# DISCRETE-TIME FRACTIONAL-ORDER CONTROLLERS

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## Abstract

The theory of fractional calculus goes back to the beginning of the theory of differential calculus but its inherent complexity postponed the application of the associated concepts. In the last decade the progress in the areas of chaos and fractals revealed subtle relationships with the fractional calculus leading to an increasing interest in the development of the new paradigm. In the area of automatic control preliminary work has already been carried out but the proposed algorithms are restricted to the frequency domain. The paper discusses the design of fractional-order discrete-time controllers. The algorithms studied adopt the time domain, which makes them suited for $z$-transform analysis and discrete-time implementation. The performance of discrete-time fractional-order controllers with linear and nonlinear systems is also investigated.

*Mathematics Subject Classification*: 26A33, 93C15, 93C55, 93C80

*Key Words and Phrases*: fractional calculus, control theory, discrete-time controllers

## 1. Introduction

Fractional calculus is a natural extension of the classical mathematics. In fact, since the beginning of the theory of differential and integral calculus, mathematicians such as Euler and Liouville investigated their ideas on the calculation of non-integer order derivatives and integrals. Nevertheless, in spite of the work that has been done in the area, the application of fractional derivatives and integrals (FDIs) has been scarce until recently. In the last years, the advances in the theory of chaos revealed relations with FDIs, motivating a renewed interest in this field. The basic aspects of the fractional calculus theory and the study of its

properties can be addressed in references [1-5] while research results can be found in [6-13]. In what concerns the application of FDI concepts we can mention a large volume of research about viscoelasticity/damping [14-32] and chaos/fractals [33-36]. However, other scientific areas are currently paying attention to the new concepts and we can refer the adoption of FDIs in biology [37], electronics [38]. signal processing [39-41], system identification [42], diffusion and wave propagation [43-46], percolation [47], modeling and identification [48-51], chemistry [52], irreversibility [53] and control [54-61]. Inspired by the fractional calculus several researchers on automatic control proposed algorithms based on the frequency [54-55] and the discrete-time [60-61] domains. This work is still giving its first steps and, consequently, many aspects remain to be investigated. This paper analyses several approaches to implement FDIs in discrete-time control systems and, in this line of thought, the paper is organized as follows. Sections two and three analyze the frequency-domain and the discrete-time approximations to FDIs, respectively. Based on the proposed discrete-time FDI approximation, sections four and five investigate the performance of fractional order algorithms with linear and nonlinear control systems, respectively, from a stability and robustness point of view. Finally, section six draws the main conclusions.

## 2. Frequency-domain approximation to fractional-order derivatives

In order to analyze the frequency-based approach to $D^{\alpha}$ $(0 < \alpha < 1)$, based on the Fourier definition of a FDI, $F\{D_{\pm}^{\alpha}\varphi\} = (\pm j\omega)^{\alpha}F\{\varphi\}$, $Re(\alpha) \geq 0$, let us consider the recursive circuit represented on **Figure 1** such that [54]:

$$I = \sum_{i=1}^{n} I_i,\ R_{i+1} = \frac{R_i}{\epsilon},\ C_{i+1} = \frac{C_i}{\eta}, \tag{1}$$

where $\eta$ and $\epsilon$ are scale factors, $I$ is the current due to an applied voltage $V$ and $R_i$ and $C_i$ are the resistance and capacitance elements of the ith branch of the circuit.

The admittance $Y(j\omega)$ is given by:

$$Y(j\omega) = \frac{I(j\omega)}{V(j\omega)} = \sum_{i=0}^{n} \frac{j\omega C\epsilon^i}{j\omega CR + (\eta\epsilon)^i}. \tag{2}$$

**Figure 2** shows the asymptotic Bode diagrams of amplitude and phase of $Y(j\omega)$. The pole and zero frequencies ($\omega_i$ and $\omega_i'$) obey the recursive relationships:

$$\frac{\omega_{i+1}'}{\omega_i'} = \frac{\omega_{i+1}}{\omega_i} = \epsilon\eta,\ \frac{\omega_i}{\omega_i'} = \epsilon,\ \frac{\omega_{i+1}'}{\omega_i} = \eta. \tag{3}$$

From the Bode diagram of amplitude or of phase, the average slope $m'$ can be calculated as:

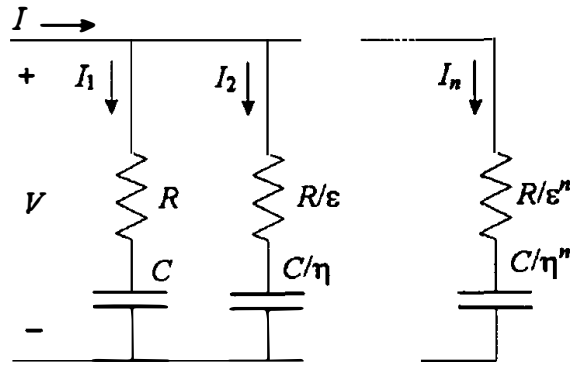


Figure 1: Electrical circuit with a recursive association of resistance and capacitance elements.

