k}\right) \tag{15a}$$

where $L^{-1}(z)$ is a minimum phase, strictly proper, rational function. It is possible to find[8] that the tracking error $e_c(k) = Y_p(k) - Y_M(k)$ and augmented error $e_a(k)$ are

$$e_c(k) = \frac{K_p}{K_M}W_M(z)(\phi^{\mathrm{T}}(k)\omega(k))$$
$$e_a(k) = e_c(k) + W_M(z)L(z)(K_1 e_2'(k) - \tilde{\omega}^{\mathrm{T}}(k)\tilde{\omega}(k)e_a(k)) \tag{15b}$$

where

$$\psi(k) = K_1(k) - \frac{K_p}{K_M}, \qquad \bar{\omega}(k) = (\bar{R}(k), \bar{V}_1(k), \bar{V}_2(k)) \in \mathbb{R}^{2n+1}$$
$$e_2'(k) = e_2(k) + e_3(k), \qquad \tilde{\omega}(k) = (\bar{\omega}(k), e_2(k)) \in \mathbb{R}^{2n+2} \tag{16}$$

From (15b) and using the above definitions, the augmented error finally becomes

$$e_a(k) = W_M(z)L(z)\left(\frac{K_p}{K_M}(\phi^{\mathrm{T}}(k)\bar{\omega}(k) + e_3(k)) - \tilde{\omega}^{\mathrm{T}}(k)\tilde{\omega}(k)e_a(k) + \psi(k)e_2'(k)\right) + \delta(k) \tag{17a}$$

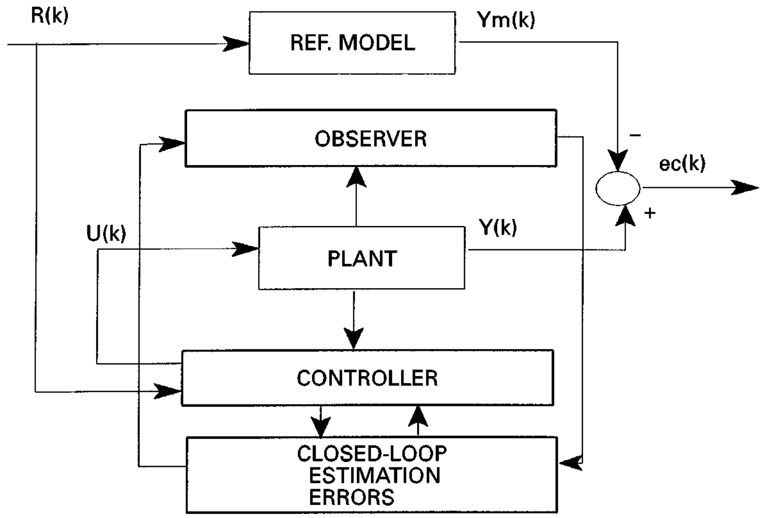

Figure 1. Discrete-time CMRAC scheme

The signal $\delta(k) \triangleq (K_p/K_M\ W_M(z)[\theta^T(k)(\bar{\omega}(k) - \omega(k))]$ is geometrically decaying. Another way of writing the augmented error is given by

$$\epsilon(k+1) = A_{MN}\epsilon(k) + b_{MN}v(k), \qquad e_a(k) = h_{MN}^T\epsilon(k) + d\,v(k)$$

$$v(k) = \frac{K_p}{K_M}(\phi^T(k)\bar{\omega}(k) + e_3(k)) + \psi(k)e'_2(k) - \tilde{\omega}^T(k)\tilde{\omega}(k)e_a(k) \tag{17b}$$

where $d + h_{MN}^T(zI - A_{MN})^{-1}b_{MN}$ is a strictly positive real (SPR) transfer function and the matrix A_{MN} and vectors b_{MN} and h_{MN} are defined in (27b). It is important to mention here that equation (17b) is a state representation of the adaptive system as shown later.[8] The meaning of each variable is explained in Part III of the Proof of Theorem 1. The CMRAC proposed here is shown in Figure 1.

To complete the design procedure, it is necessary to define the way in which the controller and identifier parameters are to be adjusted. In order to guarantee global stability of the overall adaptive scheme, the identifier adaptive laws are chosen as

$$\Delta\eta_\alpha(k) = -\gamma\frac{e_i(k)\bar{V}_2(k)}{N_i(k)} - K_M\frac{\varepsilon_{\theta_2}(k)}{N_L(k)}, \qquad \Delta\eta_\beta(k) = -\gamma\,\mathrm{sgn}(K_p)\frac{e_i(k)\bar{V}_1(k)}{K_M N_i(k)} - \frac{\varepsilon_{\theta_1}(k)}{N_k}$$

$$\Delta\eta_{K_p}(k) = -\gamma\frac{e_i(k)\bar{U}_0(k)}{K_M N_i(k)} - \frac{\theta_2^T(k)\varepsilon_{\theta_2}(k)}{N_L(k)}, \qquad \Delta\eta_{K_r}(k) = \frac{\varepsilon_{K_0}(k)}{N_k} \tag{18}$$

where $N_k > 1$ and $N_i(k)$ is a normalization factor defined as

$$N_i(k) = 1 + \bar{V}_2^T(k)\bar{V}_2^T(k) + \bar{V}_1^T(k)\bar{V}_1(k) + \bar{U}_0^2(k) \tag{19a}$$

$N_L(k)$ is defined in (9c) and γ is a real positive constant such that

$$0 < \gamma \leqslant \frac{2K_M^2}{1 + h + K_M^2} \tag{19b}$$

For the sake of simplicity we choose $W_M(z)L^{-1}(z) = 1$. Then the controller adaptive laws are chosen as

$$\Delta\phi_{\theta_2}(k) = -\,\mathrm{sgn}(K_p)\left(\gamma\frac{e_a(k)\bar{V}_2(k)}{K_M} + \frac{\varepsilon_{\theta_2}(k)}{N_L(k)}\right),$$
$$\Delta\phi_{\theta_1}(k) = -\,\gamma\,\mathrm{sgn}(K_p)\frac{e_a(k)\bar{V}_1(k)}{K_M} - \frac{\varepsilon_{\theta_1}(k)}{N_k}$$
$$\Delta\phi_{K_0}(k) = -\,\gamma\,\mathrm{sgn}(K_p)\frac{e_a(k)\bar{R}(k)}{K_M} - \frac{\varepsilon_{K_0}(k)}{N_k}, \qquad \Delta\psi(k) = -\,\gamma\, e_a(k)e_2'(k) \tag{20}$$

2.1. Proof of stability

A stability proof of the adaptive system shown in Figure 1 is given along the same lines as in discrete-time DMRAC[8] and dynamic IMRAC.[10] In order to prove that the closed-loop estimation errors and the controller and identifier parameter estimation errors are bounded, a lemma is first established. Subsequently, a theorem is stated so that it is proved that all the remaining signals are also bounded and the tracking error (as well as the identification and closed-loop estimation errors) tends asymptotically to zero.

