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Discrete-time modeling of Hamiltonian systems

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Abstract: The problem of discrete-time modeling of the lumped-parameter Hamiltonian systems is considered for engineering applications. Hence, a novel gradient-based method is presented, exploiting the discrete gradient concept and the forward Euler discretization under the assumption of the continuous Hamiltonian model is known. It is proven that the proposed discrete-time model structure defines a symplectic difference system and has the energy-conserving property under some conditions. In order to provide alternate discrete-time models, 3 different discrete-gradient definitions are given. The proposed models are convenient for the design of sampled-data controllers. All of the models are considered for several well-known Hamiltonian systems and the simulation results are demonstrated comparatively.

Key words: Hamiltonian systems, discrete-time control model, discrete gradient

1. Introduction

Hamiltonian systems are a well-known formalism for the modeling and analysis of some physical systems. Almost all electrical, electromechanical, and complex network systems with negligible dissipation can be described by a suitable Hamiltonian formalism. In the continuous-time literature, Hamiltonian system formalism has been extended by adding a dissipation structure and input and output ports, which yields a control model for lumped-parameter systems. This extended model structure is called a port-controlled Hamiltonian system. Note that systems with dissipation can be represented by this latter model structure. Therefore, the Hamiltonian formalism could be regarded as the most appropriate mathematical tool for systems in physical and engineering sciences. The fundamental theory of continuous Hamiltonian systems can be found in [1–3] and the references cited therein.

The widespread use of digital computers needs the discretization of continuous Hamiltonian systems for the analysis of discrete physical problems and the digital control of the complex systems, especially for systems where electrical and mechanical subsystems have to be considered together [4–6].

In mathematics literature, discrete Hamiltonian systems are considered for different purposes. In some of these works, discrete Hamiltonian systems are considered under the titles of ‘symplectic difference systems’ or ‘discrete symplectic systems’ and many analyses have been carried out for these systems [7–10]. Aside from these, some works on this subject dealt with the numerical computation of Hamiltonian dynamics and focused on integration methods. The survey in [11] summarized the already existing integration methods thoroughly, and the integration methods for Hamiltonian systems can be found in [12]. It should be noted that an alternative approach to the modeling and simulation of port-Hamiltonian systems, Hamiltonian systems with input and

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output, were offered in [13], where direct modeling at the discrete level was considered.

The existing methods in the mathematical literature are on the numerical modeling of Hamiltonian systems, most of which are too difficult to employ for digital control design purposes. Therefore, the derivation of a discrete-time model convenient for sampled-data control of the lumped-parameter Hamiltonian systems is considered in this study and a method based on the discrete gradient concept is presented, and the forward Euler discretization is chosen for the differential operator. It is shown that the discrete-time model obtained with the method proposed here defines a symplectic difference system and has an energy-conserving property under some conditions. Furthermore, 3 different discrete-gradient definitions are given to provide alternate discrete time models, all of the models are obtained for well-known Hamiltonian systems, and the simulation results are compared.

The proposed models are uncomplicated and practical to use in the design of controllers for engineering applications, whereas the existing numerical integration methods are complex and inconvenient for engineering applications. It might be noted that the discrete time models proposed here were used in the sampled-data control of port-Hamiltonian systems in the sense of passivity-based control and disturbance attenuation in [14–18], respectively. It should be emphasized that [17] reported a real application where the proposed model was easily and successfully utilized.

2. Preliminaries

For the reader's convenience, some definitions constantly used in the literature on Hamiltonian systems, discrete-gradient conditions [12], and quadratic approximation lemma [19] are restated herein.

The lumped-parameter standard Hamiltonian systems are defined as:

$$\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{p}}\mathbf{H}(\mathbf{q},\mathbf{p}) \\ -\nabla_{\mathbf{q}}\mathbf{H}(\mathbf{q},\mathbf{p}) \end{bmatrix}, \tag{1}$$

where $(\mathbf{q},\mathbf{p}) \in \mathbf{X} \subset \Re^{\mathbf{2n}}$ is a *2n*-dimensional manifold; $\boldsymbol{q}$ is the vector of the generalized positions; $\boldsymbol{p}$ is the vector of the generalized momentums; $H(q,p)$ is the Hamiltonian function, namely the total energy of the system; and the notation $\nabla_{(\bullet)}H$ is used to represent the gradient vector of the scalar function of H with respect to $(\bullet)$. As is known, $H(q,p)$ can be written in the following form:

$$\begin{aligned} H(\mathbf{q},\mathbf{p}) &= \tfrac{1}{2}\mathbf{p}^T\mathbf{M}^{-\mathbf{1}}(\mathbf{q})\mathbf{p} + V(\mathbf{q}) \\ &= \tfrac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\,\dot{\mathbf{q}} + V(\mathbf{q}) \end{aligned},$$

where $V(\mathbf{q})$ is the potential energy and $\mathbf{M}(\mathbf{q})$ is a symmetric and positive generalized inertia matrix. If $\mathbf{M}(\mathbf{q}) = \mathbf{M} \in \Re^{n\times n}$, the system is called a 'separable Hamiltonian system'; otherwise, it is called a 'nonseparable Hamiltonian system'. On the other hand, a Hamiltonian system with dissipation can be given as:

$$\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{bmatrix} = (\mathbf{J} - \mathbf{R}(\mathbf{q},\mathbf{p}))\nabla\mathbf{H}(\mathbf{q},\mathbf{p}), \tag{2}$$

where matrices J and $\mathbf{R}(\mathbf{q},\mathbf{p}) \in \Re^{\mathbf{2n}\times\mathbf{2n}}$ are the standard skew-symmetric matrix and the nonnegative symmetric matrix, respectively, given as:

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \quad \mathbf{R}(\mathbf{q},\mathbf{p}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R_1}(\mathbf{q},\mathbf{p}) \end{bmatrix}, \quad \mathbf{R_1}(\mathbf{q},\mathbf{p}) \geq 0.$$

In local coordinates, Hamiltonian systems with dissipation are defined as follows:

$$\dot{\mathbf{x}}(t) = [\mathbf{J}(\mathbf{x}) \; - \; \mathbf{R}(\mathbf{x})] \, \nabla \mathbf{H}(\mathbf{x}), \tag{3}$$

where $\mathbf{x}(t) \in \Re^n$ is the state vector, matrices $\mathbf{J}(\mathbf{x}) = \; -\mathbf{J}^T(\mathbf{x})$ and $\mathbf{R}(\mathbf{x}) = \mathbf{R}^{\mathbf{T}}(\mathbf{x}) \geq 0$ respectively determine the internal connection structure and the dissipation structure of the system, and, finally, $H(x)$ is the total energy function of system Eq. (1).

In the mathematics literature, discrete Hamiltonian systems are investigated under the titles of 'symplectic difference systems' or 'discrete symplectic systems' [7–10]. The formal definition for these systems is given below.

Definition 1 *Given a (2n × 2n)-dimension discrete system:*

$$\mathbf{z}(k+1) = \mathbf{S_k} \, \mathbf{z}(k), \quad k \in \mathbf{I}. \tag{4}$$

It is called a 'symplectic difference system' provided that for all k *,* $\mathbf{S_k}$ *are symplectic matrices, i.e. if* $\mathbf{S_k^T} \mathbf{J} \, \mathbf{S_k} = \mathbf{J}$ *or equivalently* $\mathbf{S_k} \, \mathbf{J} \, \mathbf{S_k^T} = \mathbf{J}$ *, where*

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \tag{5}$$

is a standard skew-symmetric matrix, namely $\mathbf{J} = \; -\mathbf{J^T}$*.*

Many properties of symplectic matrices can be found, e.g., in [20]. In particular, each symplectic matrix $\mathbf{S}$ has an inverse, such that $\mathbf{S^{-1}} = \mathbf{J} \, \mathbf{S^T} \mathbf{J^T}$, and matrices $\mathbf{S^T}$, $\mathbf{S^{-1}}$ are symplectic matrices.

As mentioned previously, in this study, several gradient-based discrete models are presented. To provide these alternate models, the discrete gradient definition given in [12] and restated in this section is considered.

Definition 2 *Let* $H(x)$ *be a differentiable scalar function in* x *, where* $\bar{\nabla} \mathbf{H}(\mathbf{x_k}, \mathbf{x_{k+1}})$ *is a discrete gradient of* $H(x)$ *if it is continuous in* $\boldsymbol{x}$ *and*

$$\begin{array}{l} \bar{\nabla}^{\mathbf{T}} \mathbf{H}(\mathbf{x_k}, \mathbf{x_{k+1}}) \; [\mathbf{x_{k+1}} - \mathbf{x_k}] \; = H(\mathbf{x_{k+1}}) - H(\mathbf{x_k}) \\ \bar{\nabla} \mathbf{H}(\mathbf{x_k}, \mathbf{x_k}) = \nabla \mathbf{H}(\mathbf{x_k}) \end{array} , \tag{6}$$

in which the gradient of $H(x)$ *defined as:*

$$\nabla H(x) := grad \, H(x) = \begin{bmatrix} \frac{\partial H}{\partial x^1} & \cdots & \frac{\partial H}{\partial x^n} \end{bmatrix}^T, \quad x = [x^1, \ldots, x^n]. \tag{7}$$

In this study, a systematic method to construct a discrete gradient for energy functions in general is also presented, so the quadratic approximation lemma, which is used to construct this generic discrete gradient, is restated below.

Lemma 2 (quadratic approximation) [19]

If and only if the quadratic function $f(\mathbf{z})$ *is defined by*

$$f(\mathbf{z}) = a_0 \, + \, \mathbf{a^T} \mathbf{z} \, + \, \frac{1}{2} \mathbf{z^T} \mathbf{A} \, \mathbf{z} \quad , \quad \mathbf{A^T} = \mathbf{A}, \tag{8}$$

then

$$f(\mathbf{z_1}) - f(\mathbf{z_0}) = \frac{1}{2}\left[\nabla\mathbf{f}(\mathbf{z_1}) + \nabla\mathbf{f}(\mathbf{z_0})\right]^T(\mathbf{z_1} - \mathbf{z_0}), \tag{9}$$

where $\nabla\mathbf{f}(\mathbf{z_r})$ *is the gradient vector of* $f(\mathbf{z})$ *evaluated at* z_r.

3. Discrete-time modeling

In order to obtain the discrete-time models of Hamiltonian systems, a gradient-based method is presented in this section. The discrete-time models obtained define symplectic difference systems and have an energy-conserving property for some discrete gradients with a special form and for some continuous systems with a specific structure. Moreover, 3 different discrete-gradient definitions are given in the sequel.