$$m' = \frac{\log \epsilon}{\log \epsilon + \log \eta} \,. \tag{4}$$

The fractional order of the frequency response is due to the recursive nature of the circuit. In fact, the admittance $Y(j\omega)$ follows the recursive formula:

$$Y\left(\frac{\omega}{\eta\epsilon}\right) = \frac{1}{\epsilon} Y(\omega) \tag{5}$$

with solution in accordance with (4) (where $\Psi$ is a scale factor):

$$Y(\omega) = \Psi(j\omega)^{-m'}, m' = \frac{\log \epsilon}{\log \epsilon + \log \eta} \,. \tag{6}$$

Consequently, the circuit of **Figure 1** represents an approach to $D^{\alpha}$, $0 < \alpha < 1$, with $m' = \alpha$, based on a recursive pole/zero placement in the frequency domain.

As demonstrated in [54], a second aspect of FDI algorithms can be illustrated through the elemental control system represented in **Figure 3**, with open-loop transfer function $G(s) = Ks^{-\alpha}\,(1 < \alpha < 2)$ in the forward path. The open-loop Bode diagrams (**Figure 4**) of amplitude and phase have a slope of $-20\alpha\ dB/dec$ and a constant phase of $-\alpha\pi/2\ rad$, respectively. Therefore, the closed-loop system has a constant phase margin of $\pi(1 - \alpha/2)\ rad$, that is independent of the system gain $K$. Likewise, this important property is also revealed through the root-locus depicted in **Figure 5**. In fact, when $1 < \alpha < 2$ the root-locus follows the relation $\pi - \pi/\alpha = cos^{-1}\zeta$ , where $\zeta$ is the damping ratio, independently of the gain $K$.

In conclusion, the Laplace/Fourier definition for a derivative of order $\alpha \in C$ is a 'direct' generalization of the classical integer-order scheme with the multiplication of the signal transform by the $s/j\omega$ operator. In what concerns au-

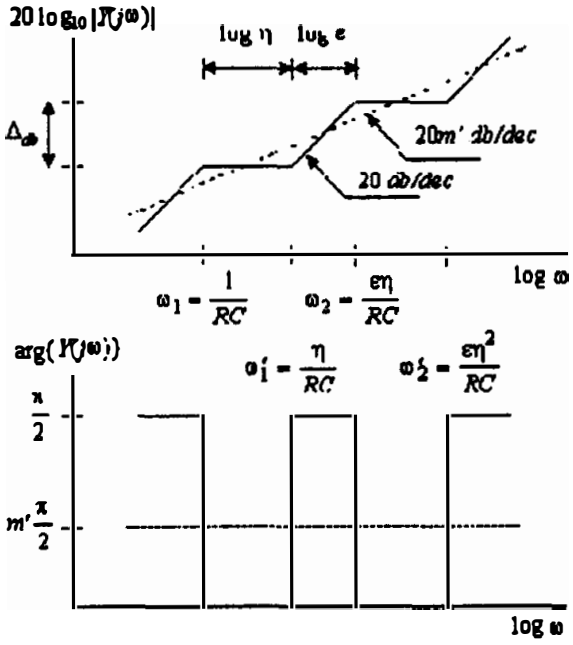


Figure 2: Bode diagrams of amplitude and phase of $Y(j\omega)$.

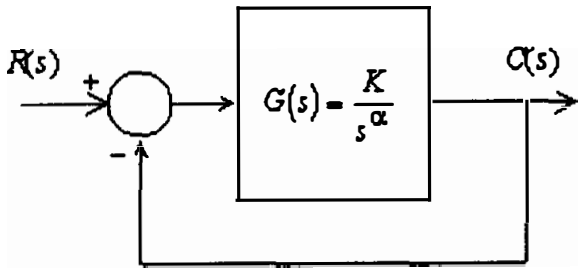


Figure 3: Block diagram for an elemental feedback control system of fractional order $\alpha$.

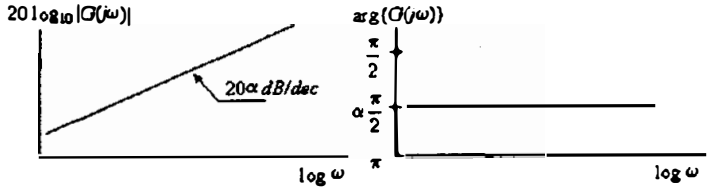


Figure 4: Open-loop Bode diagrams of amplitude and phase for a system of fractional order $1 < \alpha < 2$.