For notation purposes we define the vectors p, $\eta(k)$, $\hat{p}(k) \in \mathbb{R}^{2n+2}$ and $\phi'(k)$, $\omega'(k) \in \mathbb{R}^{2n}$ and the function $v(k)$ in the following fashion:

$$\begin{aligned}
\hat{p}(k) &= [\hat{\alpha}^T(k), \hat{\beta}^T(k), \hat{K}_p(k), \hat{K}_r(k)]^T, \qquad p = [\alpha^T, \beta^T, K_p, K_r]^T \\
\eta &= \hat{p}(k) - p = [\eta_\alpha^T(k), \eta_\beta^T(k), \eta_{K_p}(k), \eta_{K_r}(k)]^T \\
\phi'(k) &= [\phi_{\theta_1}^T(k), \phi_{\theta_2}^T(k)]^T, \qquad \omega'(k) = [V_1^T(k), V_2^T(k)]^T \\
v(k) &= \frac{K_p}{K_M}(\phi^T(k)\bar{\omega}(k) + e_3(k)) + \psi(k)e_2'(k) - \bar{\omega}^T(k)\bar{\omega}(k)e_a(k) - e_2'^2(k)e_a(k)
\end{aligned} \tag{21}$$

Lemma 1

Let us consider the adaptive system defined in Figure 1, where the identification error $e_i(k) \in \mathbb{R}$ and the augmented error $e_a(k) \in \mathbb{R}$ are given by (14) and (17a) respectively and the closed-loop estimation errors $\varepsilon_{K_0}(k)$: $\mathscr{R}^+ \to \mathscr{R}$ and $\varepsilon_{\theta 1}(k)$, $\varepsilon_{\theta 2}(k)$: $\mathscr{R}^+ \to \mathscr{R}^n$ are defined by (8b). Let us assume that the adaptive laws for the control parameters $\theta(k) \in \mathscr{R}^{2n+2}$ and $K_1(k) \in \mathscr{R}$ are given by (20) and that the identifier parameter adaptive laws for $\hat{p}(k) \in \mathscr{R}^{2n+2}$ are given by (18). If γ satisfies condition (19b) and $N_L(k)$ satisfies inequality (9c), then there exist real positive constants, M_ϕ, M_ψ, M_η, $M_{\varepsilon_{\theta_1}}$, $M_{\varepsilon_{\theta_2}}$, $M_{\varepsilon_{K_0}}$, $M_{\Delta\phi}$, $M_{\Delta\psi}$, $M_{\Delta\eta}$, M_{ϵ^2}, $M_{e_i^2}$, $M_{\varepsilon^2_{\theta_1}}$, $M_{\varepsilon^2_{\theta_2}}$, $M_{\varepsilon^2_{K_0}}$, $M_{\Delta\phi^2}$, M_{ψ^2} and $M_{\Delta\eta^2}$ such that the following inequalities are satisfied:

(i) $\|\phi(k)\| \leqslant M_\phi$, (ii) $\|\psi(k)\| \leqslant M_\psi$, (iii) $\|\eta(k)\| \leqslant M_\eta$

(iv) $\|\varepsilon_{\theta_1}(k)\| \leqslant M_{\varepsilon_{\theta_1}}$, (v) $\|\varepsilon_{\theta_2}(k)\| \leqslant M_{\varepsilon_{\theta_2}}$, (vi) $|\varepsilon_{K_0}(k)| \leqslant M_{\varepsilon_{K_0}}$

(vii) $\Delta\phi(k)\| \leqslant M_{\Delta\phi}$, (viii) $\|\Delta\psi(k)\| \leqslant M_{\Delta\psi}$, (ix) $\|\Delta\eta(k)\| \leqslant M_{\Delta\eta}$

(x) $\sum_{k=0}^{\infty} \epsilon^T(k)M\,\epsilon(k) \leqslant M_{\epsilon^2}$, (xi) $\sum_{k=0}^{\infty} \frac{e_i^2(k)}{N_i(k)} \leqslant M_{e_i^2}$, (xii) $\sum_{k=0}^{\infty} \|\varepsilon_{\theta_1}(k)\|^2 \leqslant M_{\varepsilon^2_{\theta_1}}$

(xiii) $\sum_{k=0}^{\infty} \frac{\|\varepsilon_{\theta_2}(k)\|^2}{N_L(k)} \leqslant M_{\varepsilon^2_{\theta_2}}$, (xiv) $\sum_{k=0}^{\infty} \varepsilon^2_{K_0}(k) \leqslant M_{\varepsilon^2_{K_0}}$, (xv) $\sum_{k=0}^{\infty} \|\Delta\phi(k)\|^2 \leqslant M_{\Delta\phi^2}$

(xvi) $\sum_{k=0}^{\infty} |\Delta\psi(k)|^2 \leqslant M_{\Delta\psi^2}$, (xvii) $\sum_{k=0}^{\infty} \|\Delta\eta(k)\|^2 \leqslant M_{\Delta\eta^2}$

Proof of Lemma 1

Let us consider the system $\boldsymbol{S}$ defined by equations (17b), (18) and (20). Let us choose the Lyapunov function candidate

$$V(k) = |K_{\mathrm{p}}|(\phi_{\theta_1}^{\mathrm{T}}(k)\phi_{\theta_1}(k) + \phi_{\theta_2}^{\mathrm{T}}(k)\phi_{\theta_2}(k) + \phi_{K_0}^2(k)) + |K_{\mathrm{p}}|(\eta_\beta^{\mathrm{T}}(k)\eta_\beta(k) + \eta_{K_{\mathrm{r}}}^2(k))$$
$$+ \eta_\alpha^{\mathrm{T}}(k)\eta_\alpha(k) + \eta_{K_{\mathrm{p}}}^2(k) + \psi^2(k) + 2\gamma\epsilon^{\mathrm{T}}(k)P\epsilon(k) \tag{22a}$$

where $P = P^{\mathrm{T}} > 0$ is a positive definite matrix characterized in Reference 11.