Consider the continuous-time Hamiltonian systems with dissipation:

$$\dot{x}(t) = [J(x) - R(x)]\,\nabla H(x)\,, \tag{10}$$

where $x \in \Re^n$ denotes the state vector and $J(x)=-J^T(x)$, $R(x)=R^T(x) \geq 0$. The notation $\nabla\mathbf{H}(\mathbf{x})$ is used to represent the gradient vector of the scalar function of $H(\mathbf{x})$ with respect to $\boldsymbol{x}$.

A gradient-based discrete-time model of the Hamiltonian system given in Eq. (10) can be constructed using the discrete gradient of $H(\mathbf{x})$ given in Definition 2.

When the derivatives of state variables in Eq. (10) is approximated by forward Euler as

$$\dot{\mathbf{x}} \cong \frac{\mathbf{x_{k+1}} - \mathbf{x_k}}{T} \tag{11}$$

and the discrete gradient $\bar{\nabla}\mathbf{H}(\mathbf{x}(k), \mathbf{x}(k+1))$ is substituted for the gradient term $\nabla\mathbf{H}(\mathbf{x})$, the gradient-based discrete-time model of the Hamiltonian system is obtained as follows:

$$x_{k+1} - x_k = T\,[J(x_k) - R(x_k)]\,\bar{\nabla}H, \tag{12}$$

where T is the sampling period.

For $\mathbf{R}(\mathbf{x_k}) = \mathbf{0}$, the system in Eq. (12) will have an energy conservation property if the discrete gradient of $H(\mathbf{x})$, i.e. $\bar{\nabla}\mathbf{H}$, satisfies the conditions in Definition 2, and it will define a 'symplectic difference system' when matrix $\mathbf{J}(\mathbf{x}_k) = \mathbf{J}$ is the standard skew-symmetric matrix given in Eq. (5) and the discrete gradient used has some special properties. These properties are given as a theorem in the sequel. Proposition 1 will be used to prove the theorem on the symplectic property of Eq. (12).

Proposition 1 *If matrix* F *is in the form of*

$$\mathbf{F} = [\mathbf{I} - \mathbf{JP}]^{-1}[\mathbf{I} + \mathbf{JP}]\,, \tag{13}$$

such that J *is the standard skew-symmetric matrix given in Eq. (5) and* P *is a symmetric matrix, i.e.* $\mathbf{P} = \mathbf{P^T}$, *then matrix* $\mathbf{F}$ *is a symplectic matrix.*

Proof It is easily shown that matrix $\mathbf{F}$ given in Eq. (13) holds $\mathbf{F\,J\,F^T} = \mathbf{J}$, so the proof is complete. □

Theorem 1 *Consider the Hamiltonian system without dissipation* $\dot{\mathbf{x}} = \mathbf{J}\,\nabla\mathbf{H}(\mathbf{x})$ *and if the discrete gradient of* $H(\mathbf{x})$ *has the following structure:*

$$\bar{\nabla}\mathbf{H}(\mathbf{x}) = \Phi_k\ [\mathbf{x_{k+1}} + \mathbf{x_k}]\,. \tag{14}$$

Next, the discrete-time model of this system is given in the following form:

$$x_{k+1} = F_k\, x_k, \tag{15a}$$

in which

$$\mathbf{F_k} = [\mathbf{I} - T\mathbf{J}\Phi_k]^{-1}\,[\mathbf{I} + T\mathbf{J}\Phi_k], \tag{15b}$$

where T *is the sampling period, and it then defines a symplectic difference system.*

Proof When the discrete gradient given in Eq. (14) is substituted into the discrete model given in Eq. (10) for $\mathbf{R}(\mathbf{x_k}) = \mathbf{0}$, the following relation is easily written as

$$x_{k+1} = T\,J\,F_k\ [x_{k+1} + x_k]\ + x_k,$$

and, after some algebraic operations, the difference equation given in Eq. (15) is obtained. The symplectic property of the system is proven using Proposition 1. □

Furthermore, the following theorem is on the energy-conserving property of the discrete model given in Eq. (12) for $\mathbf{R}(\mathbf{x_k}) = \mathbf{0}$.

Theorem 2 *Consider the Hamiltonian system without dissipation* $\dot{x} = J(x)\,\bar{\nabla}H(x)$ *and assume that the discrete gradient of* $H(\mathbf{x})$*, i.e.* $\bar{\nabla}\mathbf{H}$*, satisfies the conditions in Definition 2 precisely. Next, the following discrete system has the energy conservation property:*

$$x_{k+1} - x_k = \ T\,J(x_k)\,\bar{\nabla}H, \tag{16}$$

where T *is the sampling period.*

Proof If Eq. (16) is multiplied from the left by $\bar{\nabla}^{\mathbf{T}}\mathbf{H}$, one can then write the following relation:

$$\bar{\nabla}^T H\ \ [x_{k+1} - x_k]\ =\ T\,\bar{\nabla}^T H\,J(x_k)\,\bar{\nabla}H. \tag{17}$$

From the first condition of Definition 2, the following relation can be obtained:

$$\frac{H(x_{k+1}) - H(x_k)}{T} = \bar{\nabla}^{\mathbf{T}} H\,J(x_k)\,\bar{\nabla}H = 0. \tag{18}$$

The skew symmetry property of $\mathbf{J}(\mathbf{x_k})$ proves the claim. □

The following remark, which can be stated as a consequence of Theorems 1 and 2, is worth mentioning.

Remark 1 The discrete-time model obtained using the method proposed here defines a symplectic difference system whenever the form of the structure matrix of the system is the standard skew-symmetric matrix given in Eq. (5) and the discrete gradient of $H(\mathbf{x})$ has the property given in Eq. (14). However, the discrete time model always has the energy conservation property if the discrete gradient of $H(\mathbf{x})$ satisfies the first condition of Definition 2.

It is easily seen that the discrete time model of the Hamiltonian system with dissipation, namely $\mathbf{R}(\mathbf{x_k}) \neq \mathbf{0}$, which is obtained using the method proposed in this study, is obtained as

$$\mathbf{x_{k+1}} = \left[\mathbf{I} - T\left(\mathbf{J_k} - \mathbf{R_k}\right)\Phi_k\right]^{-1}\left[\mathbf{I} + T\left(\mathbf{J_k} - \mathbf{R_k}\right)\Phi_k\right]\mathbf{x_k}, \tag{19}$$

where $\mathbf{J_k} = \mathbf{J}(\mathbf{x_k})$ and $\mathbf{R_k} = \mathbf{R}(\mathbf{x_k})$.

In the sequel, 2 different discrete gradients in the form of Eq. (14) are defined such that they yield some discrete-time models, which have the structure given in Eq. (15) or Eq. (19). These discrete-gradient definitions can be used for a class of the energy functions that have a continuous gradient in the form of $\nabla\mathbf{H}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})\mathbf{x}$. Afterwards, a methodology to remove the constraint on the energy function and to obtain a generic discrete gradient is also given in this section.

The mean value theorem and the first condition of the discrete gradient imply that 2 satisfactory discrete gradients in the form of Eq. (14), which provide 2 slightly different discrete-time models in the form of Eq. (15), can be defined as follows.

Definition 3 *Consider a differentiable function in* $\boldsymbol{x}$ *given as* $H(\mathbf{x})$ *and its gradient given in the form of* $\nabla\mathbf{H}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})\mathbf{x}$*; the discrete gradient of* $H(\mathbf{x})$ *is then defined as*

$$\bar{\nabla}\mathbf{H}(\mathbf{x_k}, \mathbf{x_{k+1}}) = \Phi_k\ \left[\mathbf{x_{k+1}} + \mathbf{x_k}\right], \tag{20}$$

where

$$\Phi_k\ = \Phi(\mathbf{x_k}, \mathbf{x_{k+1}}) = \frac{1}{4}\left[\mathbf{Q}(\mathbf{x_{k+1}}) + \mathbf{Q}(\mathbf{x_k})\right]. \tag{21}$$

Definition 4 *Consider a differentiable function in* $\boldsymbol{x}$ *given as* $H(\mathbf{x})$ *and its gradient given in the form of* $\nabla\mathbf{H}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})\mathbf{x}$*; the discrete gradient of* $H(\mathbf{x})$ *is then defined as*

$$\bar{\nabla}H(x_k, x_{k+1}) = \Phi_k\ \left[x_{k+1} + x_k\right], \tag{22}$$

where

$$\Phi_k = \frac{1}{2}Q(x)|_{\frac{x_{k+1}+x_k}{2}}. \tag{23}$$

Remark 2 The discrete gradients given in Definitions 3 and 4 satisfy both of the conditions given in Definition 2 exactly when the energy function is in the form of $H(x) = x^T Z(x)\, x$ with $Z(x)=Z$, namely a constant $n \times n$ matrix. This case corresponds to nonseparable mechanical systems with quadratic potential energy. However, in general, these discrete gradients do not precisely satisfy the first condition given in Eq. (6). Therefore, the discrete-time models obtained using these definitions do not have an energy conservation property but it can easily be shown that the residual energy is insignificant for small sampling periods.

Thus far, we have defined 2 different discrete gradients to obtain discrete-time models. These discrete gradients have some drawbacks since they need a special form of energy function and/or gradient of the energy function. In the following, a systematic method is presented to construct a discrete gradient for general energy functions using the quadratic approximation lemma, which was restated in Section 2.

Considering the analogy between the first condition of Definition 2, i.e. Eq. (6), and the relation in Eq. (9) in the quadratic approximation lemma, one can conclude that there exists a discrete gradient that exactly satisfies the condition in Eq. (6) if the energy function $H(\mathbf{x})$ has the general quadratic form as in Eq. (8). Consequently, the second-order Taylor approximation of any $H(\mathbf{x})$ might be used to define a discrete gradient as follows:

$$\bar{\nabla} H = \frac{1}{2}\left[\nabla \tilde{H}(x_{k+1}) + \nabla \tilde{H}(x_k)\right],$$

where $\tilde{H}(\mathbf{x})$ is the second-order Taylor approximation of $H(\mathbf{x})$, namely

$$\tilde{H}(x) = H(x_{\mathbf{k}}) + \nabla^{\mathbf{T}} H(x_k)\,(x - x_k) + \frac{1}{2}(x - x_k)^T H_{Hess}(x_k)\,(x - x_k) \tag{24}$$

for $x_k \leq x < x_{k+1}$. As a consequence of the above analysis, the definition below is presented.