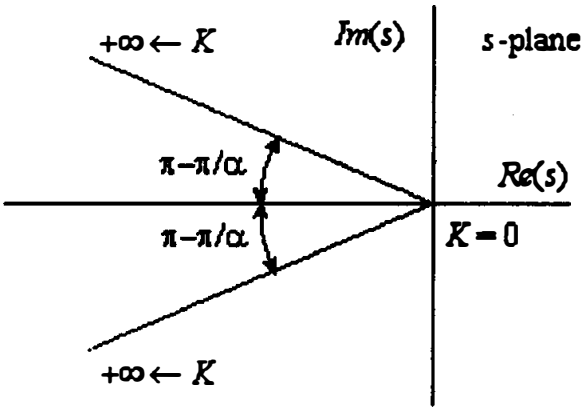


Figure 5: Root locus of a control system of fractional order $1 < \alpha < 2$.

tomatic control theory this means that frequency-based analysis methods have a straightforward adaptation to FDIs. Nevertheless, this approach has several drawbacks because the implementation of FDIs based on the Laplace/Fourier definition adopts the frequency domain and requires an infinite number of poles and zeros obeying a recursive relationship [54]. In a real approximation the finite number of poles and zeros yields a ripple in the frequency response and a limited bandwidth. Moreover, the digital conversion of the scheme requires further steps and additional approximations making difficult to analyze the final algorithm. The method is restricted to cases where a frequency response is well known and, in other circumstances, problems occur for its implementation.

## 3. Fractional-order discrete-time control algorithms

Based on the concept of fractional differential of order $\alpha$, the Grünwald-Letnikov definition (7) of a derivative of fractional order $\alpha$ of the signal $x(t)$, $D^{\alpha}[x(t)]$, leads to the expression (8):

$$\left(I^{\alpha}_{a+\varphi}\right)(x) = \frac{1}{\Gamma(\alpha)} \lim_{h \to +0} \left[ h^{\alpha} \sum_{j=0}^{\left[\frac{x-a}{h}\right]} \frac{\Gamma(\alpha+j)}{\Gamma(j+1)} \varphi(x - jh) \right], \tag{7}$$

$$D^{\alpha}[x(t)] \approx \lim_{h \to 0} \left[ \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} (-1)^{k} \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} x(t - kh) \right], \tag{8}$$

where $\Gamma$ is the gamma function and $h$ is the time increment. This formulation inspired a discrete-time FDI calculation algorithm [60, 62], based on the approximation of the time increment $h$ through the sampling period $T$, yielding the equation in the $z$ domain:

$$Z\{D^{\alpha}[x(t)]\} \approx \left[\frac{1}{T^{\alpha}}\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}z^{-k}\right]X(z) = \left(\frac{1-z^{-1}}{T}\right)^{\alpha}X(z), \quad (9)$$

where $X(z) = Z\{x(t)\}$.

A real implementation of (9) corresponds to a $n$-term truncated series (another form of the "short-memory" principle [57, 59]) given by:

$$Z\{D^{\alpha}[x(t)]\} \approx$$

$$\approx \left[\frac{1}{T^{\alpha}}\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}z^{-k}\right]X(z) = Trunc_{n}\left\{\left(\frac{1-z^{-1}}{T}\right)^{\alpha}\right\}X(z). \quad (10)$$

Nevertheless, the properties of this and other approaches must be further studied and, bearing these facts in mind, in the sequel we analyze several discrete-time approximations to FDIs. We start by considering the well-known $s \to z$ conversion schemes (also called analog to digital open-loop design methods) of Euler (or first backward difference), Tustin (or bilinear) and Simpson [63]. Note that the Grünwald-Letnikov approach (9) is similar to the Euler scheme. In our study we shall adopt for $D^{\alpha}$ expressions that are the generalization to non-integer exponents of these conversion methods as represented in **Table I**. The fractional-order conversion schemes lead to non-rational $z$-formulae. Therefore, in order to get rational expressions we expand them into Taylor series and the final algorithm corresponds to a $n$-term truncated series. These three approximations and the corresponding Taylor truncated series have distinct properties that must be analyzed before getting into a control system implementation. For example, the log-log chart of **Figure 6** shows the amplitude absolute values of the Taylor series coefficients *versus* the term order when approximating the $\alpha = \frac{1}{2}$ derivative:

Table I: Discrete-time conversion schemes

| Method | $s \to z$ conversion | Taylor series |
|---|---|---|
| Euler Grünwald-Letnikov | $s^{\alpha} \approx \left[\frac{1}{T}(1-z^{-1}\right]^{\alpha}$ | $\left(\frac{1}{T}\right)^{\alpha}\left[1-\alpha z^{-1}+\frac{\alpha(\alpha-1)}{2!}z^{-2}\ldots\right]$ |
| Tustin | $s^{\alpha} \approx \left[\frac{2}{T}\frac{(1-z^{-1}\right]^{\alpha}}{1+z^{-1}}$ | $\left(\frac{2}{T}\right)^{\alpha}\left[1-2\alpha z^{-1}+2\alpha^{2}z^{-2}\ldots\right]$ |
| Simpson | $s^{\alpha} \approx \left[\frac{3}{T}\frac{(1+z^{-1})(1-z^{-1})}{1+4z^{-1}+z^{-2}}\right]^{\alpha}$ | $\left(\frac{3}{T}\right)^{\alpha}\left[1-4\alpha z^{-1}+2\alpha(4\alpha+3)z^{-2}\ldots\right]$ |