Computing the temporal difference $\Delta V(k) = V(k+1) - V(k)$ along any trajectory of system $\boldsymbol{S}$, we get

$$\Delta V(k) \leqslant -C_1\frac{e_{\mathrm{i}}^2(k)}{N_{\mathrm{i}}(k)} - C_2\|\epsilon_{\theta_1}(k)\|^2 - C_3\epsilon_{K_0}^2(k) - C_4\frac{\|\epsilon_{\theta_2}(k)\|^2}{N_{\mathrm{L}}(k)} - 2\gamma(\epsilon^{\mathrm{T}}(k)q - \mu v(k))^2$$
$$- 2\gamma\sigma\epsilon^{\mathrm{T}}(k)M\epsilon(k) - \gamma e_{\mathrm{a}}^2(k)(C_5 e_2'^2(k) + C_6\bar{\omega}^{\mathrm{T}}(k)\bar{\omega}(k)) \leqslant 0 \tag{22b}$$

where C_1–C_6 are positive real constants defined as

$$C_1 = \gamma\left(2 - \frac{\gamma}{K_{\mathrm{M}}^2}(1 + |h| + K_{\mathrm{M}}^2)\right), \qquad C_2 = 2\frac{|h|}{N_k}\left(1 - \frac{1}{N_k}\right), \qquad C_3 = \frac{2}{N_k}\left(1 - \frac{1}{N_k}\right)$$
$$C_4 = 2 - \frac{K_{\mathrm{M}}^2 + |h| + \lambda_{\mathrm{M}}(k)}{N_{\mathrm{L}}(k)}, \qquad C_5 = 2 - \gamma, \qquad C_6 = 2 - \gamma\frac{|h|}{K_{\mathrm{M}}^2} \tag{22c}$$

σ and μ are positive real constants, M is a positive definite matrix and q is a vector defined in the Kalman–Yacubovich lemma for discrete time.[11] We will prove inequality (22b) later.

Since $V(k)$ is a Lyapunov function for system $\boldsymbol{S}$, then $\epsilon(k)$, $\theta_1(k)$, $\theta_2(k)$, $K_0(k)$, $K_1(k)$ and the estimates $\hat{\alpha}(k)$, $\hat{\beta}(k)$, $\hat{K}_{\mathrm{p}}(k)$ and $\hat{K}_{\mathrm{r}}(k)$ are bounded. Therefore assertions (i)–(iii) of Lemma 1 are demonstrated. From definitions (10) and using (i)–(iii), inequalities (iv)–(vi) are verified. Using relationships (18) and (20) together with inequalities (iv)–(vi) and using the definition of $e_{\mathrm{a}}(k)$ from (17a) and the fact that $|e_{\mathrm{i}}(k)|/\sqrt{|(N_{\mathrm{i}}(k))}$ is bounded, it is shown that propositions (vii)–(ix) are satisfied.

Adding $\Delta V(k)$ given in (22b) for $k = 0, \ldots, \infty$ and considering that $V(\infty) - V(0)$ is finite, propositions (x)–(xiv) are proved. From (18) and (20) and since assertions (x)–(xiv) are true, it is shown that inequalities (xv)–(xvii) are verified.

In order to show that (22b) is true, the following reasoning is used.

(a) Compute $\Delta V(k)$ from (22a).
(b) Replace the adaptive laws given by (18) and (20) and use the expressions for the closed loop estimation errors, identification error and augmented error given by (8b), (14) and (17a) respectively.
(c) Use the Kalman–Yacubovich lemma given in Reference 11 in the above step.
(d) Replace $v(k)$ from (21).
(e) Use the definition of the factor $N_{\mathrm{i}}(k)$ given by (19a), i.e.

$$\bar{V}_1^{\mathrm{T}}(k)\bar{V}_1(k) \leqslant N_{\mathrm{i}}(k), \qquad \bar{V}_2^{\mathrm{T}}(k)\bar{V}_2(k) \leqslant N_{\mathrm{i}}(k), \qquad \bar{U}_0^2(k) \leqslant N_{\mathrm{i}}(k)$$

to find that

$$\Delta V(k) \leqslant \frac{\gamma e_{\mathrm{i}}^2(k)}{N_{\mathrm{i}}(k)}\left(2 - \frac{\gamma}{K_{\mathrm{M}}^2}(1 + |K_{\mathrm{p}}| + K_{\mathrm{M}}^2)\right) - 2|K_{\mathrm{p}}|\frac{\varepsilon_{\theta_1}^{\mathrm{T}}(k)\varepsilon_{\theta_1}(k)}{N_k}\left(1 - \frac{1}{N_k}\right)$$
$$- 2|K_{\mathrm{p}}|\frac{\varepsilon_{K_0}^2(k)}{N_k}\left(1 - \frac{1}{N_k}\right) - \frac{\varepsilon_{\theta_2}^{\mathrm{T}}(k)\varepsilon_{\theta_2}(k)}{N_{\mathrm{L}}(k)}\left(2 - \frac{1}{N_{\mathrm{L}}(k)}(K_{\mathrm{M}}^2 + |K_{\mathrm{p}}|)\right)$$
$$+ \frac{\varepsilon_{\theta_2}^{\mathrm{T}}(k)\theta_2(k)\theta_2^{\mathrm{T}}(k)\varepsilon_{\theta_2}(k)}{N_{\mathrm{L}}^2(k)} - 2\gamma(\epsilon^{\mathrm{T}}(k)q - \mu v(k))^2 - 2\gamma\sigma\epsilon^{\mathrm{T}}(k)M\,\epsilon(k)$$
$$- \gamma e_{\mathrm{a}}^2(k)\left[(2-\gamma)e_2^2(k) + \left(2 - \gamma\frac{|K_{\mathrm{p}}|}{K_{\mathrm{M}}^2}\right)\bar{\omega}^{\mathrm{T}}(k)\bar{\omega}(k)\right] \quad (23)$$

(f) Considering that $|K_{\mathrm{p}}| \leqslant h$ and in order to keep $\Delta V(k) \leqslant 0$, then γ must satisfy the relationship (19b). Thus the constant C_1 is positive.
(g) Choosing $N_k > 1$, the constants C_2 and C_3 are positive.
(h) The matrix $\theta_2(k)\theta_2^{\mathrm{T}}(k)$ can be bounded using its maximum eigenvalue $\lambda_{\mathrm{M}}(k)$, i.e.

$$\|\theta_2(k)\theta_2^{\mathrm{T}}(k)\| \leqslant \lambda_{\mathrm{M}}(k) \quad (24)$$

It is important to note that the above condition can be written as

$$\lambda_{\mathrm{M}}(k) = \mathrm{trace}(\theta_2(k)\theta_2^{\mathrm{T}}(k)) = \|\theta_2(k)\|^2 = \sum_{i=1}^{n}\theta_{2\mathrm{i}}^2(k)$$

If we choose $N_{\mathrm{L}}(k)$ such that

$$N_{\mathrm{L}}(k) \geqslant \tfrac{1}{2}(K_{\mathrm{M}}^2 + h + \lambda_{\mathrm{M}}(k)) \quad (25)$$

then the constant C_4 is positive. From (19b) it can be seen that the constants C_5 and C_6 are positive. Thus inequality (22b) has been demonstrated.