Definition 5 *Consider a differentiable function in* $\boldsymbol{x}$ *given as* $H(\mathbf{x})$*, and then the discrete gradient of a* $H(x)$ *is defined as*

$$\overline{\nabla} H(x_k, x_{k+1}) = \nabla H(x_k) + \frac{1}{2} H_{Hess}(x_k)\,(x_{k+1} - x_k), \tag{25}$$

in which $\mathbf{H_{Hess}}(\mathbf{x})$ *is the Hessian matrix of the energy function of* $H(\mathbf{x})$*.*

When the discrete gradient given in Eq. (25) is substituted into Eq. (12), the following discrete-time model is obtained after a few algebraic operations:

$$x_{k+1} = x_k + T\,\left[I - \frac{T}{2}\,(J_k - R_k)\,H_{Hess}(x_k)\right]^{-1} (J_k - R_k)\,\nabla H(x_k), \tag{26}$$

where $J_k = J(x_k)$ and $R_k = R(x_k)$.

It should be noted that the energy function of the discrete system given in Eq. (26) is $\tilde{H}(\mathbf{x_k})$ and this energy is conserved for $R\,(x_k) = 0$. Moreover, it is easily verified that the model given in Eq. (26) defines a symplectic difference system when the skew-symmetric matrix $\mathbf{J}\,(\mathbf{x_k})$ is in the standard form given in Eq. (5) and the energy function is in the form of $H(x) = x^T Z(x)\, x$ with $Z(x) = Z$, namely a constant $n \times n$ matrix, although this discrete system is not a symplectic difference system in general.

Remark 3 The discrete-time models for port-controlled Hamiltonian systems can be given by

$$\begin{array}{l} \mathbf{x_{k+1}} - \mathbf{x_k} = T\ [\mathbf{J}(\mathbf{x_k}) - \mathbf{R}(\mathbf{x_k})]\ \bar{\nabla}\mathbf{H} + TG(x_k)u_k \\ y_k = G^T(x_k)\bar{\nabla}\mathbf{H} \end{array}, \tag{27}$$

where $\bar{\nabla}\mathbf{H}$ is the discrete gradient. The explicit models can be obtained by substituting the discrete gradients in Definitions 3, 4, and 5 presented in this study for the discrete-gradient term.

In the following section, all 3 of these models are verified on well-known Hamiltonian systems and their performances are compared.

4. Numerical experiments

This section is devoted to simulations in order to validate the discrete-time models that are obtained using the methods proposed in this study. For this purpose, it may be reasonable to give a brief review of these methods, as follows.

- The first 2 models are in the following form, from Definition 3 and 4, under the assumption of $\nabla\mathbf{H}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})\,\mathbf{x}$:

$$\mathbf{x_{k+1}} = \left[\mathbf{I} - T\,(\mathbf{J_k} - \mathbf{R_k}\,)\,\Phi_k\right]^{-1}\left[\mathbf{I} + T\,(\mathbf{J_k} - \mathbf{R_k})\,\Phi_k\right]\,\mathbf{x_k},$$

for the following Φ_k:

1. $\mathbf{\Phi_k} \; = \mathbf{\Phi}(\mathbf{x_k}, \mathbf{x_{k+1}}) = \frac{1}{4}\left[\mathbf{Q}(\mathbf{x_{k+1}}) + \mathbf{Q}(\mathbf{x_k})\right],$
2. $\mathbf{\Phi}_k = \frac{1}{2}\mathbf{Q}\;(\mathbf{x})|_{\frac{\mathbf{x_{k+1}}+\mathbf{x_k}}{\mathbf{2}}}\;.$

- The model obtained using Definition 5 is as follows:

$$\mathbf{x_{k+1}} = \mathbf{x_k} + T\;\left[\mathbf{I} - \frac{T}{2}\;(\mathbf{J_k} - \mathbf{R_k}\,)\;\mathbf{H_{Hess}}(\mathbf{x_k})\right]^{-1}(\mathbf{J_k} - \mathbf{R_k}\,)\;\nabla\mathbf{H}(\mathbf{x_k}),$$

in which $\mathbf{H_{Hess}}(\mathbf{x})$ is the Hessian matrix of the energy function of $H(\mathbf{x})$.

It should be noted that it is used as a simple approximation for $\mathbf{x_{k+1}}$ in the calculation of $\mathbf{\Phi_k} = \Phi(\mathbf{x_{k+1}}, \mathbf{x_k}) \cong \Phi(\hat{\mathbf{x}}_{\mathbf{k+1}}, \mathbf{x_k})$ as $\mathbf{x_{k+1}} \cong \hat{\mathbf{x}}_{\mathbf{k+1}} = \mathbf{F}(\mathbf{x_{k-1}})\mathbf{x_k}$ for the first 2 models.

Example 1 (van der Pol oscillator)

As is well known, the dynamic equations of the van der Pol oscillator are as given below:

$$\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = \mu\,(1 - x_1^2)x_2 - \,x_1 \end{array}\;.$$

The Hamiltonian model of the system can be given as follows:

$$\dot{\mathbf{x}} = \left[\mathbf{J} - \mathbf{R}(\mathbf{x})\right]\;\nabla\mathbf{H}\,,$$

where

$$H(\mathbf{x}) = \frac{1}{2}\mathbf{x^T}\mathbf{I_2}\,\mathbf{x}$$

is the energy function of the system and

$$\mathbf{R}(\mathbf{x}) = \left[\begin{array}{ll} 0 & 0 \\ 0 & -\mu\,(1 - x_1^2) \end{array}\right]$$

is the dissipation structure. Note that the gradients of $H(\mathbf{x})$ can be written as

$$\nabla\mathbf{H} = \mathbf{Q}(\mathbf{x})\mathbf{x} = \mathbf{I_2}\,\mathbf{x},\;\;\mathbf{Q}(\mathbf{x}) = \mathbf{P}(\mathbf{x}) = \mathbf{I_2}\;=\;\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right].$$

In this example, $\mu = 1$ is considered and the sampling period is taken as $T = 0.01$. Since the matrix $\mathbf{Q}(\mathbf{x})$ is constant, the 3 proposed discrete gradients are equivalent for this special case. Calculations for the proposed discrete gradients and corresponding discrete-time models are given in the sequel.

The discrete gradient proposed in Definition 3 is obtained as follows:

$$\begin{aligned}
\mathbf{\Phi_k} &= \mathbf{\Phi}(\mathbf{x_k}, \mathbf{x_{k+1}}) = \tfrac{1}{4}\left[\mathbf{Q}(\mathbf{x_{k+1}}) + \mathbf{Q}(\mathbf{x_k})\right] = \tfrac{1}{4}\left[I_2 + I_2\right] = \tfrac{1}{2} I_2 \\
\bar{\nabla}\mathbf{H}(\mathbf{x_k}, \mathbf{x_{k+1}}) &= \Phi_k \ \left[\mathbf{x_{k+1}} + \mathbf{x_k}\right] = \tfrac{1}{2}\left[\mathbf{x_{k+1}} + \mathbf{x_k}\right];
\end{aligned}$$

thus, the corresponding discrete-time dynamics is obtained as:

$$\begin{aligned}
\mathbf{x_{k+1}} &= \left[\mathbf{I} - T\left(\mathbf{J_k} - \mathbf{R_k}\right)\Phi_k\right]^{-1}\left[\mathbf{I} + T\left(\mathbf{J_k} - \mathbf{R_k}\right)\Phi_k\right]\mathbf{x_k} \\
\mathbf{x_{k+1}} &= \left[\mathbf{I}_2 - 0.01 S_k\right]^{-1}\left[\mathbf{I}_2 + 0.01\, S_k\right]\mathbf{x_k} \qquad , \\
\mathbf{x_{k+1}} &= F(\mathbf{x_k})\,\mathbf{x_k}
\end{aligned} \tag{28}$$

where

$$S_k = S(x_k) = \left(\mathbf{J_k} - \mathbf{R_k}\right)\Phi_k = \left(\mathbf{J} - \begin{bmatrix} 0 & 0 \\ 0 & -\left(1 - x_1^2(k)\right) \end{bmatrix}\right)\frac{1}{2} I_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & \left(1 - x_1^2(k)\right) \end{bmatrix}.$$

Note that $\mathbf{x_{k+1}}$ does not appear on the left-hand side of Eq. (28). The expression in Eq. (28) is in the explicit form; therefore, it can be directly coded in MATLAB or other environments symbolically.

The discrete gradient proposed in Definition 4 is obtained as follows:

$$\begin{aligned}
\Phi_k &= \Phi(x_k, x_{k+1}) = \tfrac{1}{2} Q\,(x)\big|_{\frac{x_{k+1}+x_k}{2}} = \tfrac{1}{2} Q\left(\tfrac{x_{k+1}+x_k}{2}\right)\Big| = \tfrac{1}{2} I_2 \\
\bar{\nabla}\mathbf{H}(\mathbf{x_k}, \mathbf{x_{k+1}}) &= \Phi_k \ \left[\mathbf{x_{k+1}} + \mathbf{x_k}\right] = \tfrac{1}{2}\left[\mathbf{x_{k+1}} + \mathbf{x_k}\right].
\end{aligned}$$

This expression is the same as that the discrete gradient that Definition 3 yields; therefore, the system dynamics obtained will also be the same, which is as given in Eq. (28). Definition 5 yields the same discrete-gradient expression as above by the following calculations.

$$\begin{aligned}
H_{Hess}(x_k) &= I_2 \\
\overline{\nabla} H(x_k, x_{k+1}) &= \nabla H(x_k) + \tfrac{1}{2} H_{Hess}(x_k)\left(x_{k+1} - x_k\right) \\
&= I_2 x_k + \tfrac{1}{2} I_2 [x_{k+1} - x_k] = \tfrac{1}{2}\left[\mathbf{x_{k+1}} + \mathbf{x_k}\right]
\end{aligned}$$

Therefore, the simulations are run using the discrete-time model in Eq. (28). The results are given in Figure 1 for the initial state $x_0 = \left[\begin{array}{cc} 0.5; & 0.5 \end{array}\right]$.

The simulation results for $\mu = 1$ are given in Figure 1 for the initial state $x_0 = \left[\begin{array}{cc} 0.5; & 0.5 \end{array}\right]$ and sampling period $T = 0.01$s.