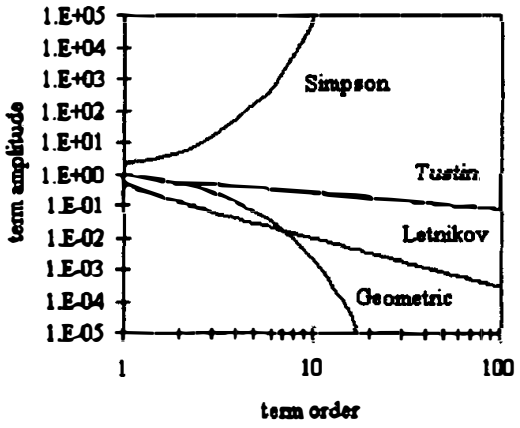


Figure 6: Amplitude of the Taylor series coefficients versus the term order when approximating the $D^{\frac{1}{2}}$ with the Grünwald-Letnikov, Tustin and Simpson algorithms.

Grünwald-Letnikov: $Z\left\{D^{\frac{1}{2}}[x(t)]\right\} \approx$

$$\approx \left[\sqrt{\frac{1}{T}}\left(1-\frac{1}{2}z^{-1}-\frac{1}{8}z^{-2}-\frac{1}{16}z^{-3}\frac{5}{128}z^{-4}\frac{7}{256}z^{-5}\cdots\right)\right]X(z). \quad (11)$$

Tustin: $Z\left\{D^{\frac{1}{2}}[x(t)]\right\} \approx$

$$\approx \left\{\sqrt{\frac{2}{T}}\left[(1-z^{-1})+\frac{1}{2}(z^{-2}-z^{-3})+\frac{3}{8}(z^{-4}-z^{-5})\cdots\right]\right\}X(z). \quad (12)$$

Simpson: $Z\left\{D^{\frac{1}{2}}[x(t)]\right\} \approx$

$$\approx \left[\sqrt{\frac{3}{T}}\left(1-2z^{-1}+5z^{-2}-16z^{-3}+\frac{105}{2}z^{-4}-177z^{-5}\cdots\right)\right]X(z). \quad (13)$$

For simplicity, in the chart the gains of the approximations are not represented. Analyzing the results we conclude that:

- While an integer-order derivative implies simply a finite series, the fractional-order derivative requires an infinite number of terms. This means that integer derivatives are 'local' operators in opposition with fractional derivatives that have, implicitly, a 'memory' of all past events.

- The 'memory'property of the fractional derivatives is highlighted when we study the magnitude of the series coefficients. For comparison purposes in **Figure 6** it is also plotted the coefficients of a geometric series having the three initial terms similar to those of the Tustin series, that is:

  Geometric: $1 - z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{4}z^{-3} + \frac{1}{8}z^{-4} - \frac{1}{16}z^{-5} - \cdots = \frac{2-z^{-1}}{2+z^{-1}}$.

  The term coefficients of the geometric series decay rapidly while those of the Tustin approximation for the fractional-order derivative have a constant diminishing. Therefore, FDIs have a kind of logarithmic-time memory that gives a higher importance to past events.

- The Tustin and Simpson approximations $D^{\frac{1}{2}}$ seem problematic. In the first case, the coefficients decay with the term order but they appear in pairs of similar magnitude. Therefore, a series truncation of even or odd order will reveal distinct characteristics and, consequently, poor convergence properties. The Simpson approach requires a series with increasing coefficients showing, clearly, convergence problems.

The alternative of the FDI 'direct' implementation in the z-domain (the so-called discrete-time system design method) leads to poor results. For an open-loop system with transfer function G($s$), a first-order sample/hold and a $D^{\alpha}(0 < \alpha < 1)$ controller, we get:

$$Z\{D^{\alpha}[x(t)]\} = \frac{Z\left[s^{\alpha}G(s)\right]}{Z\left[\frac{1-e^{-sT}}{s}G(s)\right]}X(z). \tag{14}$$

For example, with $G(s) = \frac{1}{s^2}$ it yields:

$$\begin{aligned} Z\{D^{\alpha}[x(t)]\} &= \frac{Z\left(\frac{1}{s^{2-\alpha}}\right)}{Z\left(\frac{1-e^{-sT}}{s}\frac{1}{s^2}\right)}X(z) \\ &= \left[\frac{2}{T^{1+\alpha}\Gamma(2-\alpha)}\frac{(1-z^{-1})^2}{1+z^{-1}}\left(1 + 2^{1-\alpha}z^{-1} + 3^{1-\alpha}z^{-2} + \cdots\right)\right]X(z). \end{aligned} \tag{15}$$

Adopting $\alpha = \frac{1}{2}$ in (16), for comparison purposes, the Taylor series expansion results:

$$Z\left\{D^{\frac{1}{2}}[x(t)]\right\} \approx \left\{\frac{4}{T^{\frac{3}{2}}\sqrt{\pi}}\left[1 - (3-\sqrt{2})z^{-1} + (4+\sqrt{3}-3\sqrt{2})z^{-2} - \cdots\right]\right\}X(z). \tag{16}$$

The series coefficients diminish very slowly showing convergence problems that were confirmed in the $z$-domain root-locus. Moreover, for a different transfer

function $G(s)$ we need to recalculate the expressions in (16) and (17). Therefore, this method will not be considered in the next section, where the properties of **Table I** formulae will be further analyzed from a control system perspective.