From (22b) we can conclude that $e_{\mathrm{i}}^2(k)/N_{\mathrm{i}}(k)$, $\epsilon^{\mathrm{T}}(k)M\epsilon(k)$, $\varepsilon_{K_0}^2(k)$, $\varepsilon_{\theta_1}^{\mathrm{T}}(k)\varepsilon_{\theta_1}(k)$, $\varepsilon_{\theta_2}^{\mathrm{T}}(k)\varepsilon_{\theta_2}(k)/N_{\mathrm{L}}(k)$ and $(e_{\mathrm{a}}(k)\tilde{\omega}(k))^{\mathrm{T}}(e_{\mathrm{a}}(k)\tilde{\omega}(k))$ belong to $\mathscr{L}^2$ or the signals $e_{\mathrm{i}}(k)/\sqrt{(N_{\mathrm{i}}(k))}$, $\epsilon(k)$, $\varepsilon_{K_0}(k)$, $\varepsilon_{\theta_1}(k)$, $\varepsilon_{\theta_2}(k)/\sqrt{(N_{\mathrm{L}}(k))}$ and $e_{\mathrm{a}}(k)\tilde{\omega}(k)$ belong to $\mathscr{L}^1$; therefore

$$\lim_{k\to\infty}\left\{\frac{e_{\mathrm{i}}(k)}{\sqrt{(N_{\mathrm{i}}(k))}}, \epsilon(k), \varepsilon_{K_0}(k), \varepsilon_{\theta_1}(k), \frac{\varepsilon_{\theta_2}(k)}{\sqrt{(N_{\mathrm{L}}(k))}}, e_{\mathrm{a}}(k)\tilde{\omega}(k)\right\} = 0 \quad (26a)$$

It can be shown from (25) that $N_{\mathrm{L}}(k) \in \mathscr{L}^\infty$, since $\theta_2(k) \in \mathscr{L}^\infty$. From this fact and using (26a), we get $\epsilon_{\theta_2}(k) \in \mathscr{L}^1$. Therefore all closed-loop estimation errors tend to zero as k tends to infinity, i.e.

$$\lim_{k\to\infty}(\varepsilon_{\theta_1}(k), \varepsilon_{\theta_2}(k), \varepsilon_{K_0}(k)\} = 0 \quad (26b)$$

We still have to prove that the rest of the signals of the adaptive system remain bounded. This will be done through the following theorem.

Theorem 1

Let us consider the system **S** described in Lemma 1. Then there exist finite positive real constants, $M_{\hat{y}_{\mathrm{p}}}$, M_{V_1}, M_{V_2}, $M_{y_{\mathrm{p}}}$, M_u, $M_{e_{\mathrm{i}}}$, $M_{e_{\mathrm{a}}}$ and $M_{e_{\mathrm{c}}}$ such that the following inequalities are verified:

(i) $|\hat{y}_p(k)| \leqslant M_{\hat{y}_p}$, (ii) $\| V_1(k) \| \leqslant M_{V_1}$, (iii) $V_2(k) \| \leqslant M_{V_2}$, (iv) $|y_p(k)| \leqslant M_{y_p}$,

(v) $|u(k)| \leqslant M_u$, (vi) $|e_i(k)| \leqslant M_{e_i}$, (vii) $|e_a(k)| \leqslant M_{e_c}$, (viii) $|e_c(k)| \leqslant M_{e_c}$

and also

(ix) $\lim_{k \to \infty} (e_i(k), e_a(k), e_2(k), e_c(k)\} = 0$

Proof of Theorem 1

The overall adaptive system whose stability is to be analysed can be represented by a complex vector and non-linear difference equation. For convenience, as in discrete-time DMRAC,[8] the analysis will be separated into three parts as shown in Figure 2.

Part I—Plant feedback loop. The plant plus controller can be represented by the set of equations[12]

$$X(k+1) = (A_{MN} + b_{MN}\phi^{T}(k)C)X(k) + b_{MN}(K_0^*)R(k), \qquad \omega(k) = C\,X(k) \tag{27a}$$

where

$$X(k) = (x_p^T(k), V_1^T(k), V_2^T(k))^T \in \mathscr{R}^{3n}$$

$$A_{MN} = \begin{bmatrix} A_p & b_p\theta_1^{*T} & b_p\theta_2^{*T} \\ 0 & \Lambda + l\theta_1^{*T} & l\theta_2^{*T} \\ lh^T & 0 & \Lambda \end{bmatrix} \in \mathscr{R}^{3n \times 3n}, \qquad b_{MN} = \begin{bmatrix} b_p \\ l \\ 0 \end{bmatrix}, \qquad h_{MN} = \begin{bmatrix} h^T \\ 0 \\ 0 \end{bmatrix} \in \mathscr{R}^{3n} \tag{27b}$$

$X_p(k) \in \mathscr{R}^n$ is the state of the plant, $U(k)$, $Y_p(k) \in \mathscr{R}$ are the plant input and output respectively and $V_1(k)$ and $V_2(k)$ are defined in (5).

The matrix $C \in \mathscr{R}^{2n \times 3n}$ is defined as

$$C = \begin{bmatrix} 0 & & . & . & 0 \\ . & | & - & - & - \\ . & | & & I_{2n} & \\ . & | & & & \\ 0 & | & & & \end{bmatrix}$$

where I_{2n} is the identity matrix of size $2n \times 2n$. $R(k)$ is the reference input, assumed to be uniformly bounded.

Since (27a) is a linear difference equation with bounded time-varying coefficients and since $\phi(k)$, $R(k) \in \mathscr{L}^{\infty}$, then $\| X(k) \|$ can grow at most geometrically.