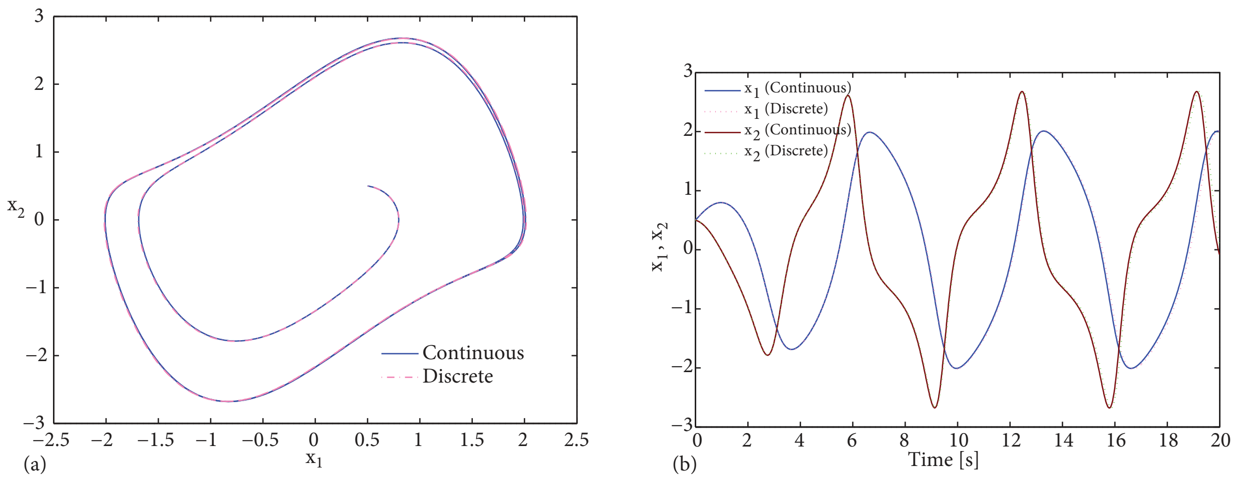

Figure 1. The discrete and continuous a) phase portraits and b) time responses of the van der Pol oscillator.

Example 2 (Henon–Heiles system)

The Henon–Heiles system is described by the following differential equations:

$$\begin{aligned} \ddot{x} &= -x - 2xy \\ \ddot{y} &= -y - x^2 + y^2 \end{aligned} .$$

The energy function of the Henon–Heiles system can be written as

$$H = \frac{1}{2}\,(p_x^2 + p_y^2) + V(x,y), \quad V(x,y) = \frac{1}{2}\,(x^2 + y^2) + x^2 y - \frac{1}{3} y^3$$

using the notation $p_x = \dot{x},\ p_y = \dot{y}$, where x, y correspond to the generalized positions and p_x, p_y correspond to the generalized momentums. Thus, the Hamiltonian model of the system can be given as:

$$\dot{x} = [J - R]\ \nabla H \quad ,$$

with $R = 0$. The calculation to obtain the discrete gradients and discrete-time models are given in the sequel. The simulations are run for a sampling period of $T = 0.002$s and the initial condition $x_0 = [0.0;\ 0.2;\ 0.03;\ \ 0.4]$, namely for the energy level $H = 0.0778$, and the results are given in Figures 2–4.

Note that $\nabla H(x)$ can be written in the form of $Q(x)x$, as given below:

$$\nabla H(x) = \begin{bmatrix} x_1 + 2x_1x_2 \\ x_2 + x_1^2 - x_2^2 \\ x_3 \\ x_4 \end{bmatrix} = Q(x)x = \begin{bmatrix} 1 & 2x_1 & 0 & 0 \\ x_1 & 1 - x_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} . \qquad (29)$$

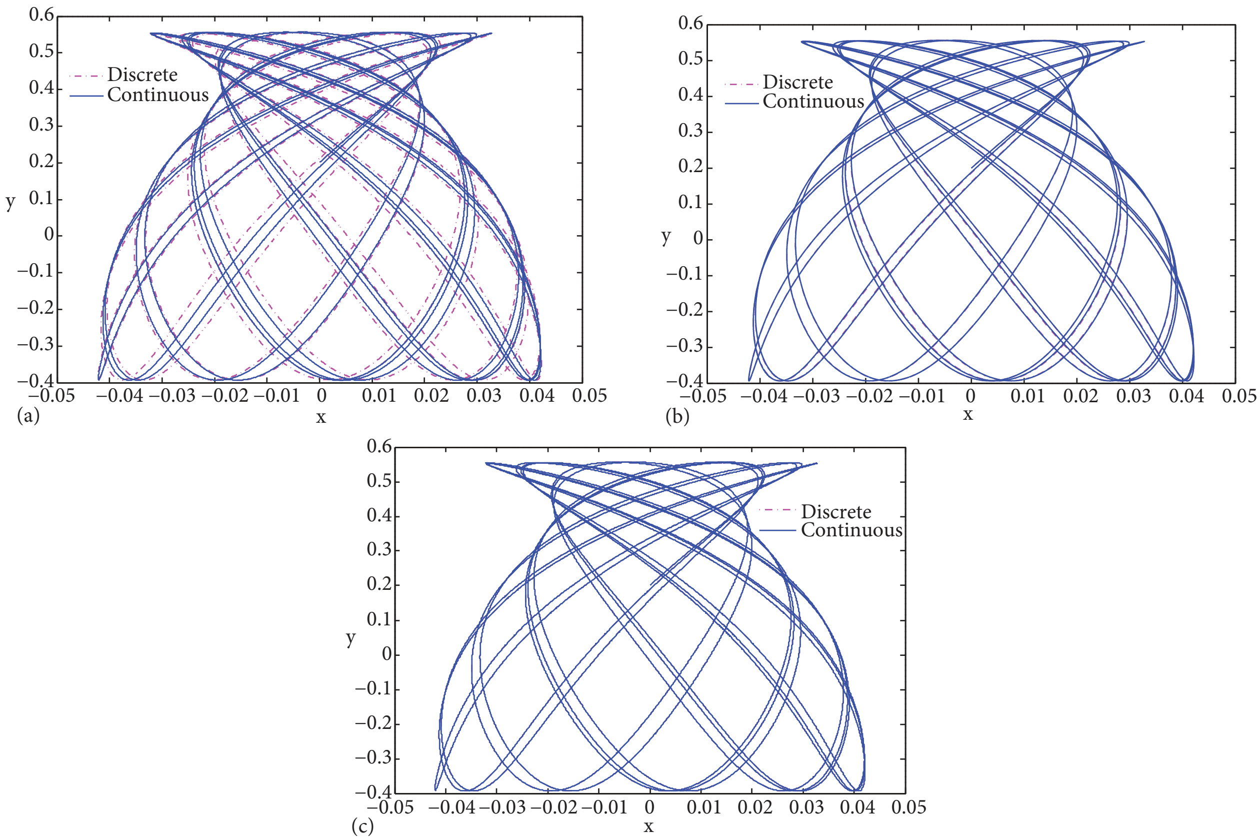

Figure 2. Phase portrait of the discrete and continuous $\boldsymbol{x}$ and $\boldsymbol{y}$ variables of the Henon–Heiles system for the discrete gradient given in Definition 3 (a), Definition 4 (b), and Definition 5 (c).

Hence, Definition 3 yields the following discrete gradient:

$$
\begin{aligned}
\bar{\nabla}\mathbf{H}(\mathbf{x_k}, \mathbf{x_{k+1}}) &= \mathbf{\Phi}_k\,[\mathbf{x_{k+1}} + \mathbf{x_k}] \\
&= \frac{1}{4}\begin{bmatrix} 2 & 2x_1(k+1)+2x_1(k) & 0 & 0 \\ x_1(k+1)+x_1(k) & 2-x_2(k+1)-x_2(k) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(k+1)+\mathbf{x}_1(k) \\ \mathbf{x}_2(k+1)+\mathbf{x}_2(k) \\ \mathbf{x}_3(k+1)+\mathbf{x}_3(k) \\ \mathbf{x}_4(k+1)+\mathbf{x}_4(k) \end{bmatrix}, \\
&= \frac{1}{4}\begin{bmatrix} 2\left(\mathbf{x}_1(k+1)+\mathbf{x}_1(k)\right) + 2\left(x_1(k+1)+x_1(k)\right)\left(\mathbf{x}_2(k+1)+\mathbf{x}_2(k)\right) \\ \left(x_1(k+1)+x_1(k)\right)^2 + \left(2-x_2(k+1)-x_2(k)\right)\left(\mathbf{x}_2(k+1)+\mathbf{x}_2(k)\right) \\ \mathbf{x}_3(k+1)+\mathbf{x}_3(k) \\ \mathbf{x}_4(k+1)+\mathbf{x}_4(k) \end{bmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_k &= \Phi(x_k, x_{k+1}) = \tfrac{1}{4}\left[Q(x_{k+1}) + Q(x_k)\right] \\
&= \frac{1}{4}\begin{bmatrix} 2 & 2x_1(k+1)+2x_1(k) & 0 & 0 \\ x_1(k+1)+x_1(k) & 2-x_2(k+1)-x_2(k) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