## 4. Performance of FDI approximations in linear control systems

A mass $M$ with a time delay $T_D$ may be considered as a simple prototype system. Therefore, in order to study the performance of the FDI approximations in control algorithms we adopt a system with transfer function:

$$G(s) = \frac{e^{-sT_D}}{Ms^2}. \tag{17}$$

An important property to be tested in the FDI approximations for control consists in the stability of the resulting closed-loop system.

**Figure 7** shows the root-locus, in the $z$ domain, for the three FDI schemes when implementing a $D^{\frac{1}{2}}$ controller, without any series truncation, for the case of $M = 1$ and $T_D = 0$ in (18).

For an infinite series the Grünwald-Letnikov algorithm seems inferior while the Simpson method looks preferable. However, for a $5^{th}$ order series truncation we get the results of **Figure 8**. As pointed out in the previous section, the Grünwald-Letnikov algorithm is 'robust' in what concerns the series truncation while the root-locus reveals increasing stability problems when passing to the Tustin and Simpson schemes. In fact, this conclusion can be confirmed tacking other values of $\alpha$ in the control algorithm and analyzing both the root-locus and the time responses.

A second aspect to be tested from the control viewpoint is the controller performance when confronted with system parameter deviations. Therefore, in **Figure 9** we compare the system time response with a Grünwald-Letnikov-based $D^{\frac{1}{2}}$ control algorithm for time delays of $T_D = 0$ sec and $T_D = 0.1$ sec. The sampling period is $T = 0.1$ sec and the controller gain is $K = 10T^{\frac{1}{2}}$. Moreover, in order to analyze the response for distinct series truncation orders, namely for $n$= 3, 4 and 5.

Clearly, the higher the order of the series truncation the better the system performance and the closer the system response with and without time delay in the loop. It should be pointed out that the adoption of a $D^{\frac{1}{2}}$ controller is just for comparison purposes and, in fact, the development of systematic design procedures for FDI-based algorithms is currently under investigation.