Part II—Prefilter. The second part of the system shown in Figure 2 consists of a diagonal matrix transfer function relating $\bar{\omega}(k)$ *and* $\omega(k)$ as follows:

$$\bar{\omega}(k) = W_M(z) I_{2n+1}\omega(k) \tag{28}$$

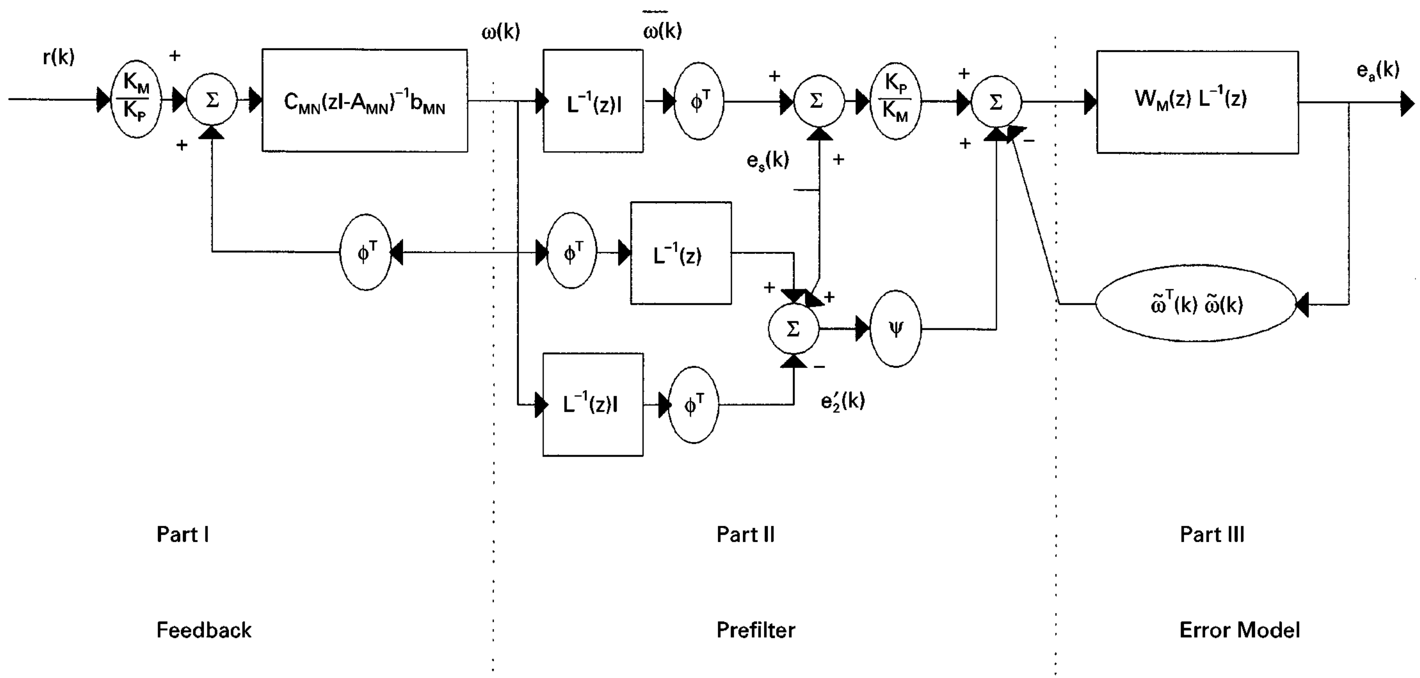

Figure 2. Decomposition of adaptive system

where $W_M(z)$ is the model reference transfer function, but in general we only need a minimum phase asymptotically stable transfer function of relative degree $n-m$. Here we have to consider the part corresponding to the generation of $e'_2(k)$ and $e_3(k)$, since in our analysis K_p is unknown and in general $\psi(k)$ is different from zero.

Part III—Error model. The last part corresponds to an error model equation whose main properties are described in References 8, 12 and 13. It consists of a strictly positive real transfer function $W_M(z)L^{-1}(z)$ with an input term $(K_p/K_M)(\phi^T(k)\bar{\omega}(k)+e_3(k))+\psi(k)e'_2(k)$ and a feedback term $\tilde{\omega}^T(k)\tilde{\omega}(k)e_a(k)$ as shown in Figure 2. Thus the complete adaptive system can be represented by a vector difference equation of the form (17b).

The state error $\epsilon(k)\in\mathscr{R}^{3n}$ is defined as $\epsilon(k)=X(k)-X^*(k)$, where $X^*(k)\in\mathscr{R}^{3n}$ is the ideal state vector obtained from (27a) when $\phi(k)=0$.

To prove that all the adaptive system signals remain bounded, we use reasoning by contradiction. From now on we assume that $\omega(k)$ and $X(k)$ grow in an *unbounded* fashion.

(i) *Model error* (*Part III*). The error model described by (27a) together with the adaptive laws (20) is of type III and is analysed in detail in References 8, 12 and 13. The input $\bar{\omega}(k)$ can be either bounded or unbounded, but in any case the properties mentioned in References 8, 12 and 13 are satisfied, in particular

$$\phi^T(k)\bar{\omega}(k)=o[\|\bar{\omega}(k)\|] \tag{29a}$$

$$\Delta\psi(k),\ \Delta\phi(k)\to 0 \quad \text{when} \quad k\to\infty \tag{29b}$$

(ii) *Prefilter* (*Part II*). From equation (28) we have

$$W_M(z)I_{2n}\omega(k)=\bar{\omega}(k)$$

where the transfer function $W_M(z)$ is of minimum phase and asymptotically stable. Applying the discrete-time version of Corollary 2 of Reference 14, (28), (29a) and (29b), we have

$$\phi^T(k)\omega(k)=o[\|\omega(k)\|_s] \tag{30}$$

(iii) *Plant feedback loop* (*Part I*). The feedback loop of Figure 2 is described by the difference equation (27a), where $R(k)$ is uniformly bounded and A_{MN} is an asymptotically stable matrix. Since we are assuming that $\omega(k)$ is not bounded, from (27a) we can conclude that[12]

$$X(k)\leqslant p_1|\phi^T(k)\omega(k)|_s+p_2, \quad p_1,p_2>0 \tag{31}$$

From (30) we have that $\phi^T(k)\omega(k)=o[\|\omega(k)\|_s]$, and considering (31), we have that