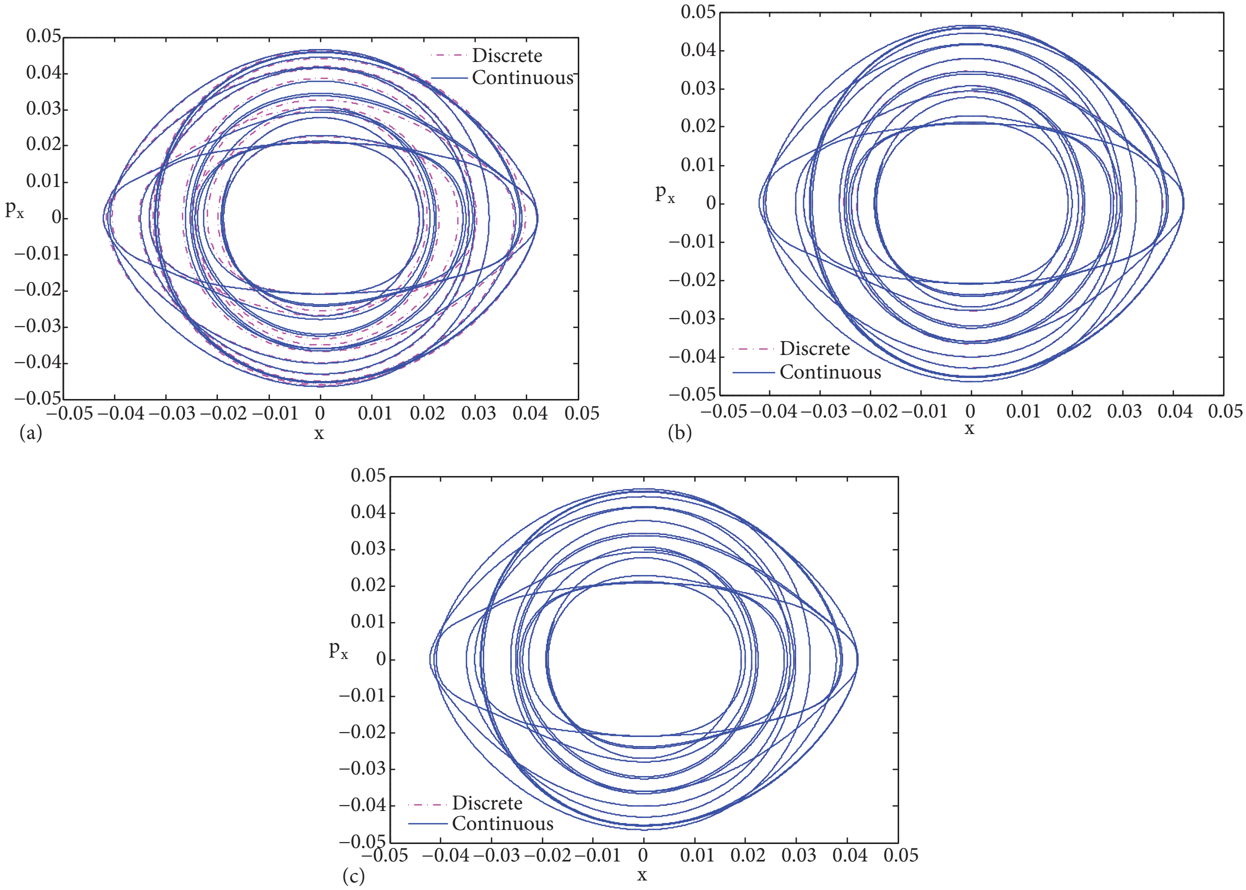

Figure 3. Phase portrait of the discrete and continuous $\boldsymbol{x}$ and $\boldsymbol{p}_x$ variables of the Henon–Heiles system for the discrete gradient given in Definition 3 (a), Definition 4 (b), and Definition 5 (c).

and

$$Q(x_{k+1}) = \begin{bmatrix} 1 & 2x_1(k+1) & 0 & 0 \\ x_1(k+1) & 1 - x_2(k+1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Q(x_k) = \begin{bmatrix} 1 & 2x_1(k) & 0 & 0 \\ x_1(k) & 1 - x_2(k) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, the corresponding discrete-time model is obtained as:

$$\mathbf{x_{k+1}} = \left[\mathbf{I} - T\left(\mathbf{J_k} - \mathbf{R_k}\right)\Phi_k\right]^{-1}\left[\mathbf{I} + T\left(\mathbf{J_k} - \mathbf{R_k}\right)\Phi_k\right]\mathbf{x_k}$$

$$\mathbf{x_{k+1}} = \left[\mathbf{I}_2 - 0.002 S_k\right]^{-1}\left[\mathbf{I}_2 + 0.002\, S_k\right]\mathbf{x_k}, \tag{30}$$

where

$$\begin{aligned} S_k = S(k) &= \left(J_{\mathbf{k}} - \mathbf{R_k}\right)\Phi_k = \mathbf{J}\Phi_k \\ &= \frac{1}{4}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -\left(2x_1(k+1) + 2x_1(k)\right) & 0 & 0 \\ -\left(x_1(k+1) + x_1(k)\right) & -\left(2 - x_2(k+1) - x_2(k)\right) & 0 & 0 \end{bmatrix}. \end{aligned} \tag{31}$$

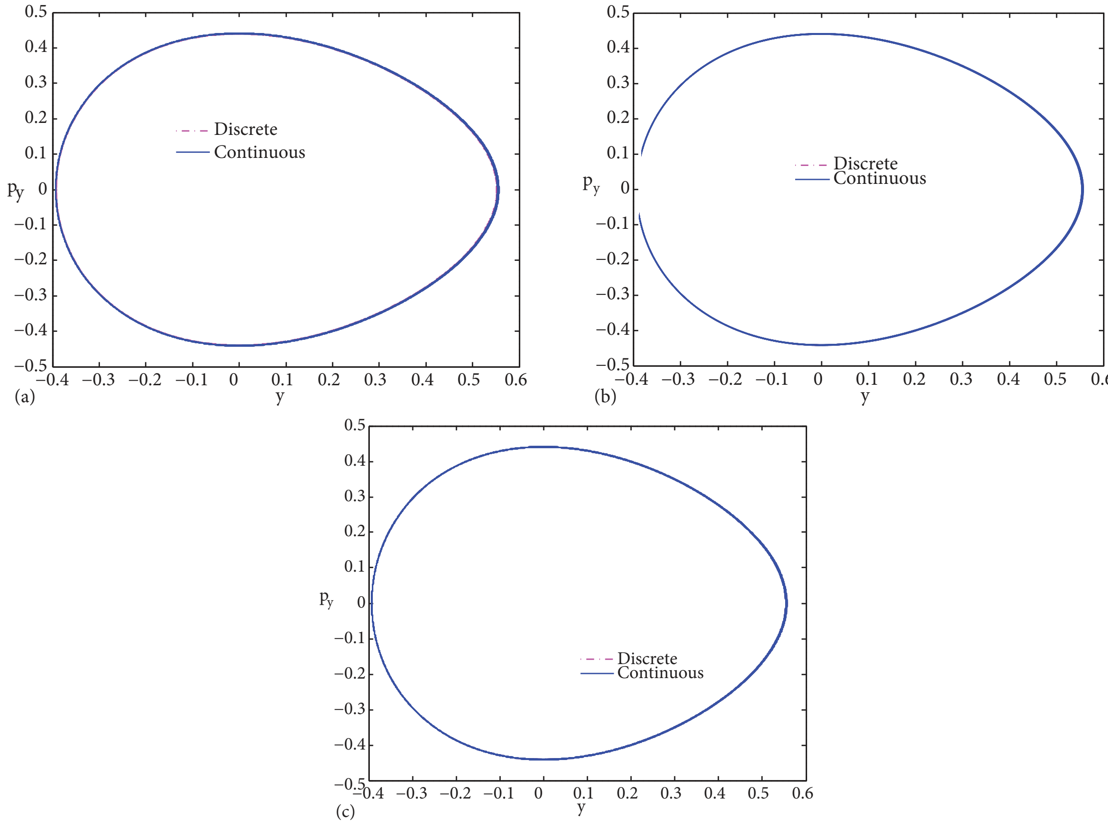

Figure 4. Phase portrait of the discrete and continuous $\boldsymbol{y}$ and $\boldsymbol{p}_y$ variables of the Henon–Heiles system for the discrete gradient given in Definition 3 (a), Definition 4 (b), and Definition 5 (c).

For the realization of this expression, the approximation $x_{k+1} \cong \hat{x}_{k+1} = F(x_{k-1})x_{\mathbf{k}}$ is used for $\mathbf{x_{k+1}}$ on the left-hand side of Eqs. (30) and (32).

Note that after obtaining $Q(x)$, as in Eq. (29), all of the above calculations can be performed in MATLAB by symbolically writing these expressions:

$$\begin{aligned}
&\Phi_k \;= \Phi(x_k, x_{k+1}) = \tfrac{1}{4}\left[Q(x_{k+1}) + Q(x_k)\right],\\
&\bar{\nabla}H(x_k, x_{k+1}) = \Phi_k \;\left[x_{k+1} + x_k\right],\\
&\mathbf{x_{k+1}} = \left[\mathbf{I} - T\left(\mathbf{J_k} - \mathbf{R_k}\right)\Phi_k\right]^{-1}\left[\mathbf{I} + T\left(\mathbf{J_k} - \mathbf{R_k}\right)\Phi_k\right]\mathbf{x_k}.
\end{aligned}$$

Above, explicit calculations are given to provide the reader better insight.

Likewise, the discrete gradient proposed in Definition 4 is obtained as follows.

$$\begin{aligned}
\Phi_k = \Phi(x_k, x_{k+1}) \quad &= \quad \tfrac{1}{2}Q\,(x)|_{\frac{x_{k+1}+x_k}{2}} = \tfrac{1}{2}Q\,\left(\tfrac{x_{k+1}+x_k}{2}\right)\Big|\\
&= \begin{bmatrix} 1 & x_1(k+1) + x_1(k) & 0 & 0 \\ \frac{x_1(k+1)+x_1(k)}{2} & 1 - \left(\frac{x_1(k+1)+x_1(k)}{2}\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}\bar{\nabla}H(x_k,x_{k+1}) &= \Phi_k\ [x_{k+1}+x_k] \\ &= \begin{bmatrix} 1 & x_1(k+1)+x_1(k) & 0 & 0 \\ \frac{x_1(k+1)+x_1(k)}{2} & 1-\left(\frac{x_1(k+1)+x_1(k)}{2}\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k+1)+x_1(k) \\ x_2(k+1)+x_2(k) \\ x_3(k+1)+x_3(k) \\ x_4(k+1)+x_4(k) \end{bmatrix} \\ &= \begin{bmatrix} x_1(k+1)+x_1(k)+(x_1(k+1)+x_1(k))\,(x_2(k+1)+x_2(k)) \\ \frac{(x_1(k+1)+x_1(k))^2}{2}+\left(1-\left(\frac{x_1(k+1)+x_1(k)}{2}\right)\right)(x_2(k+1)+x_2(k)) \\ x_3(k+1)+x_3(k) \\ x_4(k+1)+x_4(k) \end{bmatrix}\end{aligned}$$

Similar calculations can be performed as the one for the previous discrete gradient to obtain the corresponding discrete-time model, since this discrete gradient yields a discrete-time model in the structure of Eqs. (30) and (32).