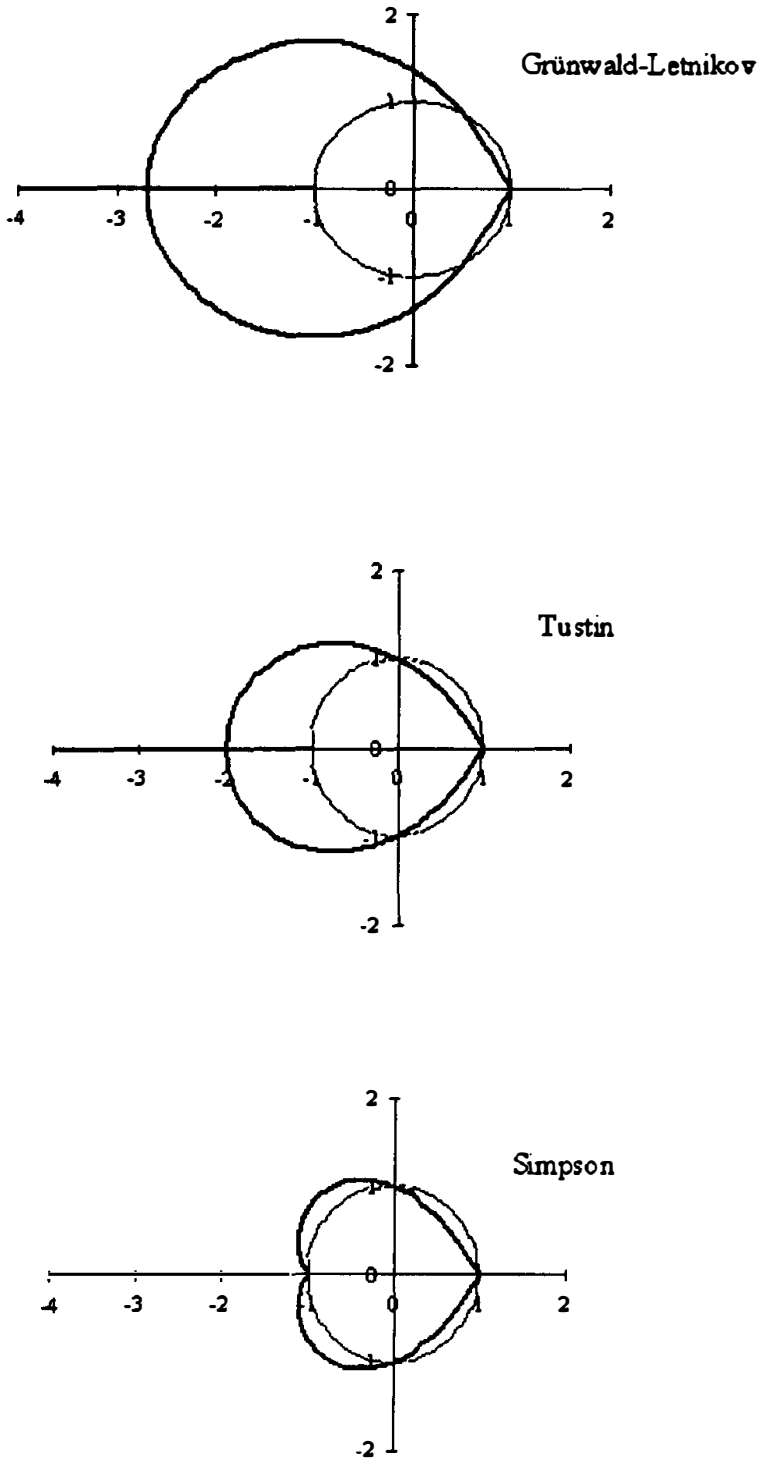


Figure 7: Root-locus for $G(s) = e^{-sT_D}/Ms^2$ $(M = 1, T_D = 0)$ under the control of a infinite series $D^{\frac{1}{2}}$ algorithm based on the approach of Grünwald-Letnikov, Tustin and Simpson

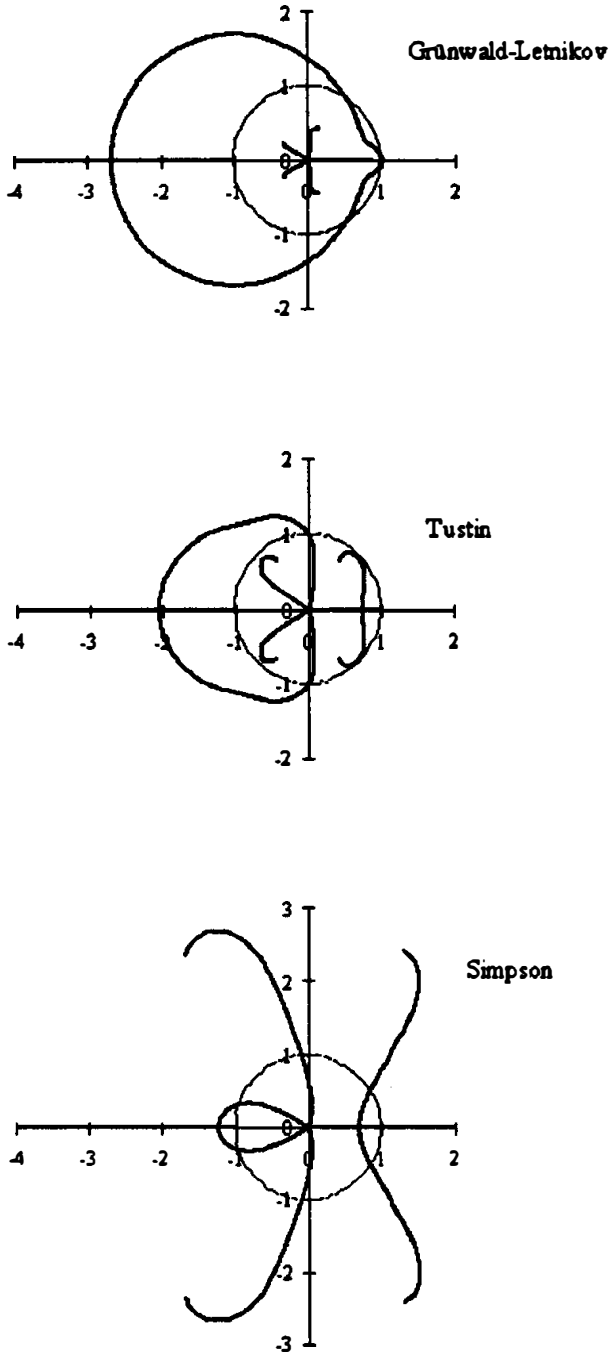


Figure 8: Root-locus for $G(s) = e^{-sT_D}/Ms^2$ $(M = 1, T_D = 0)$ under the control of a n = 5 truncation order series of $D^{\frac{1}{2}}$ algorithm based on the approach of Grünwald-Letnikov, Tustin and Simpson

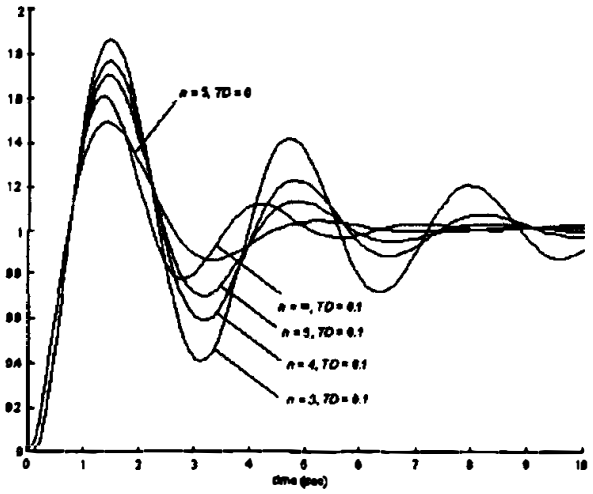

Figure 9: Time response for $G(s) = e^{-sT_D}/Ms^2$ ($M = 1, T_D = 0$ and $T_D = 0.1$) under the control of a Grünwald-Letnikov-based approximation of $D^{\frac{1}{2}}$ with truncation orders of $n = 3$, 4 and 5.