$$X(k)=o[\|CX(k)\|_s]\leqslant o[\|X(k)\|_s]$$

which contradicts the hypothesis that $X(k)$ is not bounded. Thus we conclude that $X(k)$, $\omega(k)$, $\bar{\omega}(k)$ and $\tilde{\omega}(k)$ are bounded. From this result it is easy to show that all the other signals of the adaptive system are also bounded. Therefore assertions (i)–(viii) of Theorem 1 are true and the adaptive system is globally stable. The auxiliary error $e'_2(k)$ defined in (16) can be expressed in the following manner:

$$e'_2(k)=[\phi^T(k)\bar{\omega}(k)-W_M(z)(\phi^T(k)\omega(k))]-\tilde{\omega}^T(k)\tilde{\omega}(k)e_a(k) \\ -\left(\frac{\bar{R}(k)\varepsilon_{K_0}(k)}{N_k}+\frac{V_2^T(k)\varepsilon_{\theta_2}(k)}{\operatorname{sgn}(K_p)N_L(k)}+\frac{V_1^T(k)\varepsilon_{\theta_1}(k)}{N_k}\right) \tag{32}$$

Considering (29a), (30) and the fact that $\bar{\omega}(k)$ and $\omega(k)$ are bounded, we conclude from (32) that $\phi^{T}(k)\bar{\omega}(k)$ and $\phi^{T}(k)\omega(k)$ tend to zero asymptotically. Moreover, it was shown in (26b) that the closed-loop estimation errors as well as $e_{a}(k)$ go to zero when k tends to infinity (property (ii) of error model III given in References 8, 12 and 13). Therefore, from (32), $e_2'(k)\rightarrow 0$ when $k\rightarrow\infty$.

From (15b) with $e_2'(k)\rightarrow 0$ and $e_a(k)\rightarrow 0$ we have

$$e_c(k) = \frac{K_p}{K_M} W_M(z)(\phi^{T}(k)\omega(k))$$

and since $\phi^{T}(k)\omega(k)\rightarrow 0$, then $e_c(k)\rightarrow 0$ when $k\rightarrow\infty$. From equation (15a) we conclude that $e_3(k)\rightarrow\infty$ when $k\rightarrow\infty$. Since $e_3(k)$ and $e_2'(k)\rightarrow 0$ when $k\rightarrow\infty$, from (16) we obtain that $e_2(k)$ converges to zero.

In order to prove that $e_i(k)\rightarrow 0$, when $k\rightarrow\infty$, we replace the definitions of the closed-loop estimation errors given by (10) in the output of the identifier given by (9b) to get

$$\hat{Y}_p(k) = \frac{\hat{K}_p(k)}{K_M}\varepsilon_{K_0}(k)\bar{R}(k) + \bar{R}(k) + \frac{\hat{K}_p(k)}{K_M}\left(1-\frac{1}{N_k}\right)\varepsilon_{\theta_1}^{T}(k)\bar{V}_1(k) + \frac{1}{K_M}\left(1-\frac{K_M^2}{N_L(k)}\right)\varepsilon_{\theta_2}^{T}(k)\bar{V}_2(k)$$
$$-\frac{\varepsilon_{\theta_2}^{T}(k)\theta_2(k)(\theta^{T}(k)\bar{\omega}(k)+\hat{\beta}^{T}(k)-\varepsilon_{\theta_1}^{T}(k)N_k^{-1}}{K_M N_L(k)}$$

Expressed in a condensed form, the above equation reads

$$\hat{Y}_p(k) \triangleq \varepsilon^{T}(k)\hat{P}(k)W_M(z)\omega(k) + W_M(z)R(k) \tag{33}$$

where

$$\hat{P}(k) = \begin{pmatrix} \dfrac{\hat{K}_p(k)}{K_M} & 0 & 0 \\ 0 & \dfrac{\hat{K}_p(k)}{K_M}\left(1-\dfrac{1}{N_k}\right)I_n & 0 \\ -\dfrac{\theta_2(k)K_0(k)}{K_M N_L(k)} & -\dfrac{\theta_2(k)}{K_M N_L(k)}\left(\theta_1(k)+\hat{\beta}(k)-\dfrac{\varepsilon_{\theta_1}}{N_k}\right)^{T} & \dfrac{1}{K_M}\left(1-\dfrac{K_M^2}{N_k}\right)I_n - \dfrac{\theta_2(k)\theta_2^{T}(k)}{N_L(k)} \end{pmatrix}$$

Using equation (33) and since all components of the vector $\varepsilon(k)\rightarrow 0$ as $k\rightarrow\infty$ (property (ii) of error model III given in References 8, 12 and 13), then $\hat{Y}_p(k)\rightarrow Y_M(k)$. Since $e_c(k)\rightarrow 0$, i.e. $Y_p(k)\rightarrow Y_M(k)$, then $\hat{Y}_p(k)\rightarrow Y_p(k)$; that is to say, $e_i(k)\rightarrow 0$ when $k\rightarrow\infty$. Thus proposition (ix) of Theorem 1 is true and the theorem is completely proved.

3. COMPUTER SIMULATIONS

In this section a set of simulations of a second-order plant is presented to verify the theoretical properties of the discrete-time CMRAC scheme.

An unstable second-order plant was simulated to test the CMRAC scheme under ideal conditions. The difference equation describing the plant is

$$Y_p(k+2) - 0{\cdot}1\,Y_p(k+1) - 1{\cdot}56\,Y_p(k) = 0{\cdot}9(U(k+1) - 0{\cdot}3U(k))$$