In order to construct the discrete gradient in Definition 5, it is necessary to calculate the Hessian matrix of the energy function, which is:

$$H_{Hess} = \begin{bmatrix} 1+2x_2 & 2x_1 & 0 & 0 \\ 2x_1 & 1-2x_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Substituting

$$H_{Hess}(x_k) = \begin{bmatrix} 1+2x_2(k) & 2x_1(k) & 0 & 0 \\ 2x_1(k) & 1-2x_2(k) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \nabla H(x_k) = \begin{bmatrix} x_1(k)+2x_1(k)x_2(k) \\ x_2(k)+x_1^2(k)-x_2^2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} \tag{32}$$

into Eq. (25) yields

$$\begin{aligned}\overline{\nabla}H(x_k,x_{k+1}) &= \nabla H(x_k)+\tfrac{1}{2}H_{Hess}(x_k)\,(x_{k+1}-x_k) \\ &= \begin{bmatrix} x_1(k)+2x_1(k)x_2(k) \\ x_2(k)+x_1^2(k)-x_2^2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1+2x_2(k) & 2x_1(k) & 0 & 0 \\ 2x_1(k) & 1-2x_2(k) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k+1)+x_1(k) \\ x_2(k+1)+x_2(k) \\ x_3(k+1)+x_3(k) \\ x_4(k+1)+x_4(k) \end{bmatrix} \\ &= \begin{bmatrix} x_1(k)+\frac{1}{2}x_2(k+1)+\frac{1}{2}x_1(k+1)+4x_1(k)x_2(k)+x_1(k)x_2(k+1)+x_1(k+1)x_2(k) \\ \frac{3}{2}x_2(k)+\frac{1}{2}x_2(k+1)+2x_1^2(k)-2x_2^2(k)+x_1(k)x_1(k+1)+x_2(k+1)x_2(k)+ \\ \frac{1}{2}x_3(k+1)+\frac{3}{2}x_3(k) \\ \frac{1}{2}x_4(k+1)+\frac{3}{2}x_4(k) \end{bmatrix}.\end{aligned}$$

The corresponding discrete-time model is obtained by substituting the expressions given in Eq. (32) for the Hessian matrix and the gradient of the energy function calculated at time instance k in the system equations:

$$x_{k+1} = x_k + T\left[I - \frac{T}{2}\,(J_k - R_k)\,H_{Hess}(x_k)\right]^{-1}(J_k - R_k)\,\nabla H(x_k),$$

where $R_k = 0_{2x2}$ and $T = 0.002$.

Example 3 (double pendulum system)

The double pendulum system shown in Figure 5 is defined with the total energy

$$H(\mathbf{q},\mathbf{p}) = \frac{1}{2}\mathbf{p^T M^{-1}(q)p} + V(\mathbf{q}),$$

where

$$\mathbf{M} = \begin{bmatrix} l_1^2(m_1+m_2) & m_2 l_1 l_2 cos(q_1-q_2) \\ m_2 l_1 l_2 cos(q_1-q_2) & l_2^2 m_2 \end{bmatrix},$$

$$V(\mathbf{q}) = -(m_2 l_1 l_2 cos q_2 + (m_1+m_2) g l_1 cos q_1),$$

with the following dissipation structure matrix:

$$\mathbf{R} = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 I_2 \end{bmatrix}.$$

The system parameters are taken as $m_1 = m_2 = 1$ kg, $l_1 = 0.2$m, $l_2 = 0.3$m, and $g = 0.98\ \mathrm{ms}^{-1}$, and the variables are assigned as $q_1 = \theta_1, q_2 = \theta_2$. The simulation is carried out with a sampling time of $T = 0.005$s and the initial state of $x_0 = [0.5;\ 0.3;\ 0.005;\ 0.005]$. The results are illustrated in Figures 6–8.

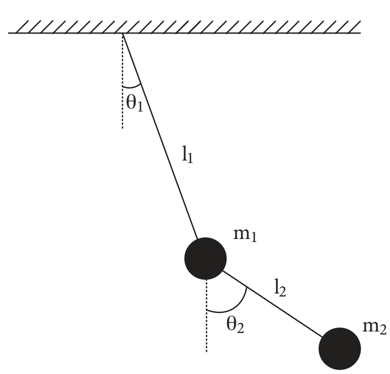

Figure 5. The double pendulum system.

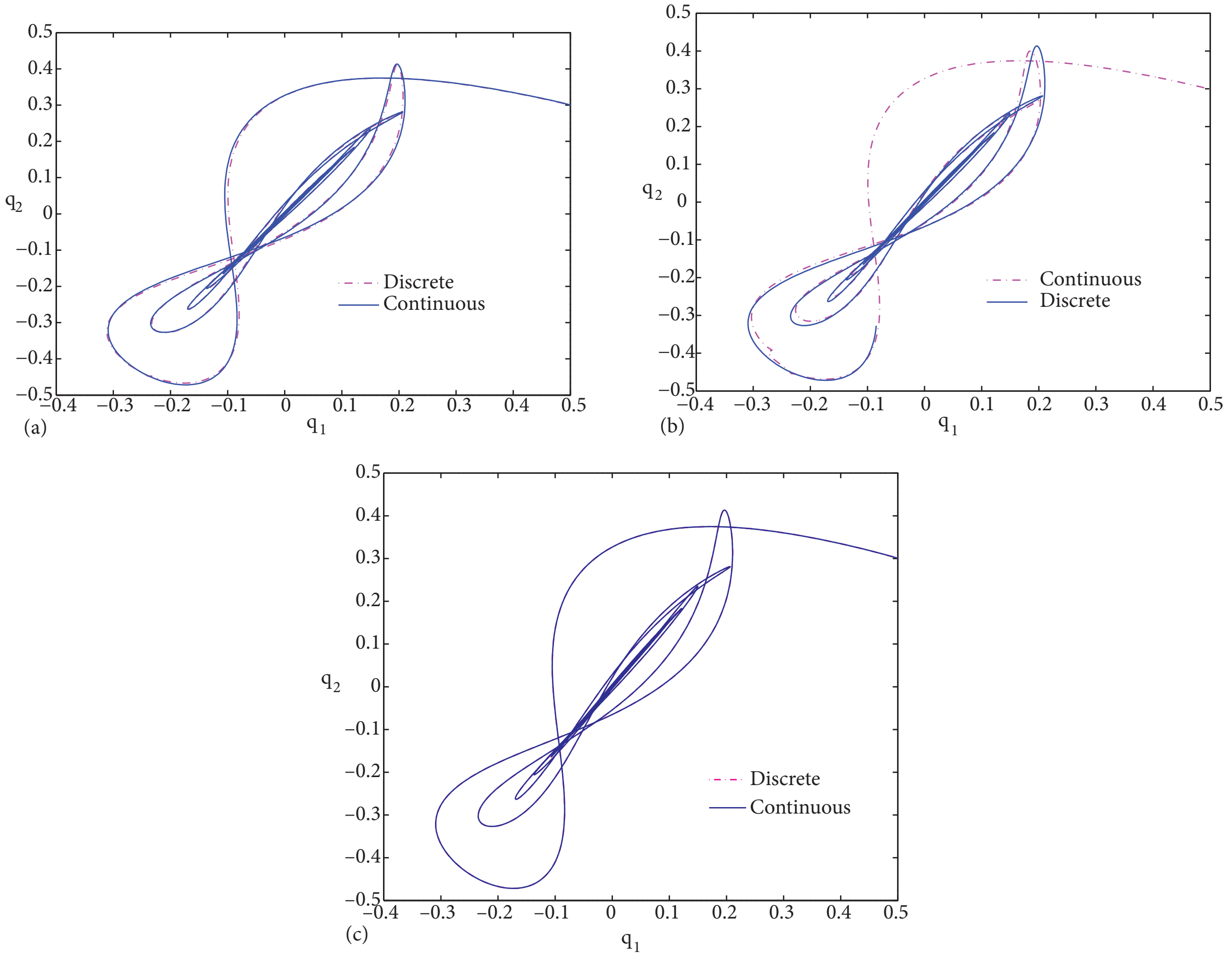

Figure 6. Phase portrait of the discrete and continuous $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ variables of the double pendulum system for the discrete gradient given in Definition 3 (a), Definition 4 (b), and Definition 5 (c).

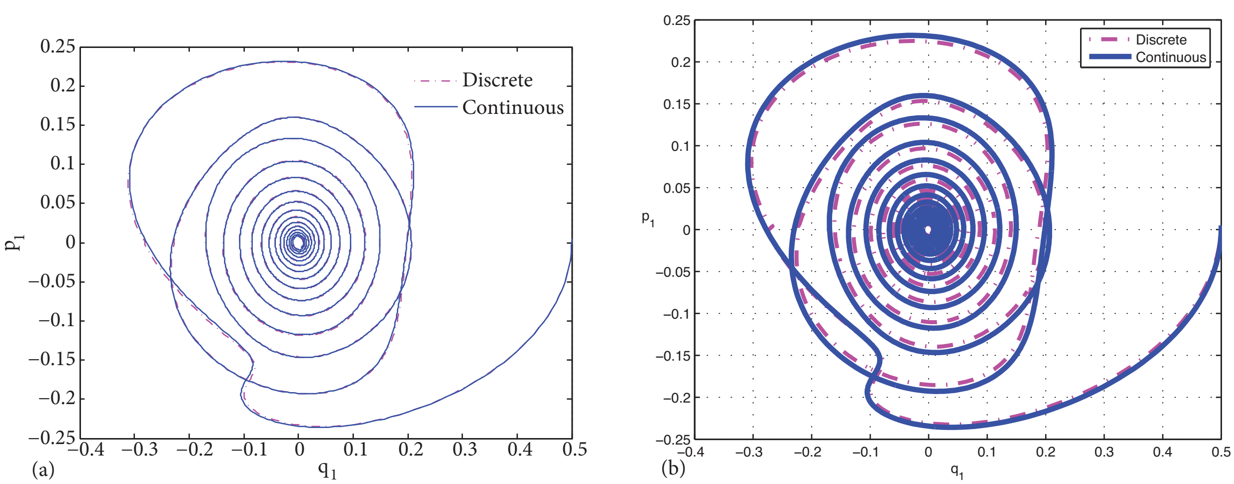

Figure 7. Phase portrait of the discrete and continuous $\boldsymbol{q}_1$ and $\boldsymbol{p}_1$ variables of the double pendulum systems for the discrete gradient given in Definition 3 (a), and Definition 4 (b).