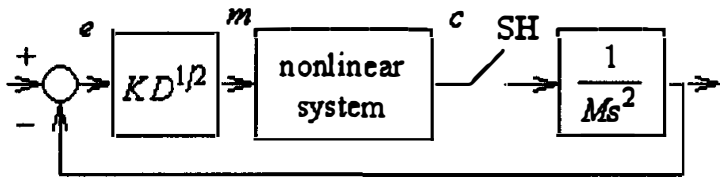


Figure 10: A $D^{\frac{1}{2}}$ controller for a system with a mass $M$ and a nonlinear actuator.

## 5. Performance of FDI approximations in nonlinear control systems

In order to investigate the robustness of the FDI-based control algorithms we introduced a nonlinear block in the forward path (**Figure 10**) and a system with a mass $M$.

Four different phenomena in the actuator are considered for this system: saturation (slope $\mu = 1, \Delta = 1$), deadzone (slope $\mu = 1, \lambda = 0.1$), hysteresis (slope $\mu = 1, \lambda = 0.1$) and relay ($\gamma = 1$) with the characteristics depicted in **Figure 11**. In all the cases, the parameters adopted in the experiments are $K = 10T^{\frac{1}{2}}$, $M$=1 and a sample and hold time $T$=0.1.

**Figure 12** shows the linear system response (*i.e.* without the nonlinear block), for a unity step input and a $n^{th}$ order ($1 \leq n \leq 7$) approximation to $D^{\frac{1}{2}}$ according with (10). It is clear that the higher the order of the approximation the better the response.

The robustness of the fractional algorithm over classical control actions is highlighted in the presence of a nonlinear actuator. **Figure 13** shows that the response for the FDI controller with $n = 1$ is very sensitive to the saturation effect

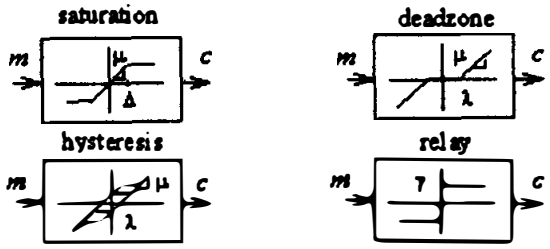


Figure 11: Nonlinear phenomena at the actuator: saturation ($\mu = 1, \Delta = 1$), deadzone ($\mu = 1, \lambda = 0.1$), hysteresis ($\mu = 1, \lambda = 0.1$) and relay ($\gamma = 1$).

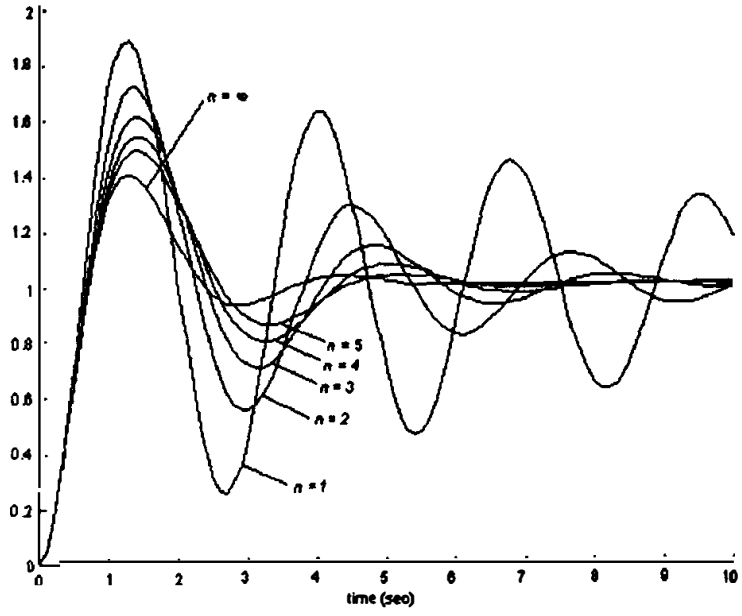


Figure 12: Linear system response for a $n = 1, \cdots, 7$ order truncation series approximation of $D^{\frac{1}{2}}$.