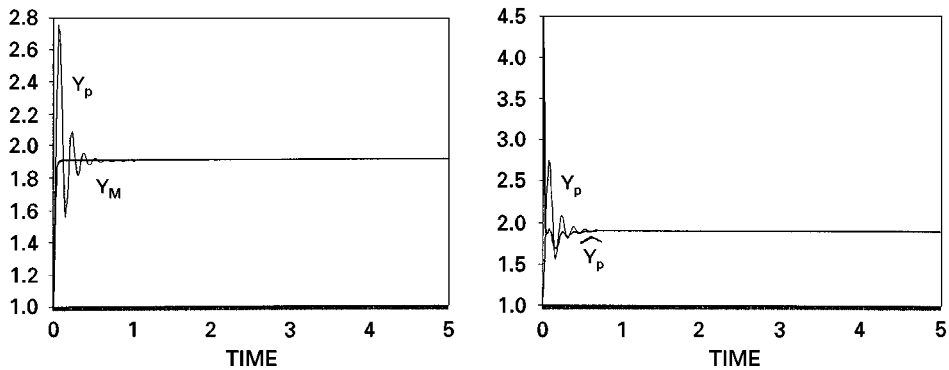

Figure 3. Combined MRAC for second-order plant and constant reference $R(k) = 2$

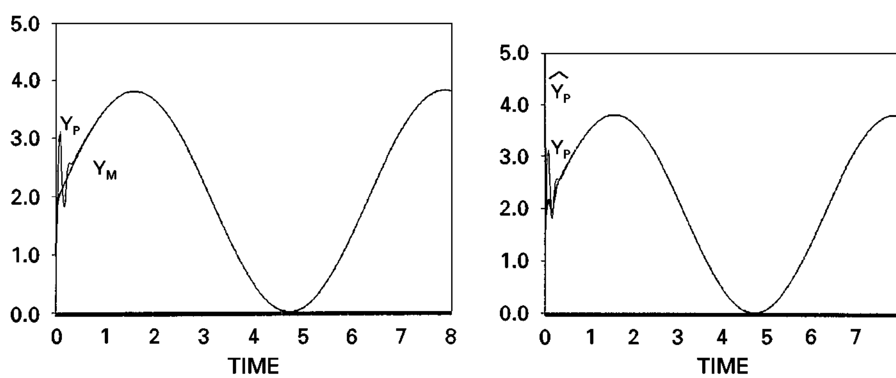

Figure 4. Combined MRAC for second-order plant and sinusoidal reference $R(k) = 2 + 2\sin(k)$

where $Y_p(0) = 1$ and $Y_p(1) = 1$. The model reference was chosen as

$$Y_M(k+2) - 0{\cdot}7\, Y_M(k+1) + 0{\cdot}12\, Y_M(k) = 0{\cdot}8(R(k+1) - 0{\cdot}5\, R(k))$$

where $Y_M(0) = 1{\cdot}1$ and $Y_M(1) = 1{\cdot}1$. The simulation results are shown in Figures 3 and 4 for different types of reference. The following numerical values for the parameters and initial conditions were chosen in all simulation: $N_k = 25{\cdot}0$, $\gamma = 0{\cdot}3$, $\theta_{11}(0) = -0{\cdot}4$, $\theta_{12}(0) = 0{\cdot}6$, $\theta_{21}(0) = -0{\cdot}5$, $\theta_{22}(0) = -0{\cdot}6$, $K_0(0) = 0{\cdot}6$, $\hat{\alpha}_{11}(0) = 0{\cdot}8$, $\hat{\alpha}_{12}(0) = 1{\cdot}2$, $\hat{\beta}_{11}(0) = 0{\cdot}2$, $\hat{\beta}_{12}(0) = -0{\cdot}2$, $\theta_{11}(1) = -0{\cdot}4$, $\theta_{12}(1) = 0{\cdot}6$, $\theta_{21}(1) = -0{\cdot}5$, $\theta_{22}(1) = -0{\cdot}6$, $K_0(1) = 0{\cdot}6$, $\hat{\alpha}_{11}(1) = 0{\cdot}8$, $\hat{\alpha}_{12}(1) = 1{\cdot}2$ and $\hat{\beta}_{11}(1) = 0{\cdot}2$.

From the above simulations it can be seen that theoretical results are verified. In particular, in all simulations, $e_c(k) \to 0$ as $k \to \infty$ without persistent excitation.

An interesting point is that parametric convergence (controller and identifier) is achieved if only parameter controller convergence is obtained. In fact, let us assume that the persistent excitation is such that the controller parameter errors are driven to zero. Then, since the closed-loop estimation errors given by (10) tend to zero, it can be concluded that the plant parameter errors are also driven to zero. This means that from the persistent excitation viewpoint the DMRAC and CMRAC schemes are equivalent.

The CMRAC scheme can also perform well under external perturbations by modifying the controller as well as the identifier adaptive laws to include four standard modifications. The general form of the adaptive laws is now

$$\begin{aligned}
\Delta\eta_\alpha(k) &= -\gamma\frac{e_i(k)\bar{V}_2(k)}{N_i(k)} - K_M\frac{\varepsilon_{\theta_2}(k)}{N_L(k)} - \hat{\alpha}(k)f(\hat{\alpha})\\
\Delta\eta_\beta(k) &= -\gamma\,\mathrm{sgn}(K_p)\frac{e_i(k)\bar{V}_1(k)}{K_M N_i(k)} - \frac{\varepsilon_{\theta_1}(k)}{N_k} - \hat{\beta}(k)f(\hat{\beta})\\
\Delta\eta_{K_p}(k) &= -\gamma\frac{e_i(k)\bar{U}_0(k)}{K_M N_i(k)} - \frac{\theta_2^T(k)\varepsilon_{\theta_2}(k)}{N_L(k)} - \hat{K}_p(k)f(\hat{K}_p)\\
\Delta\eta_{K_r}(k) &= \frac{\varepsilon_{K_0}(k)}{N_k} - \hat{K}_r(k)f(\hat{K}_r)
\end{aligned} \tag{34}$$

$$\begin{aligned}
\Delta\phi_{\theta_2}(k) &= -\mathrm{sgn}(K_p)\left(\gamma\frac{e_a(k)\bar{V}_2(k)}{K_M} + \frac{\varepsilon_{\theta_2}(k)}{N_L(k)}\right) - \theta_2(k)f(\theta_2)\\
\Delta\phi_{\theta_1}(k) &= -\gamma\,\mathrm{sgn}(K_p)\frac{e_a(k)\bar{V}_1(k)}{K_M} - \frac{\varepsilon_{\theta_1}(k)}{N_k} - \theta_1(k)f(\theta_1)\\
\Delta\phi_{K_0}(k) &= -\gamma\,\mathrm{sgn}(K_p)\frac{e_a(k)\bar{r}(k)}{K_M} - \frac{\varepsilon_{K_0}(k)}{N_k} - K_0(k)f(K_0)\\
\Delta\psi(k) &= -\gamma\, e_a(k)e_2(k) - K_1(k)f(K_1)
\end{aligned} \tag{35}$$

3.1. Dead-zone modification 15–17

In this case the modified adaptive laws for the identifier and controller have the form indicated in (34) and (35) with

$$\begin{aligned}
f(\hat{\alpha}) = f(\hat{\beta}) = f(\hat{K}_p) = f(\hat{K}_r) = 0 &\quad \text{if } |e_i| \geqslant P_{eo} + \delta\\
f(\hat{\alpha}) = f(\hat{\beta}) = f(\hat{K}_p) = f(\hat{K}_r) = \gamma = 0 &\quad \text{if } |e_i| \leqslant P_{eo} + \delta
\end{aligned}$$

and

$$\begin{aligned}
f(\theta_2) = f(\theta_1) = f(K_0) = f(K_1) = 0 &\quad \text{if } |e_a| \geqslant P_{eo} + \delta\\
f(\theta_2) = f(\theta_1) = f(K_0) = f(K_1) = \gamma = 0 &\quad \text{if } |e_a| \leqslant P_{eo} + \delta
\end{aligned}$$

Where P_{eo} is related to the bound of the external perturbation.