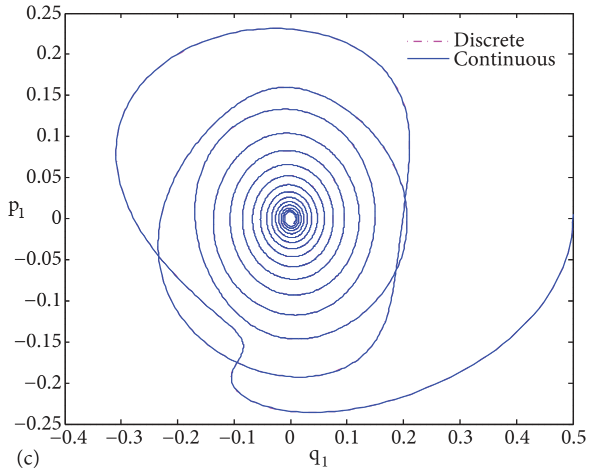

Figure 7. Phase portrait of the discrete and continuous $\boldsymbol{q}_1$ and $\boldsymbol{p}_1$ variables of the double pendulum systems for the discrete gradient given in Definition 5 (c).

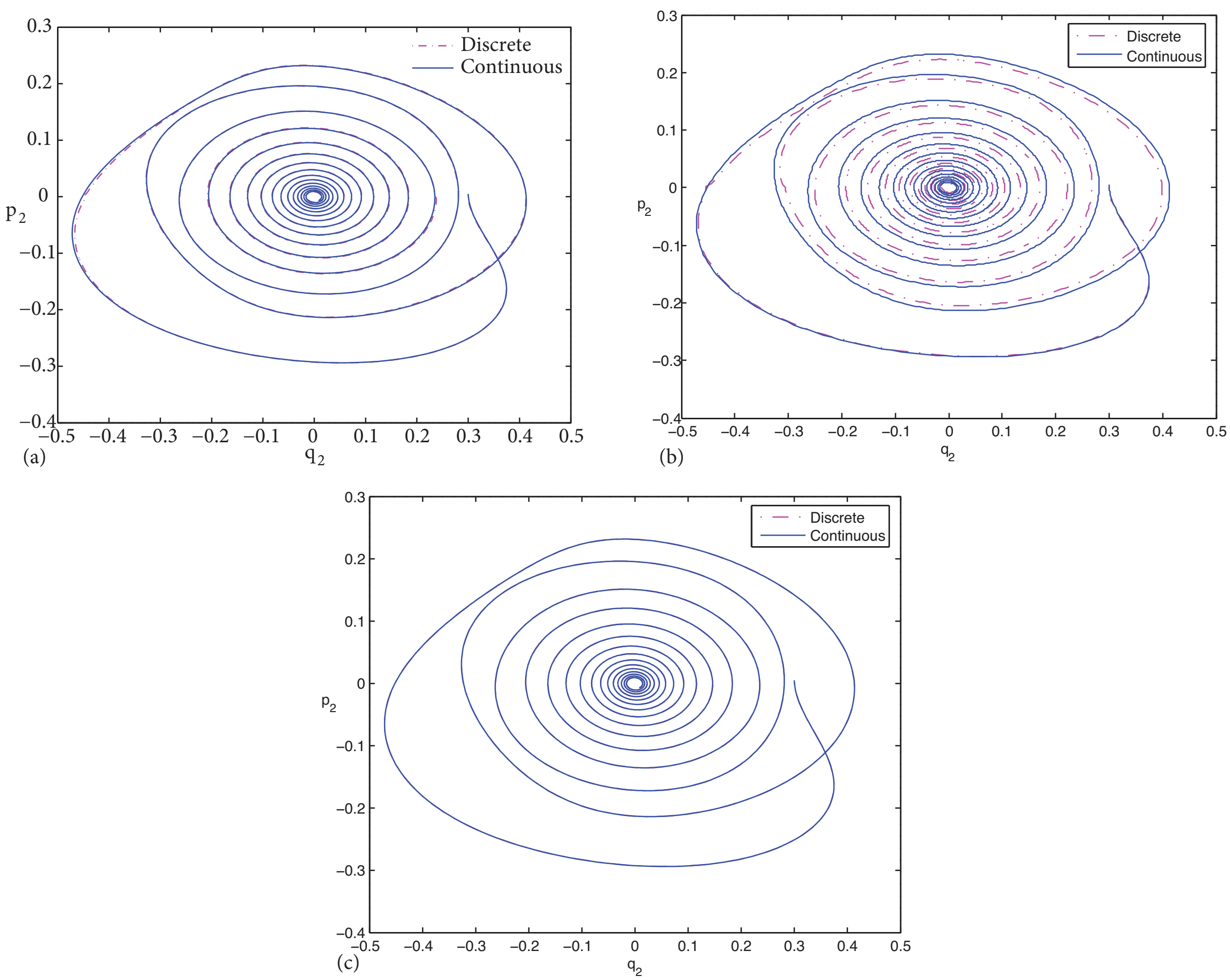

Figure 8. Phase portrait of the discrete and continuous $\boldsymbol{q}_2$ and $\boldsymbol{p}_2$ variables of the double pendulum system for the discrete gradient given in Definition 3 (a), Definition 4 (b), and Definition 5 (c).

Example 4 (cart and pendulum)

We also consider the cart and pendulum system shown in Figure 9 with

$$H(\mathbf{q},\mathbf{p}) = \frac{1}{2}\mathbf{p}^{\mathbf{T}}\mathbf{M}^{-\mathbf{1}}(\mathbf{q})\mathbf{p} + V(\mathbf{q}),$$

where

$$\mathbf{M} = \begin{bmatrix} ml^2 & mlcos(q_1) \\ mlcos(q_1) & M_s + m \end{bmatrix},$$

$$V(\mathbf{q}) = mglcos(q_1),$$

where $q_1 = \theta$ is the pendulum angle from its upright position and $q_2 = s$ is the cart position. We assume that the system has a dissipation structure as follows:

$$\mathbf{R} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R_1} \end{bmatrix} \quad , \quad \mathbf{R_1} = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}.$$

The simulation results for system parameters $L = 0.2\,\text{m}$, $M_s = 0.15\,\text{kg}$, $m = 0.45\,\text{kg}$, $g = 9.8\,\text{ms}^{-1}$, $K_1 = 0.02$, $K_2 = 0.01$, initial condition $x_0 = [0.7;\ 2.0;\ 0.005;\ 0.03]$, and sampling time $T = 0.005\,\text{s}$ are given in Figures 10–12.

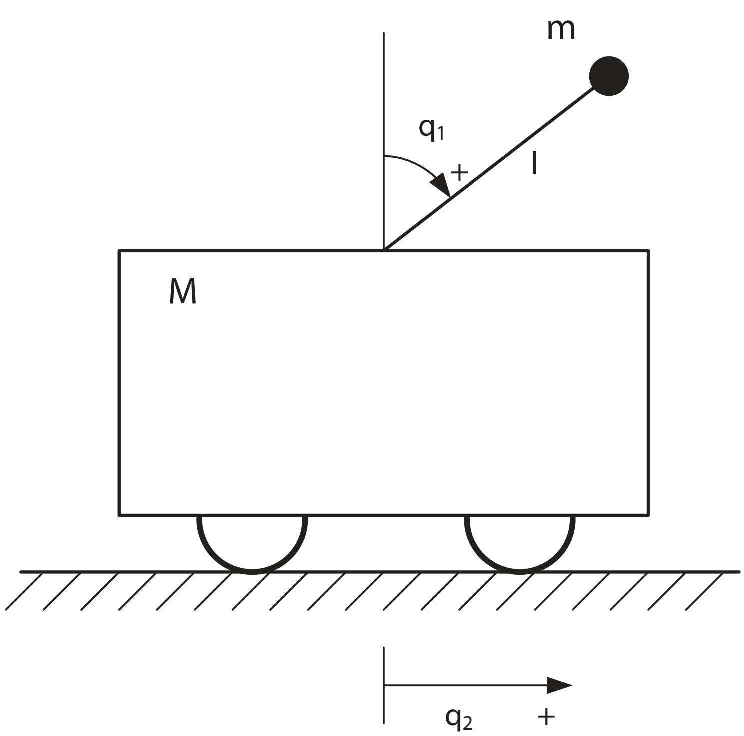

Figure 9. The cart and pendulum system.