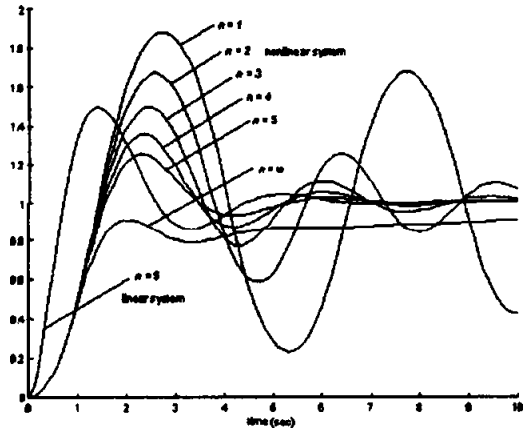

Figure 13: System response with saturation, for a $n = 7$ order truncation series approximation of $D^{\frac{1}{2}}$.

while for higher values of $n$ becomes more stable. In the same line of thought, **Figures 14, 15 and 16** reveal that the $n = 7$ truncation order approximation to the $D^{\frac{1}{2}}$ controller is robust for a large range of nonlinear phenomena. By other words, we get a better approximation to the fractional-derivative the higher the order of series truncation. In this line of thought, for comparison, the charts depict also the systems responses for $n = \infty$. Nevertheless, from the perspective of controller performance, the tuning of $K, n$ and $\alpha$ require an optimization which will depend on the system dynamics. A systematic procedure for the algorithm design in the presence on non-linear phenomena needs still further research.

## 6. Conclusions

The recent progress in the area of chaos reveals promising aspects for future developments and application of the theory of fractional calculus. In the area of automatic control some preliminary work has been proposed but the algorithms are restricted to the frequency domain. In this paper several methods for the discrete-time FDI approximation were presented and compared. The new algorithms adopt the time domain, making them well adapted for $z$-transform analysis and computer calculation. The properties of the Grünwald-Letnikov, Tustin and Simpson schemes are studied in terms of robustness and system stability, revealing that the first approach is preferable. For a simple prototype system the control algorithms based on the fractional-order concepts are simple to implement and reveal good robustness.

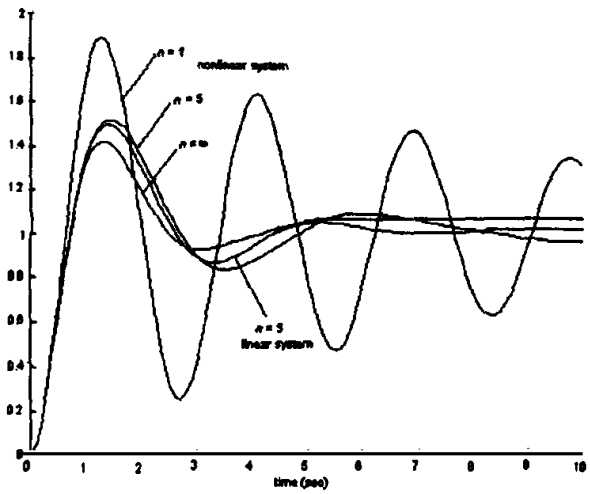


Figure 14: System response with deadzone, for a $n = 7$ order truncation series approximation of $D^{\frac{1}{2}}$.

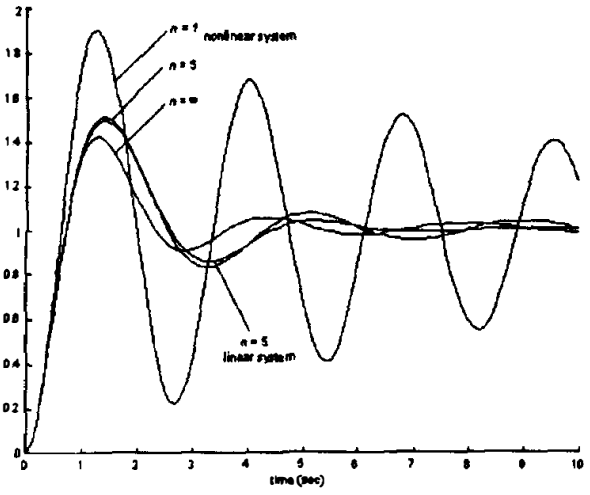


Figure 15: System response with hysteresis, for a $n = 7$ order truncation series approximation of $D^{\frac{1}{2}}$.

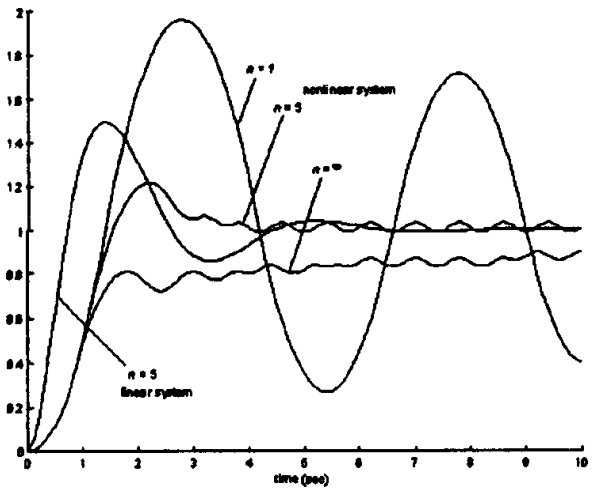


Figure 16: System response with relay, for a $n = 7$ order truncation series approximation of $D^{1/2}$ .

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