3.2. Knowledge of a bound on true parameters[18]

Let the function $f(\cdot)$ be defined as

$$f(\tau) = \begin{cases} (1 - \|\tau\|/\tau^*_{max})^2 & \text{if } \|\tau\| > \tau^*_{max}\\ 0 & \text{otherwise} \end{cases}$$

where τ is a parameter vector and τ^*_{max} is an upper bound on its norm.

The modified adaptive laws have the form indicated in (34) and (35) with the function $f(\cdot)$ just defined.

3.3. $\sigma\theta(k)$ Modification[19]

With this modification the identifier and controller adaptive laws have the form indicated in (34) and (35) with

$$f(\hat{\alpha}) = f(\hat{\beta}) = f(\hat{K}_p) = f(\hat{K}_r) = \sigma_i \qquad f(\theta_2) = f(\theta_1) = f(K_0) = f(K_1) = \sigma_c$$

where $\sigma_i > 0$, $\sigma_c > 0$.

3.4. $|e|\,\theta(k)$ Modification[20]

The adaptive laws for the identifier and controller are those indicated in (34) and (35), modified in the following fashion:

$$f(\hat{\alpha}) = f(\hat{\beta}) = f(\hat{K}_p) = f(\hat{K}_r) = \delta_i |e_i(k)| \qquad f(\theta_2) = f(\theta_1) = f(K_0) = f(K_1) = \delta_c |e_a(k)|$$

where $\delta_i > 0$, $\delta_c > 0$.

Extensive simulations (not shown here for reasons of space) show that CMRAC also performs well under external perturbations using all the above adaptive laws.

4. CONCLUSIONS

The discrete-time version of CMRAC has been presented. The method uses the direct and indirect approaches coupled by the closed-loop estimation errors as well as dynamical adjustments of the control and identification parameters. Ideal conditions for the global stability of CMRAC were derived which are different from those obtained for the continuous-time case. The robustness of discrete-time CMRAC was discussed and it was found that modifications of the adaptive laws can make the algorithm robust with respect to external perturbation.

ACKNOWLEDGEMENTS

The work reported here was supported by CONICYT under project FONDECYT 1950502.

REFERENCES

1. Duarte, M. A. and K. S. Narendra, 'Combined direct and indirect adaptive control', *IEEE Trans. Automatic Control*, **AC-34**, 1071–1075 (1989).
2. Duarte, M. A. and K. S. Narendra, 'A new approach to model reference adaptive control', *Int. J. Adapt. Control Signal Process.* **3**, 53–73 (1989).
3. Kreisselmeier, G., 'On adaptive state regulation', *IEEE Trans. Automatic Control*, **AC-27,** 3–17 (1982).
4. Kreisselmeier, G., 'Adaptive control via adaptive observation and asymptotic feedback matrix synthesis', *IEEE Trans. Automatic Control*, **AC-25,** 717–722 (1980).
5. Kreisselmeier, G. and M. C. Smith, 'Stable adaptive regulation of arbitrary *n*th order plant', *IEEE Trans. Automatic Control*, **AC-31**, 299–305 (1986).
6. Morse, A. S., 'Towards a unified theory of parameter adaptive control. Part II: Certainty equivalence and implicit tuning', *IEEE Trans. Automatic Control*, **AC-37**, 15–29 (1992).
7. Ortega, R., 'On Morse's new adaptive controller: parameter convergence and transient performance', *IEEE Trans. Automatic Control*, **AC-38**, 1191–1202 (1993).
8. Narendra, K. S. and Y. H. Lin, 'Stable discrete adaptive control', *IEEE Trans. Automatic Control*, **AC-25**, 456–461 (1980).
9. Chen, C. T., *Linear System Theory and Design*, Holt, Rinehart and Winston, New York, 1984.
10. Ponce, R. F., 'CARM indirecto dinámico y combinado en tiempo discreto', *Electrical Engineering Thesis*, Electrical Engineering Department, University of Chile, Santiago, 1993 (in Spanish).

11. Neumann, S. P., 'Frequency domain stability criteria in nonlinear automatic control', *Doctoral Dissertation*, Harvard University, Cambridge, MA, 1967.
12. Lin, Y. H., 'Global stability of model reference adaptive control system', *Ph.D. Thesis*, Yale University, New Haven, CT, 1980.
13. Lin, Y. H. and K. S. Narendra, 'A new error model for adaptive system', *IEEE Trans. Automatic Control*, **AC-25**, 585–587 (1980).
14. Narendra, K. S. and A. M. Annaswamy, 'A general approach to the stability analysis of adaptive system', *Int. J. Control*, **41**, 193–216 (1985).
15. Egardt, B., *Stability of Adaptive Controllers*, Springer, Berlin, 1979.
16. Peterson, B. B. and K. S. Narendra, 'Bounded error adaptive control', *IEEE Trans. Automatic Control*, **AC-27**, 1161–1168 (1982).
17. Samson, C., 'Stability analysis of adaptively controlled system subject to bounded disturbance, *Automatica*, **19**, 81–86 (1983).
18. Kreisselmeier, G. and K. S. Narendra, 'Stable model reference adaptive control in the presence of bounded disturbance', *IEEE Trans. Automatic Control*, **AC-27**, 1169–1175 (1982).
19. Ioannou, P. A. and P. V. Kokotovic, *Adaptive System with Reduced Models*, Springer, New York, 1983.
20. Narendra, K. S. and A. M. Annaswamy, 'A new adaptive law for robust adaptive control without persistent excitation', *IEEE Trans. Automatic Control*, **AC-32**, 134–145 (1987).
21. Monopoli, R. V., 'Model reference adaptive control with an augmented error signal', *IEEE Trans. Automatic Control*, **AC-22,** 2–8 (1977).