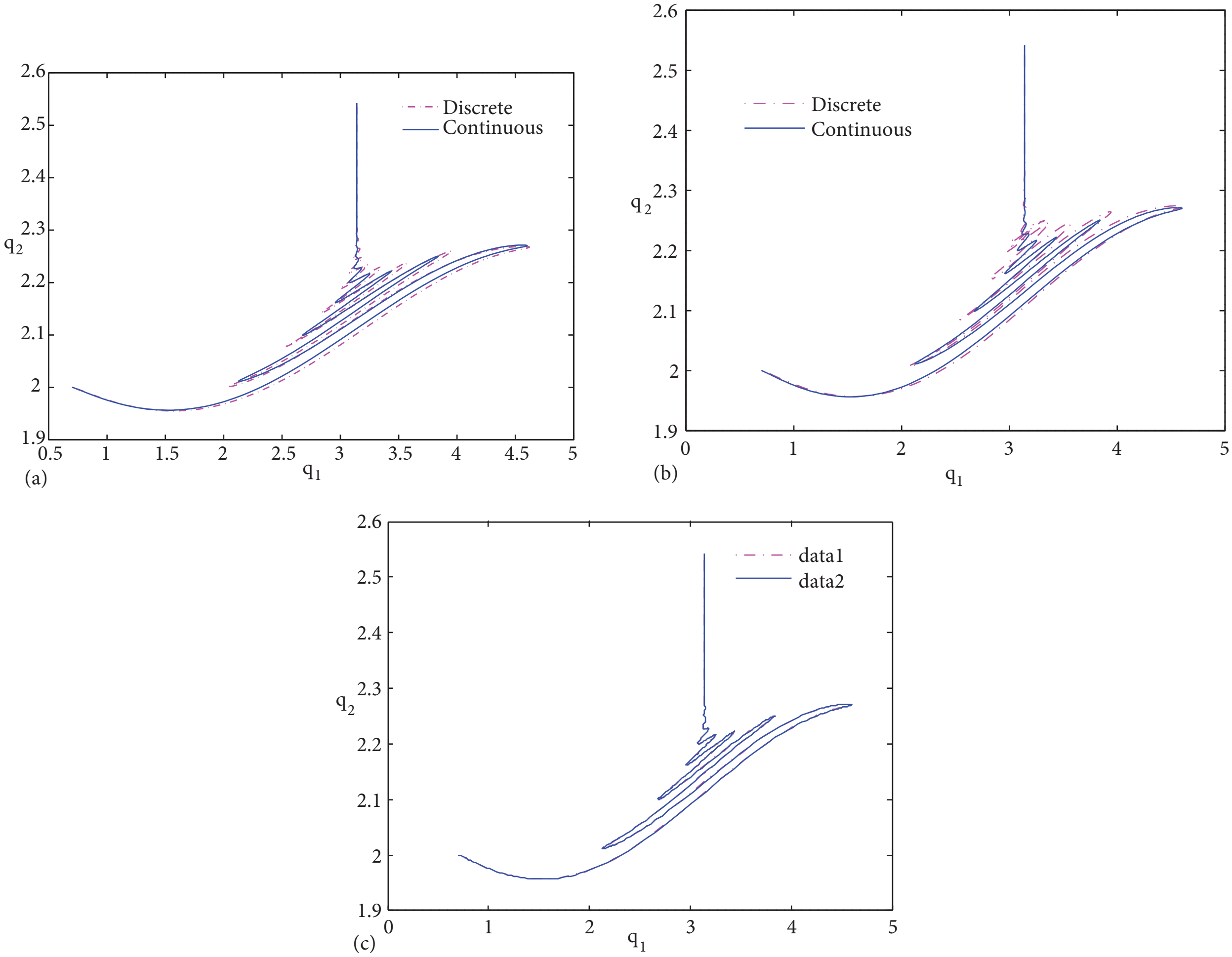

Figure 10. Phase portrait of the discrete and continuous $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ variables of the double pendulum for the discrete gradient given in Definition 3 (a), Definition 4 (b), and Definition 5 (c).

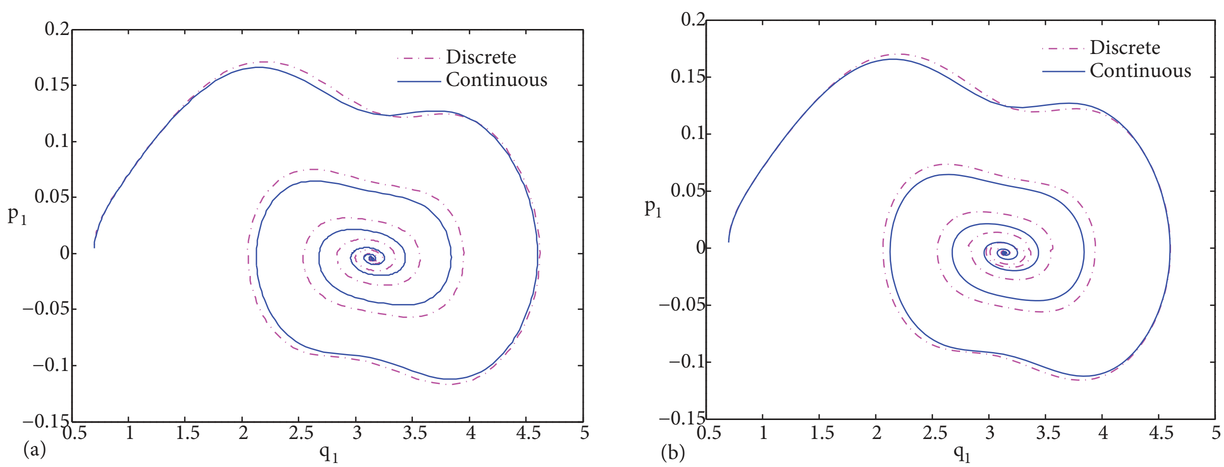

Figure 11. Phase portrait of the discrete and continuous $\boldsymbol{q}_1$ and $\boldsymbol{p}_1$ variables of the double pendulum for the discrete gradient given in Definition 3 (a), and Definition 4 (b).

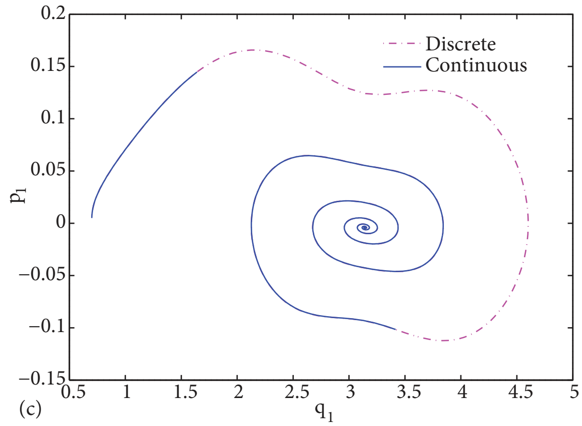

Figure 11. Phase portrait of the discrete and continuous $\boldsymbol{q}_1$ and $\boldsymbol{p}_1$ variables of the double pendulum for the discrete gradient given in Definition 5 (c).

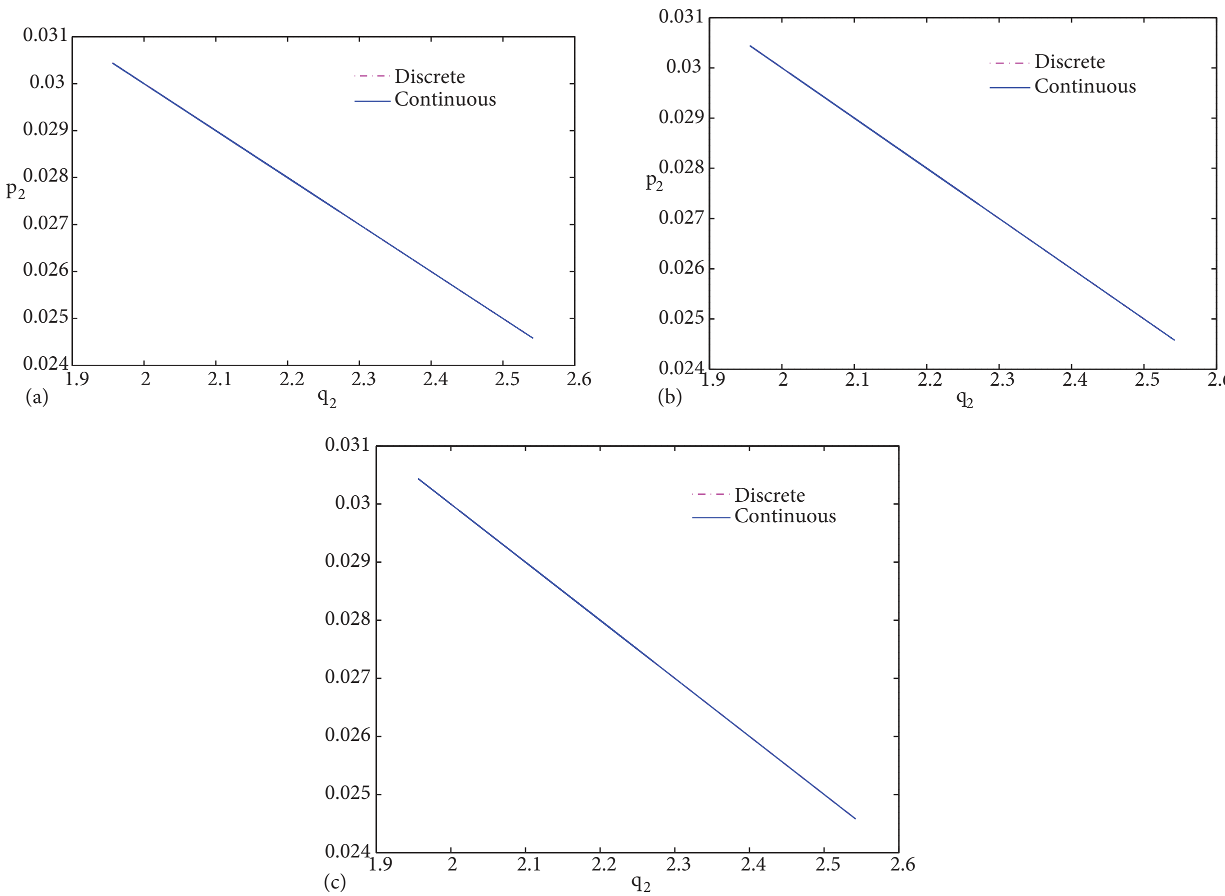

Figure 12. Phase portrait of the discrete and continuous $\boldsymbol{q}_2$ and $\boldsymbol{p}_2$ variables of the double pendulum for the discrete gradient given in Definition 3 (a), Definition 4 (b), and Definition 5 (c).

5. Conclusions

A gradient-based method was presented to achieve the discrete-time modeling of lumped-parameter Hamiltonian systems, which is convenient to use in engineering applications. The models obtained using the proposed method have an energy conservation property whenever the discrete gradient of $H(\mathbf{x})$, i.e. $\bar{\nabla}H$, satisfies the conditions in Definition 2, and they define the symplectic difference systems when the structure matrices are in the form of standard skew-symmetric matrix. The discrete gradients used also have the special form given in Eq. (14) in general. As is already known, preserving the symplectic structure and the energy conservation property simultaneously is impossible for the general case [11,12]. The first 2 discrete gradients presented here, Definitions 3 and 4, were defined under the restriction of $\nabla\mathbf{H}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})\,\mathbf{x}$. These 2 slightly different discrete gradients, which provide models in the same form as in Eq. (15), were proposed with motivation from the mean value theorem and midpoint concept. The properties of these models were given in Remark 2. Moreover, a novel discrete gradient definition was offered by taking inspiration from the quadratic approximation lemma. The related discrete model was given and the analysis of the properties of this model was done. It should be noted that all of the models proposed are equivalent when the energy function of the Hamiltonian system is in the form of $\mathbf{H}(\mathbf{x}) = \mathbf{x}^{\mathbf{T}}\mathbf{Z}(\mathbf{x})\,\mathbf{x}$ with $\mathbf{Z}(\mathbf{x}) = \mathbf{Z}$, namely a constant $n \times n$ matrix. As the expressions are handy and quite simple in structure, the proposed discrete-time port-Hamiltonian models in the form of Eq. (27) were successfully used utilizing the 3 proposed discrete gradients, where the explicit models are given by Eq. (19) with (21) or (23) and (26) in [14–18], and, specifically, [17] presents results of a real application of an overhead crane system. The simulation results demonstrate that the proposed discrete-time models satisfactorily represent the continuous time dynamics of the considered class of systems and they are convenient to use for engineering application purposes.

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